ESE 531: Digital Signal Processing

Lec 19: April 7, 2020

Fast Fourier Transform

Last Time

- Discrete Fourier Transform
  - Linear convolution through circular convolution
    - Overlap and add
    - Overlap and save
  - Circular convolution through DFT
- Today
  - The Fast Fourier Transform

Fast Fourier Transform (FFT)

Fast Fourier Transform Algorithms

- We are interested in efficient computing methods for the DFT and inverse DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad k = 0, \dots, N-1$$

$$x[n] = \sum_{k=0}^{N-1} X[k]W_N^{-kn}, \quad n = 0, \dots, N-1$$

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}.$$

Reminder: Inverse DFT via DFT

- Recall that we can use the DFT to compute the inverse DFT:

$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$

  - Hence, we can just focus on efficient computation of the DFT.
- Straightforward computation of an N-point DFT (or inverse DFT) requires N^2 complex multiplications.

Computation Order

- *Fast Fourier transform algorithms* enable computation of an N-point DFT (or inverse DFT) with the order of just $N \cdot \log_2 N$ complex multiplications.
  - This can represent a huge reduction in computational load, especially for large N.

N	N^2	$N \cdot \log_2 N$	$\frac{N^2}{N \cdot \log_2 N}$
16	256	64	4.0
128	16,384	896	18.3
1,024	1,048,576	10,240	102.4
8,192	67,108,864	106,496	630.2

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8,192	67,108,864	106,496	630.2
6×10^6	36×10^{12}	135×10^6	2.67×10^5

* 6Mp image size

Eigenfunction Properties

- Most FFT algorithms exploit the following properties of W_N^{kn}:
  - Conjugate Symmetry

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$$

  - Periodicity in n and k

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

  - Power

$$W_N^2 = W_{N/2}$$

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$$W_N^{k(N-n)} = \left(e^{-j\frac{2\pi}{N}}\right)^{k(N-n)}$$

$$= \left(e^{-j\frac{2\pi}{N}(N-n)}\right)^k = \left(e^{-j2\pi}e^{j\frac{2\pi}{N}n}\right)^k$$

$$= \left(e^{j\frac{2\pi}{N}}\right)^{nk} = \left(W_N^*\right)^{nk} = \left(W_N^{nk}\right)^*$$

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$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

$$W_N^{kn} = \left(e^{-j\frac{2\pi}{N}}\right)^{kn}\left(e^{-j\frac{2\pi}{N}}\right)^{kN} = \left(e^{-j\frac{2\pi}{N}}\right)^{kn}\left(e^{-j\frac{2\pi}{N}}\right)^{nN}$$

$$= W_N^{kn} = W_N^{kn}$$

Eigenfunction Properties

- Most FFT algorithms exploit the following properties of W_N^{kn}:
  - Power

$$W_N^2 = W_{N/2}$$

$$W_N^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2 = \left(e^{-j\frac{2\pi}{N}\times 2}\right) = \left(e^{-j\frac{2\pi}{N/2}}\right)$$

$$= W_{N/2}$$

FFT Algorithms via Decimation

- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - Decimation-in-time algorithms decompose x[n] into successively smaller subsequences.
  - Decimation-in-frequency algorithms decompose X[k] into successively smaller subsequences.
- Note: Assume length of x[n] is power of 2 ($N = 2^\nu$). If not, zero-pad to closest power of 2.

Decimation-in-Time FFT

- We start with the DFT

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$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

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  - These are two DFTs, each with half the number of samples (N/2)

Decimation-in-Time FFT

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

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$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k}$$

Decimation-in-Time FFT

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Note that:

$$W_N^{2rk} = e^{-j\left(\frac{2\pi}{N}\right)(2rk)} = e^{-j\left(\frac{2\pi}{N/2}\right)rk} = W_{N/2}^{rk}$$

Remember this trick, it will turn up often.

