

ESE 531: Digital Signal Processing

Lec 19: April 7, 2020
Fast Fourier Transform




Last Time

- ❑ Discrete Fourier Transform
 - Linear convolution through circular convolution
 - Overlap and add
 - Overlap and save
 - Circular convolution through DFT

- ❑ Today
 - The Fast Fourier Transform

Fast Fourier Transform (FFT)



Fast Fourier Transform Algorithms

- We are interested in efficient computing methods for the DFT and inverse DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, \dots, N-1$$

$$x[n] = \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad n = 0, \dots, N-1$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}.$$



Reminder: Inverse DFT via DFT

- Recall that we can use the DFT to compute the inverse DFT:

$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N} (\mathcal{DFT}\{X^*[k]\})^*$$

- Hence, we can just focus on efficient computation of the DFT.
- Straightforward computation of an N-point DFT (or inverse DFT) requires N^2 complex multiplications.

Computation Order

- ❑ *Fast Fourier transform algorithms* enable computation of an N -point DFT (or inverse DFT) with the order of just $N \cdot \log_2 N$ complex multiplications.
 - This can represent a huge reduction in computational load, especially for large N .

N	N^2	$N \cdot \log_2 N$	$\frac{N^2}{N \cdot \log_2 N}$
16	256	64	4.0
128	16,384	896	18.3
1,024	1,048,576	10,240	102.4
8,192	67,108,864	106,496	630.2

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16	256	64	4.0
128	16,384	896	18.3
1,024	1,048,576	10,240	102.4
8,192	67,108,864	106,496	630.2
6×10^6	36×10^{12}	135×10^6	2.67×10^5

* 6Mp image size



Eigenfunction Properties

- ❑ Most FFT algorithms exploit the following properties of W_N^{kn} :

- Conjugate Symmetry

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$$

- Periodicity in n and k

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

- Power

$$W_N^2 = W_{N/2}$$

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 - Conjugate Symmetry

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$$

$$\begin{aligned} W_N^{k(N-n)} &= \left(e^{-j\frac{2\pi}{N}} \right)^{k(N-n)} \\ &= \left(e^{-j\frac{2\pi}{N}(N-n)} \right)^k = \left(e^{-j2\pi} e^{j\frac{2\pi}{N}n} \right)^k \\ &= \left(e^{j\frac{2\pi}{N}} \right)^{nk} = \left(W_N^* \right)^{nk} = \left(W_N^{nk} \right)^* \end{aligned}$$



Eigenfunction Properties

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 - Periodicity in n and k

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$


$$\begin{aligned} W_N^{kn} &= \left(e^{-j\frac{2\pi}{N}} \right)^{kn} \left(e^{-j\frac{2\pi}{N}} \right)^{kN} = \left(e^{-j\frac{2\pi}{N}} \right)^{kn} \left(e^{-j\frac{2\pi}{N}} \right)^{nN} \\ &= W_N^{kn} = W_N^{kn} \end{aligned}$$

Eigenfunction Properties

- Most FFT algorithms exploit the following properties of W_N^{kn} :
 - Power

$$W_N^2 = W_{N/2}$$


$$\begin{aligned} W_N^2 &= \left(e^{-j\frac{2\pi}{N}} \right)^2 = \left(e^{-j\frac{2\pi}{N} \times 2} \right) = \left(e^{-j\frac{2\pi}{N/2}} \right) \\ &= W_{N/2} \end{aligned}$$



FFT Algorithms via Decimation

- ❑ Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
 - Decimation-in-time algorithms decompose $x[n]$ into successively smaller subsequences.
 - Decimation-in-frequency algorithms decompose $X[k]$ into successively smaller subsequences.


- ❑ Note: Assume length of $x[n]$ is power of 2 ($N = 2^v$). If not, zero-pad to closest power of 2.



Decimation-in-Time FFT

- We start with the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, \dots, N-1$$




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- Separate the sum into even and odd terms:

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$



Decimation-in-Time FFT


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- These are two DFTs, each with half the number of samples ($N/2$)



Decimation-in-Time FFT

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):




Decimation-in-Time FFT

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$



Decimation-in-Time FFT

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{2rk} \end{aligned}$$

Decimation-in-Time FFT

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Note that:

$$W_N^{2rk} = e^{-j\left(\frac{2\pi}{N}\right)(2rk)} = e^{-j\left(\frac{2\pi}{N/2}\right)rk} = W_{N/2}^{rk}$$

Remember this trick, it will turn up often.

Decimation-in-Time FFT

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{2rk} \end{aligned}$$

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk}$$


Decimation-in-Time FFT

Let $n = 2r$ (n even) and

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, \dots, N-1$$

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{2rk} \end{aligned}$$

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Decimation-in-Time FFT

Hence:

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \\ &\triangleq G[k] + W_N^k H[k], \quad k = 0, \dots, N-1 \end{aligned}$$

Decimation-in-Time FFT

Hence:

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \\ &\triangleq G[k] + W_N^k H[k], \quad k = 0, \dots, N-1 \end{aligned}$$

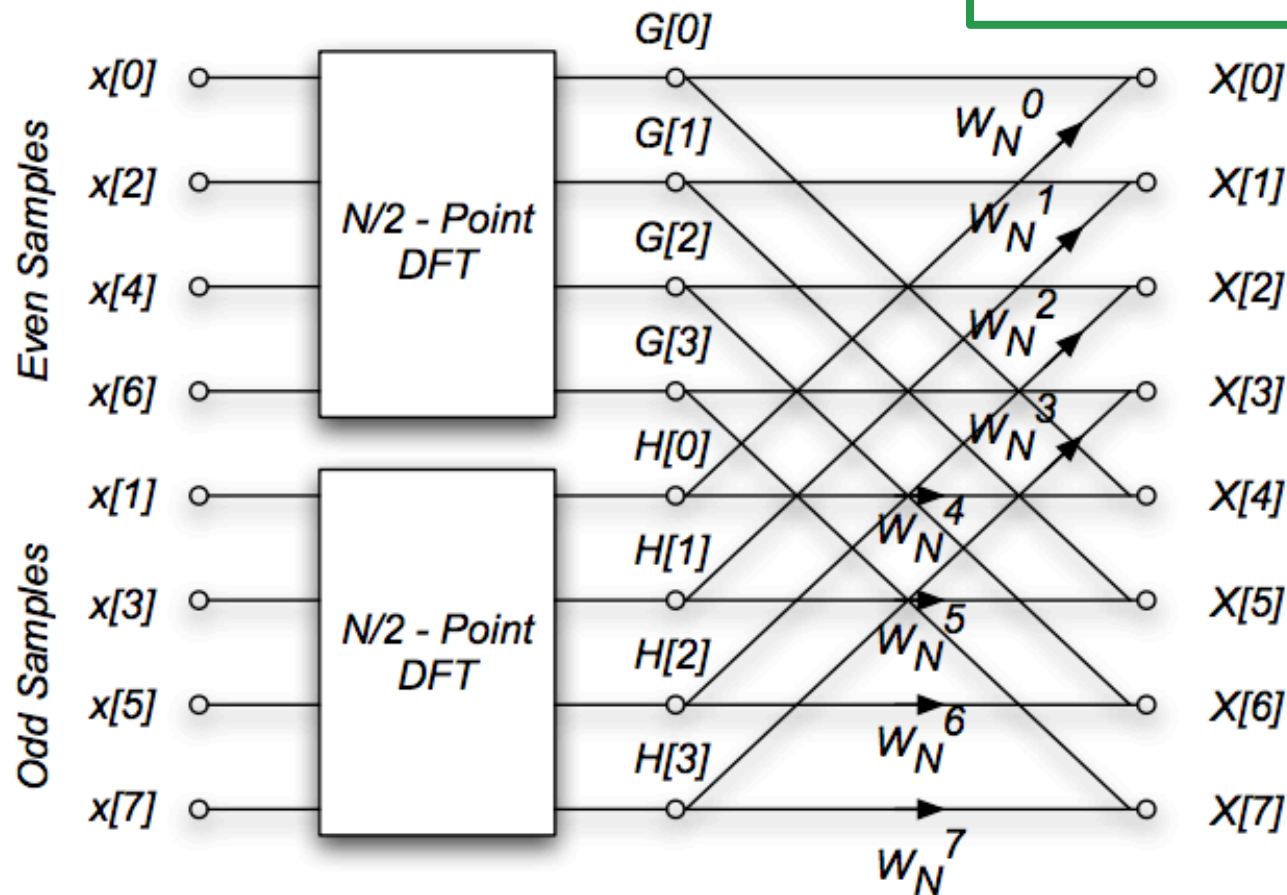
where we have defined:

$$\begin{aligned} G[k] &\triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} && \Rightarrow \text{DFT of even samples} \\ H[k] &\triangleq \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} && \Rightarrow \text{DFT of odd samples} \end{aligned}$$

