

ESE 531: Digital Signal Processing

Lec 4: January 28, 2020

Discrete Time Fourier Transform



Lecture Outline

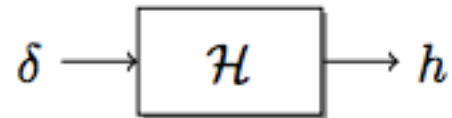
- ❑ LTI Systems
- ❑ Difference Equations
- ❑ Eigenfunctions
- ❑ Discrete Time Fourier Transform
 - Definition
 - Properties
- ❑ Frequency Response of LTI Systems

LTI Systems

DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

A block diagram representing an LTI system. An input signal x enters a rectangular block labeled h . The output of the block is a signal y .

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$



System Properties

❑ Causality

- $y[n]$ only depends on $x[m]$ for $m \leq n$

❑ Linearity

- Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
 - $Ax_1[n] + Bx_2[n] \rightarrow Ay_1[n] + By_2[n]$

❑ Memoryless

- $y[n]$ depends only on $x[n]$

❑ Time Invariance

- Shifted input results in shifted output
 - $x[n-q] \rightarrow y[n-q]$

❑ BIBO Stability

- A bounded input results in a bounded output (ie. max signal value exists for output if max)



Difference Equations



- Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$



Difference Equations

□ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

Difference Equations

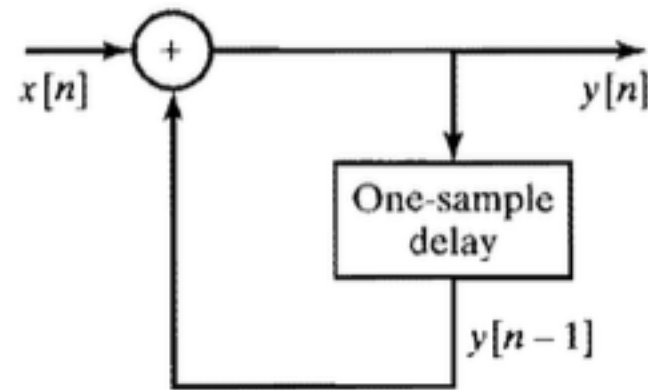
□ Accumulator example

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$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$



Example: Difference Equation

□ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$



Example: Difference Equation



- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$



Example: Difference Equation

- ❑ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- ❑ Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Example 2.13 in
text (p 37)

Eigenfunctions



Eigenfunction



□ $x[n] = e^{j\omega n}$



Eigenfunction

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

Eigenvalue (frequency response)

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
 - Re and Im
 - Mag and Phase



DT Frequency Response



□ $H(e^{j(\omega + 2\pi)n})?$

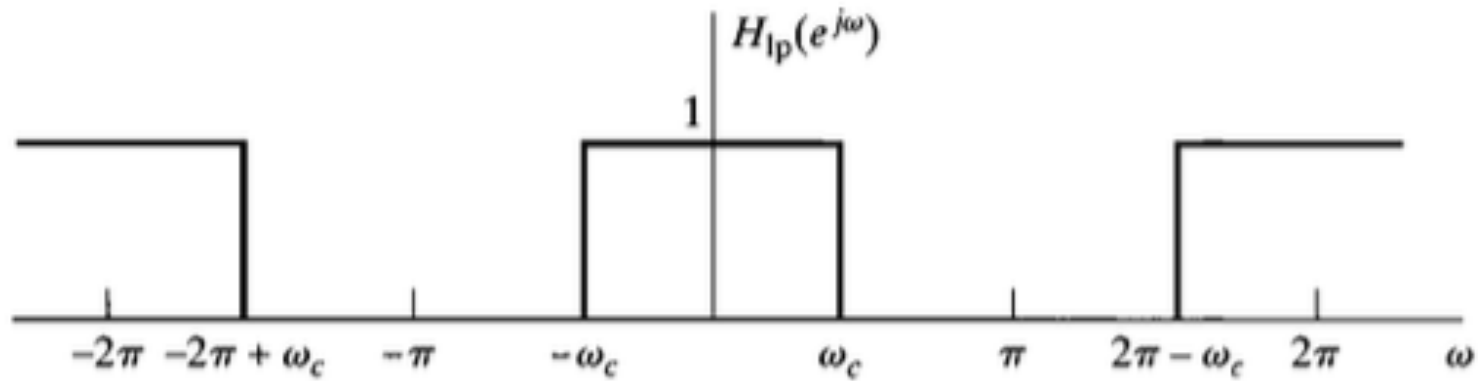


DT Frequency Response

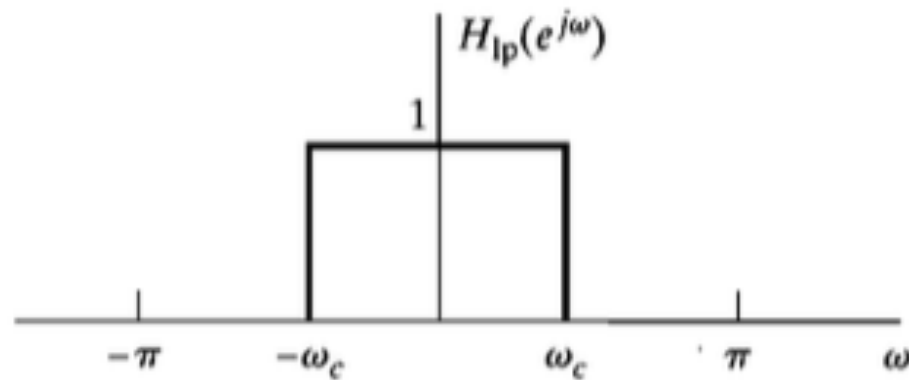
□ $H(e^{j(\omega+2\pi)n})?$

$$\begin{aligned} H(e^{j(\omega+2\pi)n}) &= \sum_{k=-\infty}^{\infty} h[k] e^{-j(\omega+2\pi)k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} e^{-j2\pi k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \\ &= H(e^{j\omega n}) \end{aligned}$$

Periodicity of Low Pass Freq Response



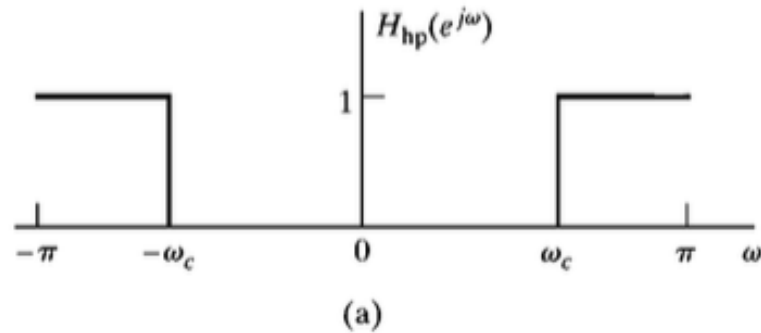
(a)