Decimation-in-Time FFT

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Decimation-in-Time FFT

Let $n = 2r$ (n even) and

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad k = 0, \ldots, N-1$$

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r]W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r]W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]W_N^{2rk} \end{aligned}$$

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Decimation-in-Time FFT

Hence:

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk} \\ &\triangleq G[k] + W_N^k H[k], \quad k = 0, \ldots, N-1 \end{aligned}$$

Decimation-in-Time FFT

Hence:

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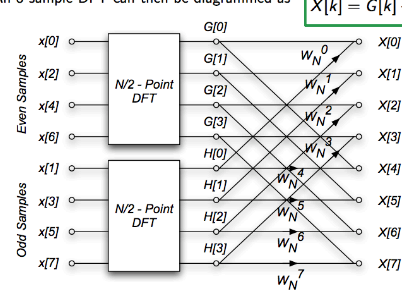
where we have defined:

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r]W_{N/2}^{rk} \quad \Rightarrow \text{DFT of even samples}$$

$$H[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r+1]W_{N/2}^{rk} \quad \Rightarrow \text{DFT of odd samples}$$

Decimation-in-Time FFT

An 8 sample DFT can then be diagrammed as $X[k] = G[k] + W_N^k H[k]$



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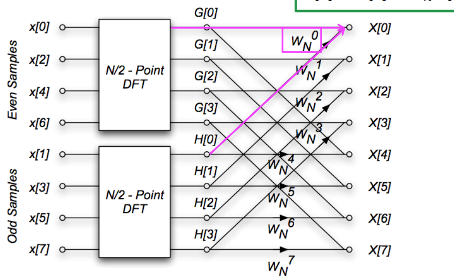

$X[0]=G[0]+W_N^0 H[0]$

Decimation-in-Time FFT

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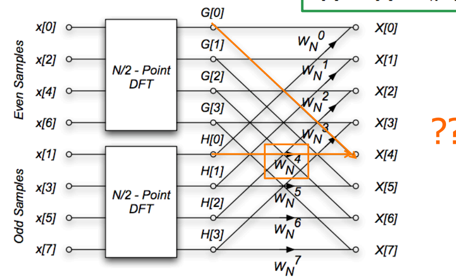

$X[4]=G[4]+W_N^4 H[4]$

Decimation-in-Time FFT

Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk}$$

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$$G[k+N/2] = \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)}$$

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$$G[k+N/2] = \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)} = \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \underbrace{W_{N/2}^{r(N/2)}}_{=1} = \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} = G[k]$$

Decimation-in-Time FFT

- So,

$$G[k+(N/2)] = G[k]$$
$$H[k+(N/2)] = H[k]$$

- The periodicity of G[k] and H[k] allows us to further simplify. For the first N/2 points we calculate G[k] and W_N^kH[k], and then compute the sum

$$X[k] = G[k] + W_N^k H[k] \qquad \forall\{k : 0 \le k < \frac{N}{2}\}.$$

Decimation-in-Time FFT

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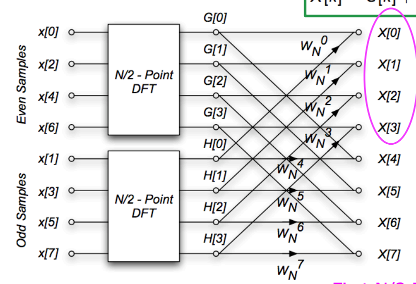
$X[k] = G[k] + W_N^k H[k]$



First N/2 DFTs
$0 \le k < N/2$

Decimation-in-Time FFT

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How does periodicity help for $\frac{N}{2} \le k < N$?