Decimation-in-Time FFT

An 8 sample DFT can then be diagrammed as

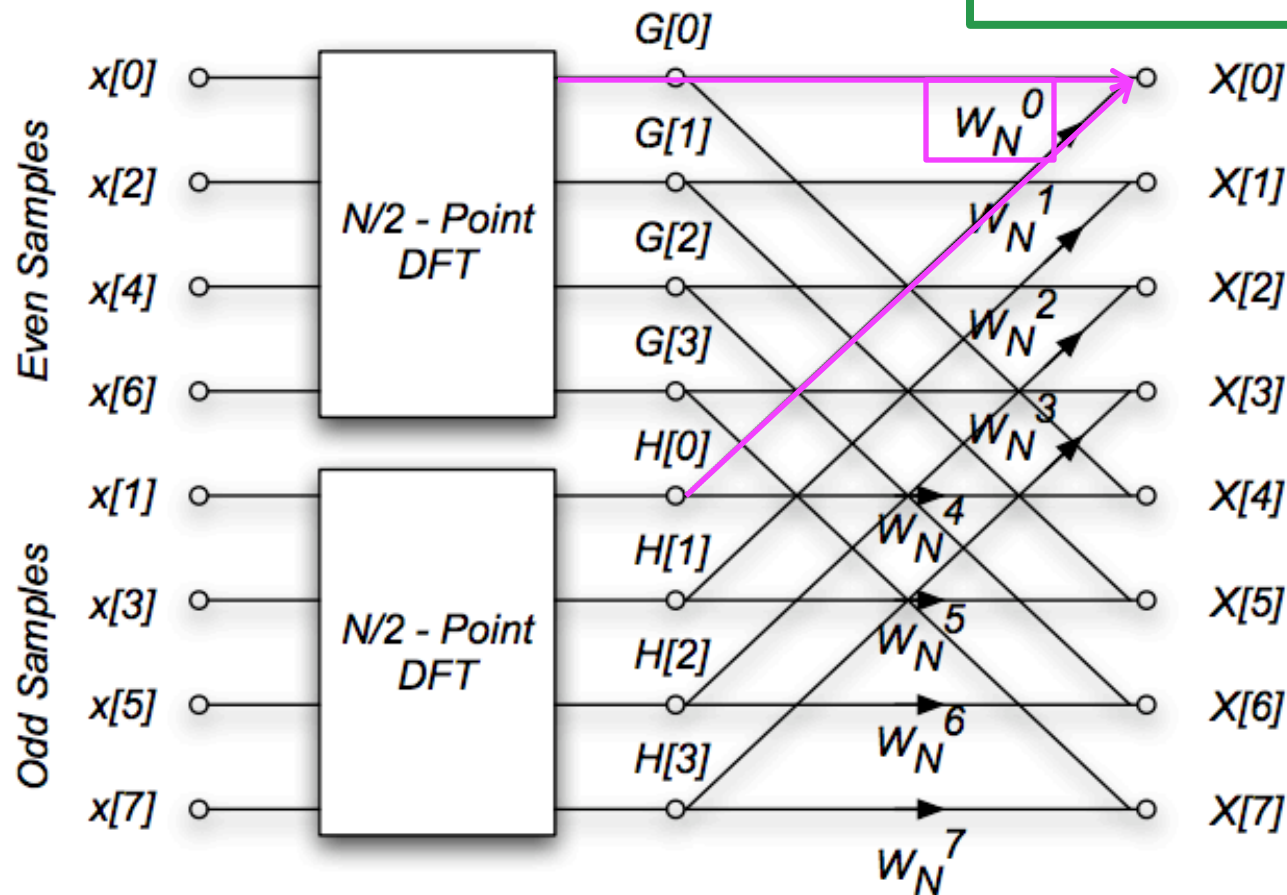
$$X[k] = G[k] + W_N^k H[k]$$



Decimation-in-Time FFT

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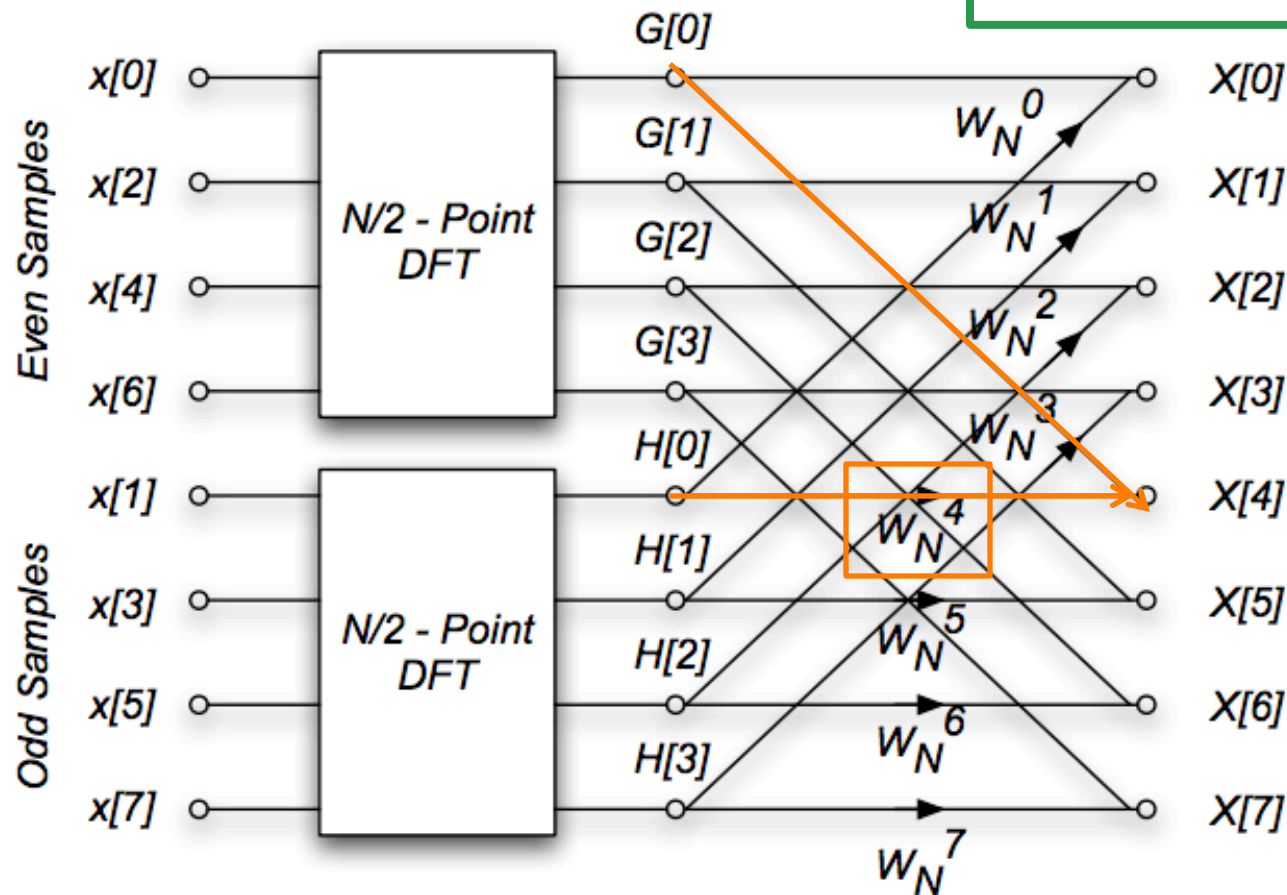


$$X[0] = G[0] + W_N^0 H[0]$$

Decimation-in-Time FFT


An 8 sample DFT can then be diagrammed as

$$X[k] = G[k] + W_N^k H[k]$$



??


$$X[4] = G[4] + W_N^4 H[4]$$



Decimation-in-Time FFT

Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk}$$




Decimation-in-Time FFT

Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk}$$

$$G[k + N/2] = \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)}$$



Decimation-in-Time FFT


Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$\begin{aligned} G[k] &\triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \\ G[k + N/2] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} W_{N/2}^{r(N/2)} \end{aligned}$$

Decimation-in-Time FFT

Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$\begin{aligned} G[k] &\triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \\ G[k + N/2] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} W_{N/2}^{r(N/2)} = 1 \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \\ &= G[k] \end{aligned}$$



Decimation-in-Time FFT

□ So,

$$\begin{aligned}G[k + (N/2)] &= G[k] \\H[k + (N/2)] &= H[k]\end{aligned}$$

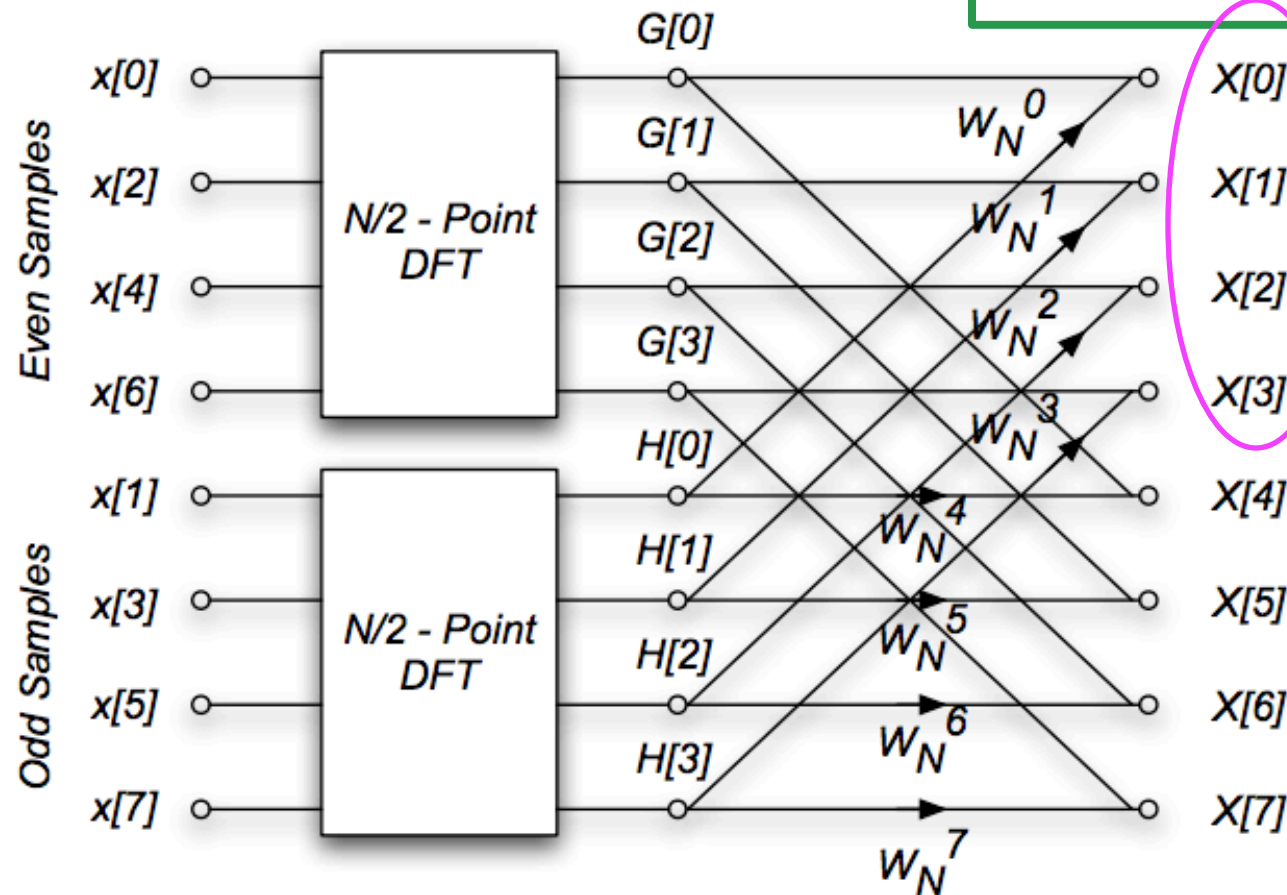
□ The periodicity of $G[k]$ and $H[k]$ allows us to further simplify. For the first $N/2$ points we calculate $G[k]$ and $W_N^k H[k]$, and then compute the sum

$$X[k] = G[k] + W_N^k H[k] \quad \forall \{k : 0 \leq k < \frac{N}{2}\}.$$

Decimation-in-Time FFT

An 8 sample DFT can then be diagrammed as

$$X[k] = G[k] + W_N^k H[k]$$



First $N/2$ DFTs
 $0 \leq k < N/2$

Decimation-in-Time FFT


□ So,

$$\begin{aligned}G[k + (N/2)] &= G[k] \\H[k + (N/2)] &= H[k]\end{aligned}$$

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How does periodicity help for $\frac{N}{2} \leq k < N$?



Decimation-in-Time FFT

$$X[k] = G[k] + W_N^k H[k] \quad \forall \{k : 0 \leq k < \frac{N}{2}\}.$$

for $\frac{N}{2} \leq k < N$:

$$W_N^{k+(N/2)} = ?$$

$$X[k + (N/2)] = ?$$

Decimation-in-Time FFT

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for $\frac{N}{2} \leq k < N$:

$$W_N^{k+N/2} = \left(e^{-j\frac{2\pi}{N}} \right)^{k+N/2}$$

$$W_N^{k+(N/2)} = ?$$

$$X[k + (N/2)] = ?$$

Decimation-in-Time FFT

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for $\frac{N}{2} \leq k < N$:

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$$\begin{aligned} W_N^{k+N/2} &= \left(e^{-j\frac{2\pi}{N}} \right)^{k+N/2} \\ &= \left(e^{-j\frac{2\pi}{N}k} \right) \left(e^{-j\frac{2\pi}{N} \times \frac{N}{2}} \right) \end{aligned}$$

$$X[k + (N/2)] = ?$$

Decimation-in-Time FFT


$$X[k] = G[k] + W_N^k H[k] \quad \forall \{k : 0 \leq k < \frac{N}{2}\}.$$

for $\frac{N}{2} \leq k < N$:

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$$\begin{aligned} W_N^{k+N/2} &= \left(e^{-j\frac{2\pi}{N}} \right)^{k+N/2} \\ &= \left(e^{-j\frac{2\pi}{N}k} \right) \left(e^{-j\frac{2\pi}{N} \times \frac{N}{2}} \right) \\ &= \left(e^{-j\frac{2\pi}{N}k} \right) \left(e^{-j\pi} \right) = -W_N^k \end{aligned}$$