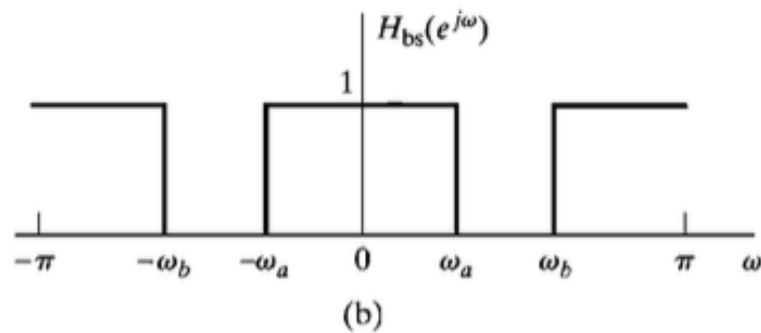
(b)

Other Filters

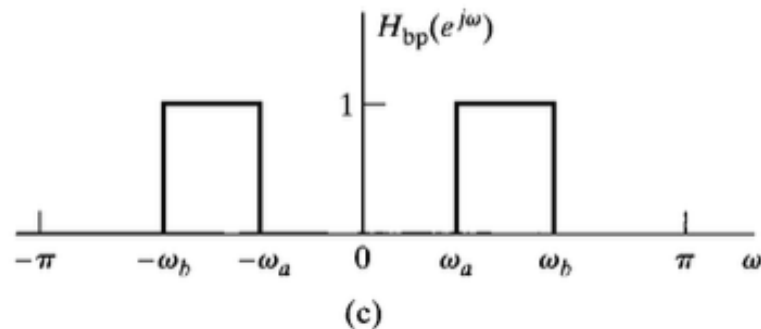
High-pass



Band-stop



Band-pass



Discrete-Time Fourier Transform (DTFT)



DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$



DTFT Definition

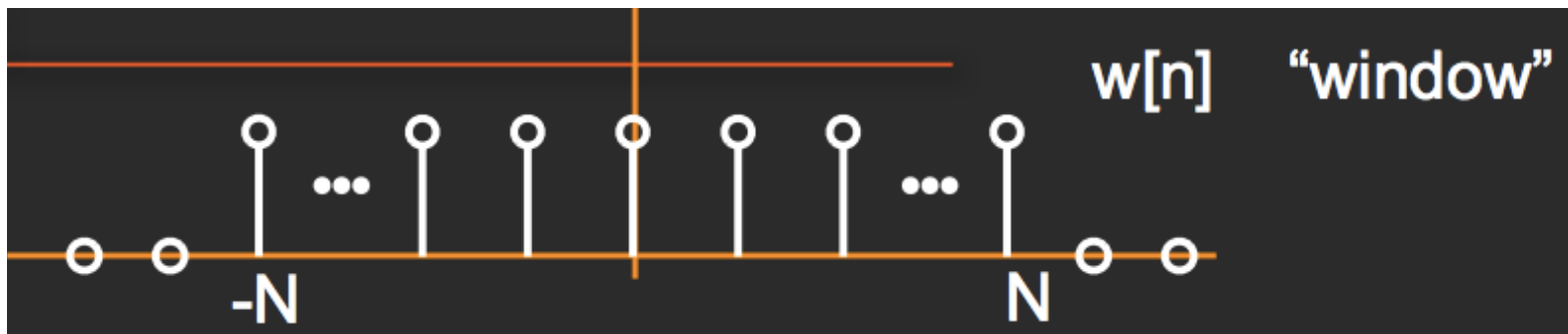
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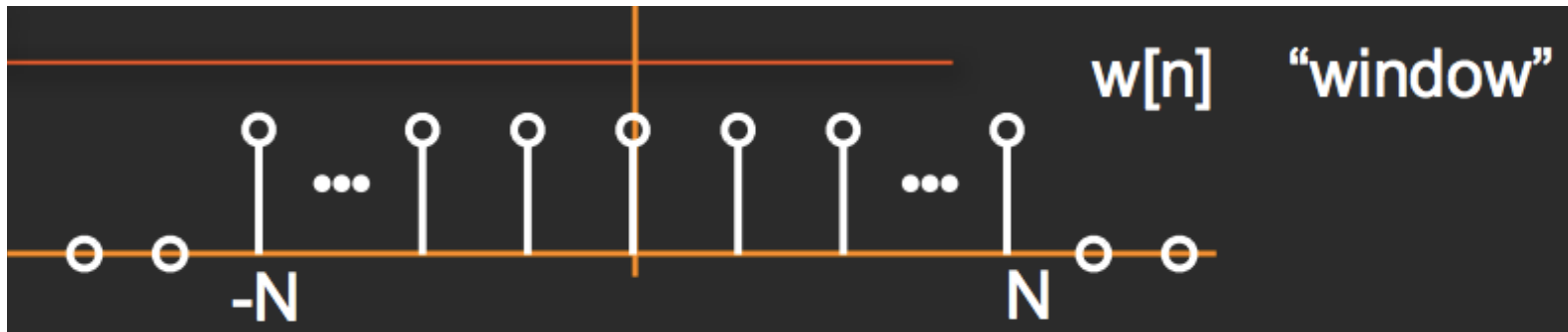
Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi f k}$$
$$x[n] = \int_{-0.5f_s}^{0.5f_s} X(f) e^{j2\pi f n} df$$

Example: Window DTFT



Example: Window DTFT



$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} w[k] e^{-j\omega k} \\ &= \sum_{k=-N}^N e^{-j\omega k} \end{aligned}$$



Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^N e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \dots + e^{j\omega 0} + \dots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Useful sum: $1 + p + p^2 + \dots + p^M = \frac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^N e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \dots + e^{j\omega 0} + \dots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Useful sum: $1 + p + p^2 + \dots + p^M = \frac{1 - p^{M+1}}{1 - p}$

$$p = e^{j\omega} \quad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}} \\ &= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \\ &= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}} \\ &= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)} \end{aligned}$$

