ESE 531: Digital Signal Processing

Lec 4: January 28, 2020

Discrete Time Fourier Transform

Lecture Outline

- LTI Systems
- Difference Equations
- Eigenfunctions
- Discrete Time Fourier Transform
  - Definition
  - Properties
- Frequency Response of LTI Systems

LTI Systems

DEFINITION: A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response

$$\delta \longrightarrow \boxed{\mathcal{H}} \longrightarrow h$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

$$x \longrightarrow \boxed{h} \longrightarrow y \qquad y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

System Properties

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Difference Equations

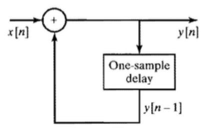
- Accumulator example

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$

$$y[n]=x[n]+\sum_{k=-\infty}^{n-1} x[k]$$

$$y[n]=x[n]+y[n-1]$$

$$\boxed{y[n]-y[n-1]=x[n]}$$



$$\boxed{\sum_{k=0}^{N} a_k y[n-k]=\sum_{m=0}^{M} b_m x[n-m]}$$

Example: Difference Equation

- Moving Average System

$$y[n]=\frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} x[n-k]$$

Example: Difference Equation

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- Let M_1=0 (i.e. system is causal)

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Example 2.13 in text (p 37)

Eigenfunctions

Eigenfunction

- $x[n]=e^{j\omega n}$

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$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

Eigenvalue (frequency response)

- $x[n]=e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
  - Re and Im
  - Mag and Phase

DT Frequency Response

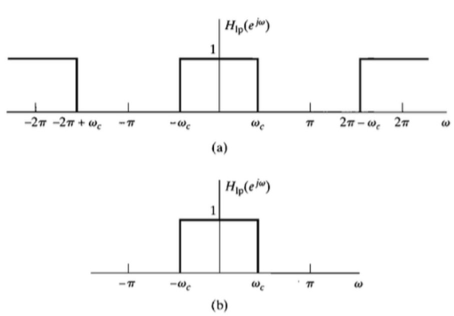
- $H(e^{j(\omega+2\pi)n})$?

DT Frequency Response

- $H(e^{j(\omega+2\pi)n})$?

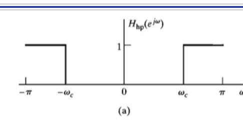
$$\begin{aligned} H(e^{j(\omega+2\pi)n}) &= \sum_{k=-\infty}^{\infty} h[k]e^{-j(\omega+2\pi)k} \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}e^{-j2\pi k} \\ &= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega n}) \end{aligned}$$

Periodicity of Low Pass Freq Response

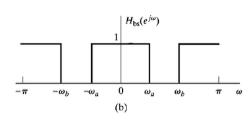


Other Filters

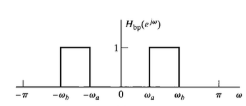
High-pass



Band-stop



Band-pass



Discrete-Time Fourier Transform (DTFT)

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

DTFT Definition

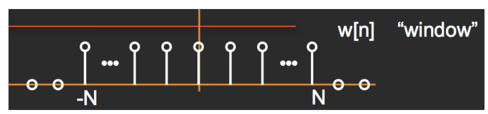
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$\boxed{\begin{aligned} X(f) &= \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk} \\ x[n] &= \int_{-0.5f_s}^{0.5f_s} X(f)e^{j2\pi fn}df \end{aligned}}$$

Example: Window DTFT



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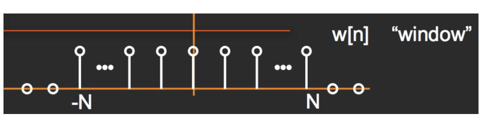


$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k]e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^{N} e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1-p^{M+1}}{1-p}$

Example: Window DTFT

$$W(e^{j\omega}) = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{j\omega N} + e^{j\omega(N-1)} + \ldots + e^{j\omega 0} + \ldots e^{-j\omega(N-1)} + e^{-j\omega N}$$

$$= e^{-j\omega N}(1 + e^{j\omega} + \ldots + e^{j\omega N} + \ldots + e^{j\omega(2N-1)} + e^{j\omega 2N})$$

