ESE 531: Digital Signal Processing

Lec 5: January 30, 2020

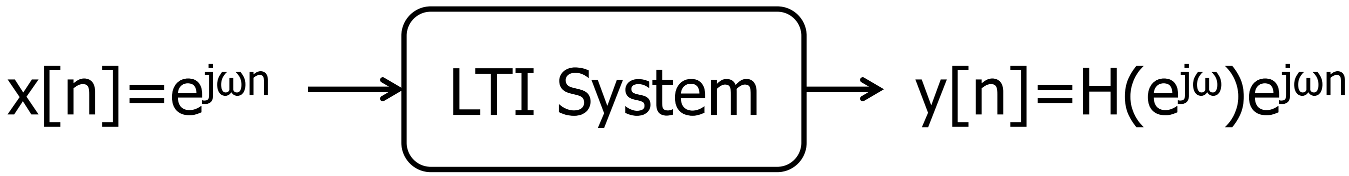
z-Transform

Lecture Outline

- LTI System Frequency Response
- z-Transform
  - Regions of convergence (ROC) & properties
  - z-Transform properties
- Inverse z-transform

LTI System Frequency Response

- (DT)Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

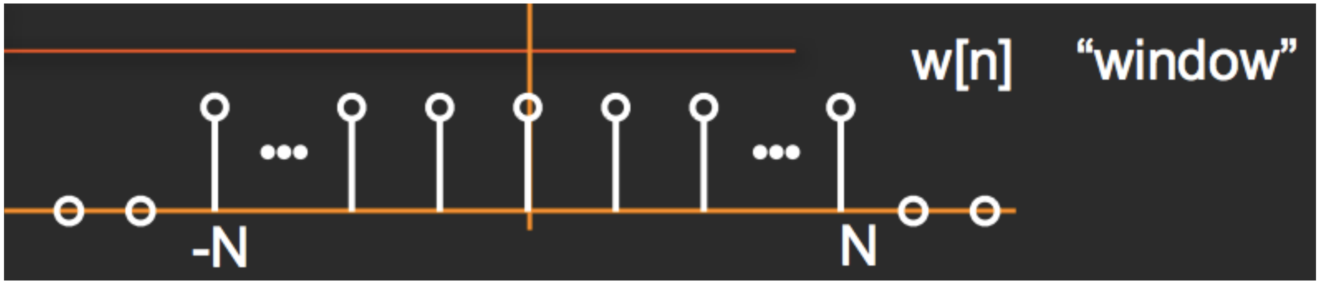
Example: Moving Average

- Moving Average Filter
  - Causal: M_1=0, M_2=M

