# ESE 531: Digital Signal Processing

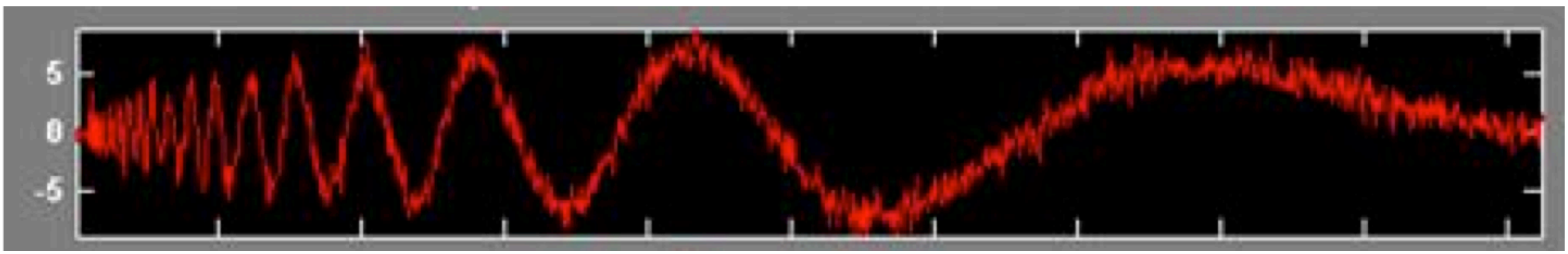
Lec 23: April 21, 2020

Wavelet Transform

# Wavelet Transform

# Motivation

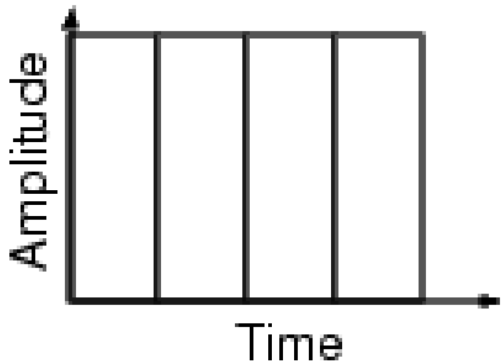
- Some signals obviously have spectral characteristics that vary with time



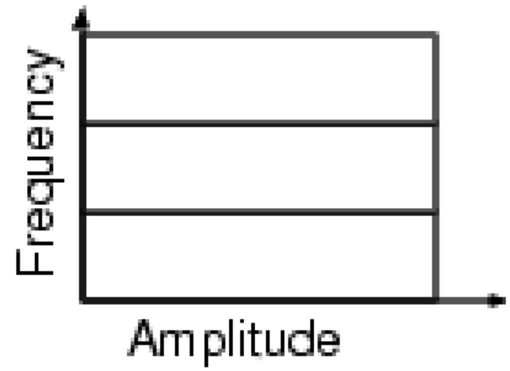
# Criticism of Fourier Spectrum

- It's giving you the spectrum of the 'whole time-series'
- Which is OK if the time-series is stationary. But what if its not?
- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

# Transform Comparison

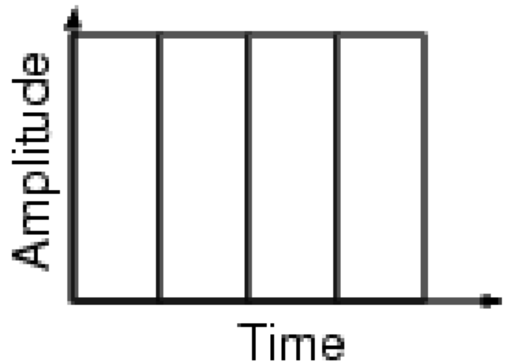



Time Domain (Shannon)

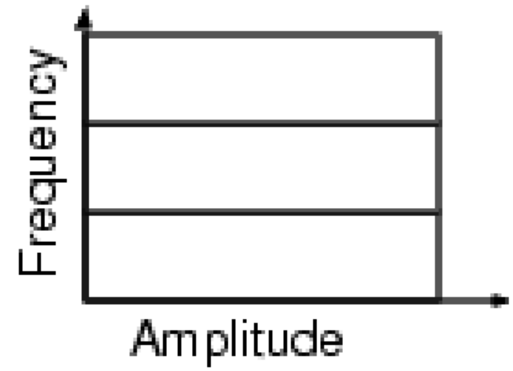



Frequency Domain (Fourier)

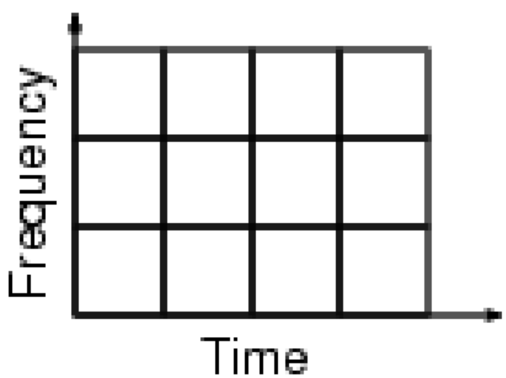
# Transform Comparison



Time Domain (Shannon)



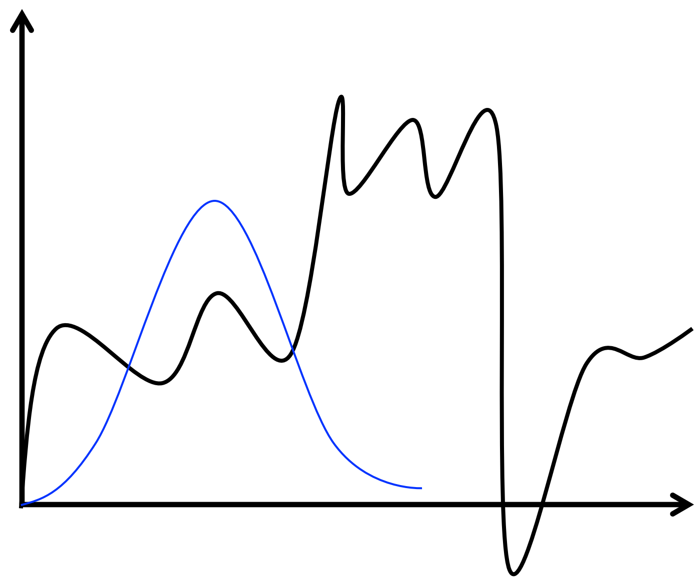
Frequency Domain (Fourier)



STFT (Gabor)

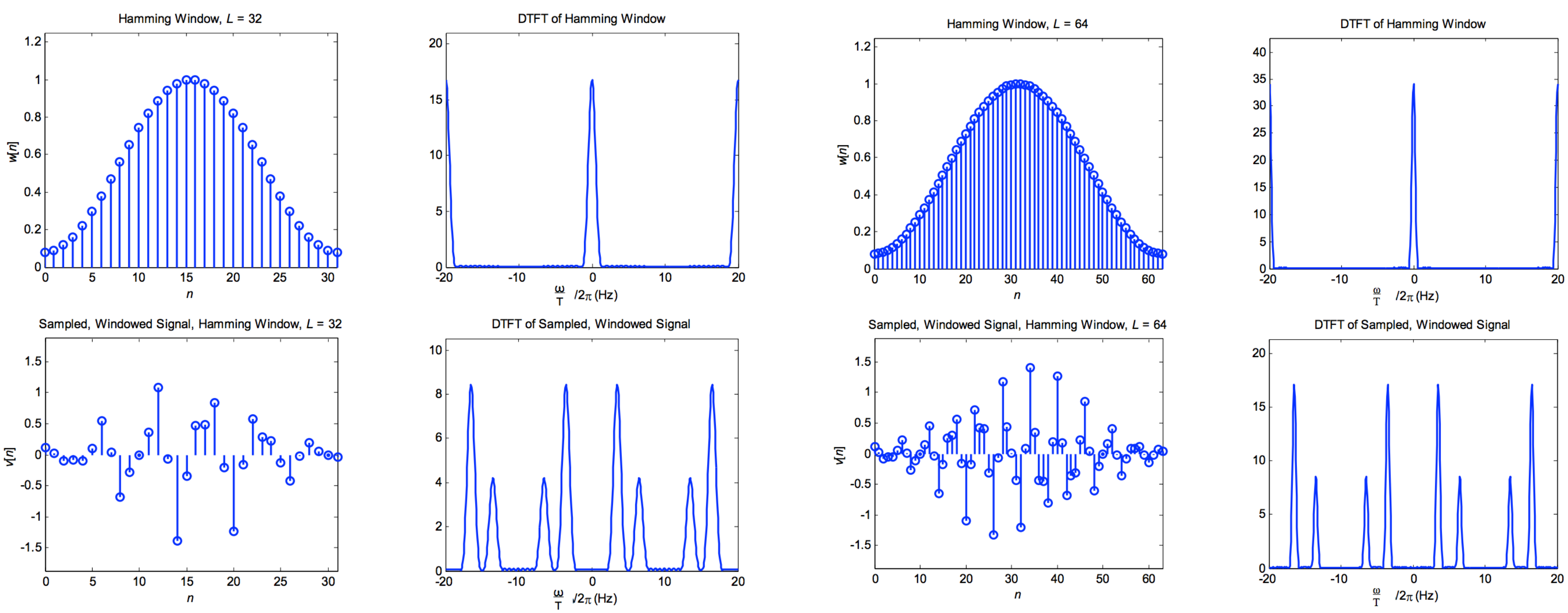
# Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)

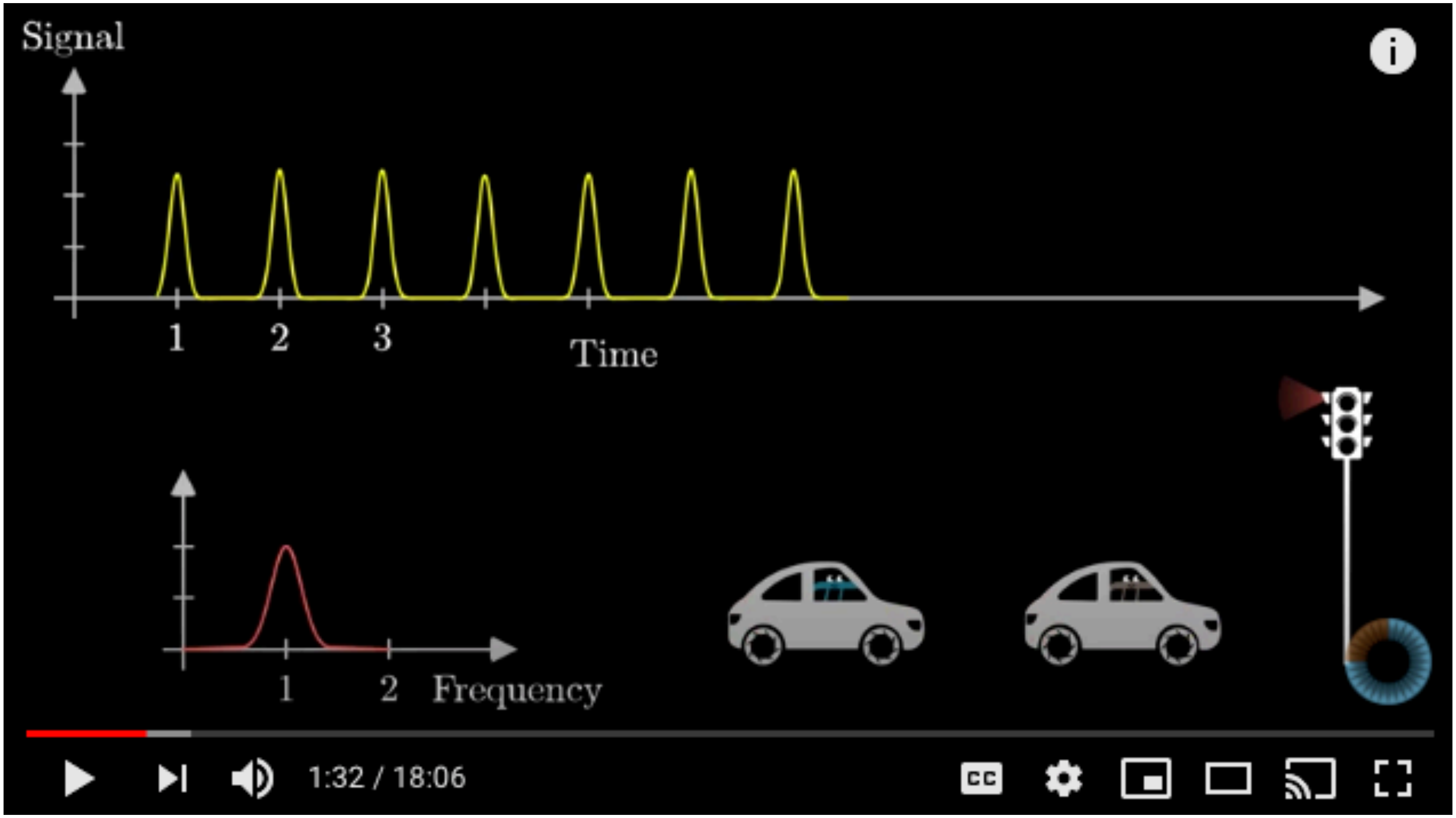


# Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi = 1/T = 20\text{Hz}$
- Hamming Window, L = 32 vs. L = 64