Decimation-in-Time FFT

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for $\frac{N}{2} \le k < N$:

$W_N^{k+(N/2)} = ?$

$X[k+(N/2)] = ?$

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$$= \left(e^{-j\frac{2\pi}{N}k}\right)\left(e^{-j\frac{2\pi}{N}\times\frac{N}{2}}\right)$$

$$= \left(e^{-j\frac{2\pi}{N}k}\right)\left(e^{-j\pi}\right) = -W_N^k$$

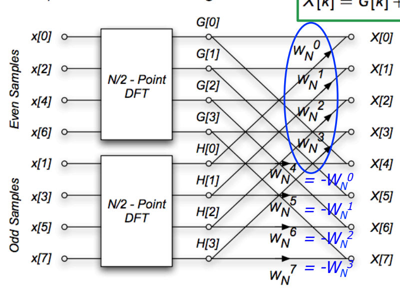
Decimation-in-Time FFT

$$X[k+(N/2)] = G[k] - W_N^k H[k]$$

- We previously calculated G[k] and $W_N^k H[k]$.
- Now we only have to compute their difference to obtain the second half of the spectrum. No additional multiplies are required.

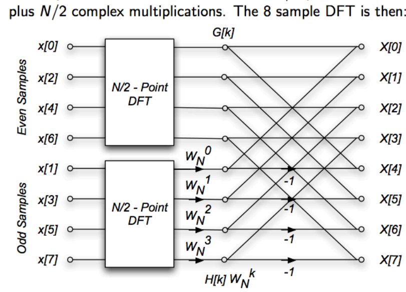
Decimation-in-Time FFT

An 8 sample DFT can then be diagrammed as $X[k] = G[k] + W_N^k H[k]$



Decimation-in-Time FFT

The N-point DFT has been reduced two $N/2$-point DFTs, plus $N/2$ complex multiplications. The 8 sample DFT is then:



Decimation-in-Time FFT

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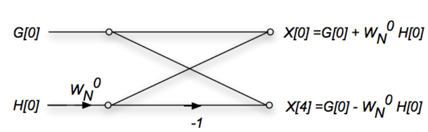
- We can define a *butterfly operation*, e.g., the computation of X[0] and X[4] from G[0] and H[0]:



Decimation-in-Time FFT

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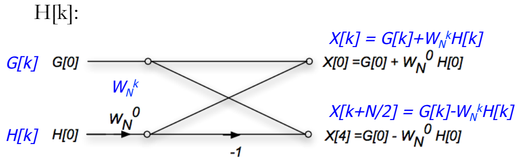
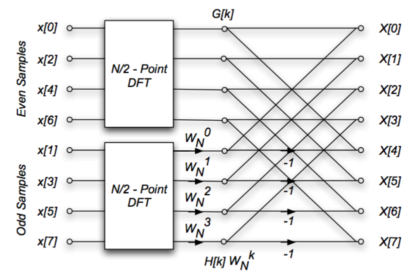


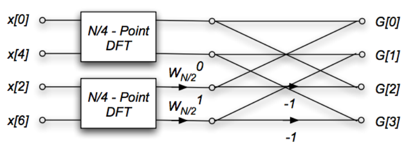
Decimation-in-Time FFT

- Still O(N²) operations…. What should we do?



Decimation-in-Time FFT

- We can use the same approach for each of the N/2 point DFT's. For the N = 8 case, the N/2 DFTs look like



*Note that the inputs have been reordered again.

Decimation-in-Time FFT

- At this point for the 8 sample DFT, we can replace the N/4 = 2 sample DFT's with a single butterfly. The fundamental eigenfunction is:

$$W_{N/4} = W_{8/4} = W_2 = e^{-j\pi} = -1$$

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$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$
$$= \sum_{n=0}^{1} x[n] W_2^{kn}$$
$$= x[0] W_2^{k\cdot 0} + x[1] W_2^{k\cdot 1}$$
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$$= x[0] + x[1] W_2^{k}$$

X[0]=x[0]+x[1]
$X[1]=x[0]+x[1]W_2^1$

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X[0]=x[0] + x[1]
X[1]=x[0] - x[1]