Useful sum: $1 + p + p^2 + \ldots + p^M = \dfrac{1 - p^{M+1}}{1 - p}$

$$p = e^{j\omega} \qquad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

Example: Window DTFT

$$W(e^{j\omega}) = e^{-j\omega N}\frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}}$$

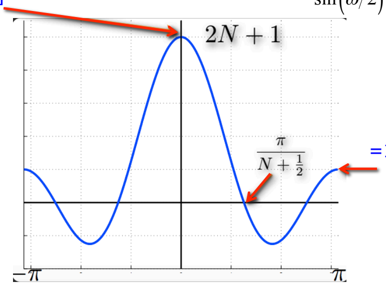
$$= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Periodic sinc

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

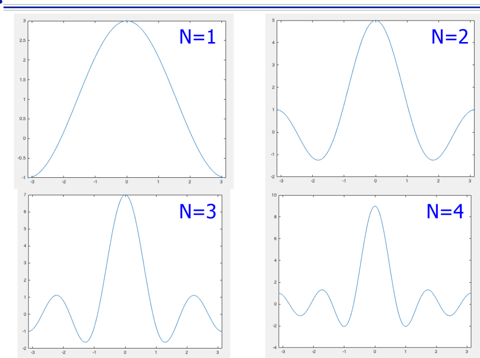
Also, Σw[n]



=1 why?

Plot for N=2

Periodic Sinc



Fourier Transform Pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Properties of the DTFT

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

- Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$

If x[n] real

$$\mathrm{Re}\{X(e^{-j\omega})\} = \mathrm{Re}\{X(e^{j\omega})\}$$

$$\mathrm{Im}\{X(e^{-j\omega})\} = -\mathrm{Im}\{X(e^{j\omega})\}$$

Properties of the DTFT

- Time Reversal:

$$x[n] \leftrightarrow X(e^{j\omega})$$
$$x[-n] \leftrightarrow X(e^{-j\omega})$$

If x[n] real

$$x[-n] \leftrightarrow X^*(e^{j\omega})$$

- Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$
$$x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$
$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

Properties of the DTFT

- Differentiation in Frequency:

$$x[n] \leftrightarrow X(e^{j\omega})$$
$$nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$
$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Fourier Transform Theorems

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

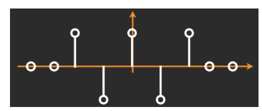
Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

Example:

- What is DTFT of:



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

Example: Windowed cos(π n)

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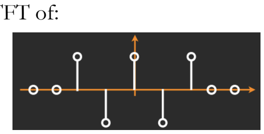

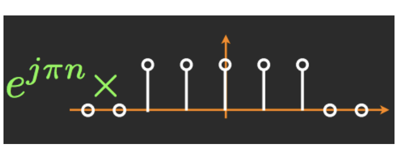

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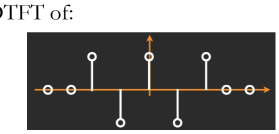
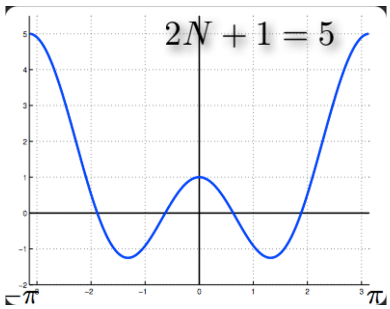


$$\boxed{e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})}$$

Example: Windowed cos(π n)



Frequency Response of LTI Systems

LTI Systems

DEFINITION: A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

$$x \longrightarrow \boxed{h} \longrightarrow y \qquad y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\,x[m]$$

$$y[n] = x[n] * h[n]$$

LTI System Frequency Response

- (DT)Fourier Transform of impulse response

$$x[n]=e^{j\omega n} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]=H(e^{j\omega})e^{j\omega n}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Example: Moving Average