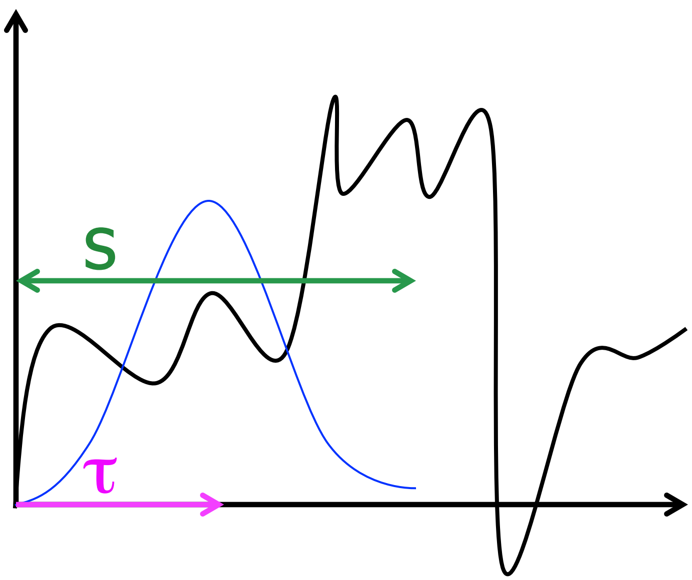
# Uncertainty Principle




https://youtu.be/MBnnXbOM5S4?t=49

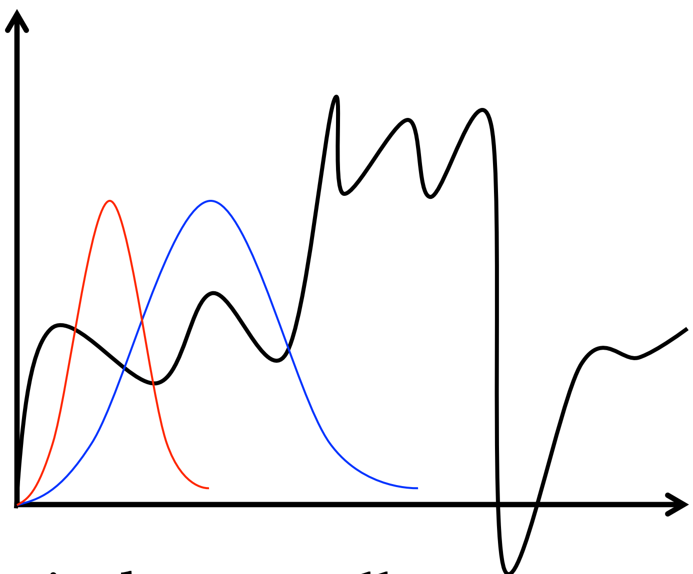
# Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)



# Discrete Time-Dependent FT

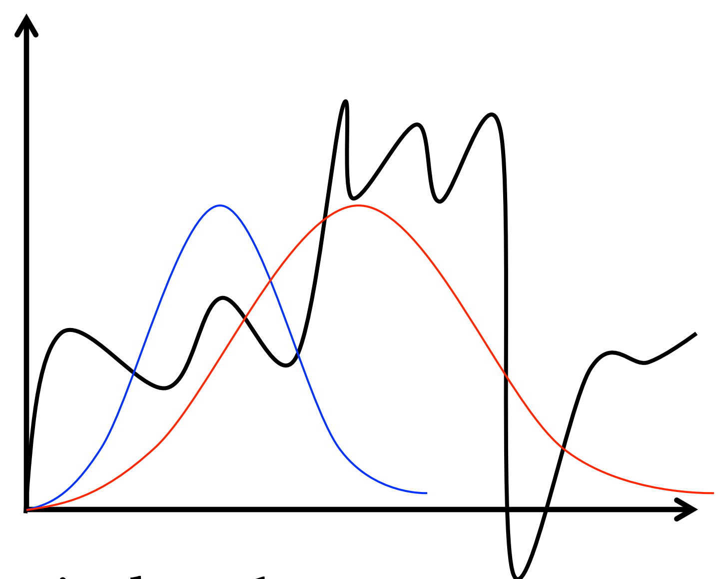
- Fixed window size, shift in time (Gabor)



- Make the window smaller
  - Better localization
  - Less spectral resolution

# Discrete Time-Dependent FT

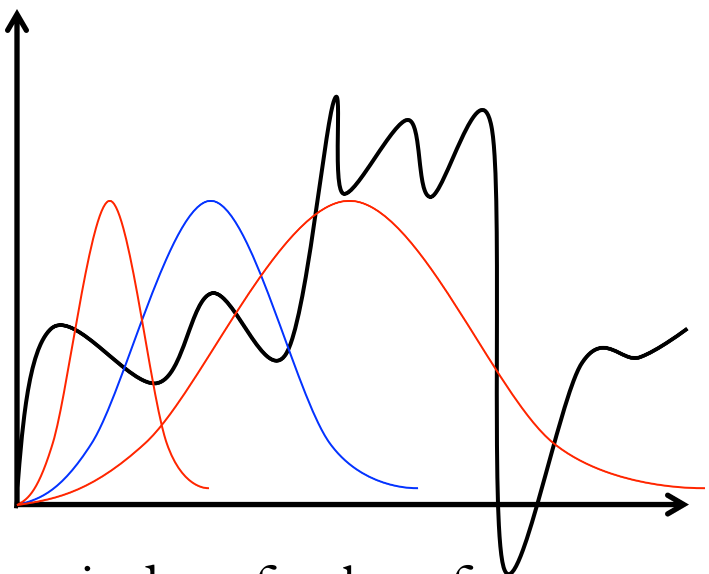
- Fixed window size, shift in time (Gabor)



- Make the window larger
  - Worse localization
  - More spectral resolution

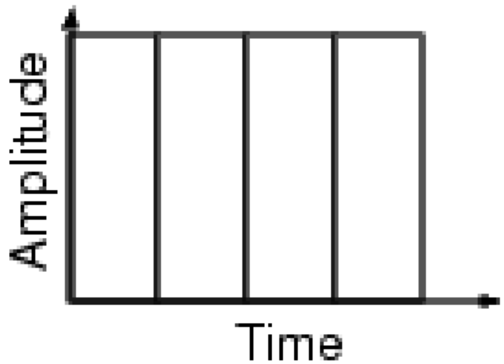
# Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)

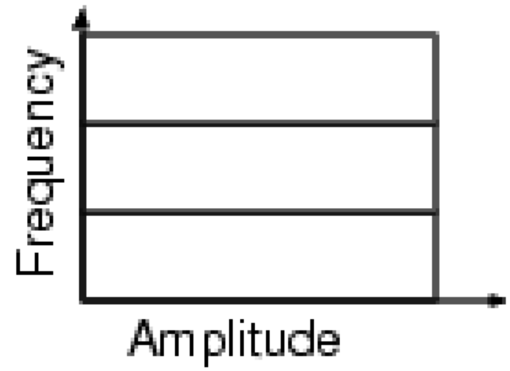


  - Use a big window for low frequency content that is not localized in time
  - Use a small window for high frequency content that is localized in time

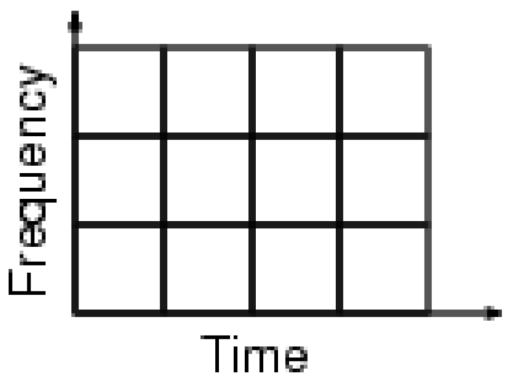
# Transform Comparison



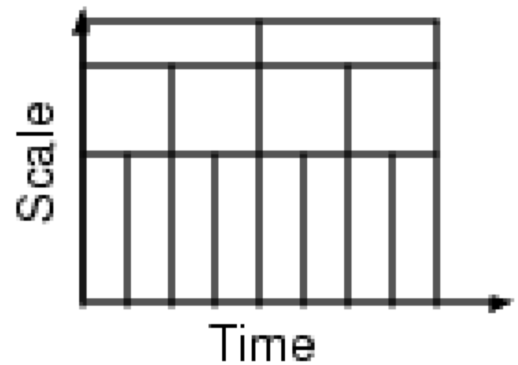

Time Domain (Shannon)




Frequency Domain (Fourier)



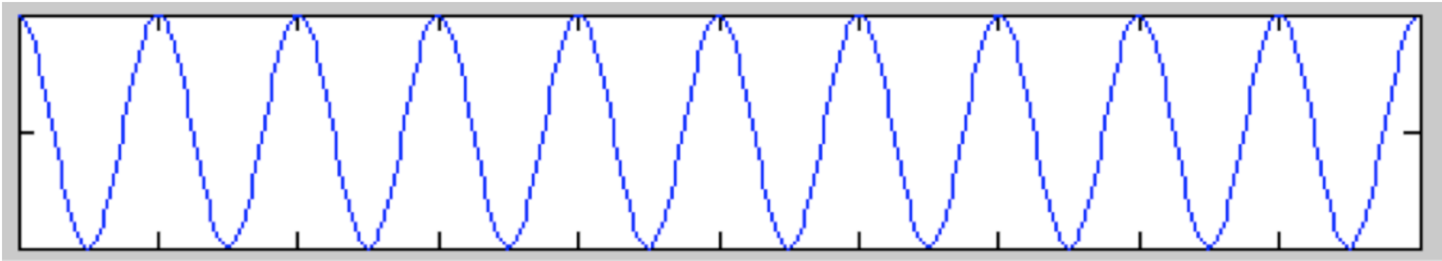

STFT (Gabor)



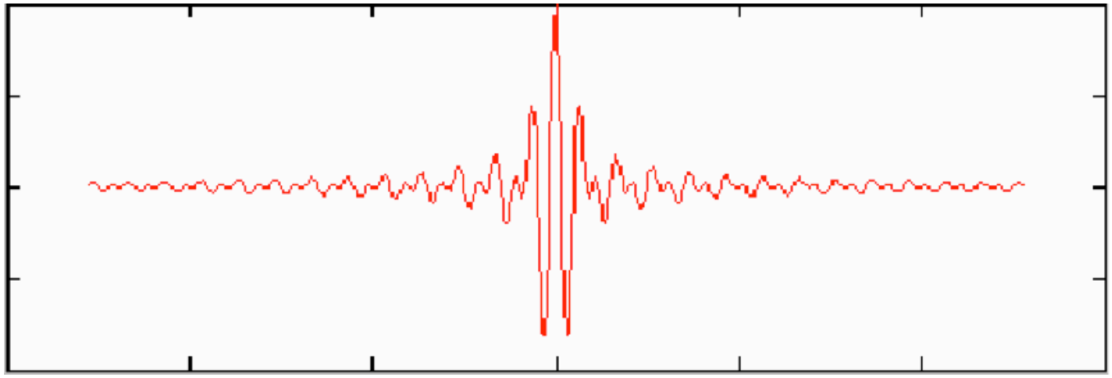

Wavelet Analysis

# Fourier vs. Wavelet

- Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



- Wavelet Analysis is based on an short duration wavelet of a specific center frequency

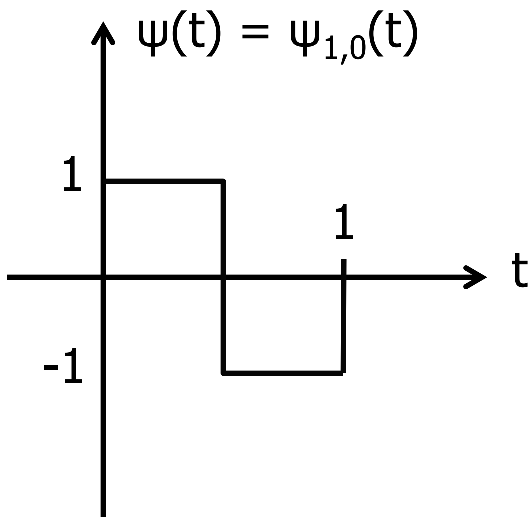


# Wavelet Transform

- All wavelets derived from *mother* wavelet

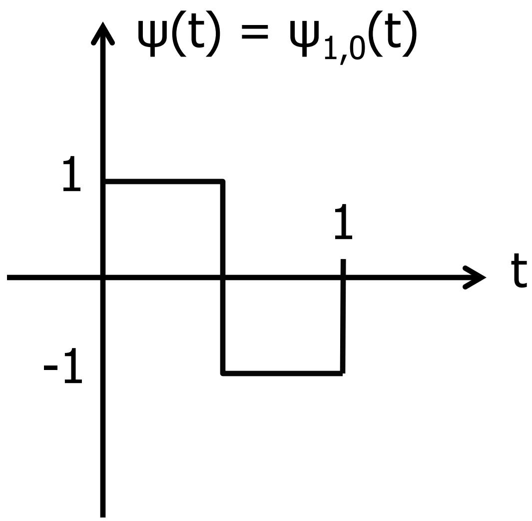
$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\psi\left(\frac{t-\tau}{s}\right)$$

