ESE 531: Digital Signal Processing

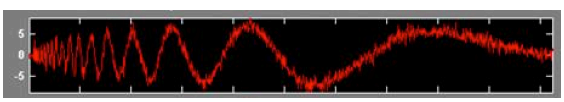
Lec 23: April 21, 2020

Wavelet Transform

Wavelet Transform

Motivation

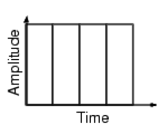
- Some signals obviously have spectral characteristics that vary with time



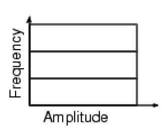
Criticism of Fourier Spectrum

- It's giving you the spectrum of the 'whole time-series'
- Which is OK if the time-series is stationary. But what if its not?
- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

Transform Comparison

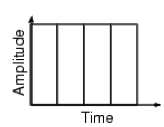


Time Domain (Shannon)

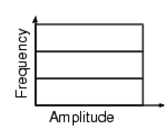


Frequency Domain (Fourier)

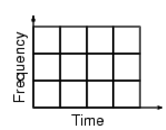
Transform Comparison



Time Domain (Shannon)



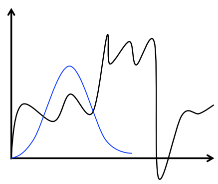
Frequency Domain (Fourier)



STFT (Gabor)

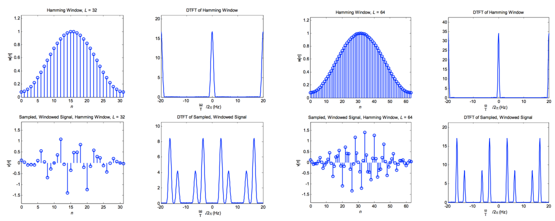
Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)

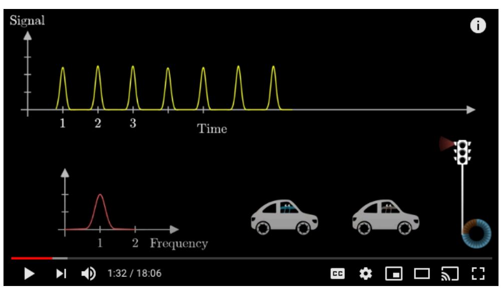


Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi = 1/T = 20Hz$
- Hamming Window, L = 32 vs. L = 64



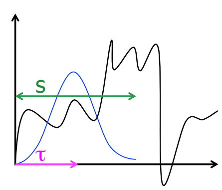
Uncertainty Principle



https://youtu.be/MBnnXbOM5S4?t=49

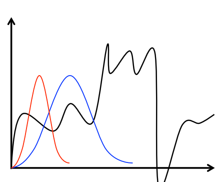
Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)



Discrete Time-Dependent FT

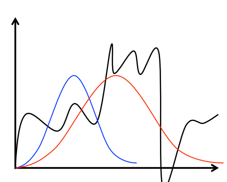
- Fixed window size, shift in time (Gabor)



- Make the window smaller
  - Better localization
  - Less spectral resolution

Discrete Time-Dependent FT

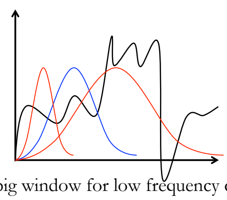
- Fixed window size, shift in time (Gabor)



- Make the window larger
  - Worse localization
  - More spectral resolution

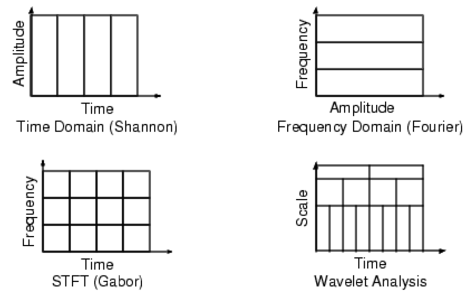
Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)



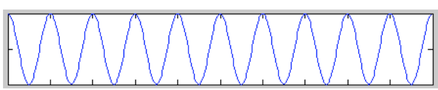
  - Use a big window for low frequency content that is not localized in time
  - Use a small window for high frequency content that is localized in time