$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

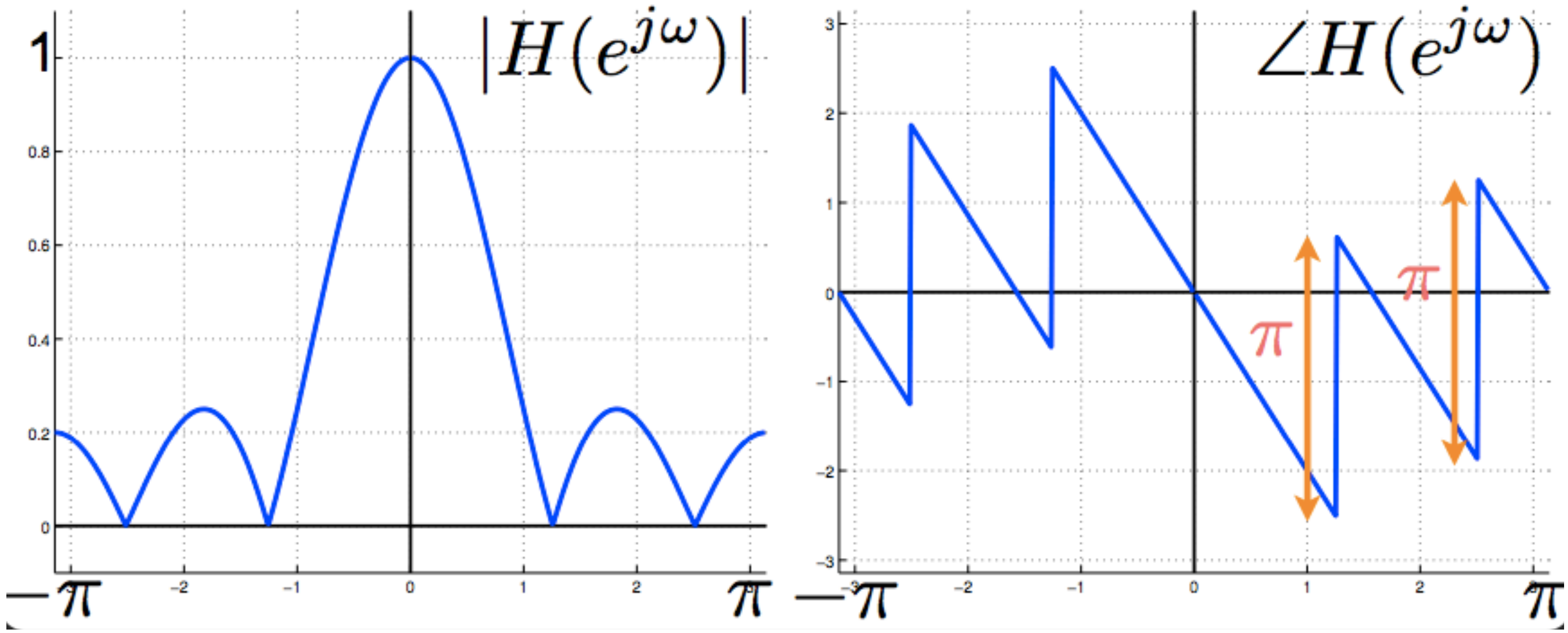
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average

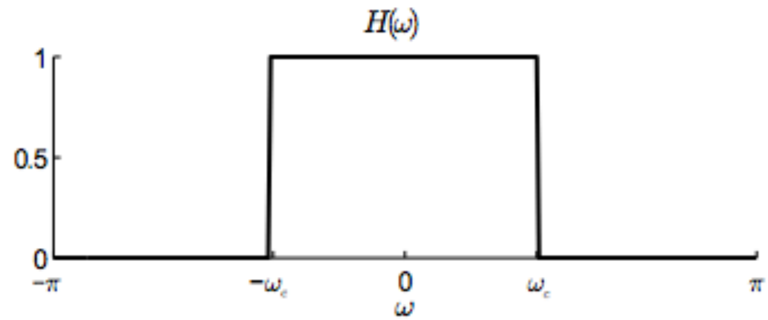


M=4
(N=2)

Example: Ideal Low-Pass Filter

- The frequency response H(ω) of the ideal low-pass filter passes low frequencies (near ω=0) but blocks high frequencies (near $\omega=\pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



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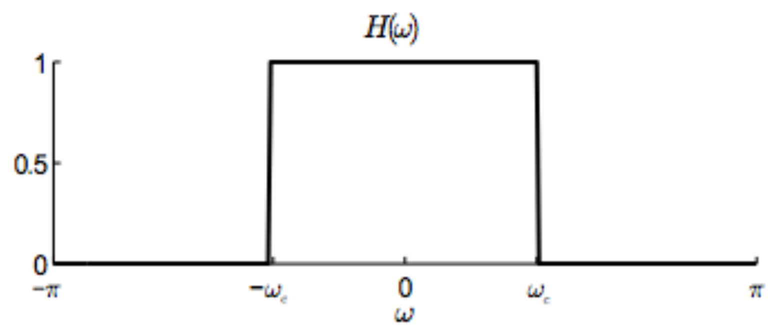