Decimation-in-Time FFT

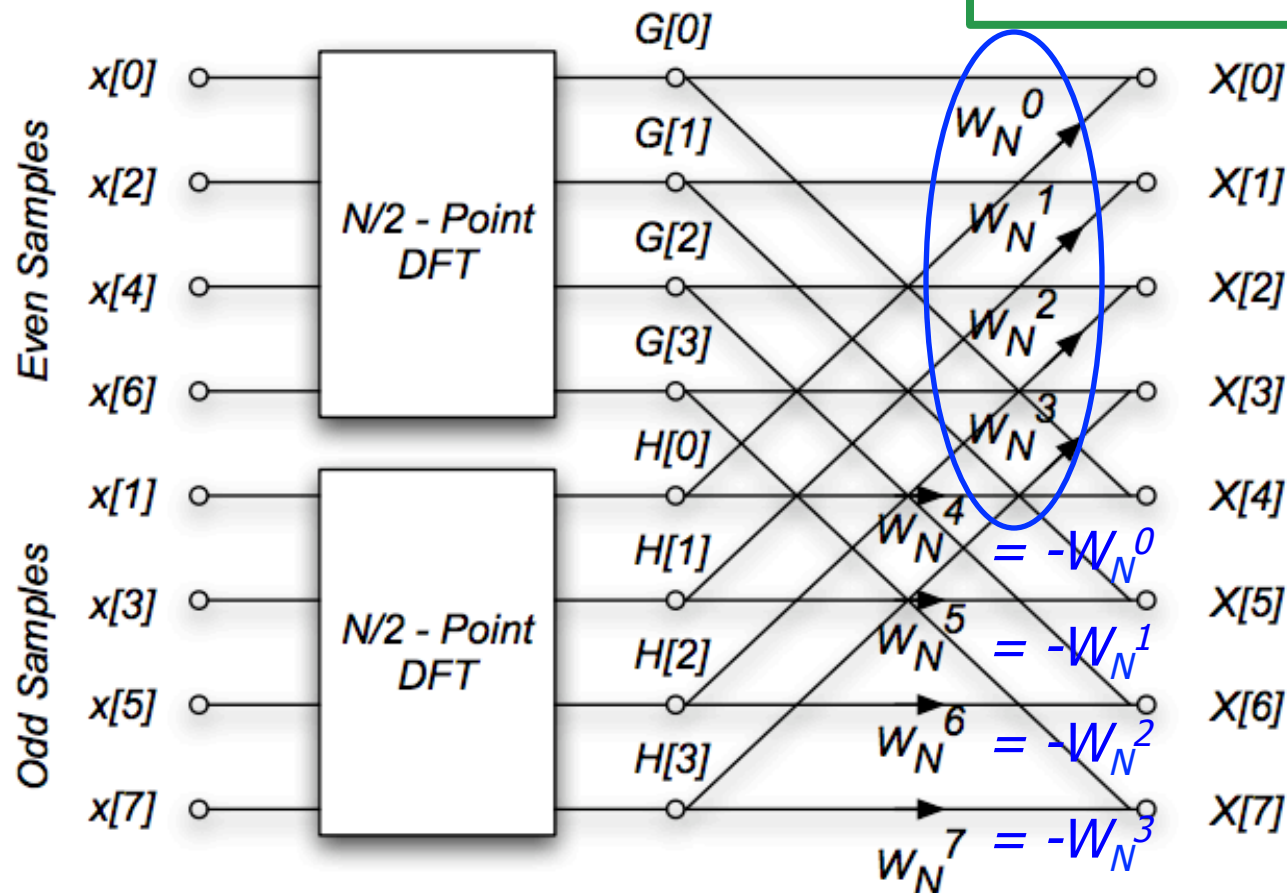
$$X[k + (N/2)] = G[k] - W_N^k H[k]$$

- We previously calculated $G[k]$ and $W_N^k H[k]$.
- Now we only have to compute their difference to obtain the second half of the spectrum. No additional multiplies are required.

Decimation-in-Time FFT

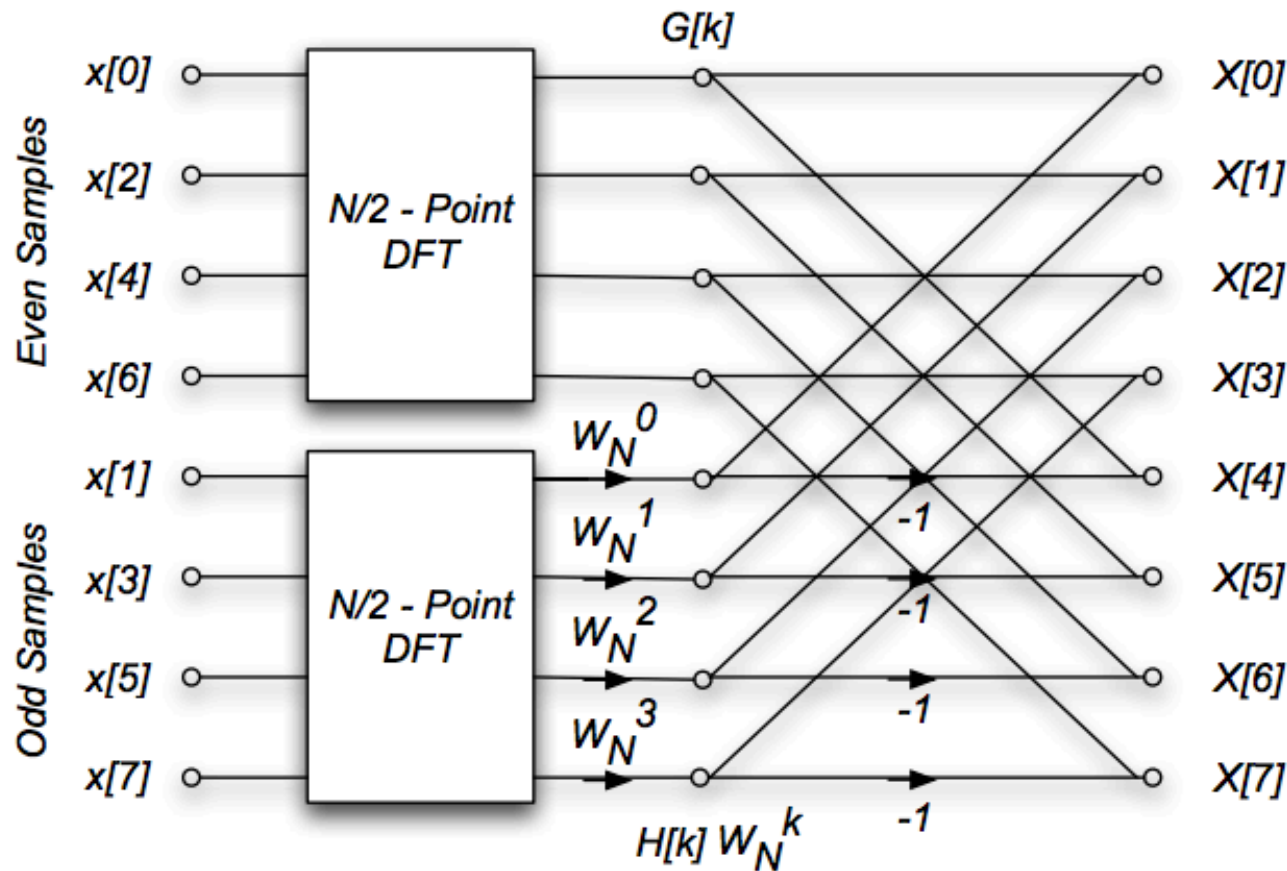
An 8 sample DFT can then be diagrammed as

$$X[k] = G[k] + W_N^k H[k]$$



Decimation-in-Time FFT

The N -point DFT has been reduced to two $N/2$ -point DFTs, plus $N/2$ complex multiplications. The 8 sample DFT is then:



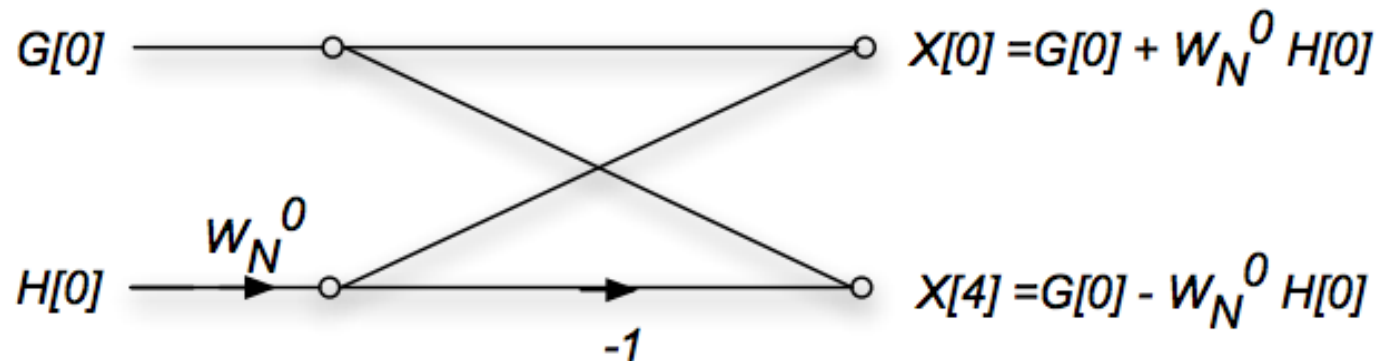


Decimation-in-Time FFT

- Note that the inputs have been reordered so that the outputs come out in their proper sequence.

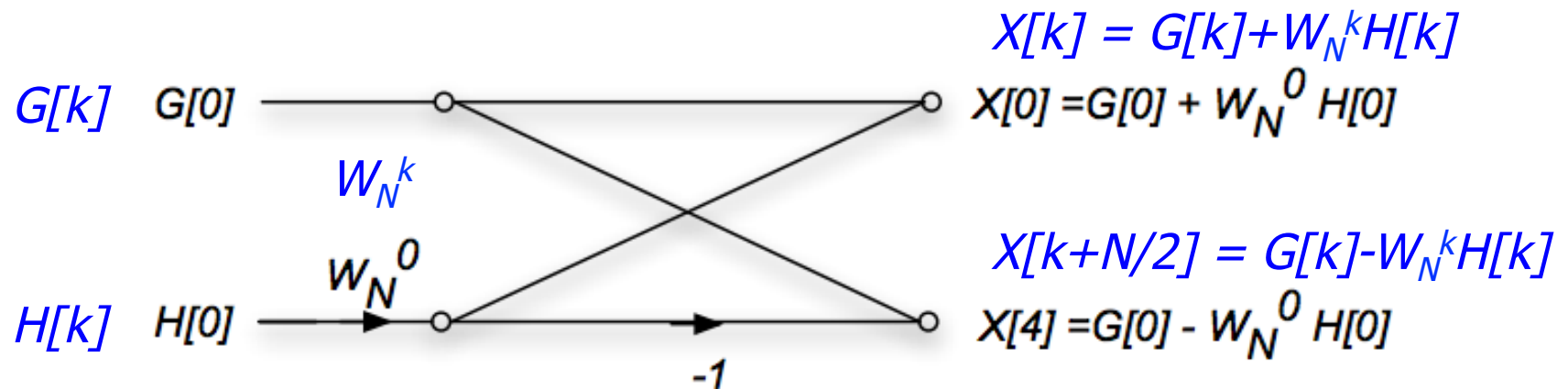
Decimation-in-Time FFT

- Note that the inputs have been reordered so that the outputs come out in their proper sequence.
- We can define a *butterfly operation*, e.g., the computation of $X[0]$ and $X[4]$ from $G[0]$ and $H[0]$:



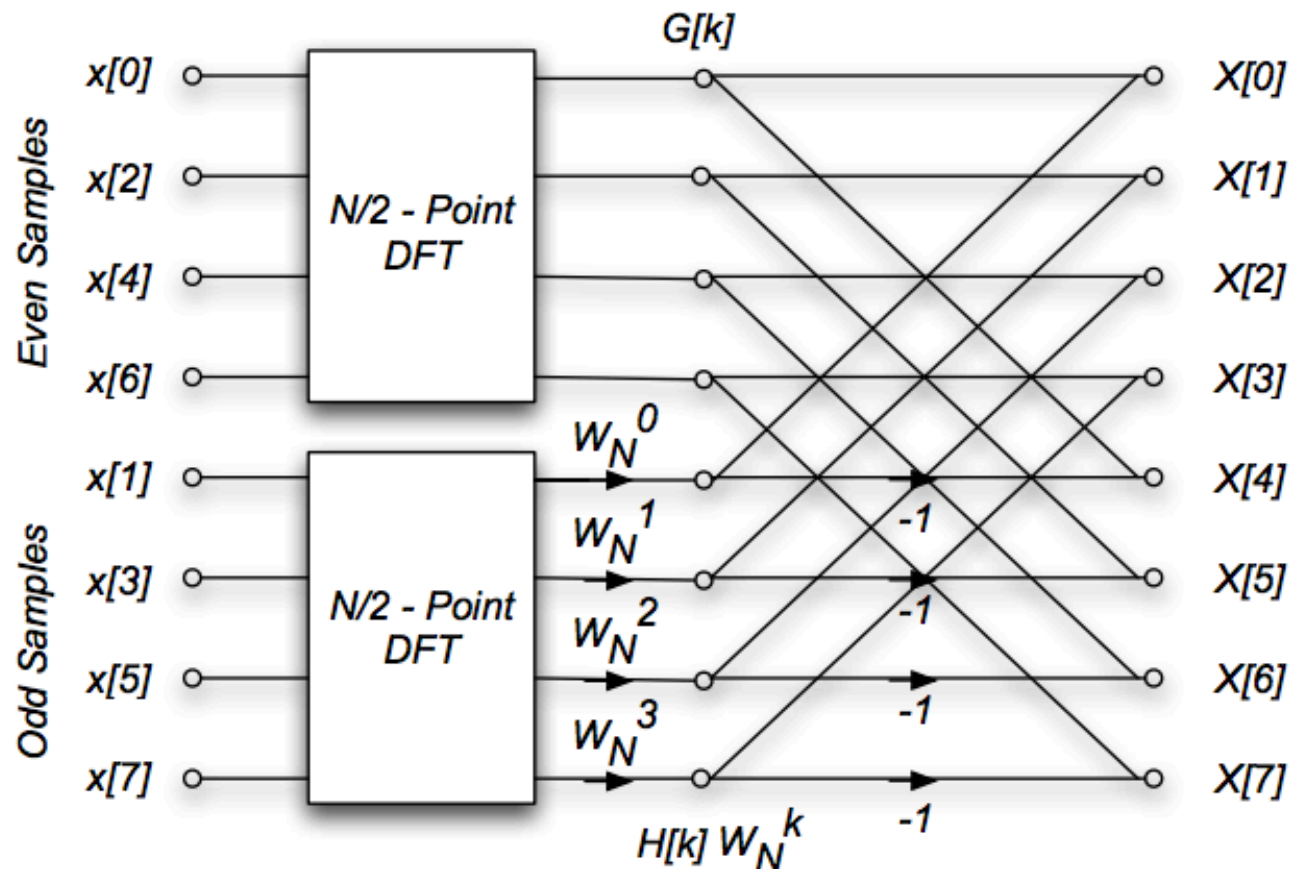
Decimation-in-Time FFT

- Note that the inputs have been reordered so that the outputs come out in their proper sequence.
- We can define a *butterfly operation*, e.g., the computation of $X[k]$ and $X[k+N/2]$ from $G[k]$ and $H[k]$:



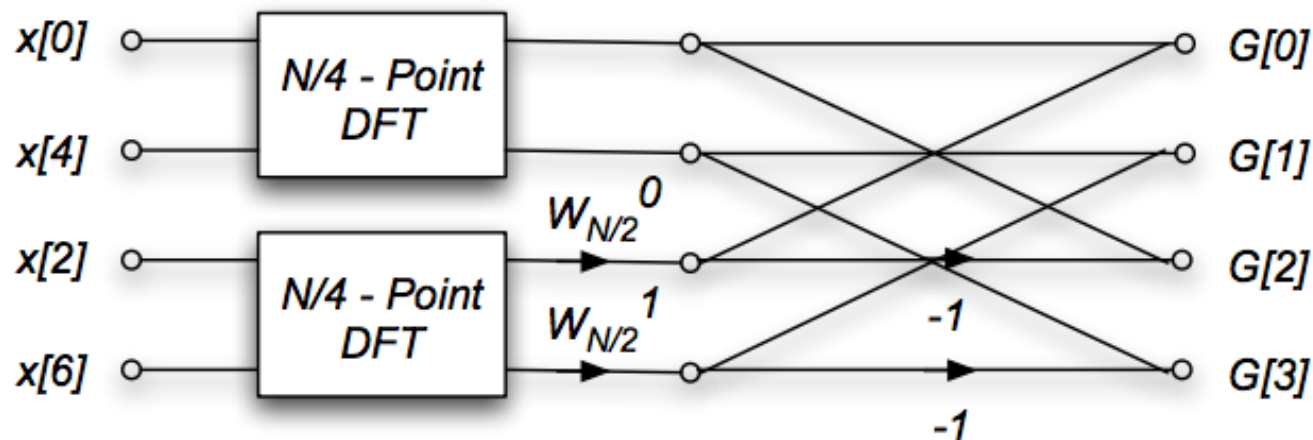
Decimation-in-Time FFT

- Still $O(N^2)$ operations.... What should we do?




Decimation-in-Time FFT

- We can use the same approach for each of the $N/2$ point DFT's. For the $N = 8$ case, the $N/2$ DFTs look like




*Note that the inputs have been reordered again.



Decimation-in-Time FFT

- At this point for the 8 sample DFT, we can replace the $N/4 = 2$ sample DFT's with a single butterfly. The fundamental eigenfunction is:

$$W_{N/4} = W_{8/4} = W_2 = e^{-j\pi} = -1$$




Decimation-in-Time FFT

- At this point for the 8 sample DFT, we can replace the $N/4 = 2$ sample DFT's with a single butterfly. The fundamental eigenfunction is:

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$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} \\ &= \sum_{n=0}^1 x[n] W_2^{kn} \\ &= x[0] W_2^{k \cdot 0} + x[1] W_2^{k \cdot 1} \\ &= x[0] + x[1] W_2^k \end{aligned}$$



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$$\begin{aligned} X[0] &= \\ X[1] &= \end{aligned}$$

Decimation-in-Time FFT

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$$\begin{aligned} X[0] &= x[0] + x[1] \\ X[1] &= x[0] + x[1] W_2^1 \end{aligned}$$

Decimation-in-Time FFT

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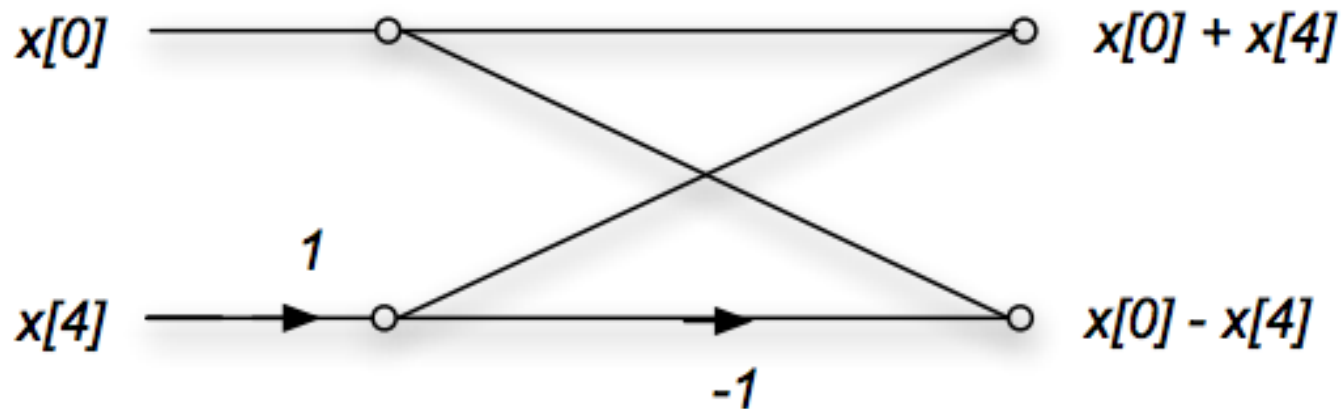
$$\begin{aligned} X[0] &= x[0] + x[1] \\ X[1] &= x[0] - x[1] \end{aligned}$$

Decimation-in-Time FFT

- At this point for the 8 sample DFT, we can replace the $N/4 = 2$ sample DFT's with a single butterfly. The fundamental eigenfunction is:

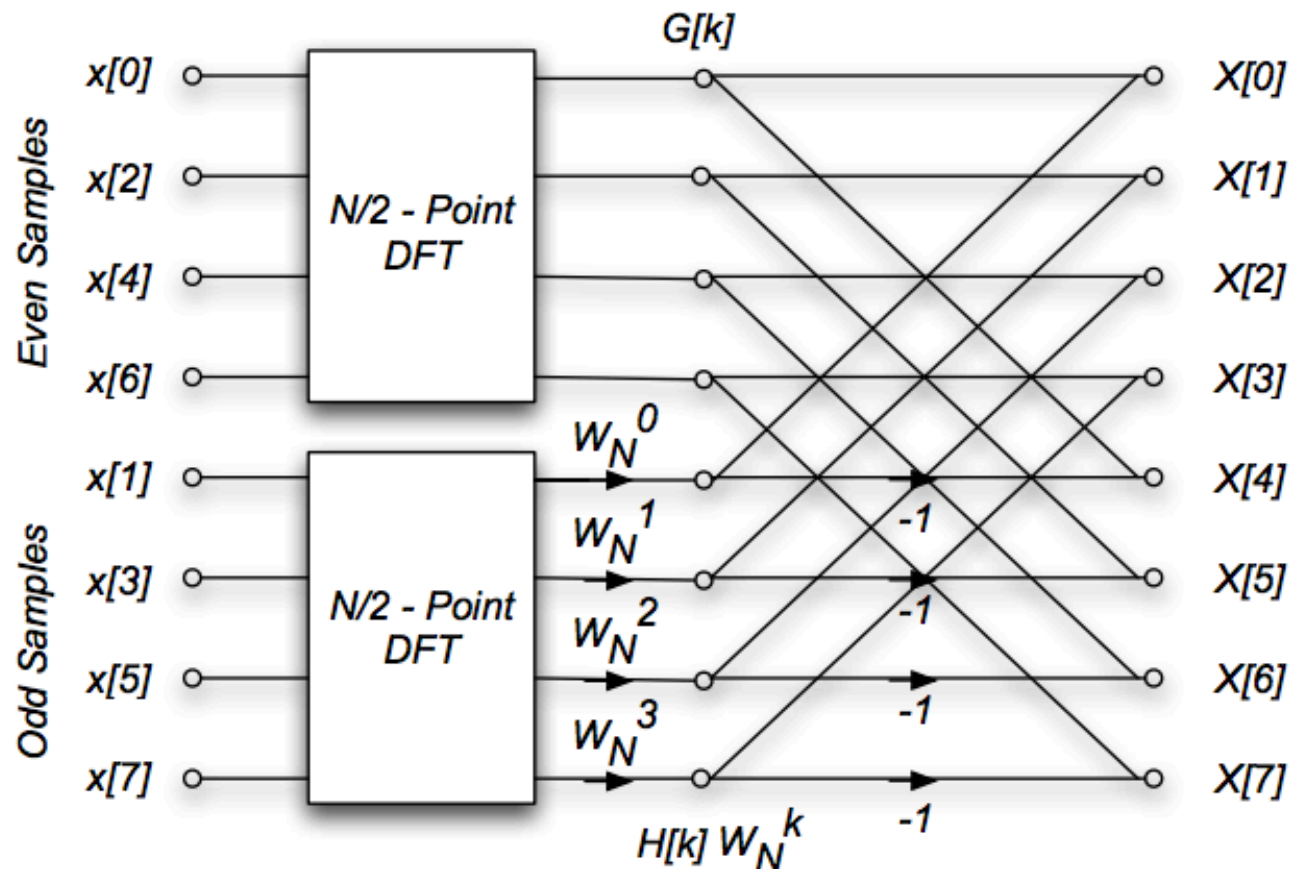
$$W_{N/4} = W_{8/4} = W_2 = e^{-j\pi} = -1$$