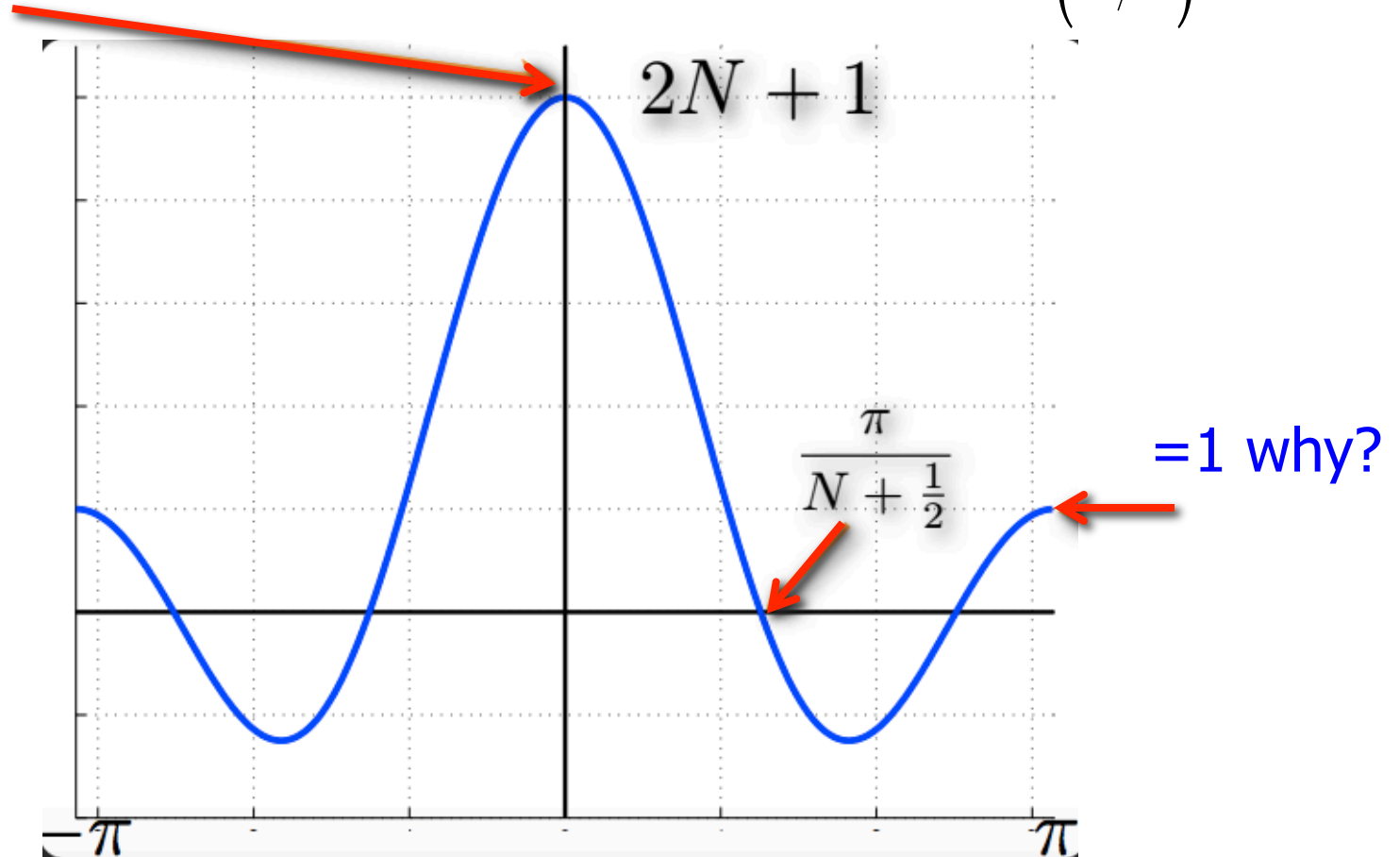
Periodic sinc

Example: Window DTFT



$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

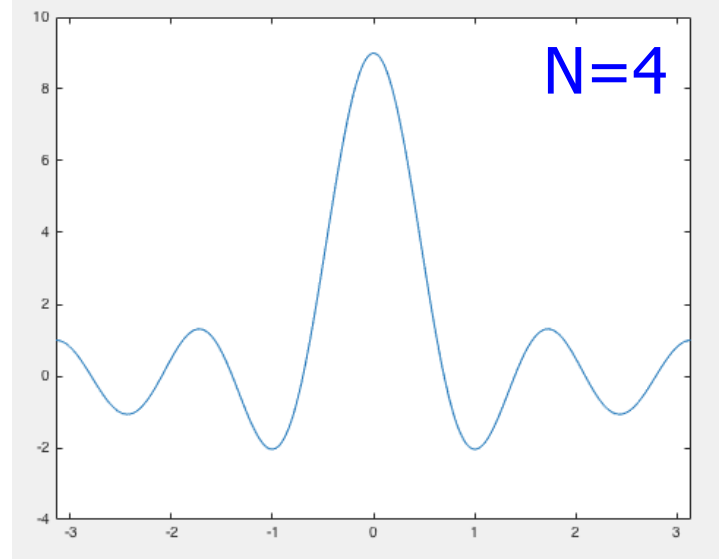
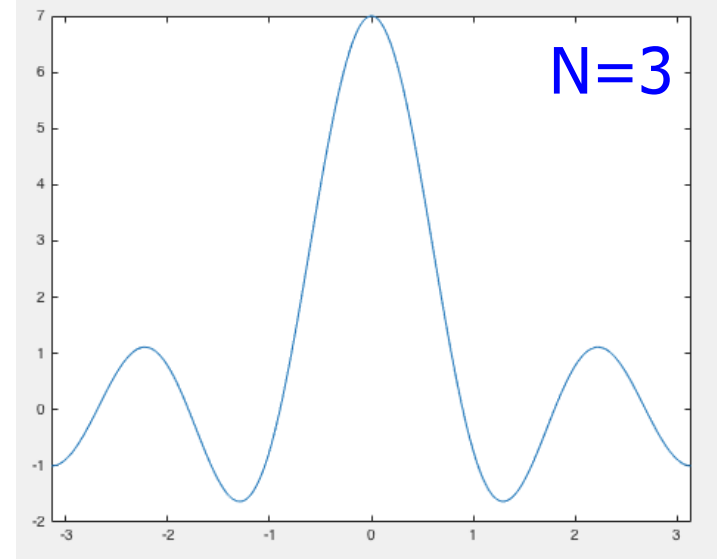
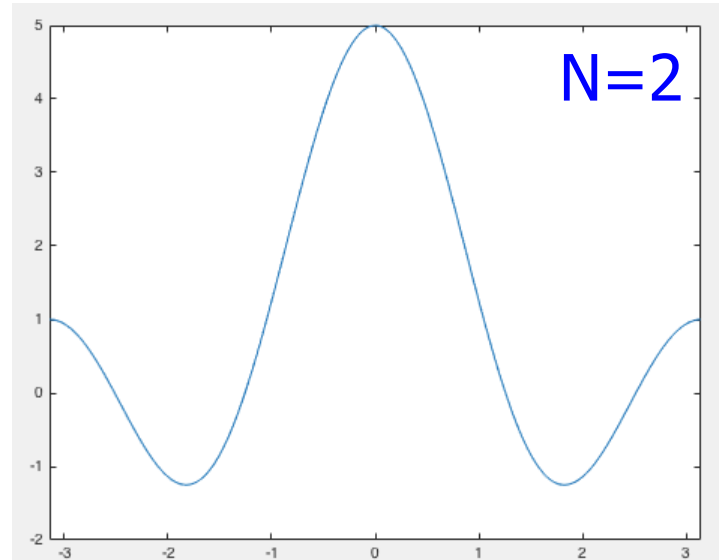
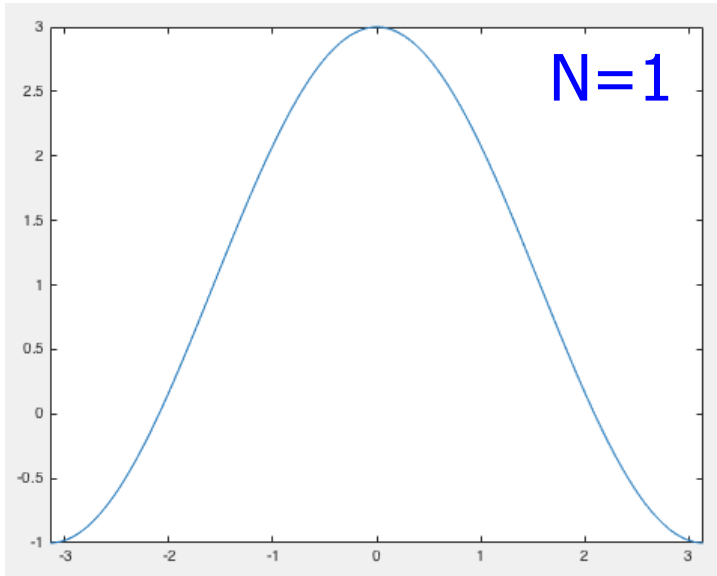
Also, $\sum w[n]$