# Example: Haar Wavelet



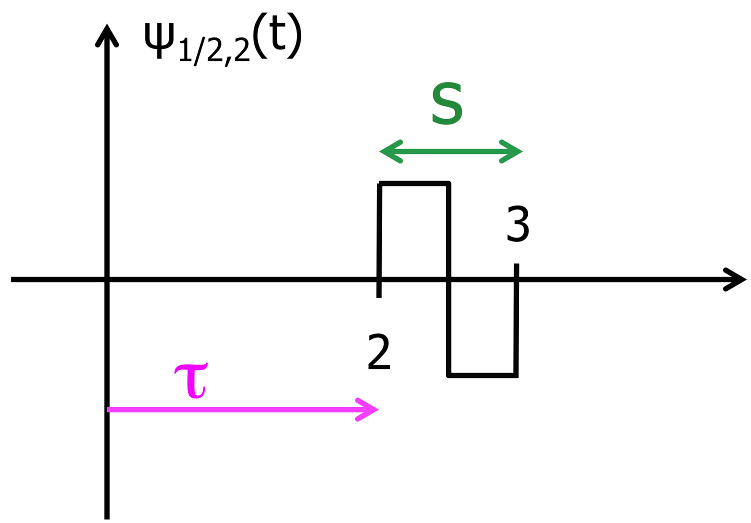

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

# Example: Haar Wavelet

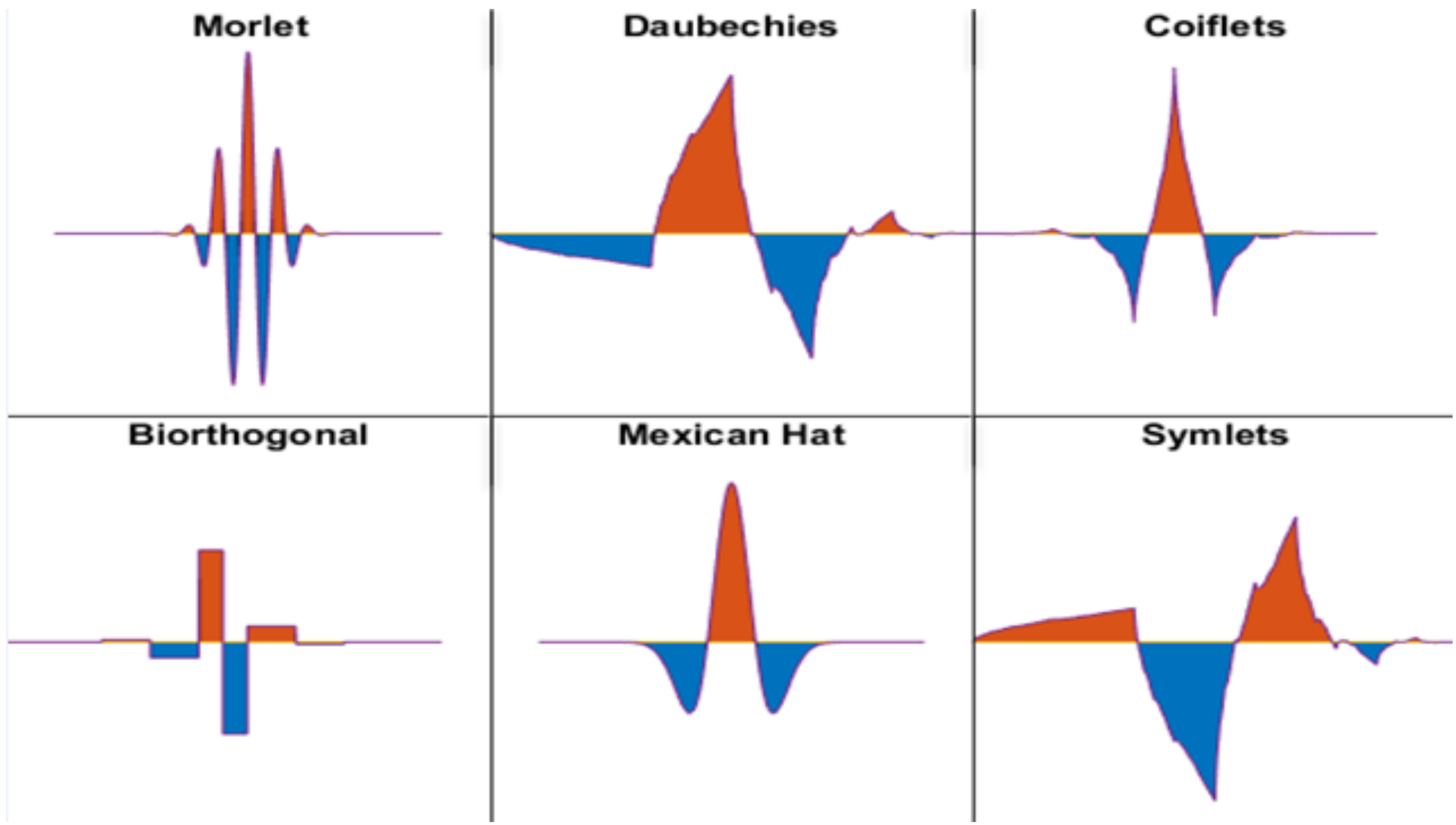



$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

$s=1/2$, $\tau=2$



# Examples of Wavelets



# Wavelet – Scaled and Shifted

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

change in scale

wavelet with
scale, s and
translation, τ

Mother wavelet

# Continuous Wavelet Transform

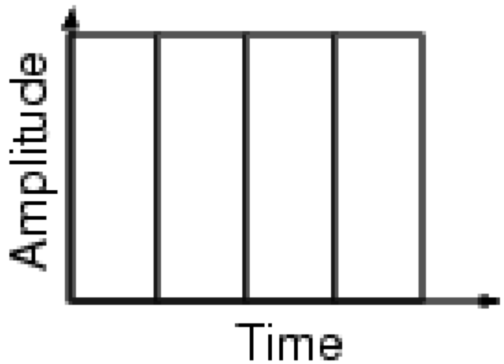
time-series

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

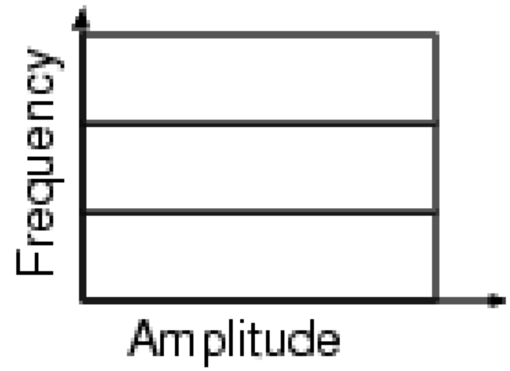
coefficient of wavelet
with
scale, s and time, τ

wavelet with
scale, s, and shift, τ

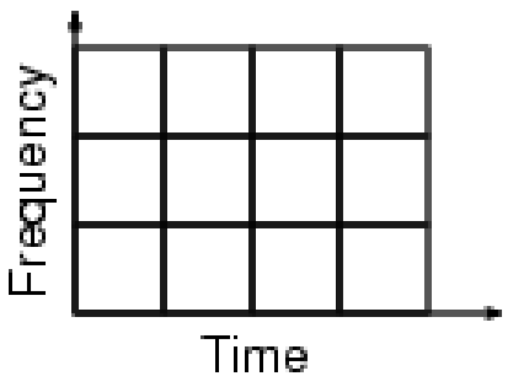
# Transform Comparison



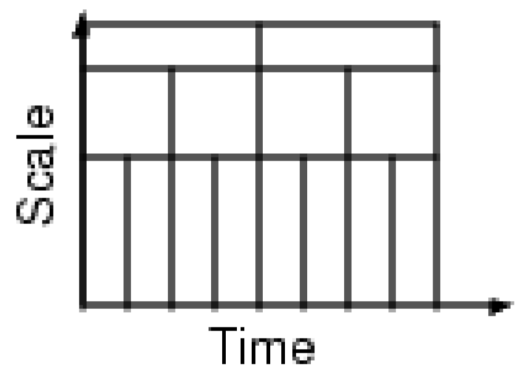
Time Domain (Shannon)



Frequency Domain (Fourier)

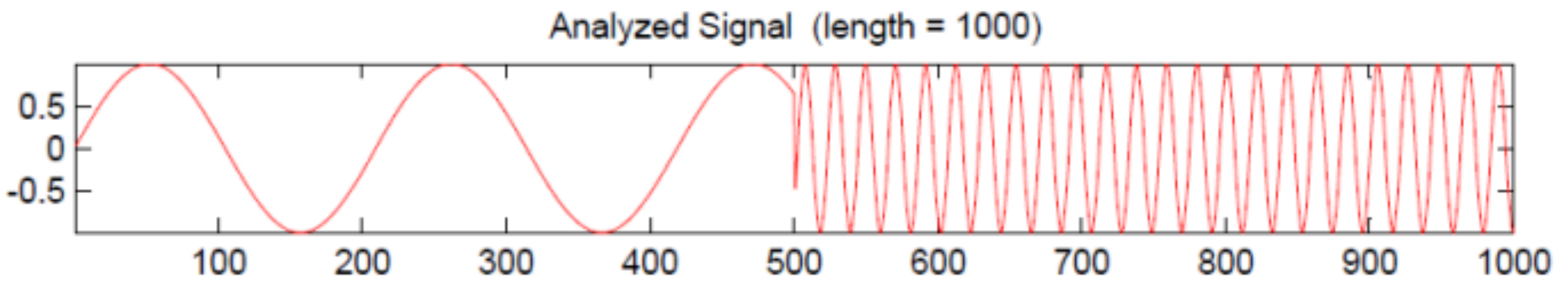


STFT (Gabor)



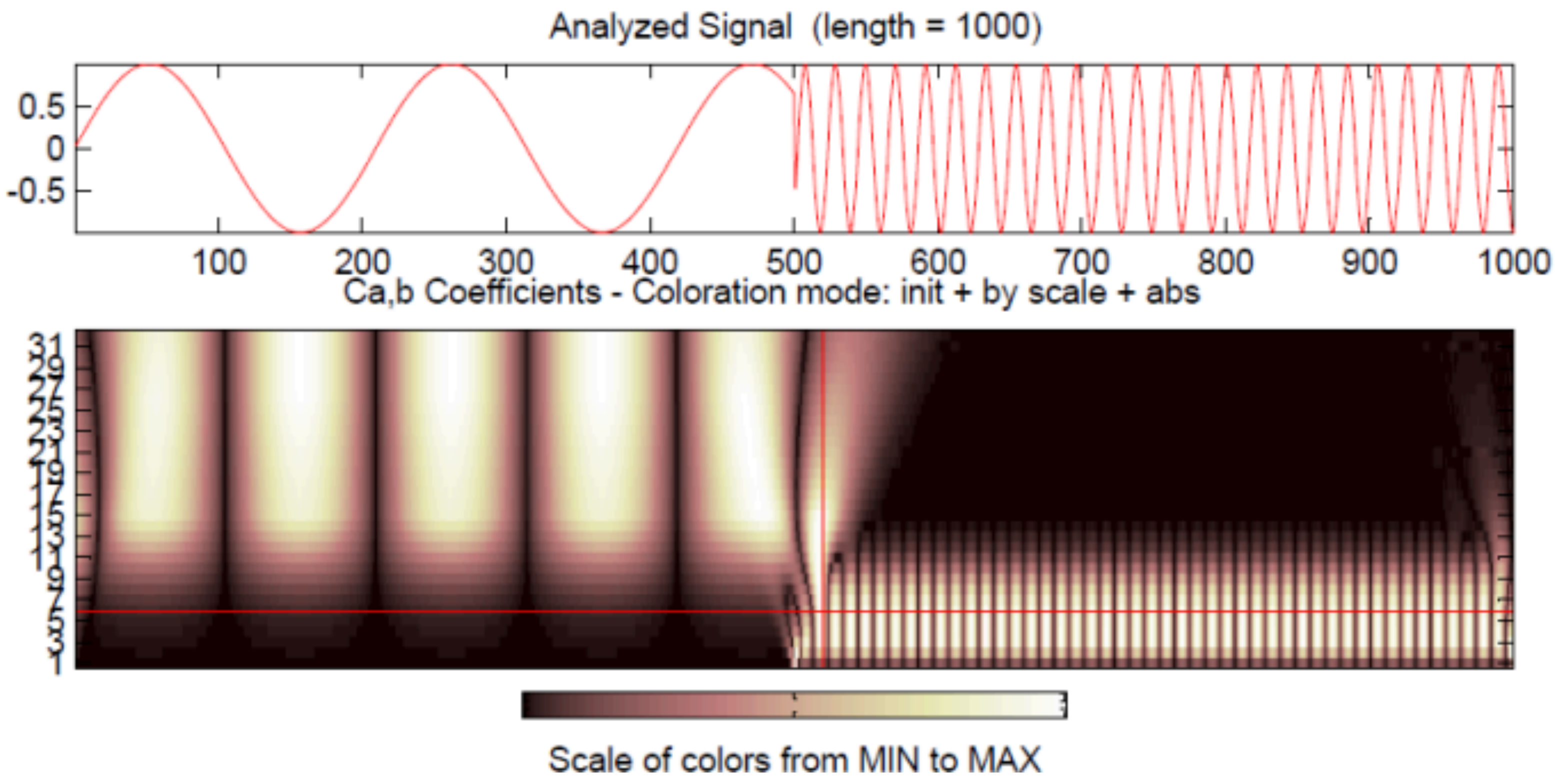
Wavelet Analysis

# Wave Demo



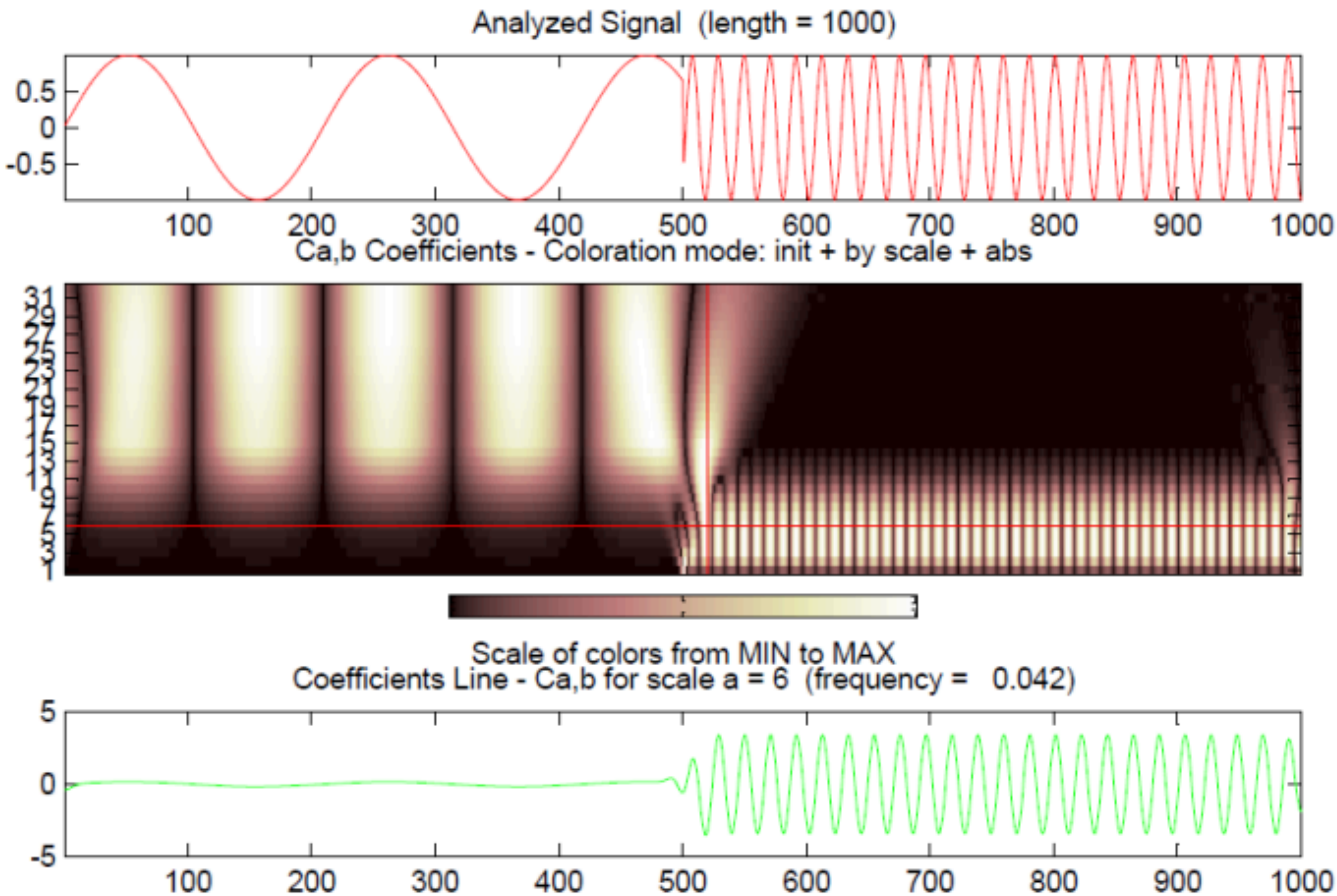
Analyzed Signal (length = 1000)
0.5
0
-0.5
100
200
300
400
500
600
700
800
900
1000