The diagram of this stage is then



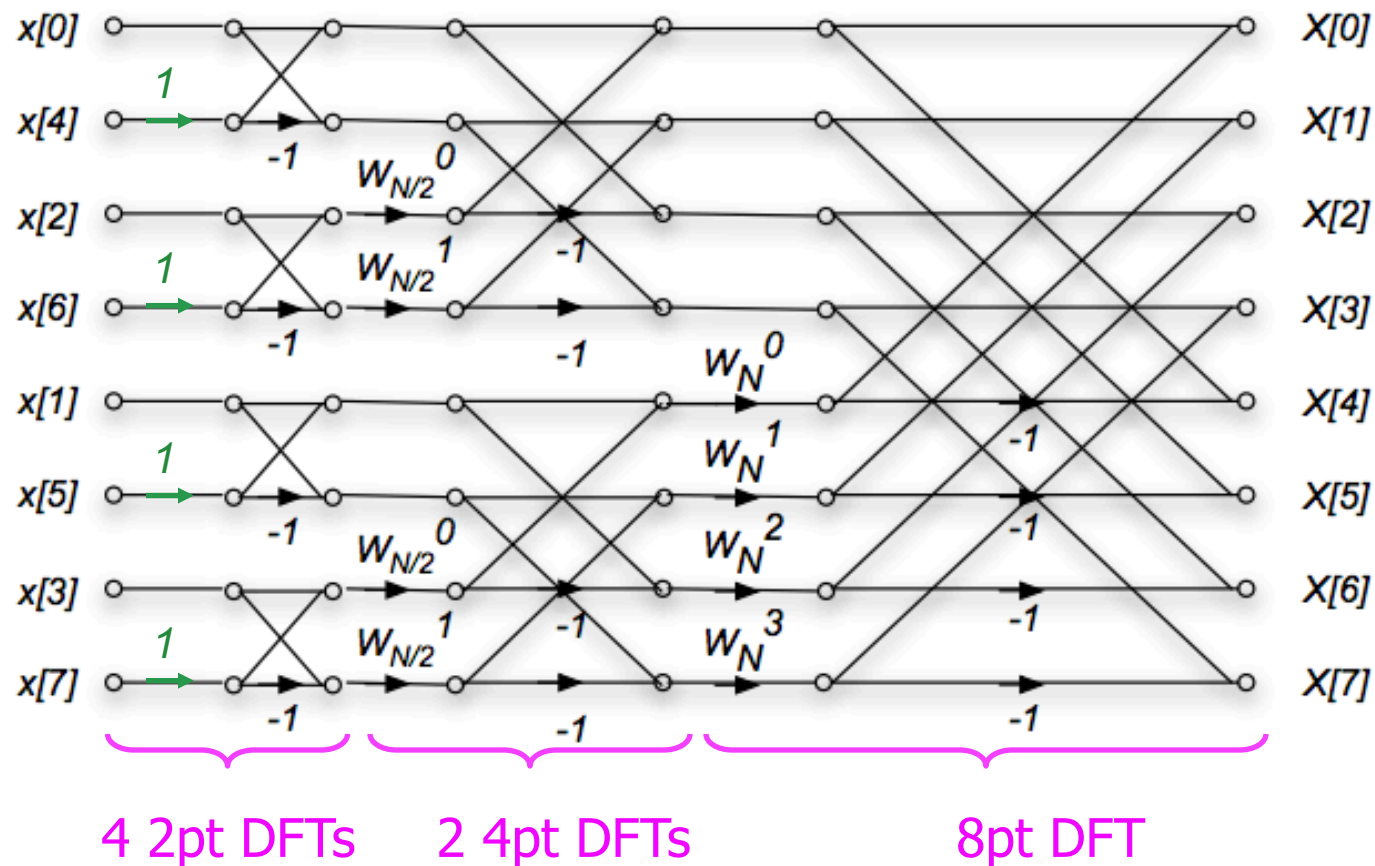
Decimation-in-Time FFT

- Replace $N/2$ -point DFT with 4-point DFT and $N/4$ -point DFT with butterfly operations



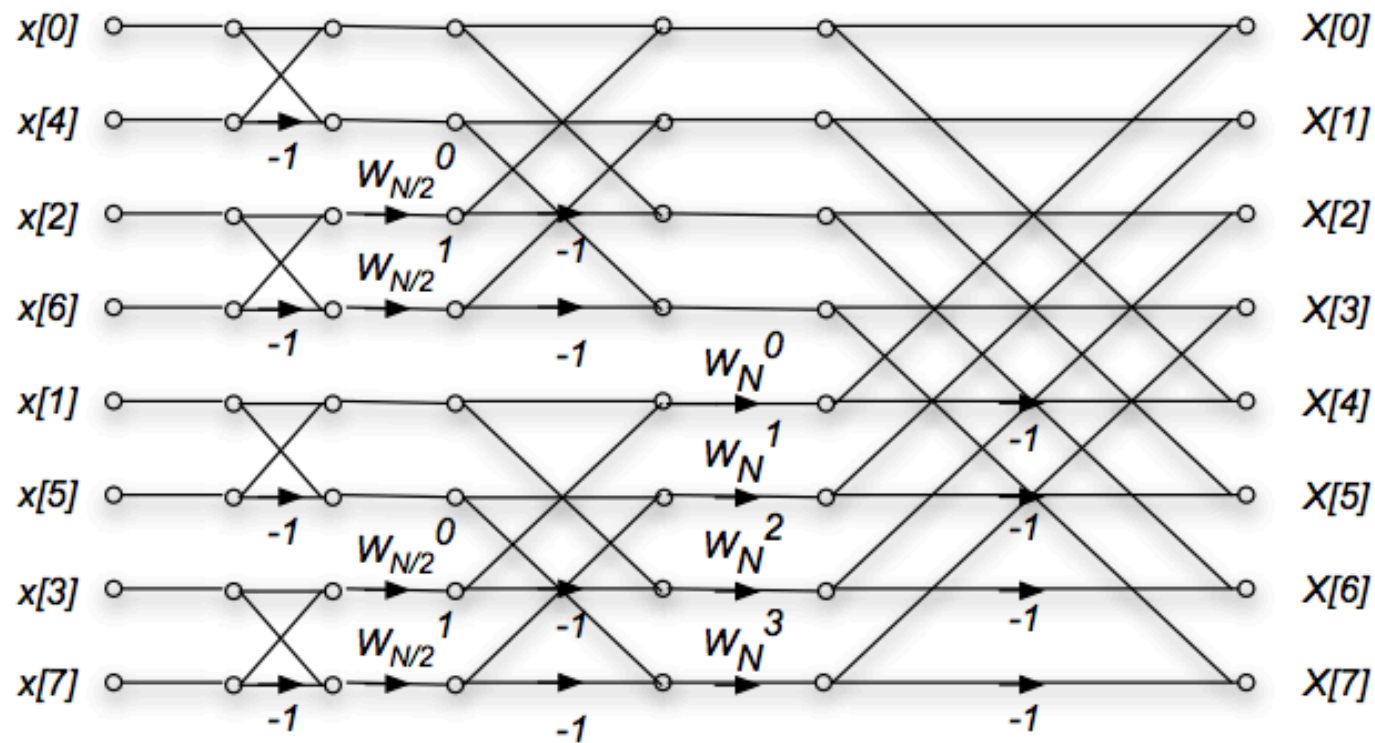
Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:




Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:




- $3 = \log_2(N) = \log_2(8)$ stages
- $4 = N/2 = 8/2$ multiplications in each stage
 - 1st stage has trivial multiplication



Decimation-in-Time FFT


- ❑ In general, there are $\log_2 N$ stages of decimation-in-time.
- ❑ Each stage requires $N/2$ complex multiplications, some of which are trivial.
- ❑ The total number of complex multiplications is $(N/2) \log_2 N$, or is $O(N \log_2 N)$



Decimation-in-Time FFT

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- ❑ Each stage requires $N/2$ complex multiplications, some of which are trivial.
- ❑ The total number of complex multiplications is $(N/2) \log_2 N$, or is $O(N \log_2 N)$


- ❑ The order of the input to the decimation-in-time FFT algorithm must be permuted.
 - First stage: split into odd and even.
 - Zero low-order address bit (LSB) first
 - Next stage repeats with next zero-lower bit
 - Net effect is reversing the bit order of indexes



Decimation-in-Time FFT

This is illustrated in the following table for $N = 8$.


Decimal	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111



Decimation-in-Time FFT

This is illustrated in the following table for $N = 8$.

Decimal	Binary	Bit-Reversed Binary
0	000	000
1	001	100
2	010	010
3	011	110
4	100	001
5	101	101
6	110	011
7	111	111



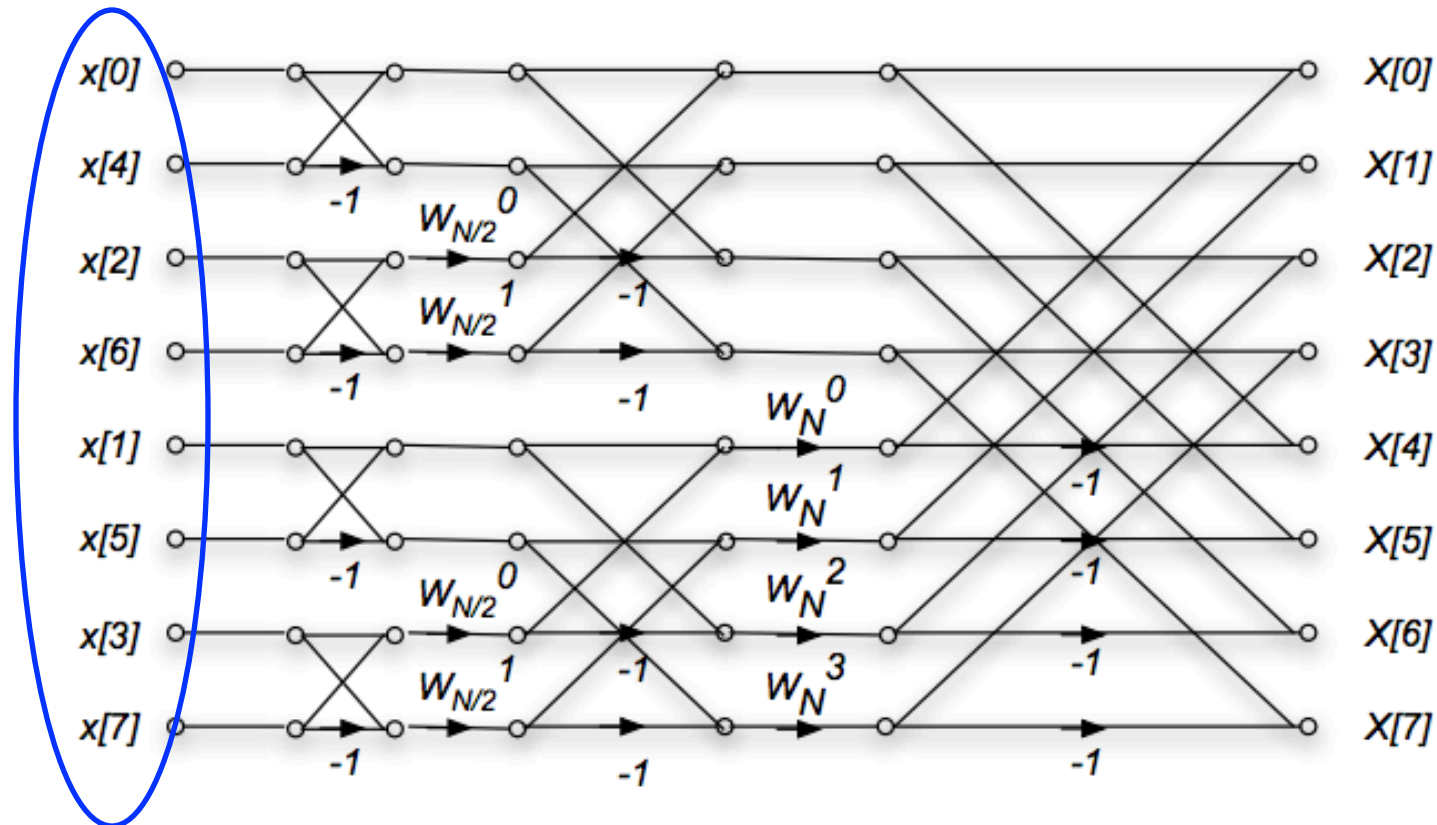
Decimation-in-Time FFT

This is illustrated in the following table for $N = 8$.

Decimal	Binary	Bit-Reversed Binary	Bit-Reversed Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:





Decimation-in-Frequency FFT

The DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$



Decimation-in-Frequency FFT

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If we only look at the even samples of $X[k]$, we can write $k = 2r$,

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)}$$



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We split this into two sums, one over the first $N/2$ samples, and the second of the last $N/2$ samples.

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=N/2}^{N-1} x[n] W_N^{2rn}$$



Decimation-in-Frequency FFT

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Decimation-in-Frequency FFT

But $W_N^{2r(n+N/2)} = W_N^{2rn} W_N^{rN} = W_N^{2rn} = W_{N/2}^{rn}$.