Plot for $N=2$



Periodic Sinc



Fourier Transform Pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. 1 $(-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi k)$
4. $a^n u[n]$ $(a < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n]$ $(a < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin \omega_p (n+1)}{\sin \omega_p} u[n]$ $(r < 1)$	$\frac{1}{1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \omega < \omega_c, \\ 0, & \omega_c < \omega \leq \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k)]$



Properties of the DTFT

□ Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

□ Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

□ Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega}) \quad \text{If } x[n] \text{ real}$$

$$\text{Re}\{X(e^{-j\omega})\} = \text{Re}\{X(e^{j\omega})\}$$

$$\text{Im}\{X(e^{-j\omega})\} = -\text{Im}\{X(e^{j\omega})\}$$



Properties of the DTFT

□ Time Reversal:

$$x[n] \leftrightarrow X(e^{j\omega})$$

If $x[n]$ real

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

$$x[-n] \leftrightarrow X^*(e^{j\omega})$$

□ Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$$



Properties of the DTFT

□ Differentiation in Frequency:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$$

□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

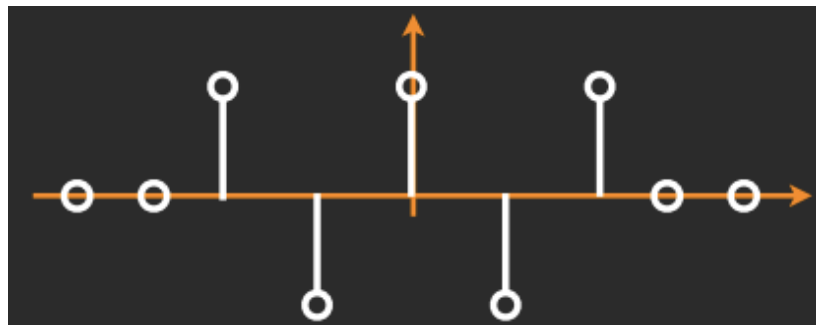
Fourier Transform Theorems

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} x[n] ^2$	$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) ^2 d\omega$
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

Example:

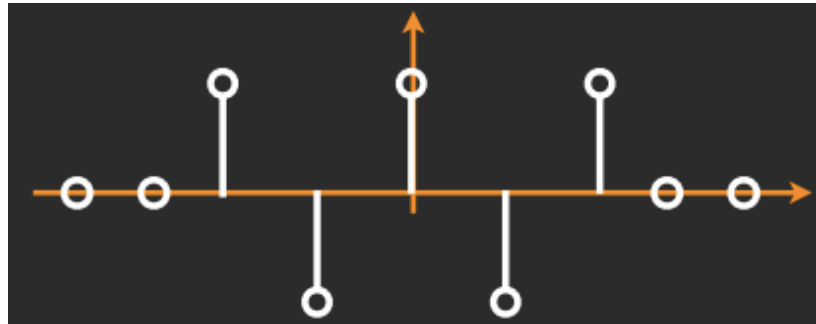
□ What is DTFT of:



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

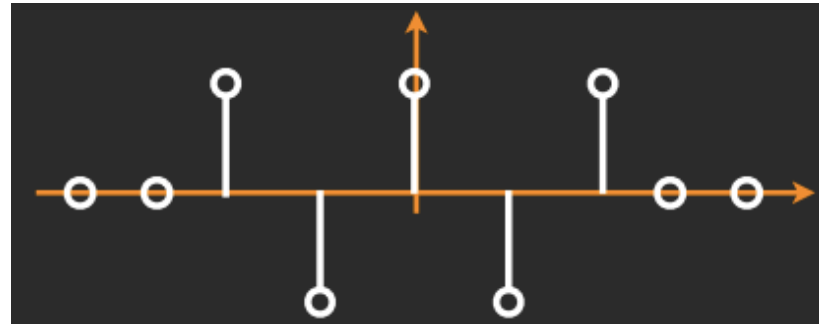
Example: Windowed $\cos(\pi n)$

□ What is DTFT of:



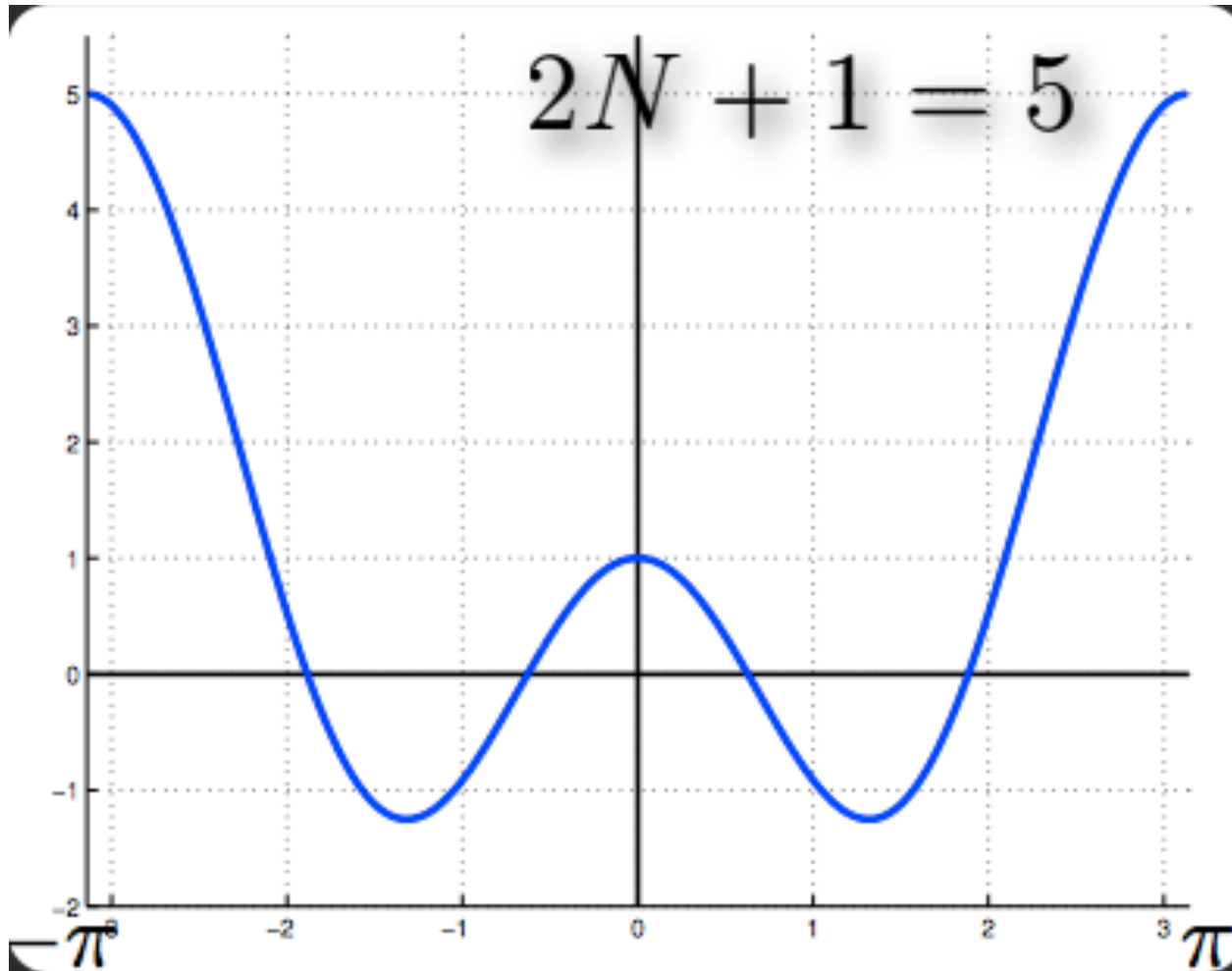
Example: Windowed $\cos(\pi n)$

□ What is DTFT of:



$$e^{j\omega_0 n} x[n] \Leftrightarrow X(e^{j(\omega - \omega_0)})$$

Example: Windowed $\cos(\pi n)$



Frequency Response of LTI Systems

LTI Systems

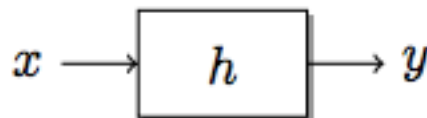
DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be characterized by its impulse response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$

LTI System Frequency Response

- (DT) Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

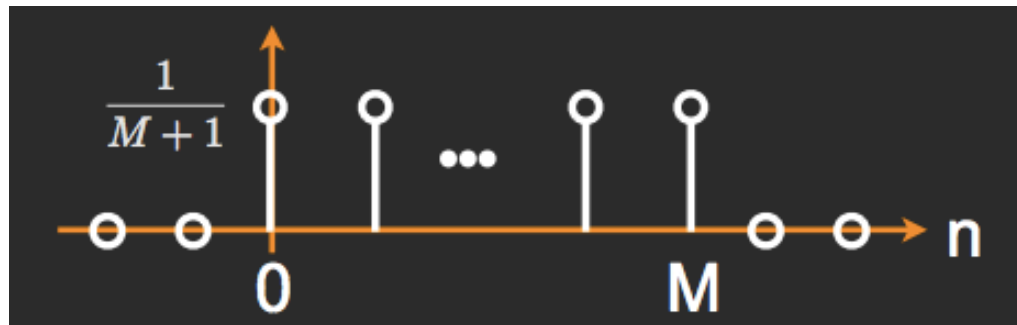
Example: Moving Average

□ Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response



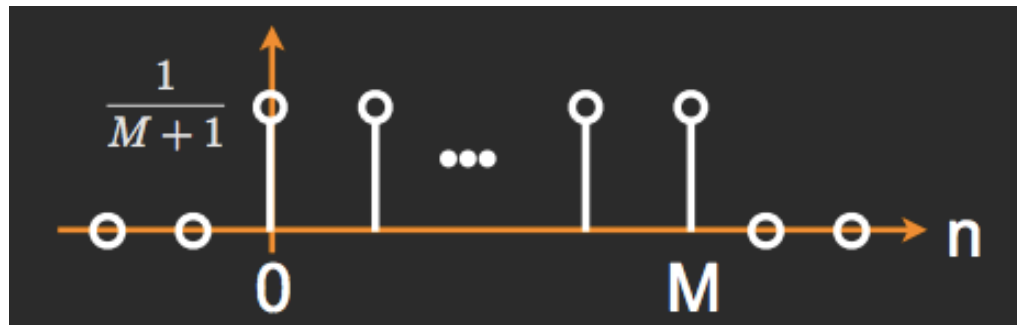
Example: Moving Average

□ Moving Average Filter

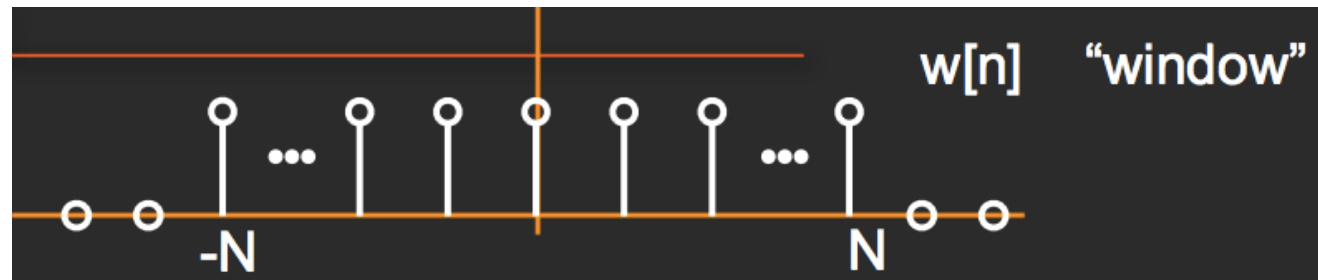
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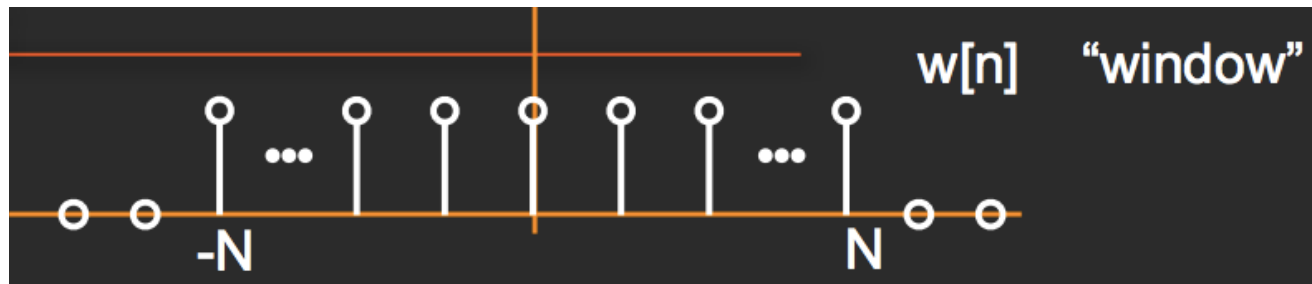
Impulse
response