# Wave Demo



Analyzed Signal (length = 1000)
0.5
0
-0.5
100
200
300
400
500
600
700
800
900
1000
Ca,b Coefficients - Coloration mode: init + by scale + abs
Scale of colors from MIN to MAX

# Wave Demo



Analyzed Signal (length = 1000)
0.5
0
-0.5
100
200
300
400
500
600
700
800
900
1000
Ca,b Coefficients - Coloration mode: init + by scale + abs
31
29
27
25
23
21
19
17
15
13
11
9
7
5
3
1
Scale of colors from MIN to MAX
Coefficients Line - Ca,b for scale a = 6 (frequency = 0.042)
5
0
-5
100
200
300
400
500
600
700
800
900
1000

# Inverse Wavelet Transform

- Build up a time-series as sum of wavelets of different scales, s, and positions, t

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)\,d\tau\,ds$$

time-series

wavelet with scale, s and time, τ

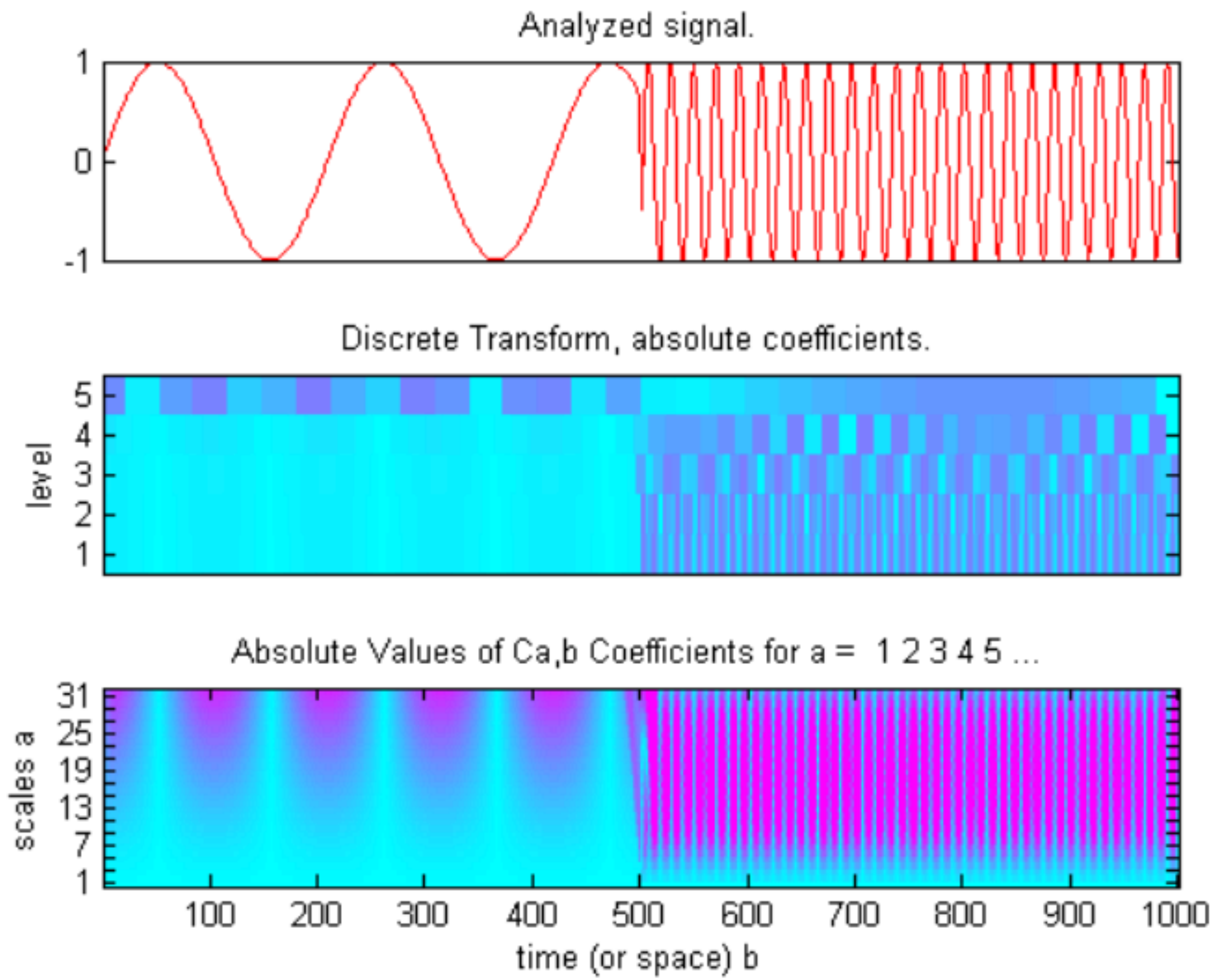
coefficients of wavelets

# Discrete wavelets:

- Scale wavelets only by integer powers of 2
  - $s_j = 2^j$
- And shifting by integer multiples of $s_j$ for each successive scale
  - $\tau_{j,k} = k2^j$
- Then $\gamma(s_j, \tau_{j,k}) = \gamma_{jk}$
  - where $j = 1, 2, \ldots\infty$, $k = -\infty\ldots -2, -1, 0, 1, 2, \ldots\infty$

$$\gamma_{j,k} = \frac{1}{\sqrt{2^j}} \int f(t)\Psi\left(\frac{t-k2^j}{2^j}\right)dt$$

# DWT vs CWT



Analyzed signal.
1
0
-1
Discrete Transform, absolute coefficients.
level
5
4
3
2
1
Absolute Values of Ca,b Coefficients for a = 1 2 3 4 5 ...
scales a
31
25
19
13
7
1
100
200
300
400
500
600
700
800
900
1000
time (or space) b

# Wavelet Transform

- Determining the wavelet coefficients for a fixed scale, *s*, can be thought of as a filtering operation

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

$$\gamma_s(t) = \int f(t)\Psi_s(t)\,dt = f(t) * \Psi_s(t)$$

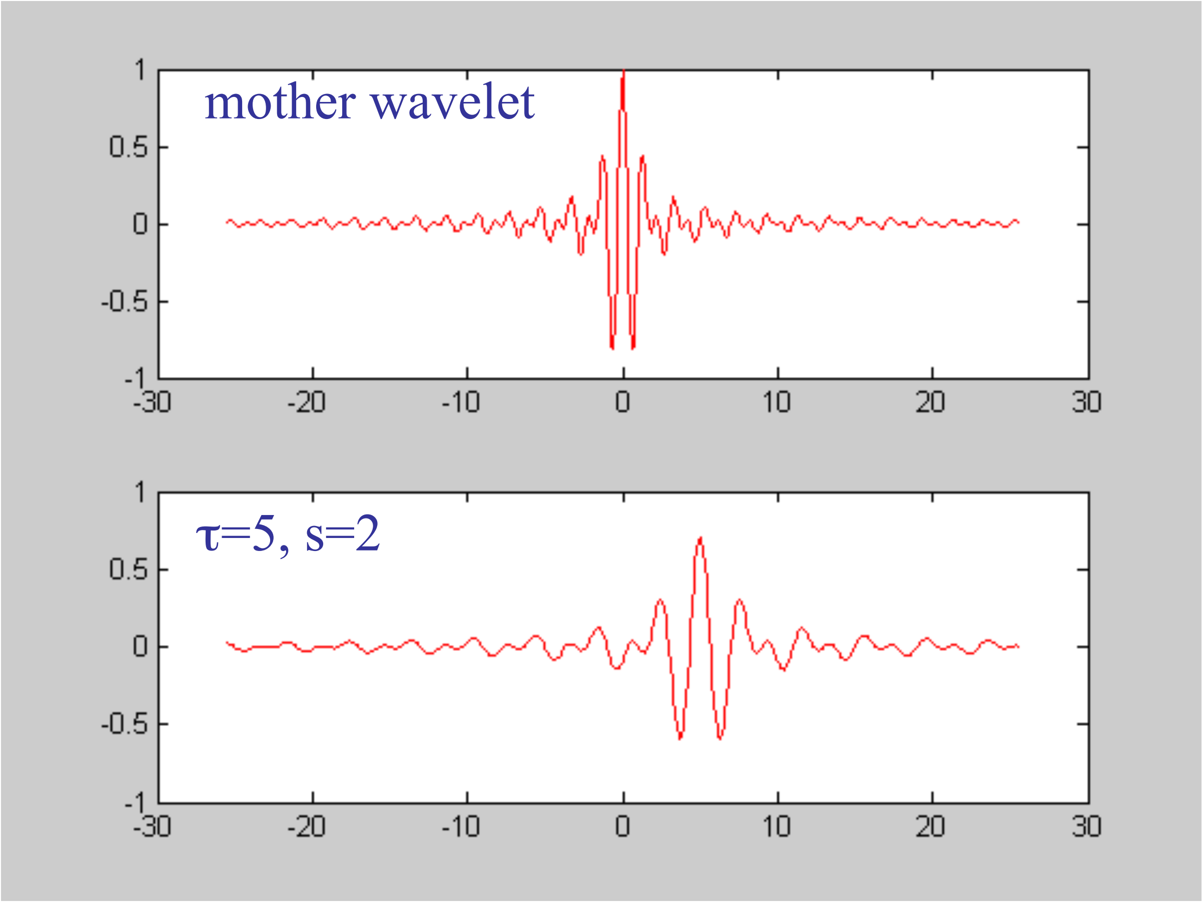
- where

$$\Psi_s(t) = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

# Shannon Wavelet

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$$

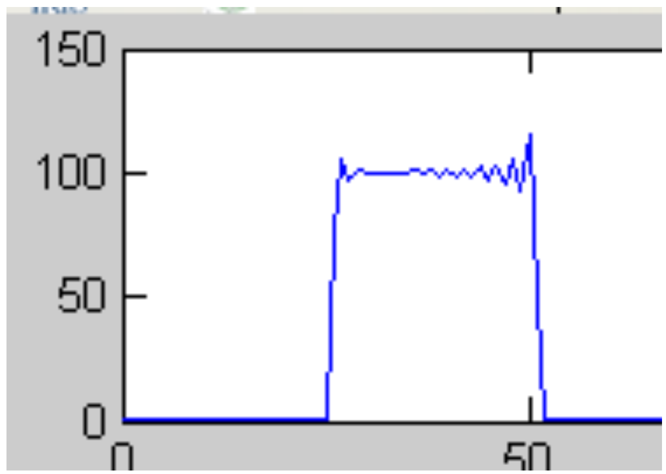
- $\Psi(t) = 2\,\text{sinc}(2t) - \text{sinc}(t)$




time

# Fourier spectrum of Shannon Wavelet

$\Psi_s(j\Omega)$




frequency, $\Omega$

- Wavelet coefficients are a result of bandpass filtering

# Discrete Wavelet Transform

- The coefficients of $\Psi$ is just the band-pass filtered time-series, where $\Psi$ is the wavelet, now viewed as the impulse response of a bandpass filter.
  - Discrete wavelet → $s = 2^j$

# Discrete Wavelet Transform

- The coefficients of $\Psi$ is just the band-pass filtered time-series, where $\Psi$ is the wavelet, now viewed as the impulse response of a bandpass filter.
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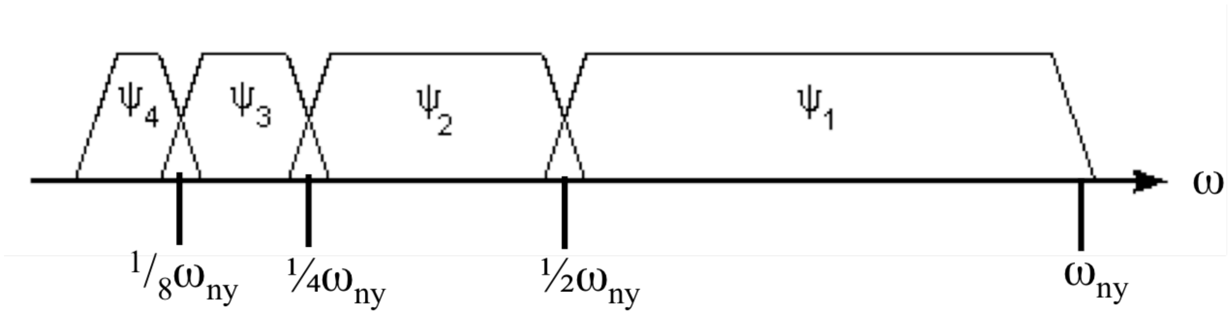