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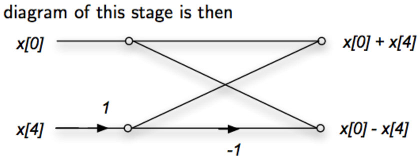
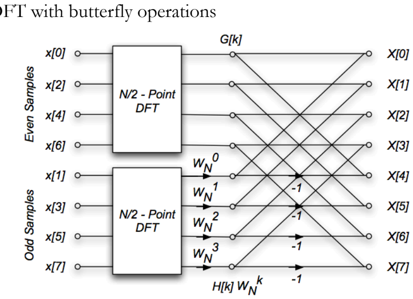
The diagram of this stage is then



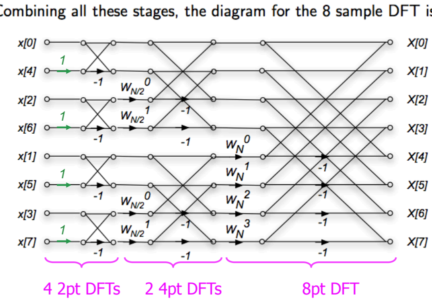
Decimation-in-Time FFT

- Replace N/2-point DFT with 4-point DFT and N/4-point DFT with butterfly operations



Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:



Decimation-in-Time FFT

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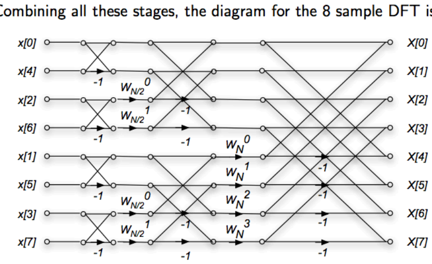

- $3=\log_2(N)=\log_2(8)$ stages
- 4=N/2=8/2 multiplications in each stage
  - 1st stage has trivial multiplication

Decimation-in-Time FFT

- In general, there are $\log_2 N$ stages of decimation-in-time.
- Each stage requires N/2 complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2) \log_2 N$, or is $O(N \log_2 N)$

Decimation-in-Time FFT

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- Each stage requires N/2 complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2) \log_2 N$, or is $O(N \log_2 N)$
- The order of the input to the decimation-in-time FFT algorithm must be permuted.
  - First stage: split into odd and even.
    - Zero low-order address bit (LSB) first
  - Next stage repeats with next zero-lower bit
  - Net effect is reversing the bit order of indexes

Decimation-in-Time FFT

This is illustrated in the following table for $N = 8$.

Decimal	Binary
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

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This is illustrated in the following table for $N = 8$.

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2	010	010
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4	100	001
5	101	101
6	110	011
7	111	111

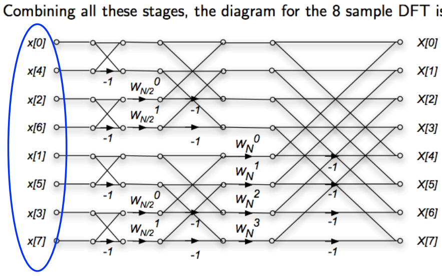
Decimation-in-Time FFT

This is illustrated in the following table for $N = 8$.

Decimal	Binary	Bit-Reversed Binary	Bit-Reversed Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Decimation-in-Time FFT

Combining all these stages, the diagram for the 8 sample DFT is:



Decimation-in-Frequency FFT

The DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

Decimation-in-Frequency FFT

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$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

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$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)}$$

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We split this into two sums, one over the first $N/2$ samples, and the second of the last $N/2$ samples.

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=N/2}^{N-1} x[n] W_N^{2rn}$$

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Decimation-in-Frequency FFT

But $W_N^{2r(n+N/2)} = W_N^{2rn} W_N^{rN} = W_N^{2rn} = W_{N/2}^{rn}$.
We can then write

Decimation-in-Frequency FFT

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Decimation-in-Frequency FFT

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$$
\begin{aligned}
X[2r] &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2r(n+N/2)} \\
&= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2rn} \\
&= \sum_{n=0}^{(N/2)-1} (x[n] + x[n+N/2])\, W_{N/2}^{rn}
\end{aligned}
$$

This is the $N/2$-length DFT of first and second half of $x[n]$ summed.