Scaled & Time
Shifted window



Example: Moving Average



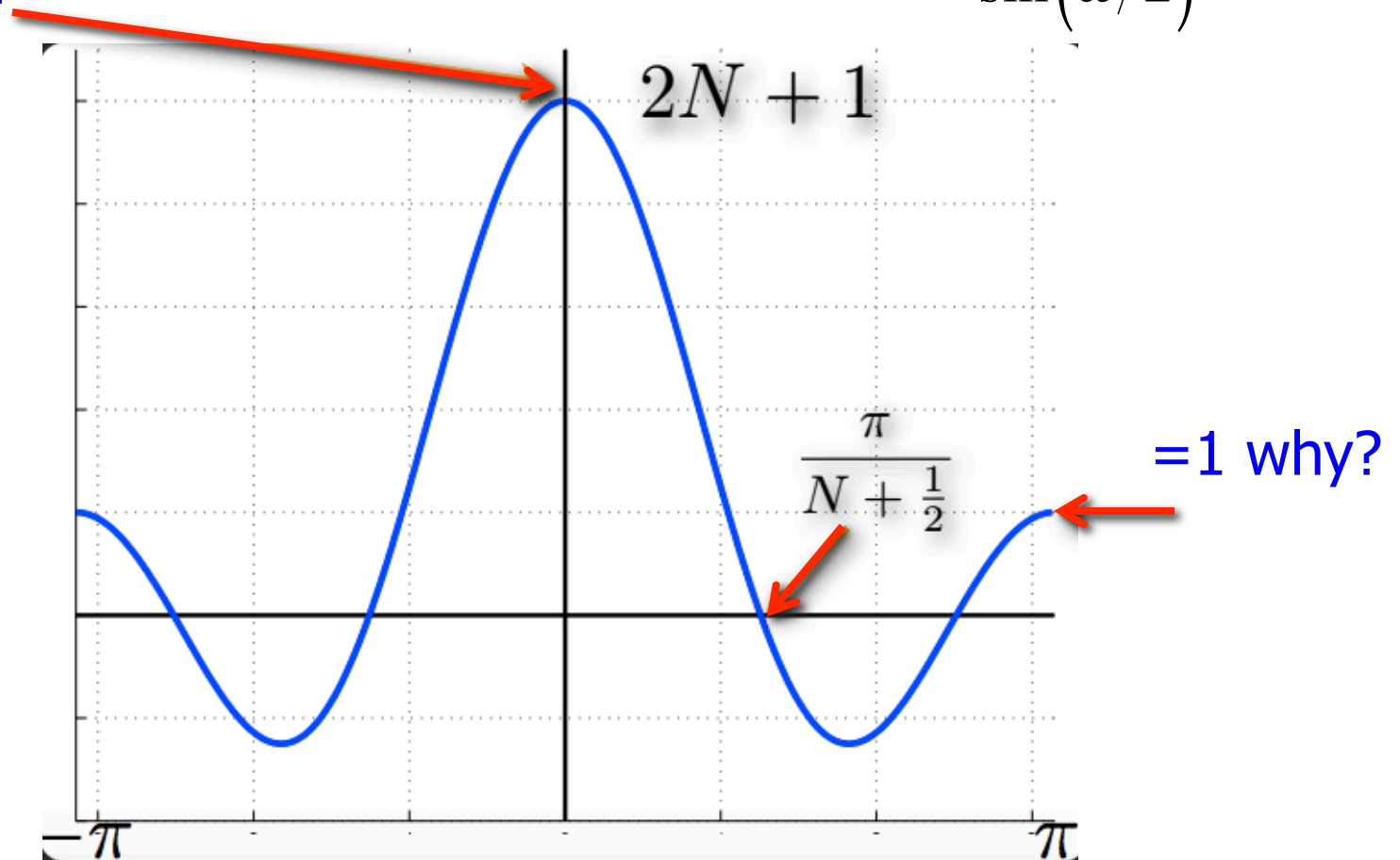
$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N + 1/2)\omega\right)}{\sin(\omega/2)}$$

Example: Window DTFT

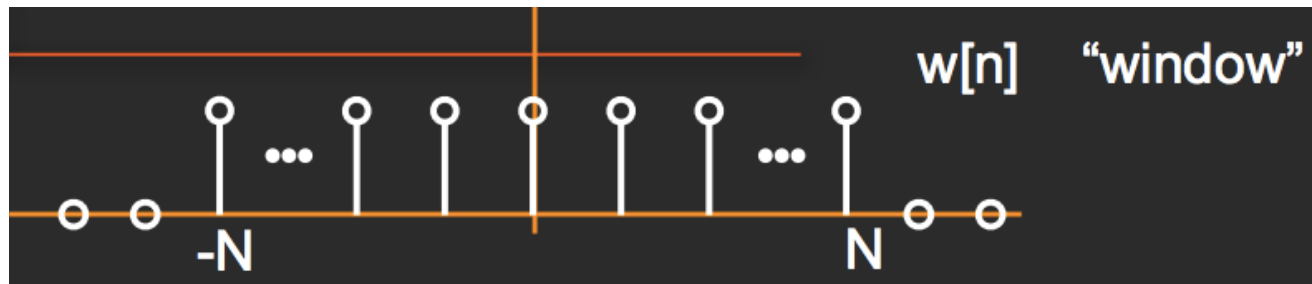
$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Also, $\sum x[n]$