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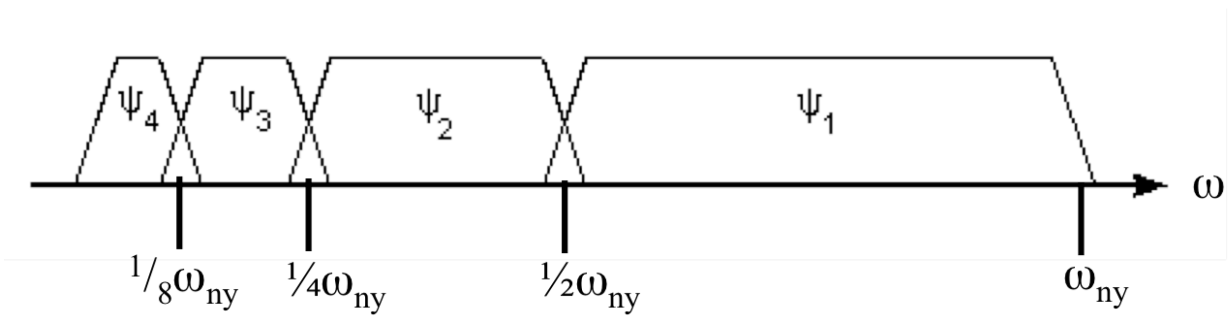
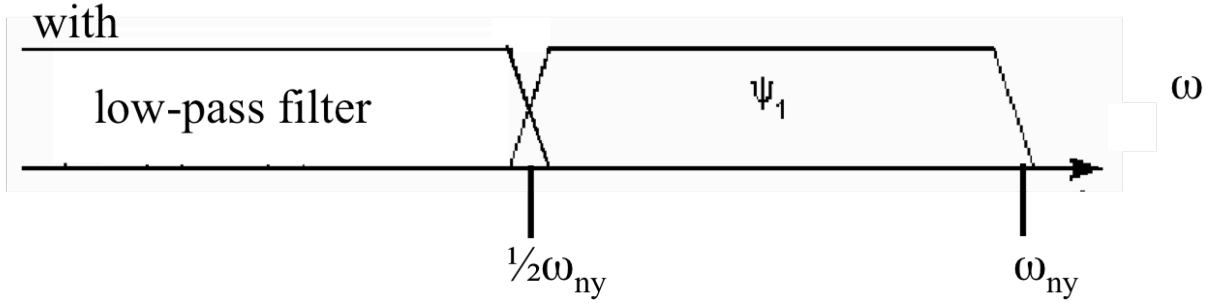
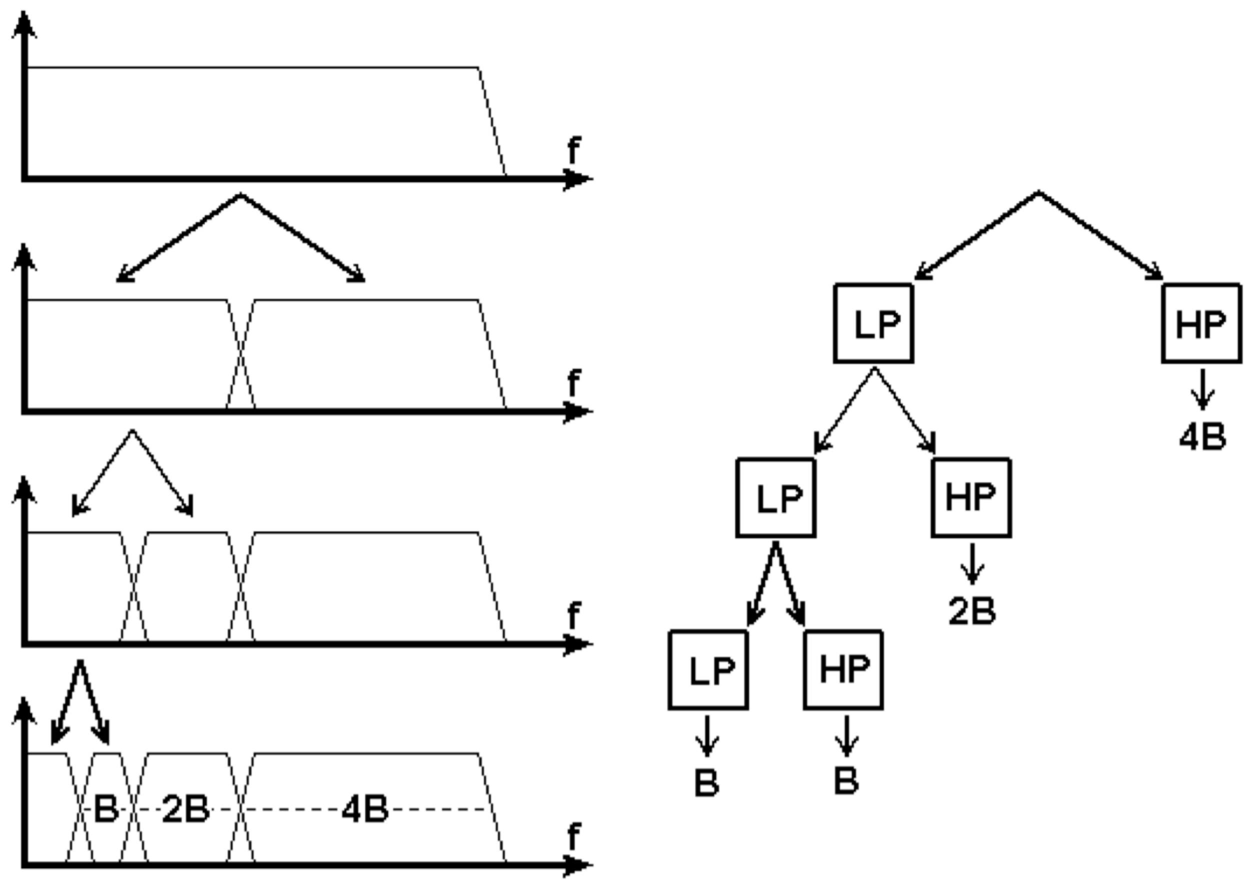



with

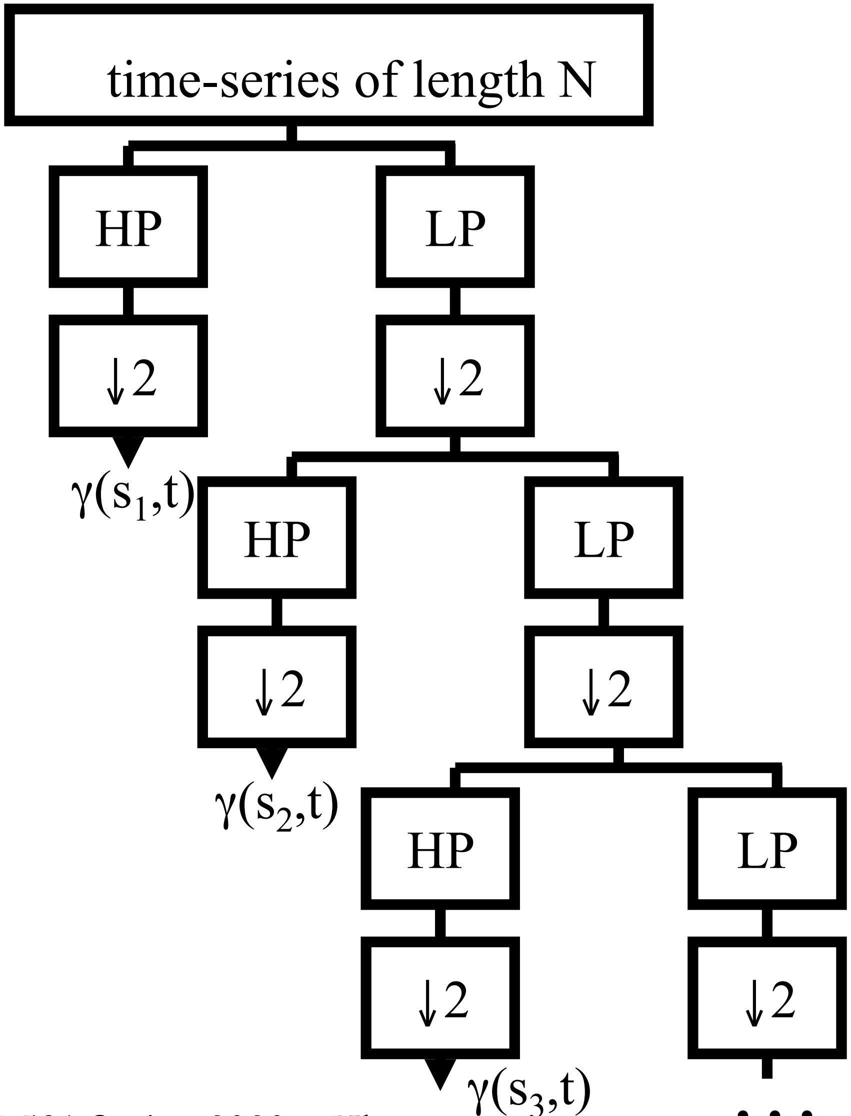


# Digital Wavelet as Multirate Filter Bank

- Repeat recursively!