Decimation-in-Frequency FFT

$$
\begin{aligned}
X[2r] &= \mathrm{DFT}_{\frac{N}{2}} \{(x[n] + x[n+N/2])\} \\
X[2r+1] &= \mathrm{DFT}_{\frac{N}{2}} \{(x[n] - x[n+N/2])\, W_N^n\}
\end{aligned}
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(By a similar argument that gives the odd samples)

Decimation-in-Frequency FFT

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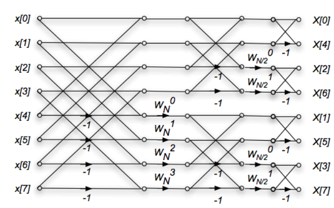
(By a similar argument that gives the odd samples)

- Continue the same approach on the N/2 DFTs, and N/4 DFTs until we reach the 2-point DFT, which is a simple butterfly operation

Decimation-in-Frequency FFT

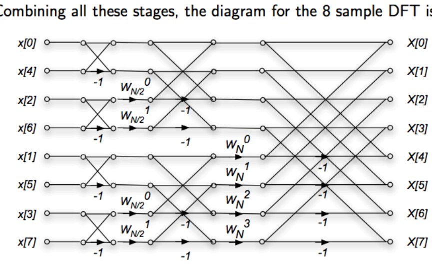
The diagram for and 8-point decimation-in-frequency DFT is as follows



This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Decimation-in-Time FFT

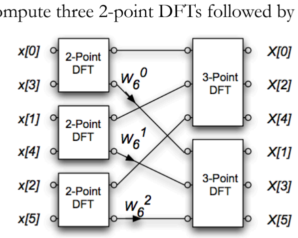
Combining all these stages, the diagram for the 8 sample DFT is:



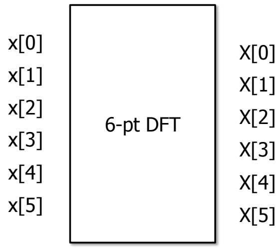
- $3=\log_2(N)=\log_2(8)$ stages
- $4=N/2=8/2$ multiplications in each stage
  - 1st stage has trivial multiplication

Non-Power-of-2 FFTs

- A similar argument applies for any length DFT, where the length N is a composite number
- For example, if N=6, with decimation-in-frequency you could compute three 2-point DFTs followed by two 3-point DFTs



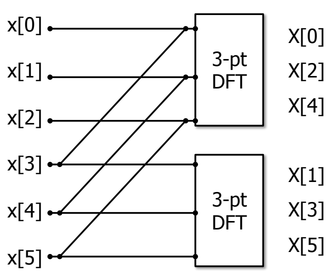
Example: Non-Power-of-2 FFTs



Decimation-in-Frequency FFT

$$X[2r] = \text{DFT}_{\frac{N}{2}}\{(x[n] + x[n+N/2])\}$$
$$X[2r+1] = \text{DFT}_{\frac{N}{2}}\{(x[n] - x[n+N/2])\, W_N^n\}$$

Example: Non-Power-of-2 FFTs



Decimation-in-Frequency FFT

$$X[2r] = \text{DFT}_{\frac{N}{2}}\{(x[n] + x[n+N/2])\}$$
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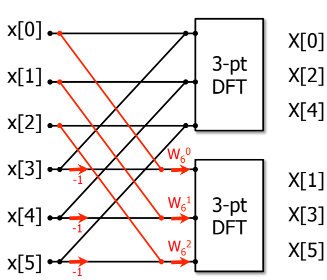
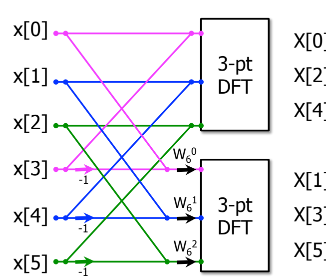
Example: Non-Power-of-2 FFTs