Transform Comparison

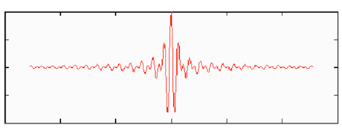


Fourier vs. Wavelet

- Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



- Wavelet Analysis is based on an short duration wavelet of a specific center frequency

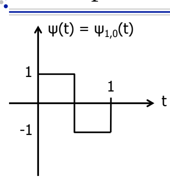


Wavelet Transform

- All wavelets derived from *mother* wavelet

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

Example: Haar Wavelet



$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

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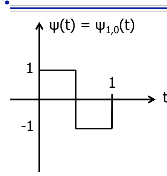
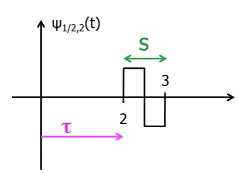
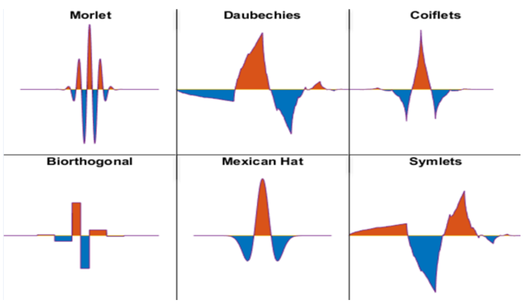


$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

$s=1/2$, $\tau=2$



Examples of Wavelets



Wavelet – Scaled and Shifted

$$\Psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t-\tau}{s}\right)$$

normalization

shift in time

change in scale

wavelet with scale, s and translation, τ

Mother wavelet

Continuous Wavelet Transform

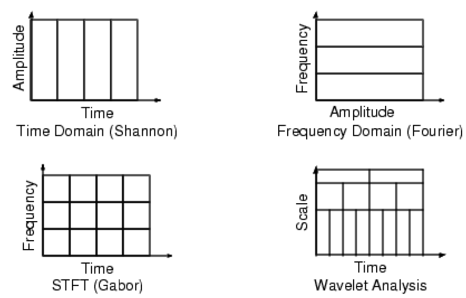
time-series

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

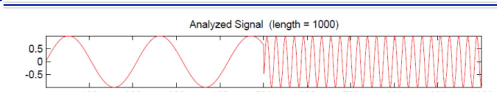
coefficient of wavelet with scale, s and time, τ

wavelet with scale, s, and shift, τ

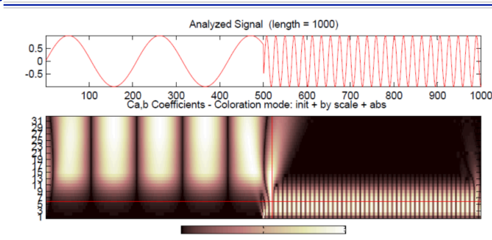
Transform Comparison



Wave Demo



Wave Demo



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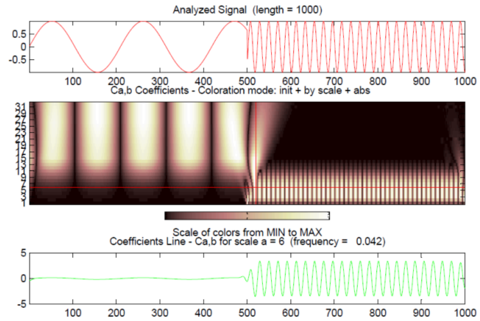


Inverse Wavelet Transform

- Build up a time-series as sum of wavelets of different scales, s, and positions, t

$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)\,d\tau\,ds$$

time-series

coefficients of wavelets

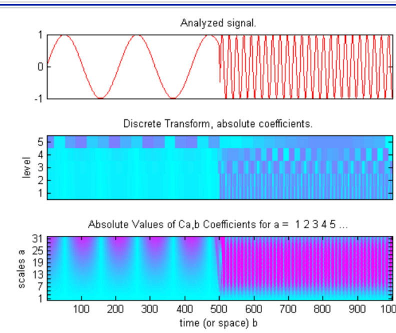
wavelet with scale, s and time, τ

Discrete wavelets:

- Scale wavelets only by integer powers of 2
  - $s_j = 2^j$
- And shifting by integer multiples of s_j for each successive scale
  - $\tau_{j,k} = k2^j$
- Then $\gamma(s_j, \tau_{j,k}) = \gamma_{jk}$
  - where j = 1, 2, …∞, k = -∞… -2, -1, 0, 1, 2, …∞

$$\gamma_{j,k} = \frac{1}{\sqrt{2^j}}\int f(t)\Psi\left(\frac{t-k2^j}{2^j}\right)dt$$

DWT vs CWT



Wavelet Transform

- Determining the wavelet coefficients for a fixed scale, *s*, can be thought of as a filtering operation

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

$$\gamma_s(t) = \int f(t)\Psi_s(t)dt = f(t) * \Psi_s(t)$$

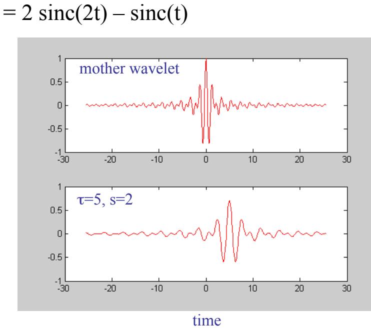
- where

$$\Psi_s(t) = \frac{1}{\sqrt{s}}\Psi\left(\frac{t}{s}\right)$$

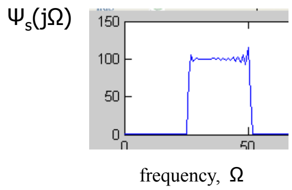
Shannon Wavelet

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

- $\Psi(t) = 2\,\text{sinc}(2t) - \text{sinc}(t)$



Fourier spectrum of Shannon Wavelet



- Wavelet coefficients are a result of bandpass filtering

Discrete Wavelet Transform

- The coefficients of Ψ is just the band-pass filtered time-series, where Ψ is the wavelet, now viewed as the impulse response of a bandpass filter.
  - Discrete wavelet → $s = 2^j$

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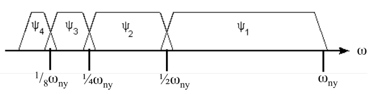

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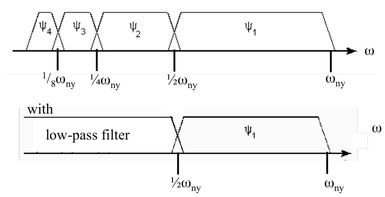
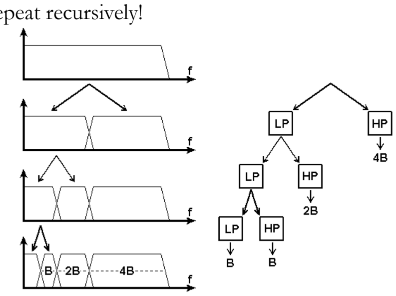
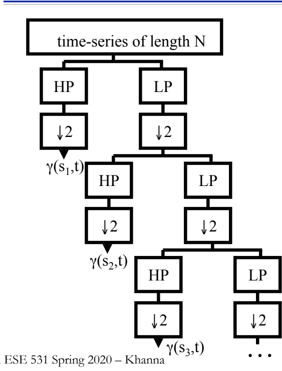


Digital Wavelet as Multirate Filter Bank

- Repeat recursively!