# Digital Wavelet as Multirate Filter Bank




$\gamma(s_1,t)$: N/2 coefficients

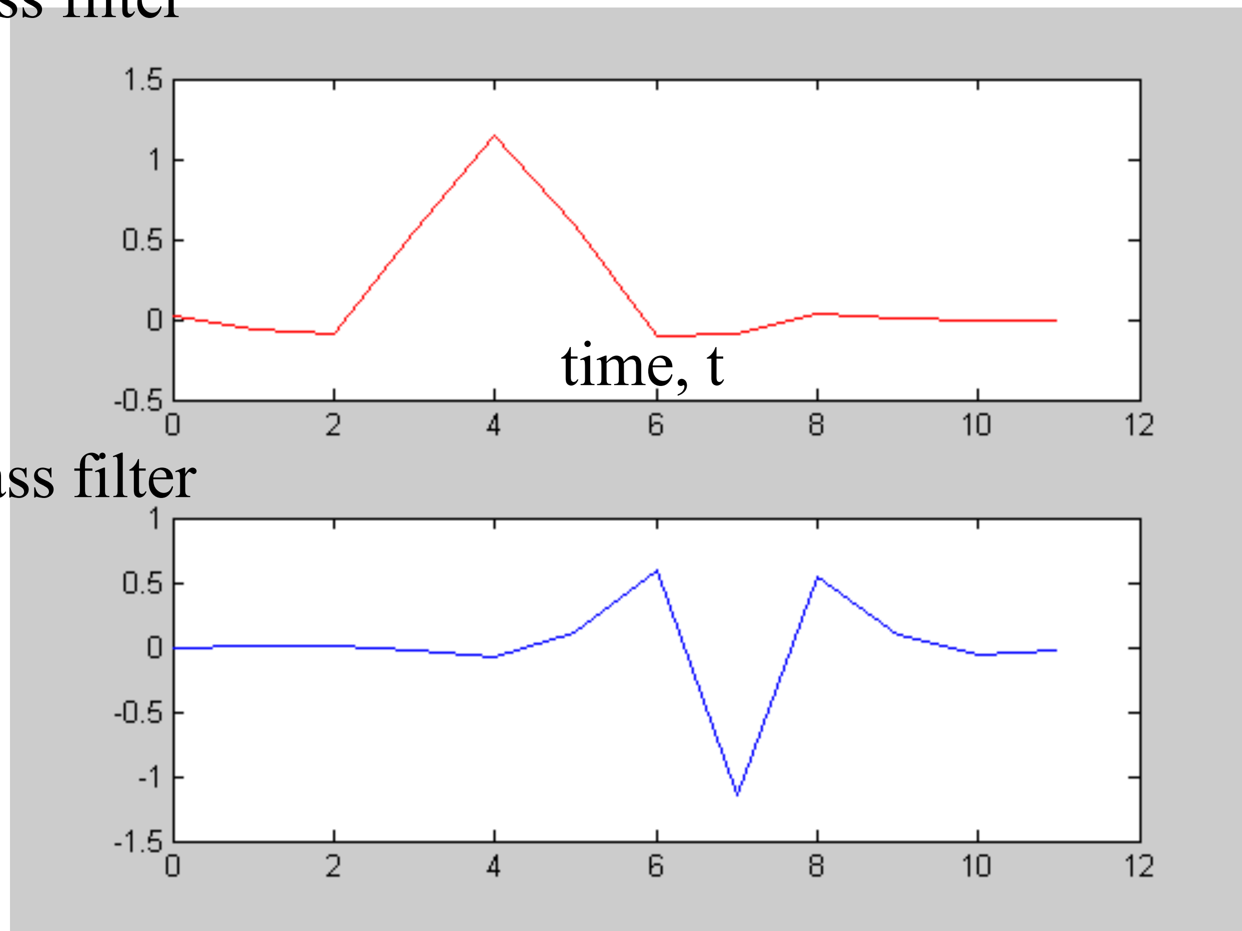
$\gamma(s_2,t)$: N/4 coefficients

$\gamma(s_2,t)$: N/8 coefficients

Total: N coefficients

# Impulse Responses

Coiflet low pass filter

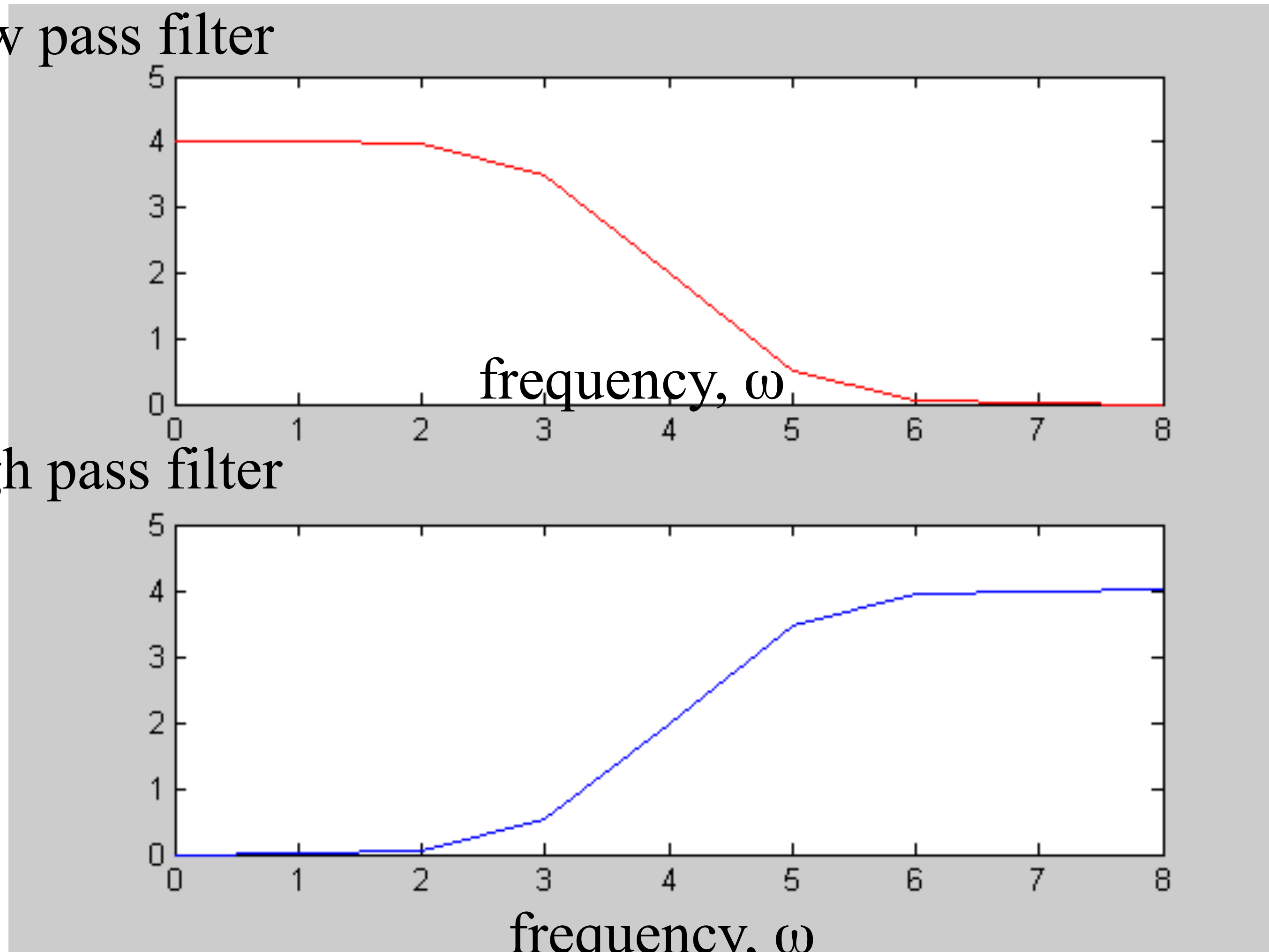



Coiflet high-pass filter

time, t

# Filter Responses

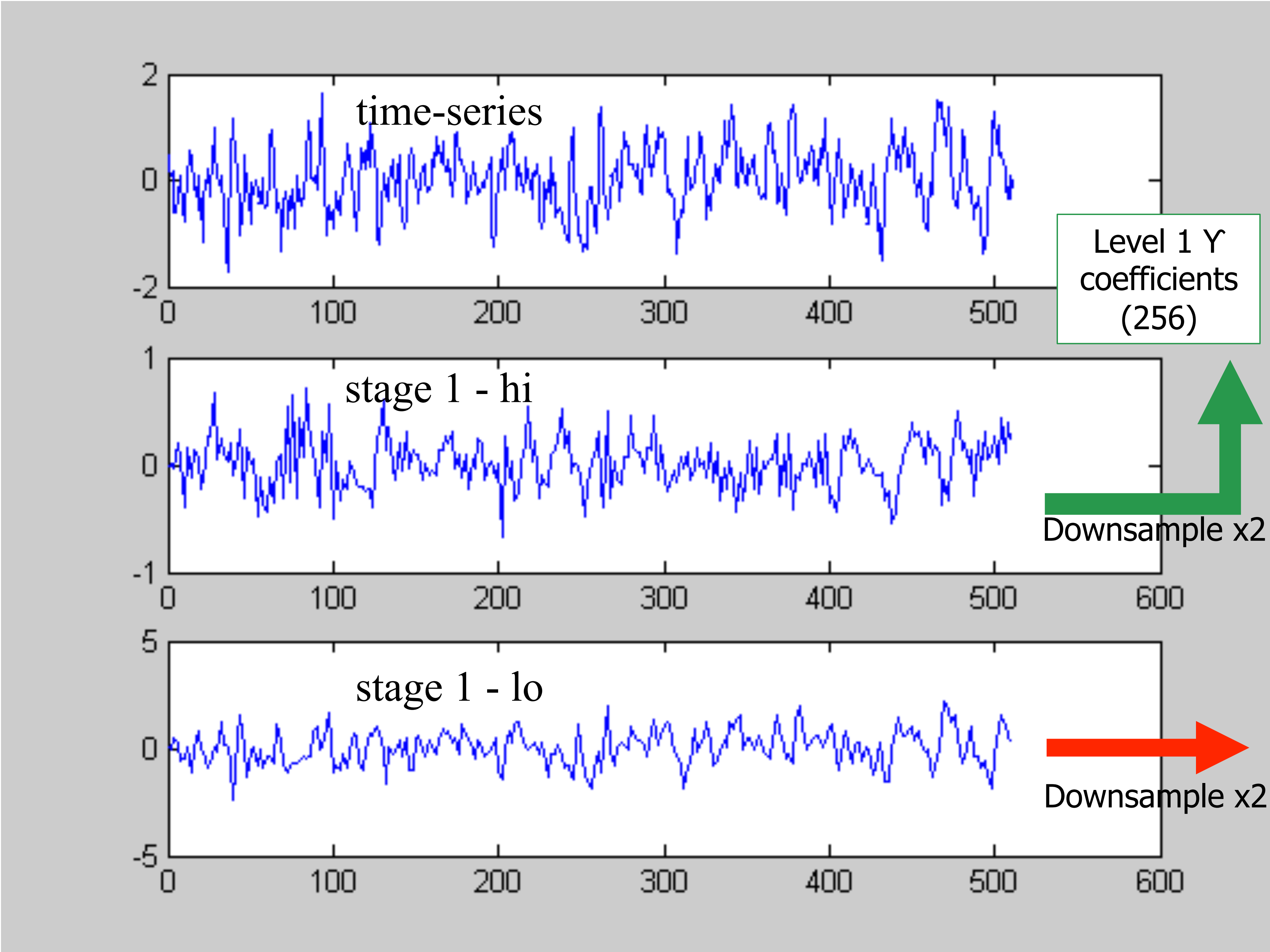
Spectrum of low pass filter

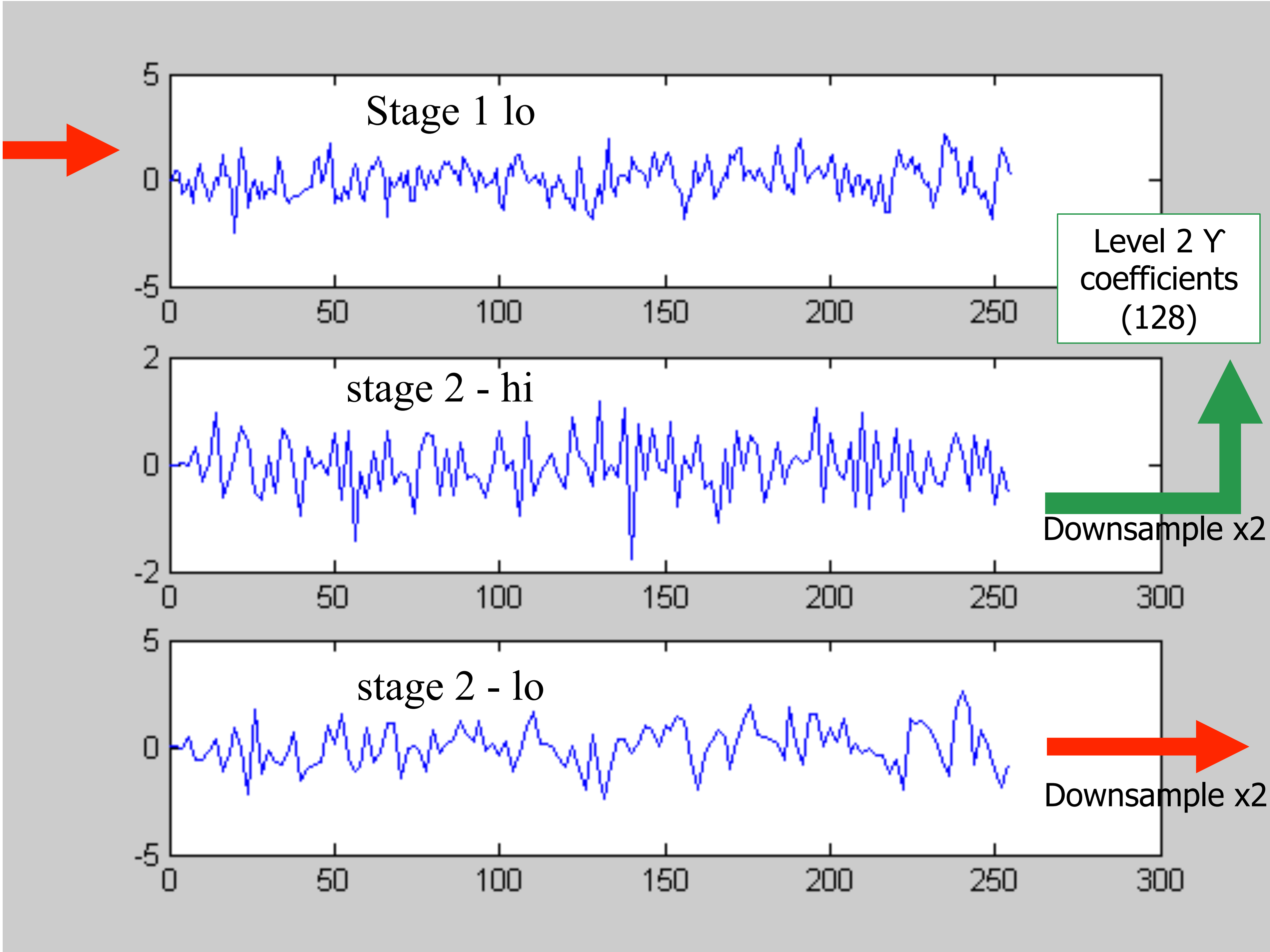