- Moving Average Filter
  - Causal: M_1=0, M_2=M

$$y[n] = \frac{x[n-M]+...+x[n]}{M+1}$$

Impulse response

Example: Moving Average

- Moving Average Filter
  - Causal: M_1=0, M_2=M

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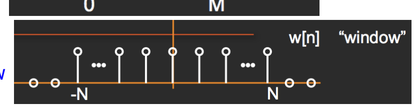
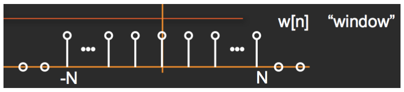
Impulse response

Scaled &Time Shifted window



Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Window DTFT

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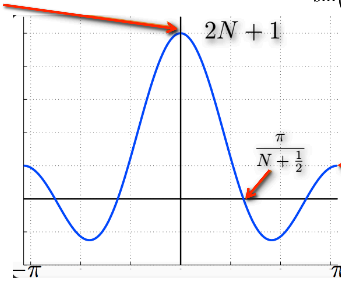
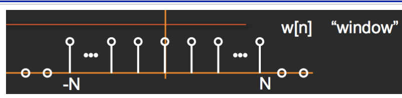
Also, Σx[n]



=1 why?

Plot for N=2

Example: Moving Average



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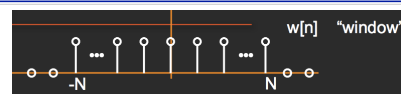
$h[n] =$

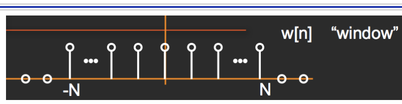
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$h[n] = \frac{1}{M+1} w[n-M/2] \leftrightarrow H(e^{j\omega}) =$$

Example: Moving Average



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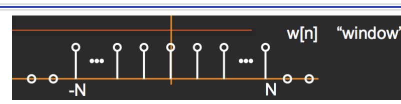
$$\boxed{x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})}$$

Example: Moving Average



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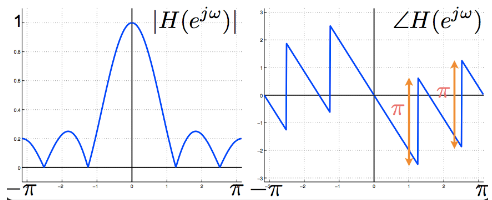
$$h[n] = \frac{1}{M+1} w[n-M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average

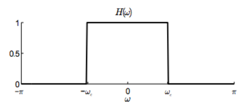


M=4
(N=2)

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$



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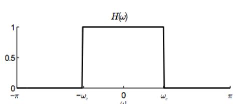

- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n}\, \frac{d\omega}{2\pi} = \left.\frac{e^{j\omega n}}{2\pi j n}\right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}$$

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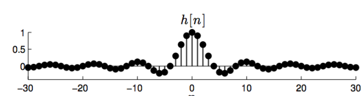
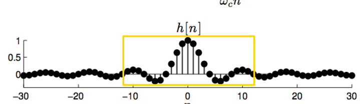


Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \le |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

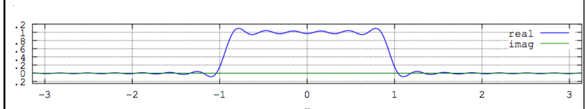
$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate and shift

$$h_{LP}[n] = w_N[n-N] \cdot h[n-N]$$

Example: Practical LP Filter



- Pass band smeared and rippled
  - Smearing determined by width of main lobe
  - Rippling determined by size of main lobes

Big Ideas

- Difference Equations
  - Help see implementation… more later with z-transform
- Discrete Time Fourier Transform
  - Represent signals as a sum of scaled and phase shifted complex sinusoids (eigenfunctions)
  - Continuous in frequency over 2π

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

- Frequency Response of LTI Systems

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

  - Frequency response of impulse response
  - Describes scaling and phase shifting of a pure frequency

$$x[n]=e^{j\omega n} \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n]=H(e^{j\omega n})e^{j\omega n}$$

Admin

- HW 1 out now
  - Due 2/2 at midnight
  - Submit in Canvas