Plot for $N=2$



Example: Moving Average

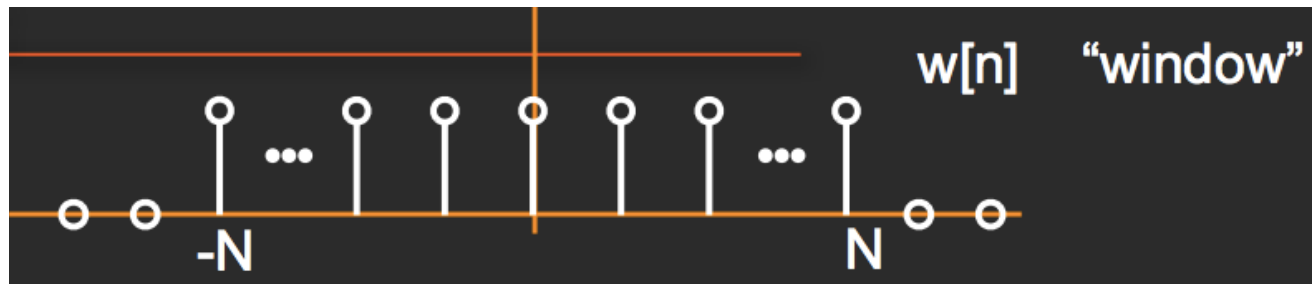


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] =$$

Example: Moving Average

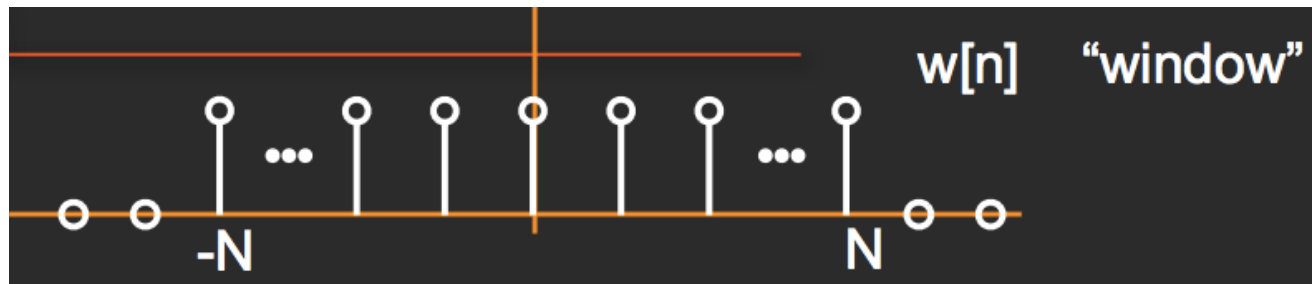


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) =$$

Example: Moving Average



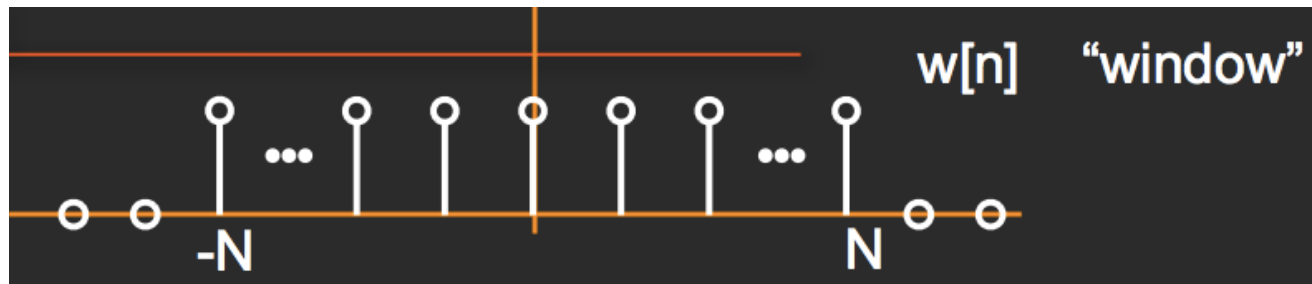
$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) =$$

$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

Example: Moving Average

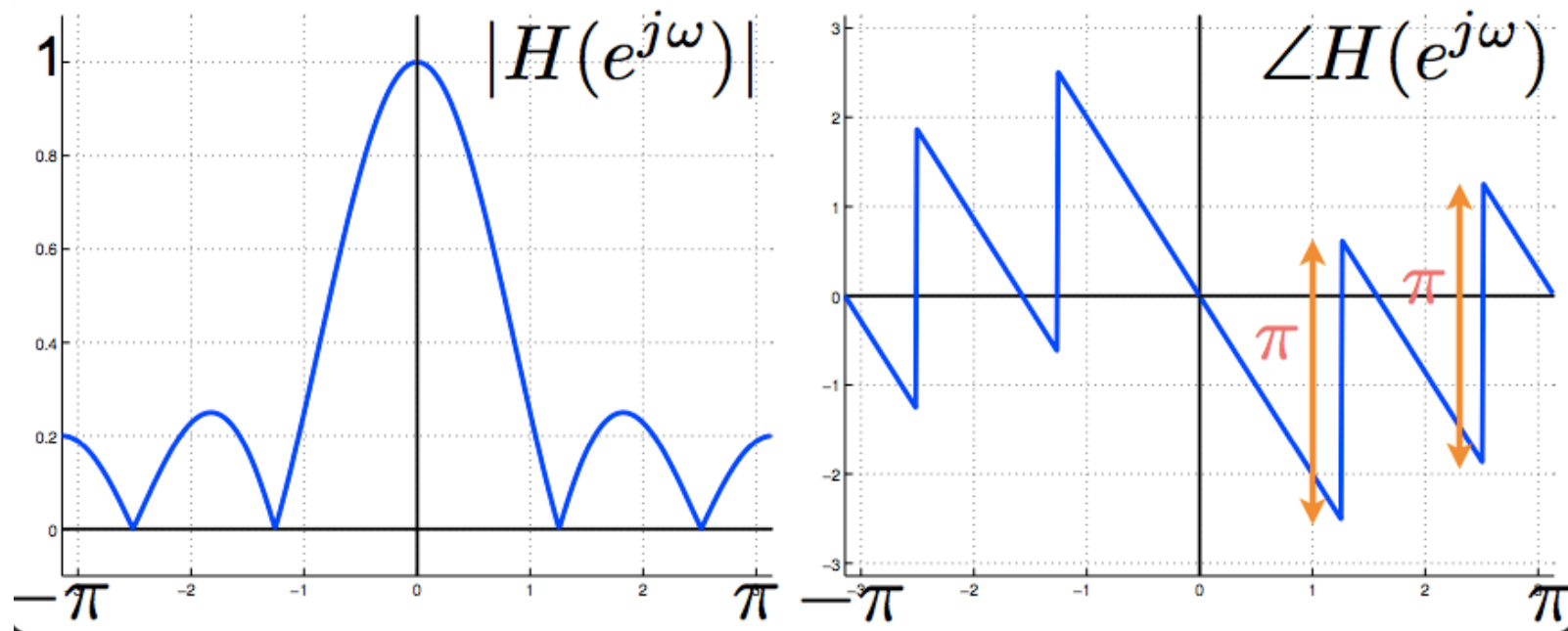


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average

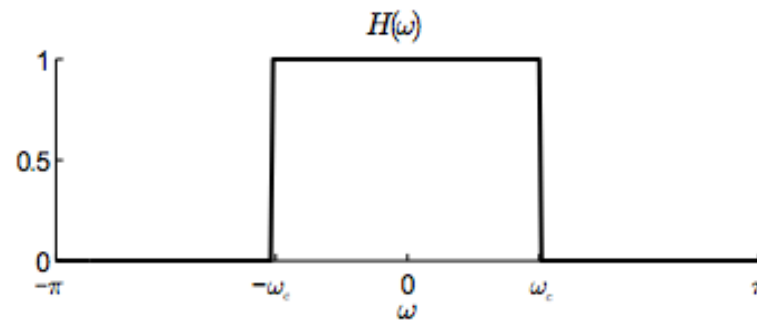


$M=4$
($N=2$)