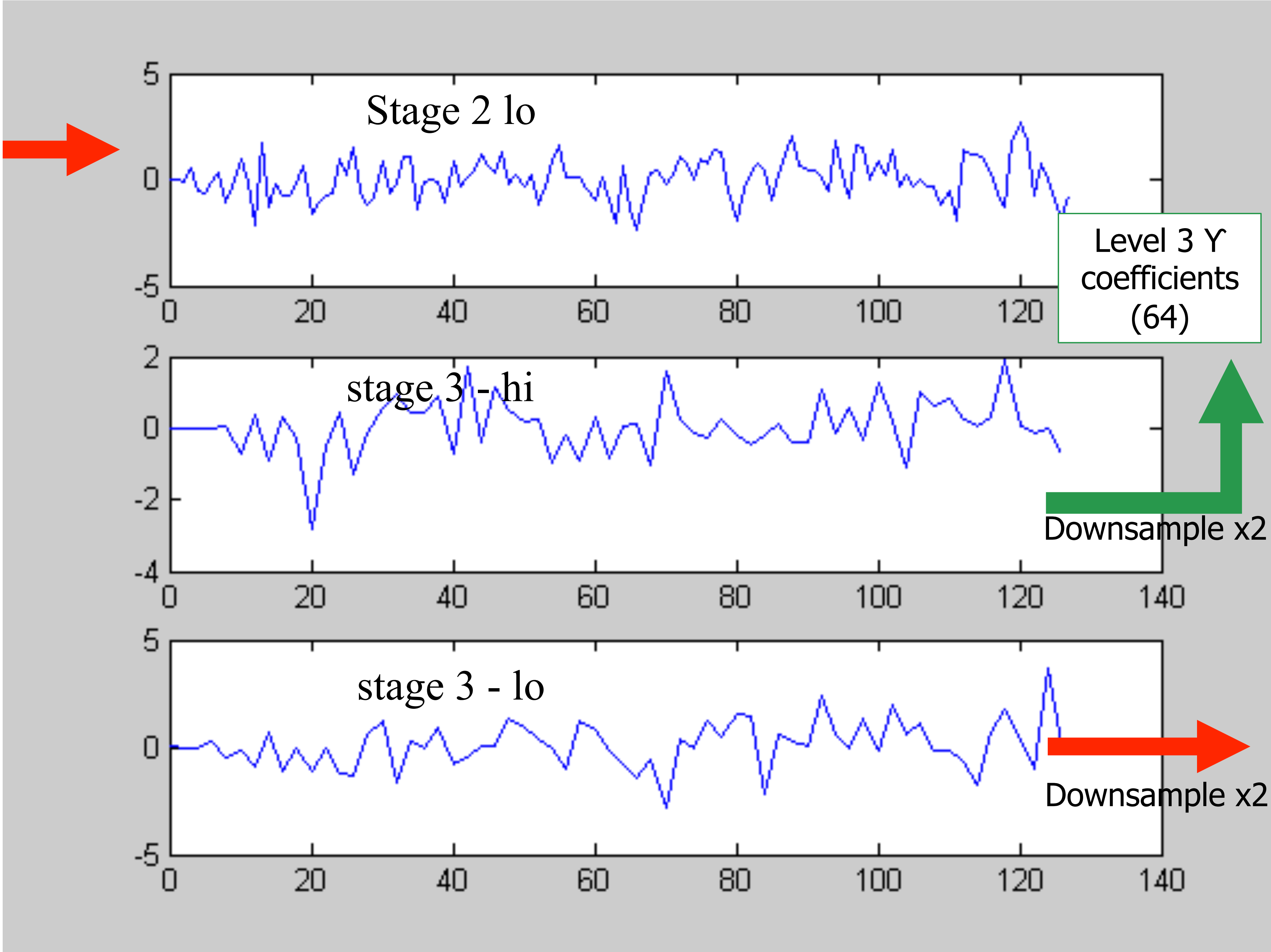

frequency, ω

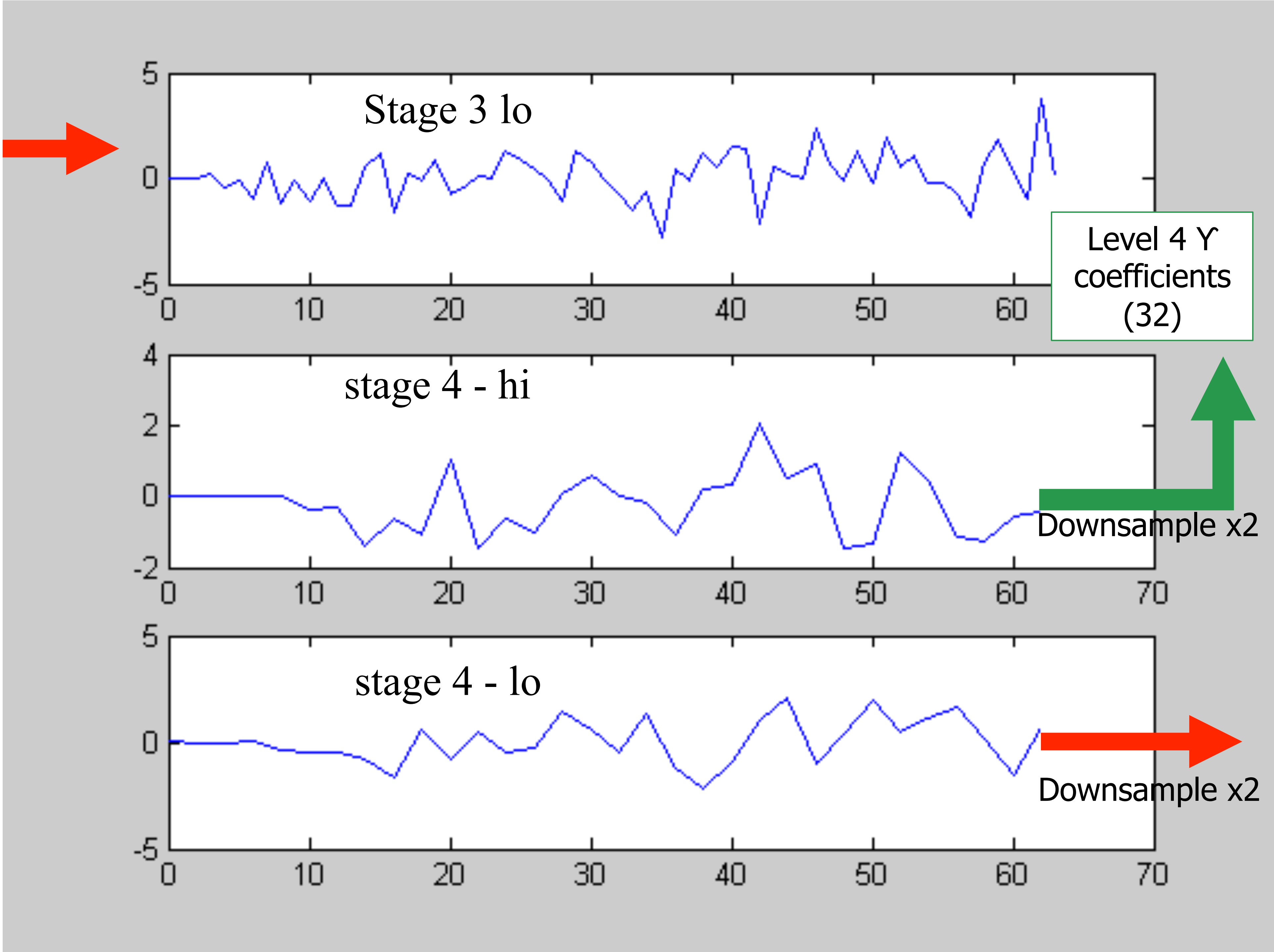
Spectrum of high pass filter

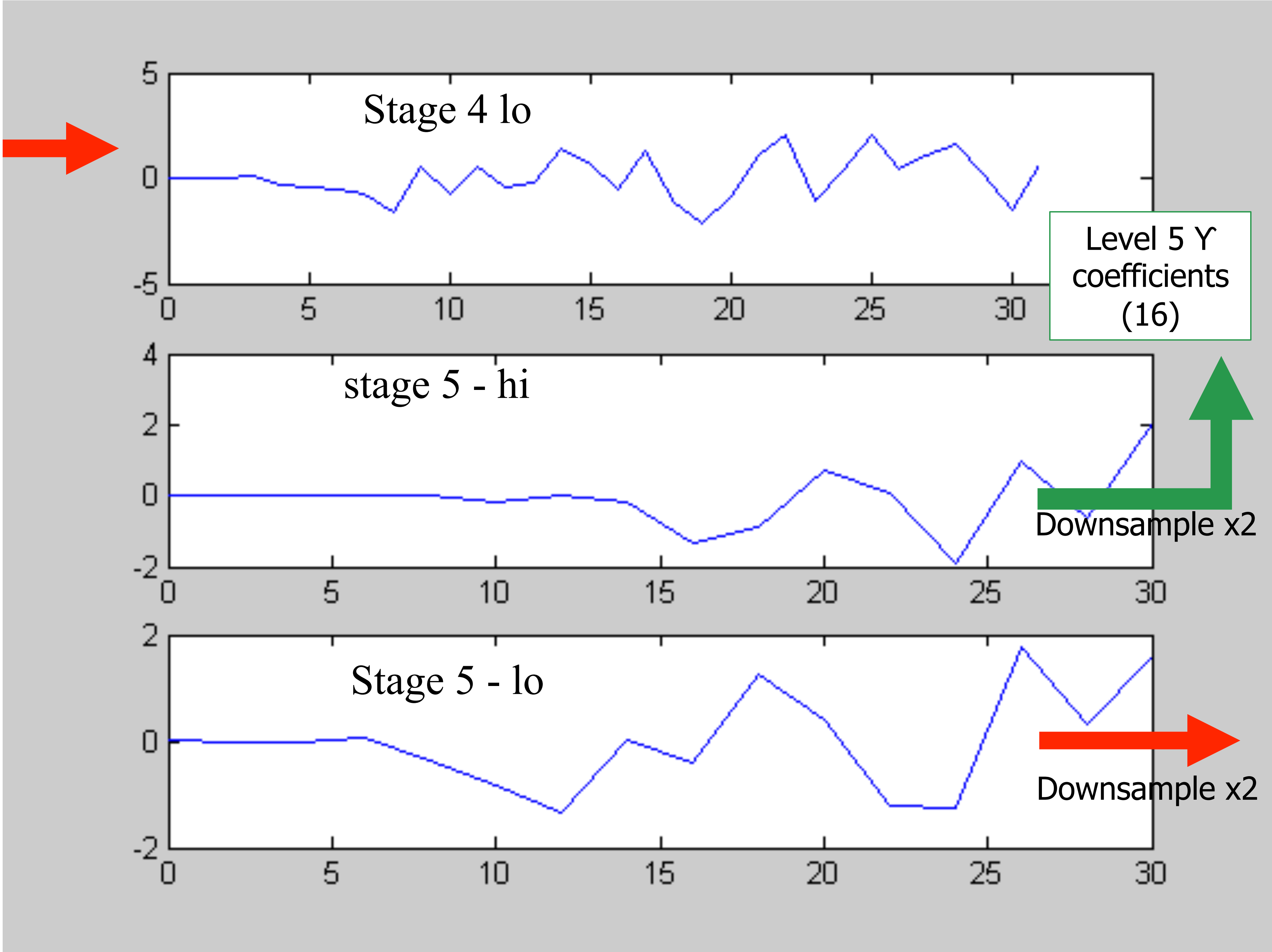
frequency, ω

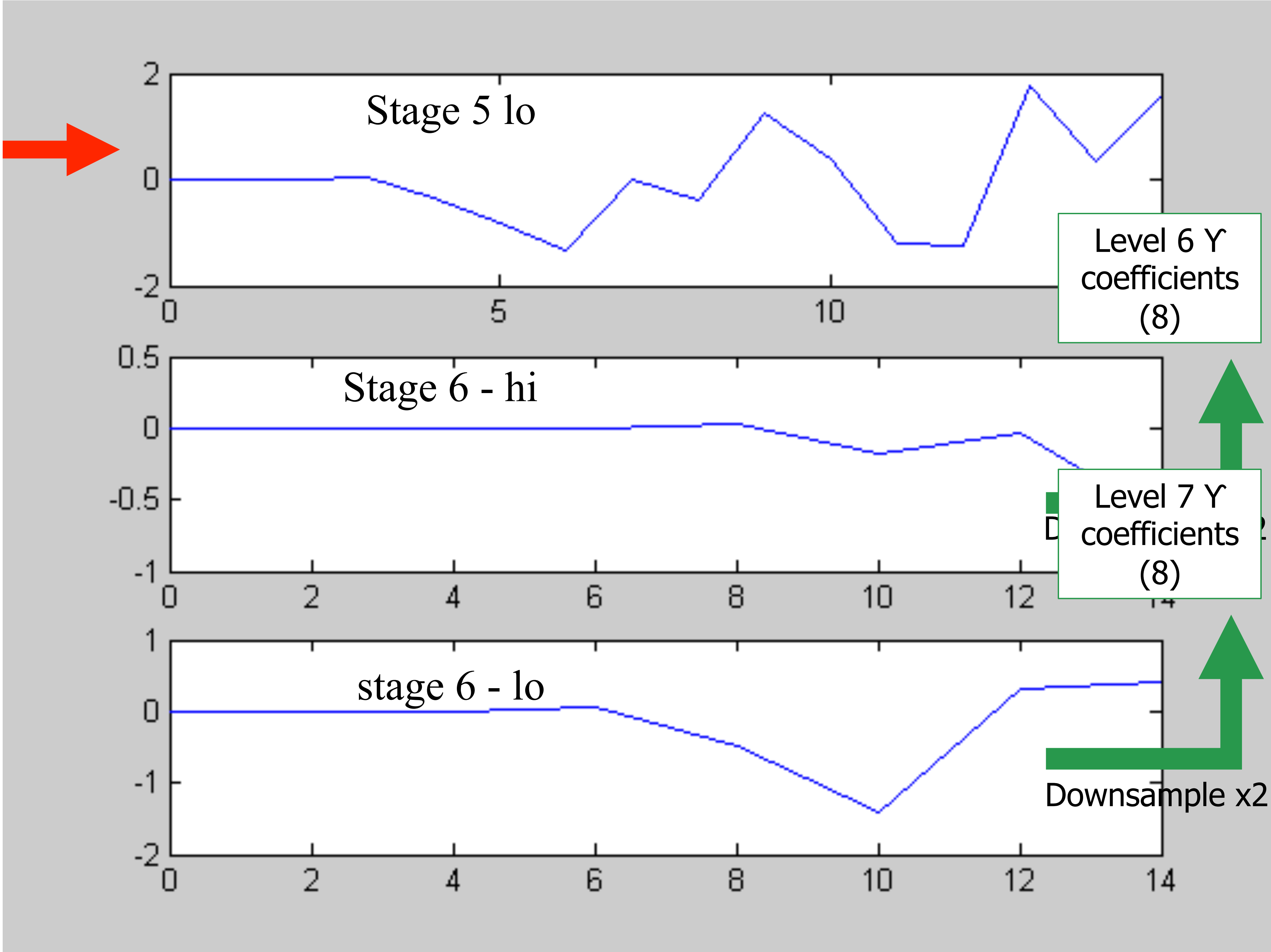

time-series
stage 1 - hi
stage 1 - lo
Level 1 Y coefficients (256)
Downsample x2
Downsample x2
2
0
-2
1
0
-1
5
0
-5
0
100
200
300
400
500
600


Stage 1 lo
stage 2 - hi
stage 2 - lo
Level 2 Y coefficients (128)
Downsample x2
Downsample x2
5
0
-5
2
0
-2
5
0
-5
0
50
100
150
200
250
300