Example: Non-Power-of-2 FFTs



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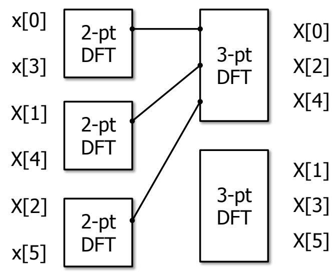

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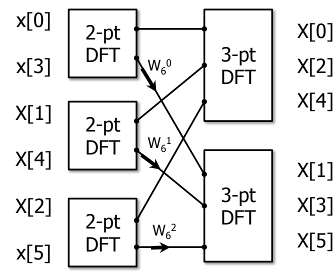


Non-Power-of-2 FFTs

- Good component DFTs are available for lengths up to 20(ish). Many of these exploit the structure for that specific length
  - For example, a factor of

$$W_N^{N/4} = e^{-j\frac{2\pi}{N}(N/4)} = e^{-j\frac{\pi}{2}} = -j$$

    Just swaps the real and imaginary components of a complex number. Hence a DFT of length 4 doesn't require any complex multiples.
  - Half of the multiples of an 8-point DFT also don't require multiplication
  - Composite length FFTs can be very efficient for any length that factors into terms of this order

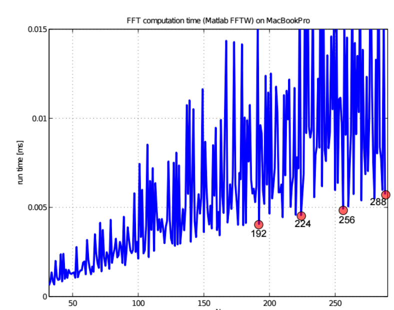
Non-Power-of-2 FFTs

- For example N = 693 factors into
  - N = (7)(9)(11)
- each of which can be implemented efficiently. We would perform
  - 9 x 11 DFTs of length 7
  - 7 x 11 DFTs of length 9, and
  - 7 x 9 DFTs of length 11

Non-Power-of-2 FFTs

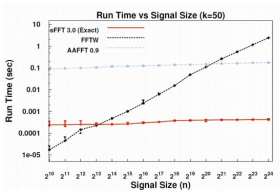
- Historically, the power-of-two FFTs were much faster (better written and implemented).
- For non-power-of-two length, it was faster to zero pad to power of two.
- Recently this has changed. The free FFTW package implements very efficient algorithms for almost any filter length. Matlab has used FFTW since version 6

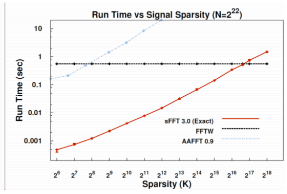
FFT Computation Time



Beyond NlogN

- What if the signal x[n] has a k sparse frequency
  - A. Gilbert et. al, "Near-optimal sparse Fourier representations via sampling
  - H. Hassanieh et. al, "Nearly Optimal Sparse Fourier Transform"
  - Others…
  - O(K Log N) instead of O(N Log N)





From: http://groups.csail.mit.edu/netmit/sFFT/paper.html

Big Ideas

- Fast Fourier Transform
  - Enable computation of an N-point DFT (or DFT^{-1}) with the order of just $N \cdot \log_2 N$ complex multiplications.
  - Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
    - Decimation-in-time algorithms
    - Decimation-in-frequency
  - Historically, power-of-2 DFTs had highest efficiency
  - Modern computing has led to non-power-of-2 FFTs with high efficiency
  - Sparsity leads to reduced computation on order $K \cdot \log N$

Admin

- HW 8 due Sunday