We can then write

Decimation-in-Frequency FFT

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Decimation-in-Frequency FFT

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We can then write

$$\begin{aligned} X[2r] &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n + N/2] W_N^{2r(n+N/2)} \\ &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n + N/2] W_N^{2rn} \\ &= \sum_{n=0}^{(N/2)-1} (x[n] + x[n + N/2]) W_{N/2}^{rn} \end{aligned}$$

This is the $N/2$ -length DFT of first and second half of $x[n]$ summed.



Decimation-in-Frequency FFT

$$\begin{aligned} X[2r] &= \text{DFT}_{\frac{N}{2}} \{ (x[n] + x[n + N/2]) \} \\ X[2r + 1] &= \text{DFT}_{\frac{N}{2}} \{ (x[n] - x[n + N/2]) W_N^n \} \end{aligned}$$

(By a similar argument that gives the odd samples)



Decimation-in-Frequency FFT

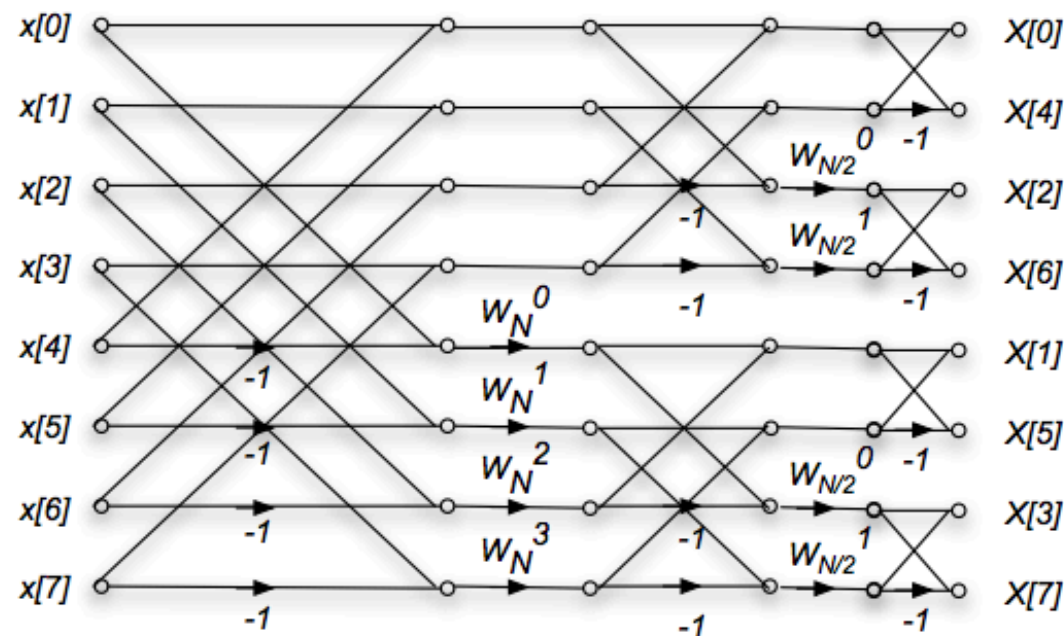
$$\begin{aligned} X[2r] &= \text{DFT}_{\frac{N}{2}} \{(x[n] + x[n + N/2])\} \\ X[2r + 1] &= \text{DFT}_{\frac{N}{2}} \{(x[n] - x[n + N/2]) W_N^n\} \end{aligned}$$

(By a similar argument that gives the odd samples)

- Continue the same approach on the $N/2$ DFTs, and $N/4$ DFTs until we reach the 2-point DFT, which is a simple butterfly operation

Decimation-in-Frequency FFT

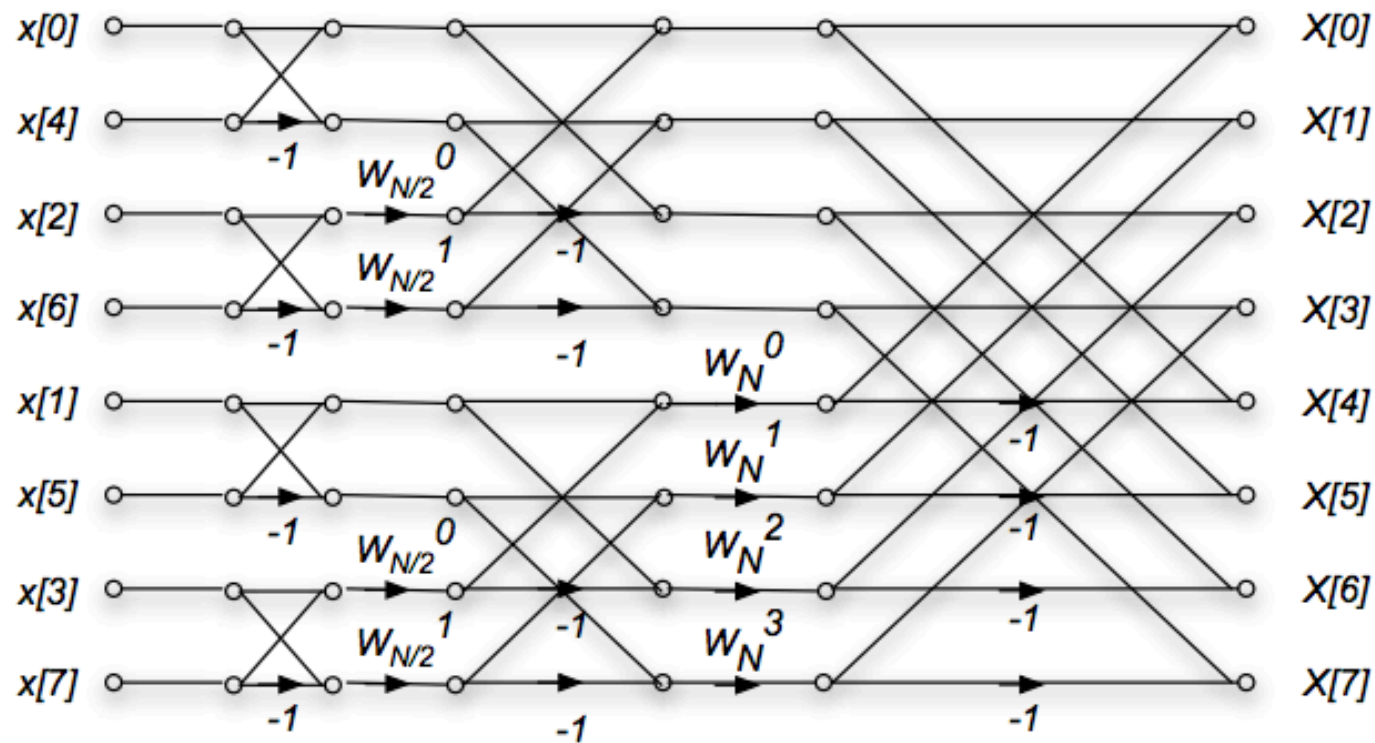
The diagram for an 8-point decimation-in-frequency DFT is as follows



This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Decimation-in-Time FFT

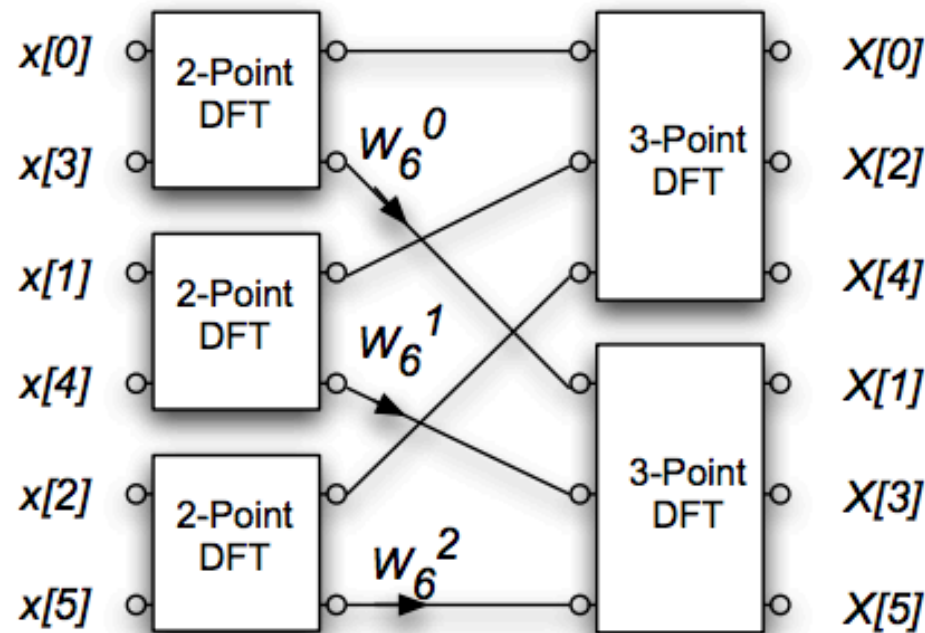
Combining all these stages, the diagram for the 8 sample DFT is:



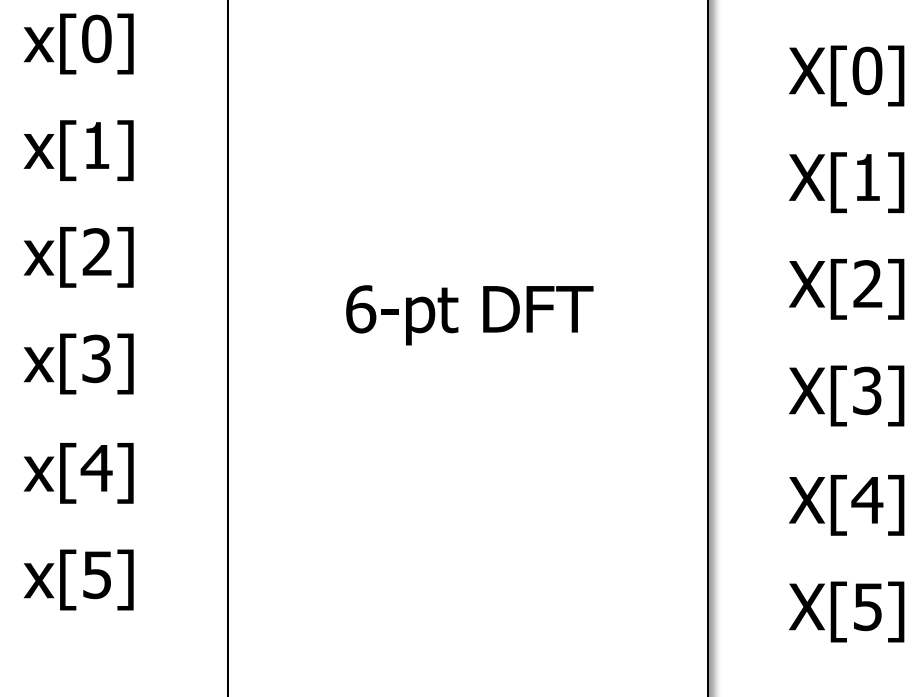
- $3 = \log_2(N) = \log_2(8)$ stages
- $4 = N/2 = 8/2$ multiplications in each stage
 - 1st stage has trivial multiplication

Non-Power-of-2 FFTs

- A similar argument applies for any length DFT, where the length N is a composite number
- For example, if $N=6$, with decimation-in-frequency you could compute three 2-point DFTs followed by two 3-point DFTs