Digital Wavelet as Multirate Filter Bank



$\gamma(s_1,t)$: N/2 coefficients

$\gamma(s_2,t)$: N/4 coefficients

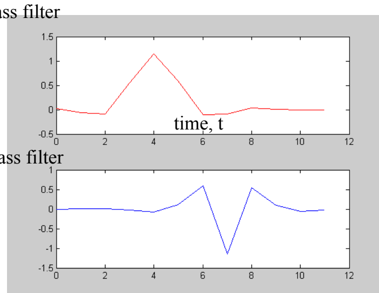
$\gamma(s_2,t)$: N/8 coefficients

Total: N coefficients

Impulse Responses

Coiflet low pass filter

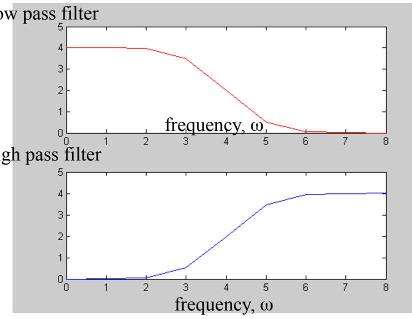
Coiflet high-pass filter

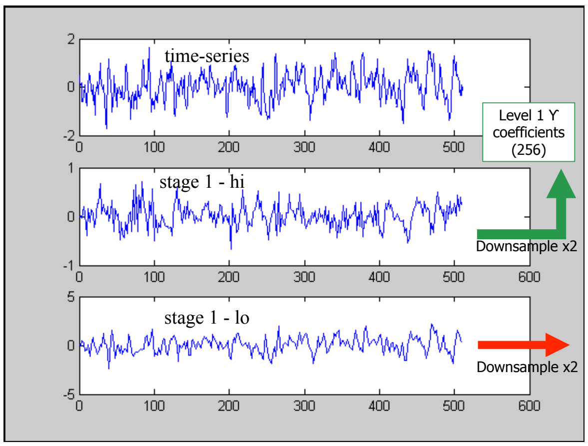


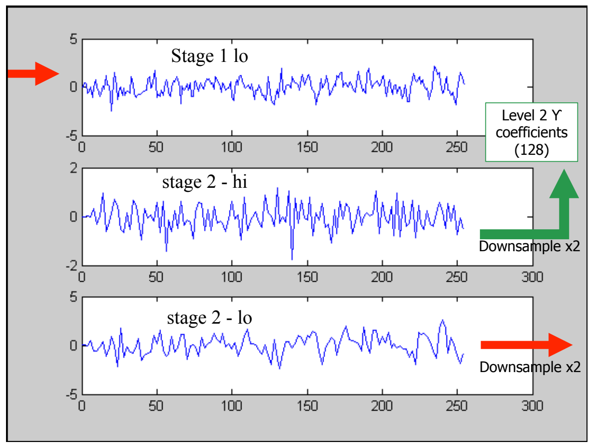
Filter Responses

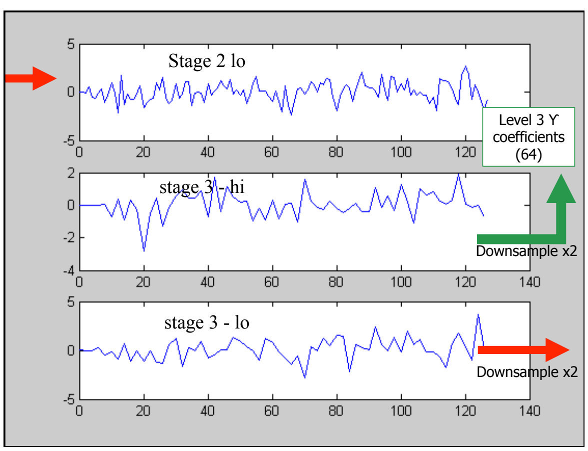
Spectrum of low pass filter

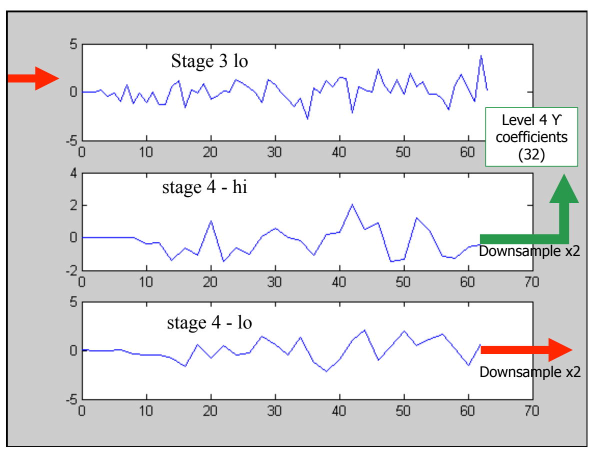
Spectrum of high pass filter

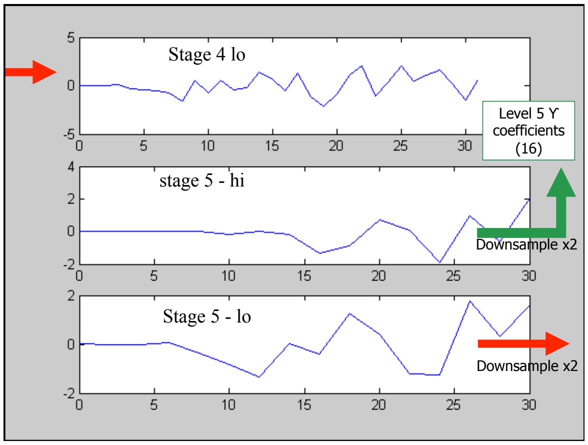


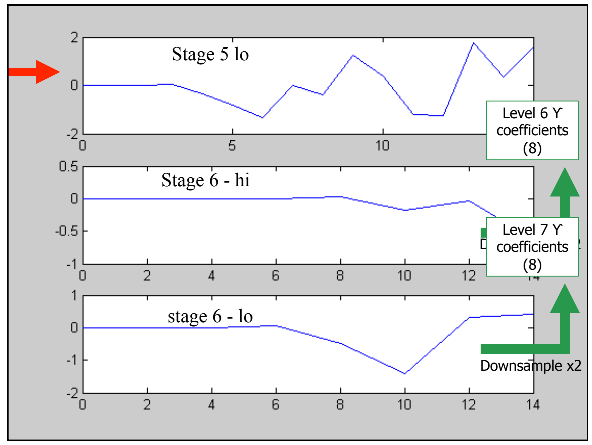


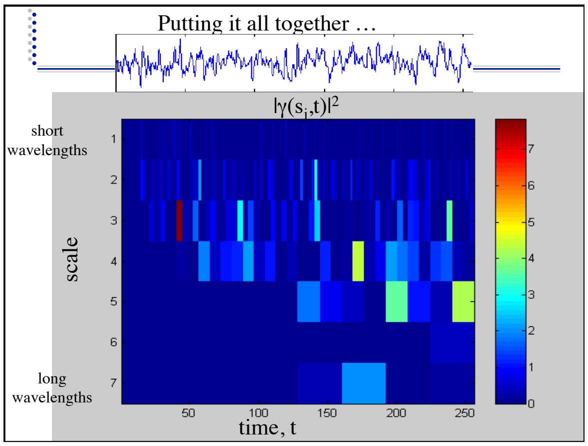




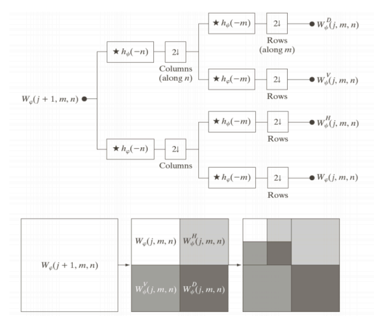








Expanding to Two Dimensions



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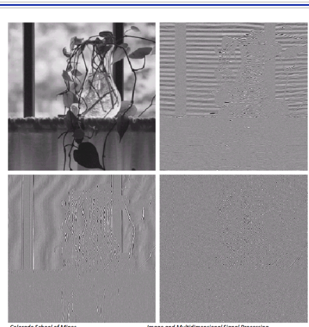


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

a	d^V
d^H	d^D

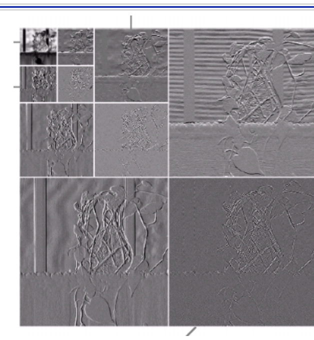
a(m,n): approximation

d^V(m,n): detail in vertical

d^H(m,n): detail in horizontal

d^D(m,n): detail in diagonal

Expanding to Two Dimensions



Big Ideas

- Wavelet transform
  - Capture temporal data with fewer coefficients than STFT
  - Use scaling and translation to get different resolution at different levels

Admin

- Project
  - Due 4/28

- Final Exam – 5/7 (3pm-5pm)
  - In Canvas
    - Will have a 2 hr window to complete within a 12 hr window
  - Open course notes and textbook, but cannot communicate with each other about the exam
    - Students will have randomized and different questions
    - Reminder, it is not in your best interest to share the exam
  - Old exams posted on old course websites
  - Covers Lec 1- 20
    - Does not include lec 12 (data converters and noise shaping) or IIR Filters