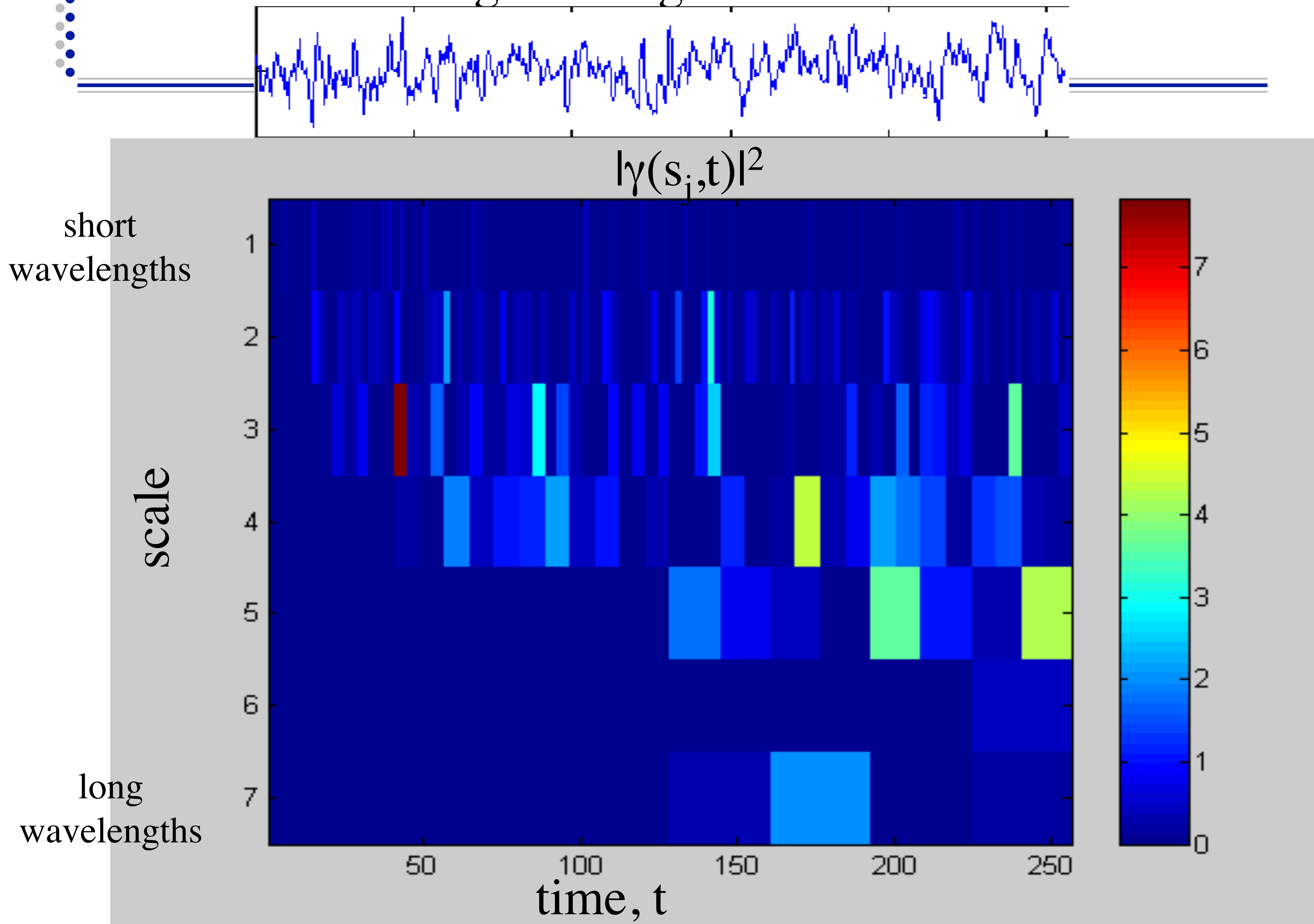

Stage 2 lo
stage 3 - hi
stage 3 - lo
Level 3 Y coefficients (64)
Downsample x2
Downsample x2


Stage 3 lo
stage 4 - hi
stage 4 - lo
Level 4 Y
coefficients
(32)
Downsample x2
Downsample x2

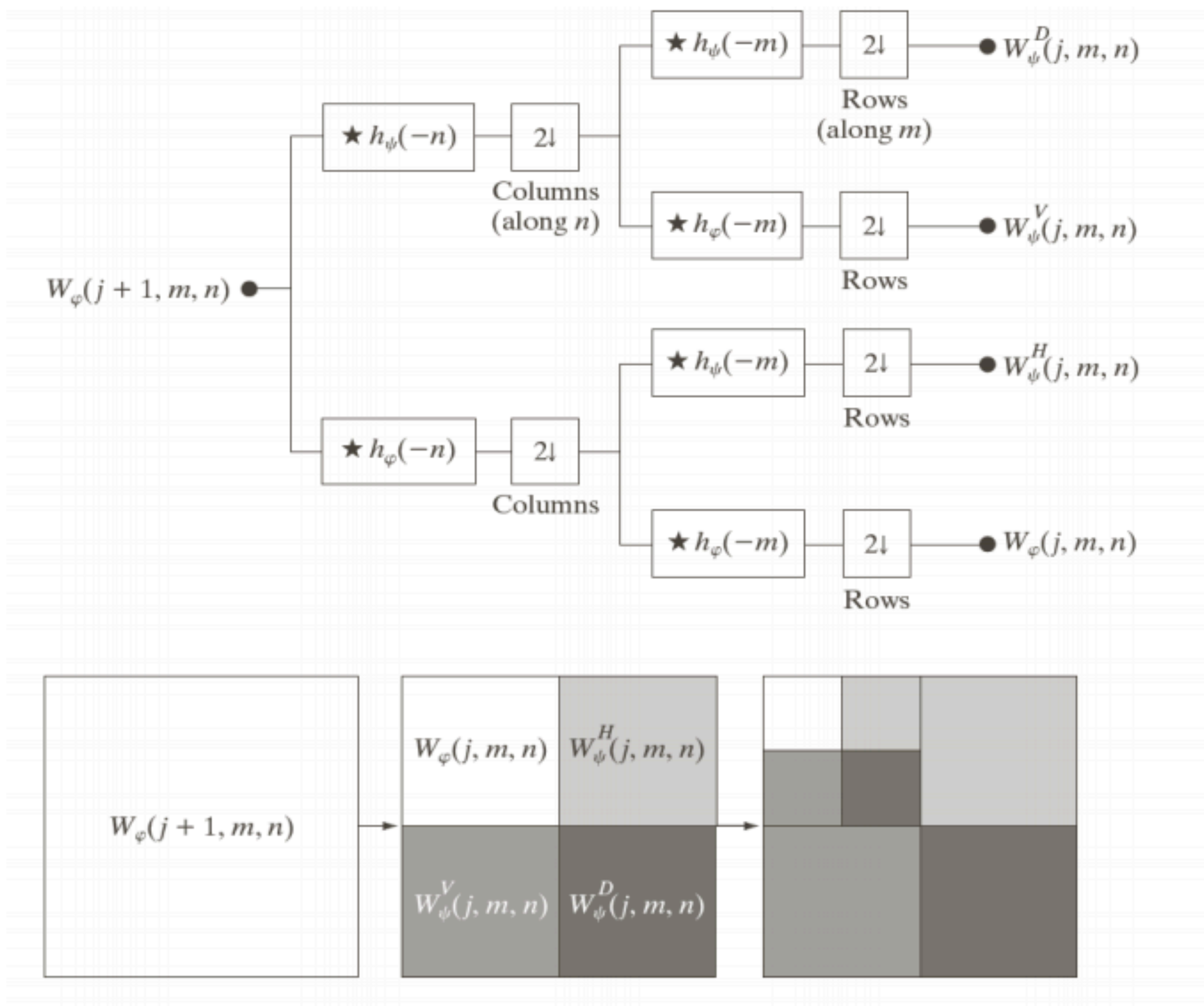

Stage 4 lo
stage 5 - hi
Stage 5 - lo
Level 5 Y coefficients (16)
Downsample x2
Downsample x2


Stage 5 lo
Stage 6 - hi
stage 6 - lo
Level 6 Y coefficients (8)
Level 7 Y coefficients (8)
Downsample x2

# Putting it all together …



# Expanding to Two Dimensions



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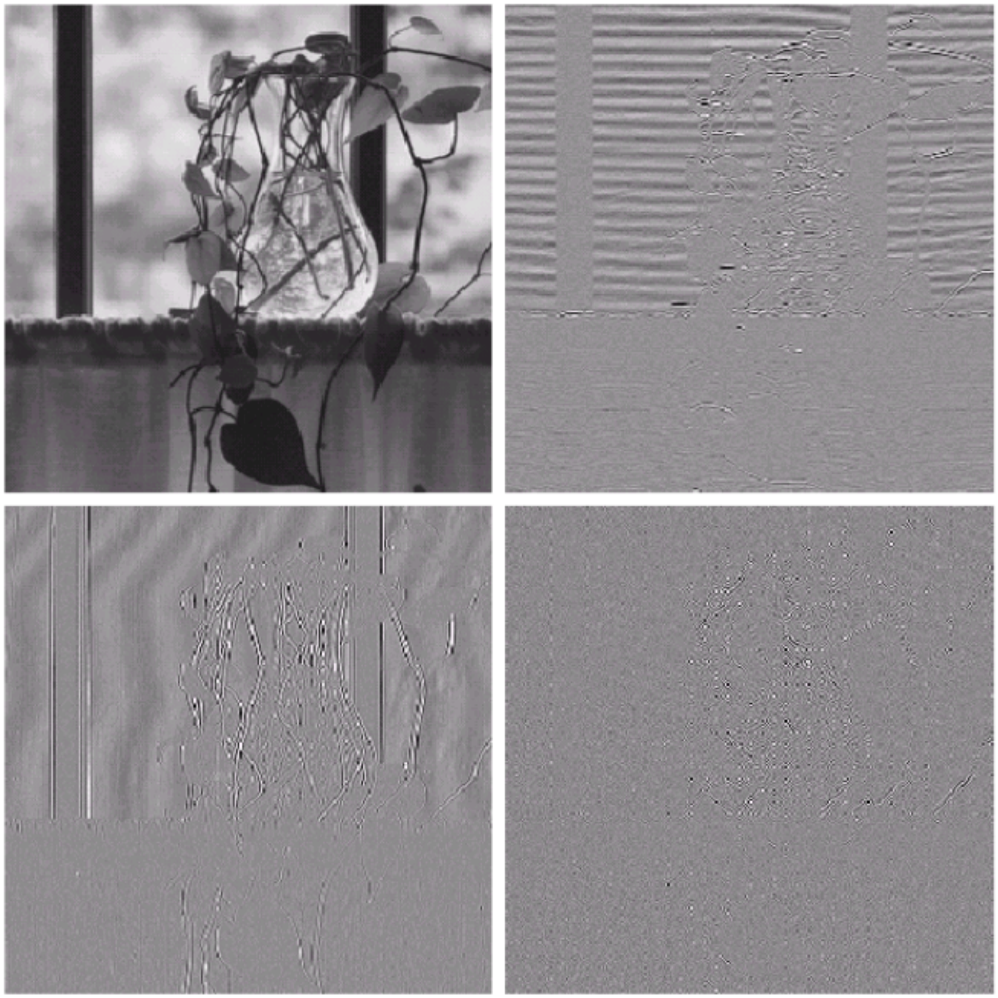


*Colorado School of Mines* *Image and Multidimensional Signal Processing*

**FIGURE 7.7** A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

| | |
|---|---|
| a | $d^V$ |
| $d^H$ | $d^D$ |

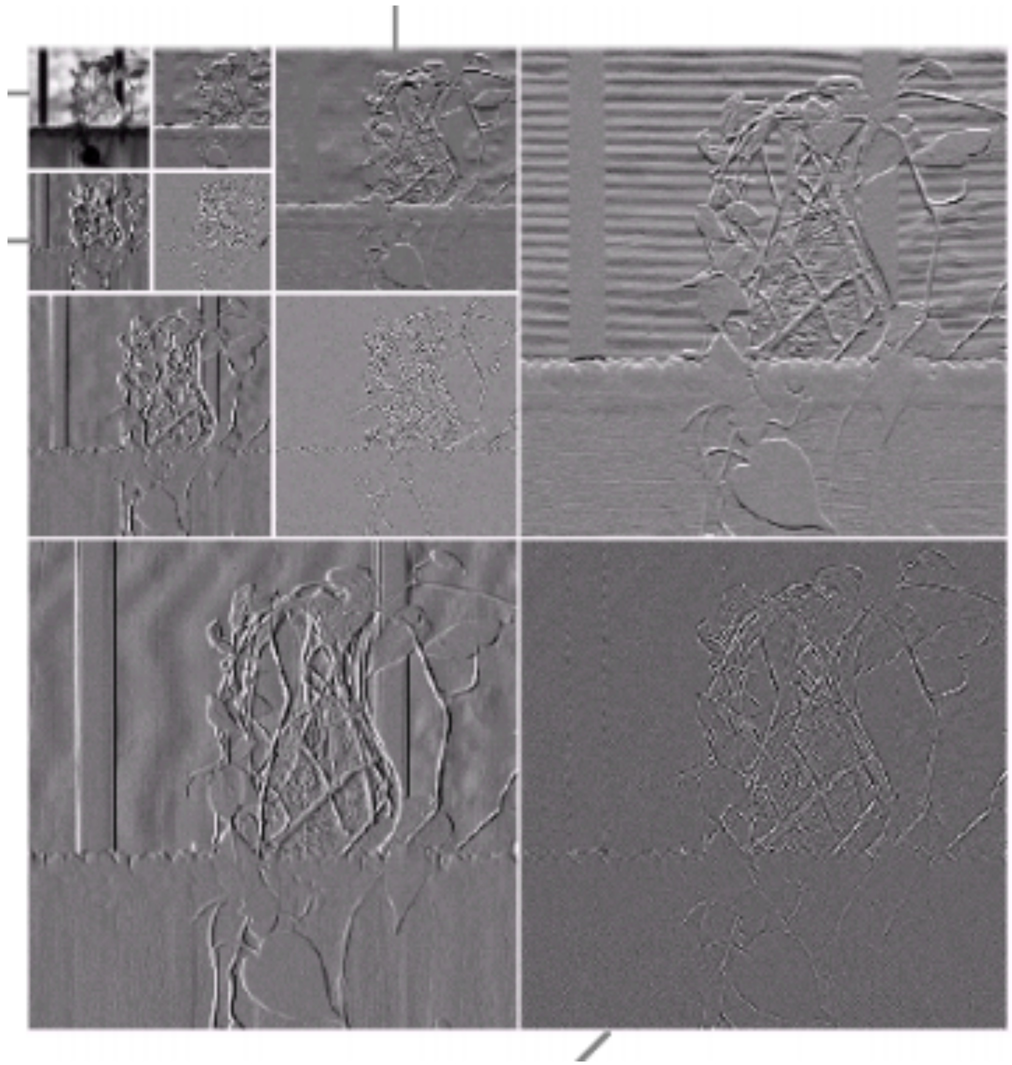
a(m,n): approximation

$d^V$(m,n): detail in vertical

$d^H$(m,n): detail in horizontal

$d^D$(m,n): detail in diagonal

# Expanding to Two Dimensions



# Big Ideas

- Wavelet transform
  - Capture temporal data with fewer coefficients than STFT
  - Use scaling and translation to get different resolution at different levels

# Admin

- Project
  - Due 4/28

- Final Exam – 5/7 (3pm-5pm)
  - In Canvas
    - Will have a 2 hr window to complete within a 12 hr window
  - Open course notes and textbook, but cannot communicate with each other about the exam
    - Students will have randomized and different questions
    - Reminder, it is not in your best interest to share the exam
  - Old exams posted on old course websites
  - Covers Lec 1- 20
    - Does not include lec 12 (data converters and noise shaping) or IIR Filters