Example: Non-Power-of-2 FFTs

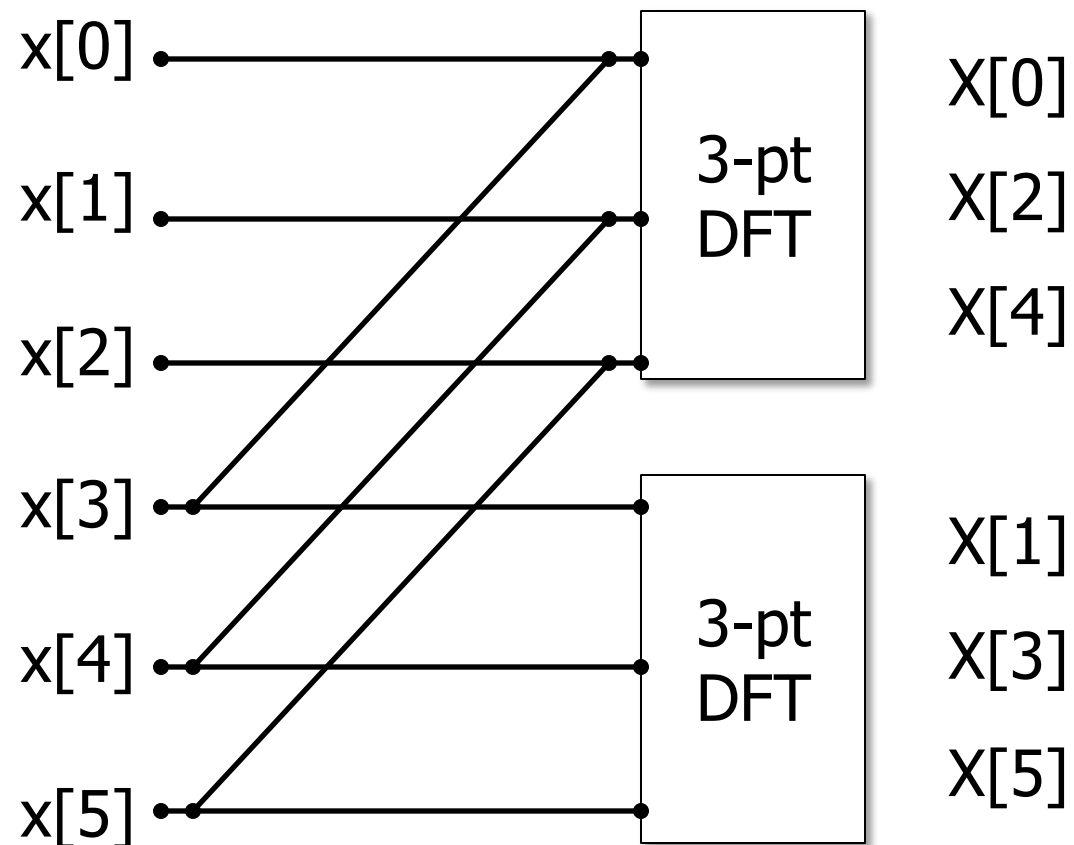




Decimation-in-Frequency FFT

$$\begin{aligned} X[2r] &= \text{DFT}_{\frac{N}{2}} \{(x[n] + x[n + N/2])\} \\ X[2r + 1] &= \text{DFT}_{\frac{N}{2}} \{(x[n] - x[n + N/2]) W_N^n\} \end{aligned}$$

Example: Non-Power-of-2 FFTs

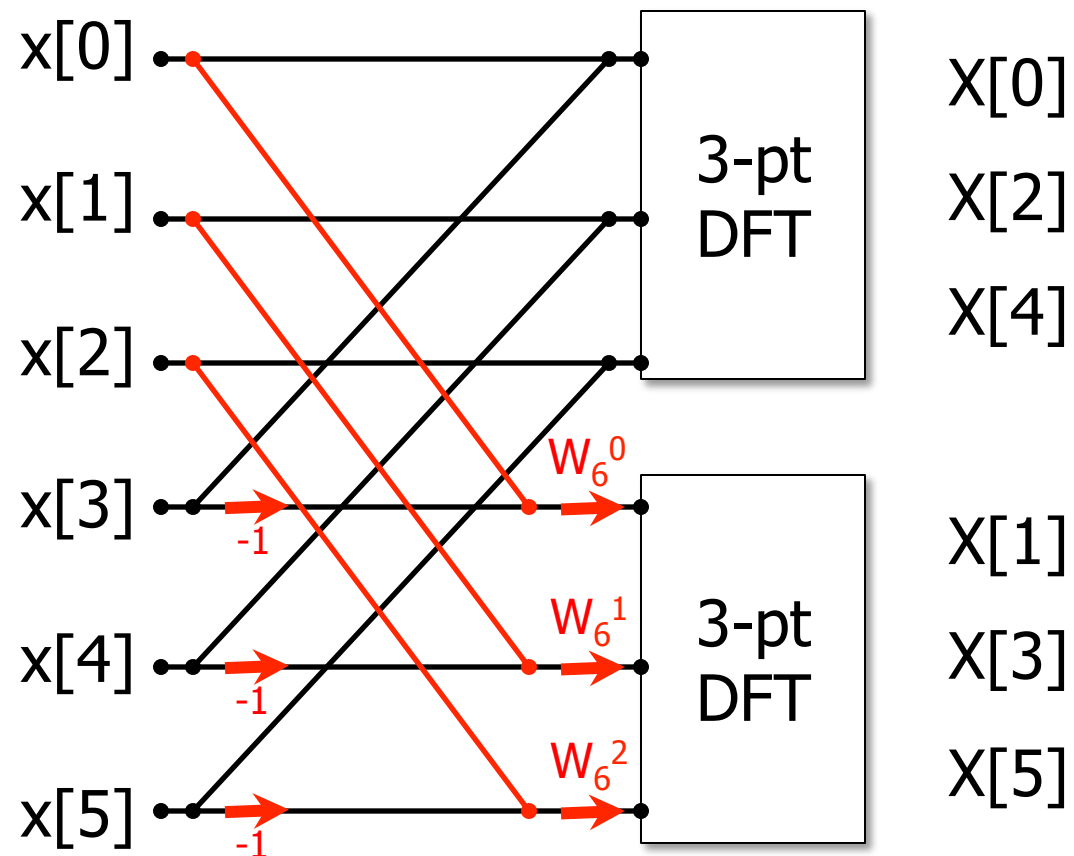




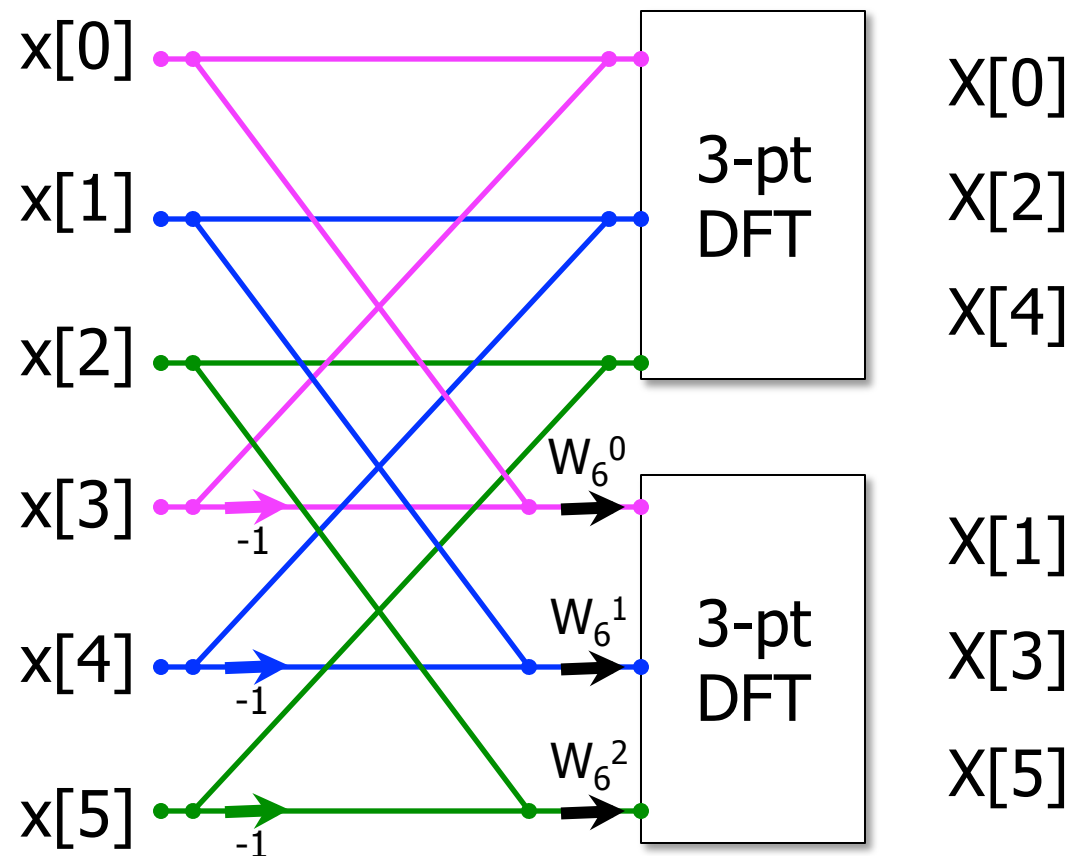
Decimation-in-Frequency FFT

$$\begin{aligned} X[2r] &= \text{DFT}_{\frac{N}{2}} \{(x[n] + x[n + N/2])\} \\ X[2r + 1] &= \text{DFT}_{\frac{N}{2}} \{(x[n] - x[n + N/2]) W_N^n\} \end{aligned}$$