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

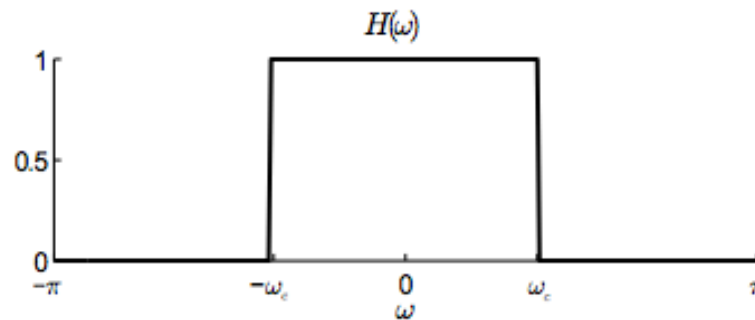
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



Example: Ideal Low-Pass Filter

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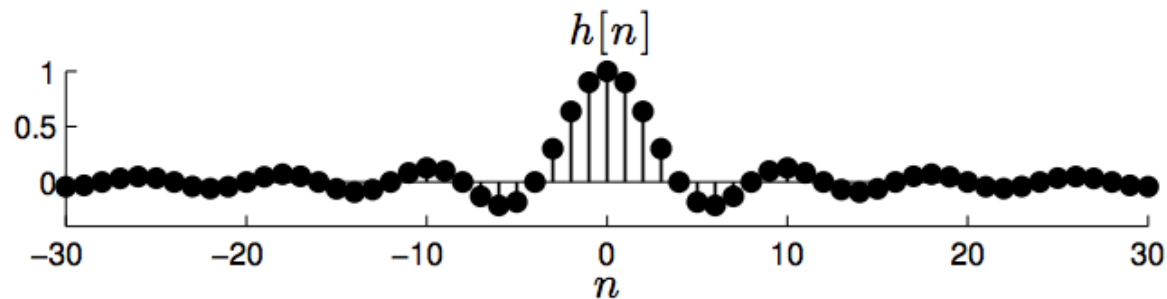
- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

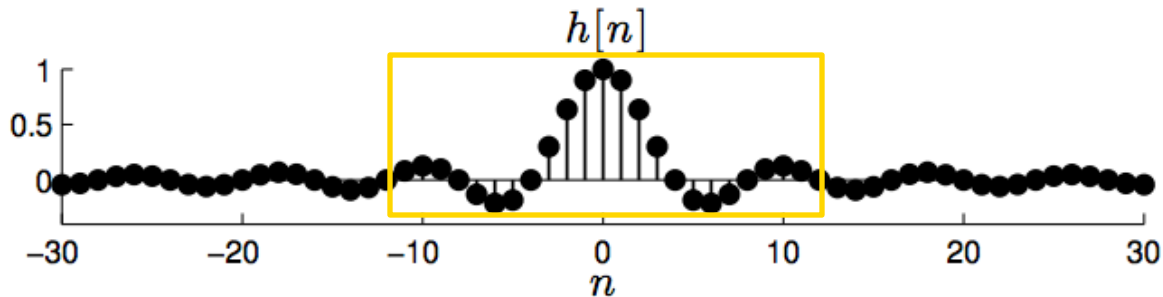


Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

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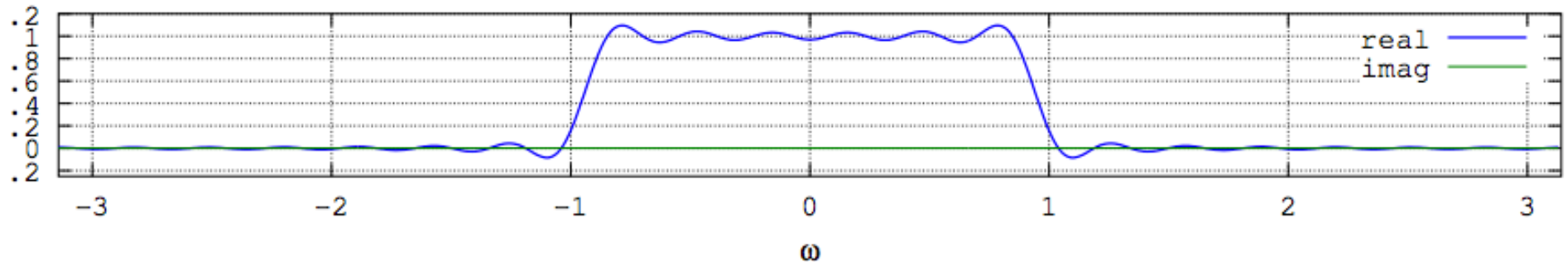
$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$

Example: Practical LP Filter



- ❑ Pass band smeared and rippled
 - Smearing determined by width of main lobe
 - Rippling determined by size of main lobes

Big Ideas

□ Difference Equations

- Help see implementation... more later with z-transform

□ Discrete Time Fourier Transform

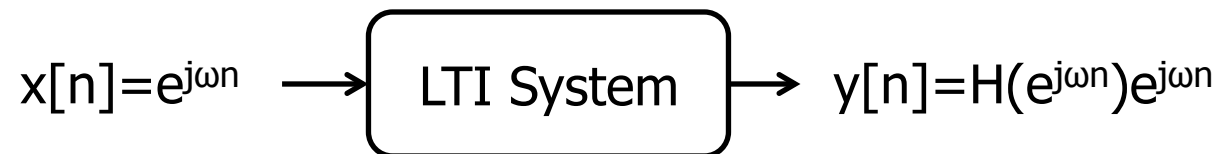
- Represent signals as a sum of scaled and phase shifted complex sinusoids (eigenfunctions)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

- Continuous in frequency over 2π

□ Frequency Response of LTI Systems $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$

- Frequency response of impulse response
- Describes scaling and phase shifting of a pure frequency





Admin

- ❑ HW 1 out now
 - Due 2/2 at midnight
 - Submit in Canvas