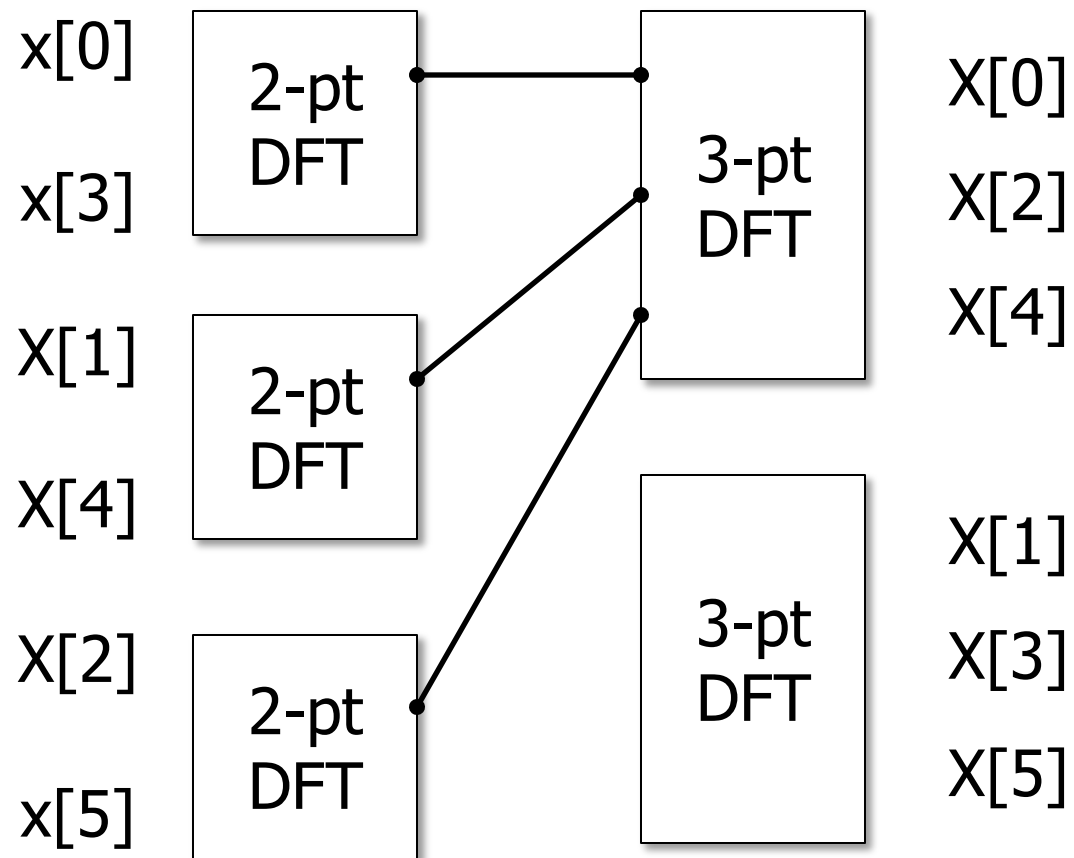
Example: Non-Power-of-2 FFTs



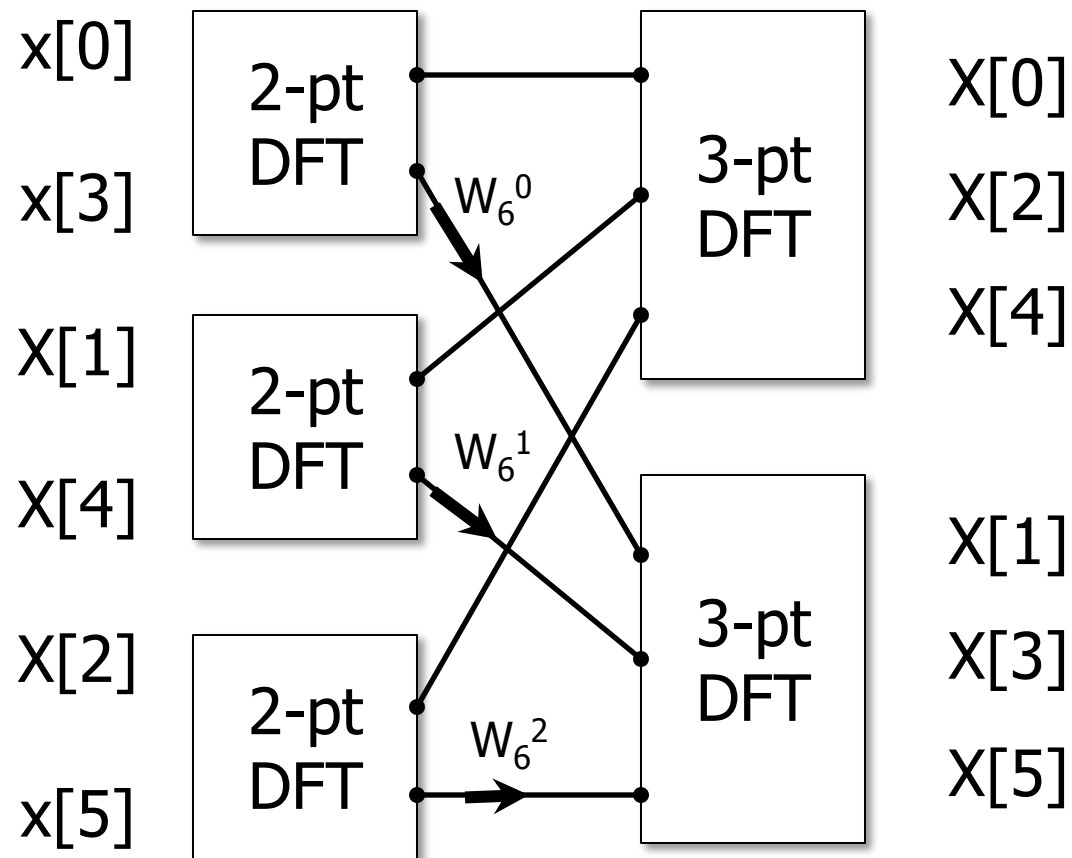
Example: Non-Power-of-2 FFTs



Example: Non-Power-of-2 FFTs



Example: Non-Power-of-2 FFTs





Non-Power-of-2 FFTs

- ❑ Good component DFTs are available for lengths up to 20(ish). Many of these exploit the structure for that specific length

- For example, a factor of

$$W_N^{N/4} = e^{-j\frac{2\pi}{N}(N/4)} = e^{-j\frac{\pi}{2}} = -j$$

Just swaps the real and imaginary components of a complex number. Hence a DFT of length 4 doesn't require any complex multiples.

- Half of the multiples of an 8-point DFT also don't require multiplication
 - Composite length FFTs can be very efficient for any length that factors into terms of this order



Non-Power-of-2 FFTs

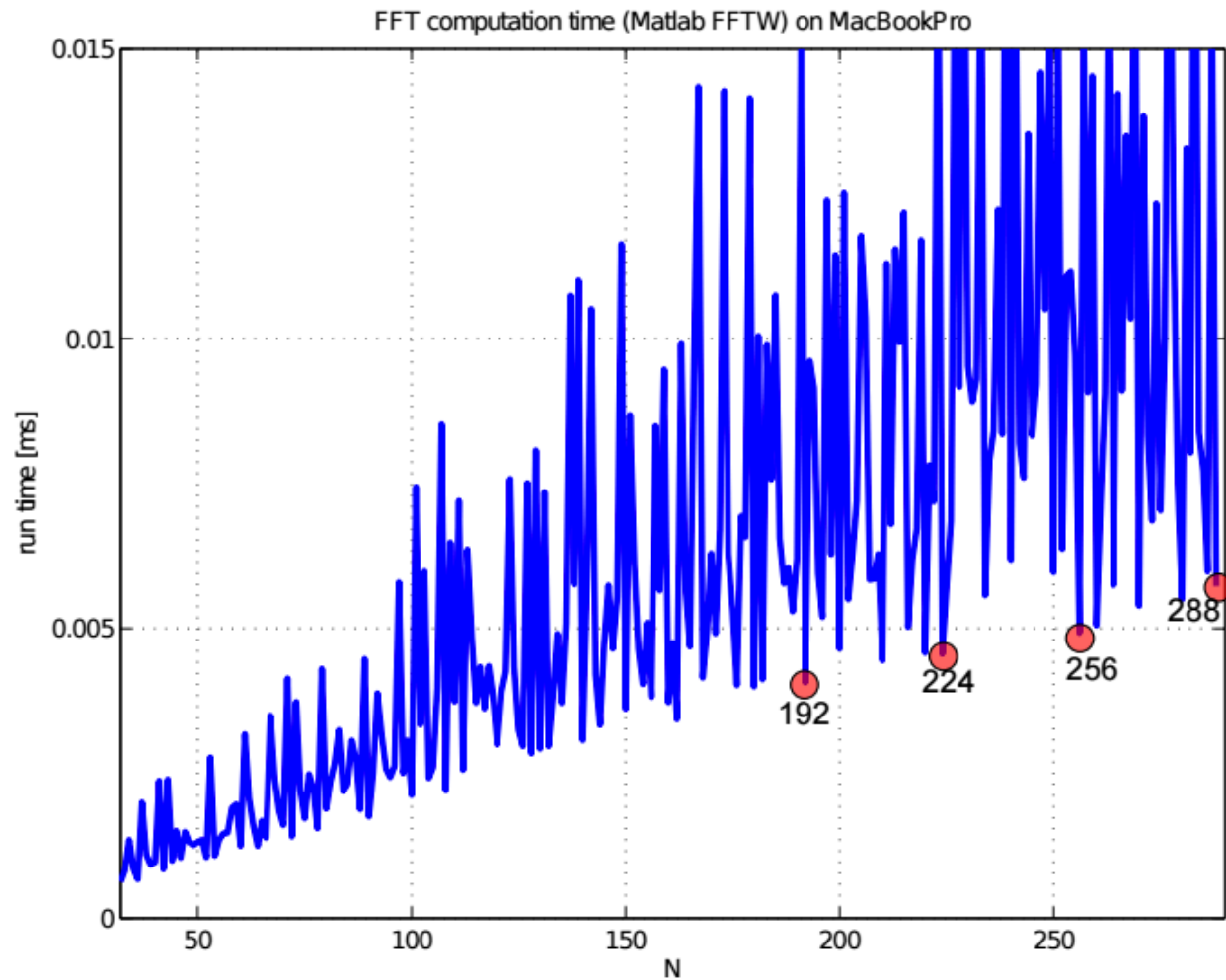
- ❑ For example $N = 693$ factors into
 - $N = (7)(9)(11)$
- ❑ each of which can be implemented efficiently. We would perform
 - 9 x 11 DFTs of length 7
 - 7 x 11 DFTs of length 9, and
 - 7 x 9 DFTs of length 11



Non-Power-of-2 FFTs

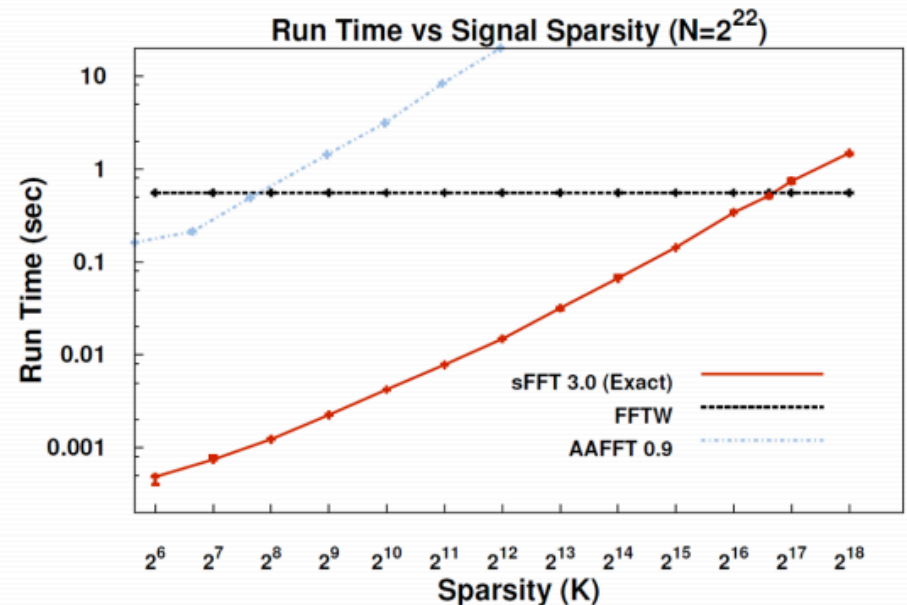
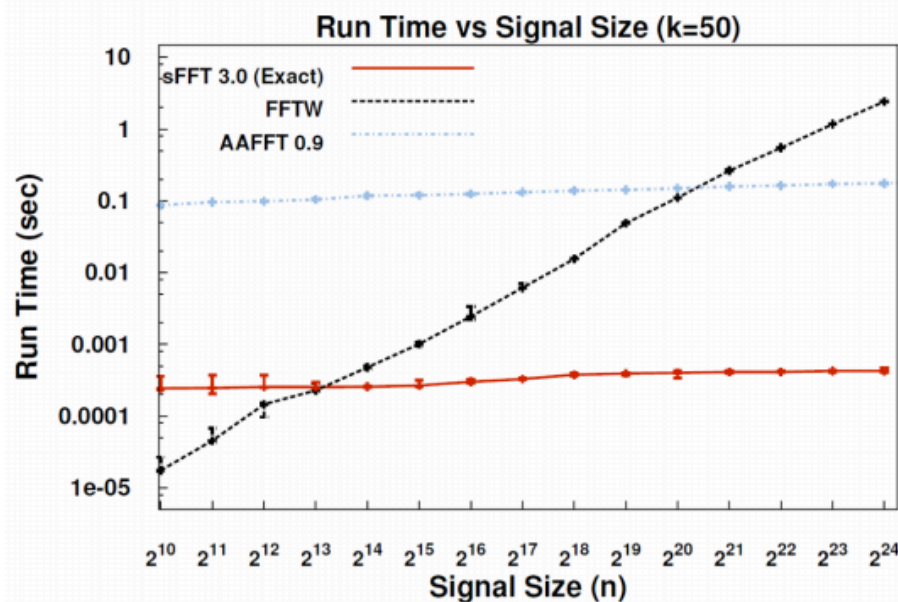
- ❑ Historically, the power-of-two FFTs were much faster (better written and implemented).
- ❑ For non-power-of-two length, it was faster to zero pad to power of two.
- ❑ Recently this has changed. The free FFTW package implements very efficient algorithms for almost any filter length. [Matlab has used FFTW since version 6](#)

FFT Computation Time



Beyond NlogN

- What if the signal $x[n]$ has a k sparse frequency
 - A. Gilbert et. al, “Near-optimal sparse Fourier representations via sampling
 - H. Hassanieh et. al, “Nearly Optimal Sparse Fourier Transform”
 - Others...
 - $O(K \log N)$ instead of $O(N \log N)$





Big Ideas

❑ Fast Fourier Transform

- Enable computation of an N -point DFT (or DFT^{-1}) with the order of just $N \cdot \log_2 N$ complex multiplications.
- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
 - Decimation-in-time algorithms
 - Decimation-in-frequency
- Historically, power-of-2 DFTs had highest efficiency
- Modern computing has led to non-power-of-2 FFTs with high efficiency
- Sparsity leads to reduced computation on order $K \cdot \log N$



Admin

- ❑ HW 8 due Sunday