- Compute the impulse response h[n] given this H(ω)
- Apply the inverse DTFT:

$$h[n] = \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

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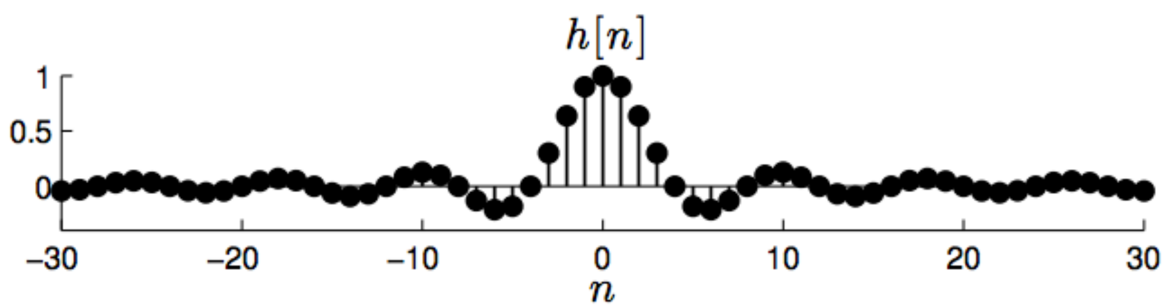

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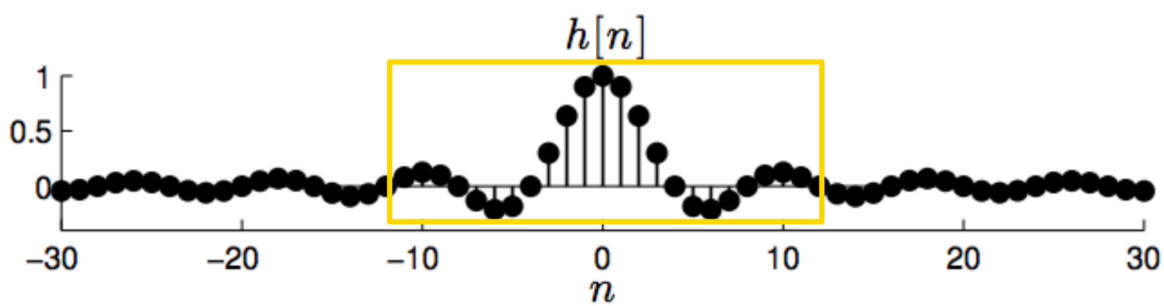
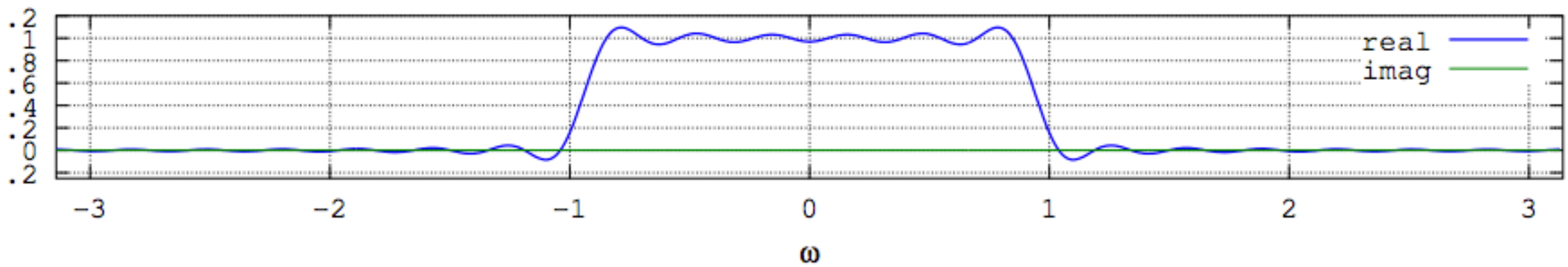


Truncate and shift

$$h_{LP}[n] = w_N[n-N] \cdot h[n-N]$$

Example: Practical LP Filter



- Pass band smeared and rippled
  - Smearing determined by width of main lobe
  - Rippling determined by size of main side lobes

z-Transform

z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats

Reminder: DTFT Definition

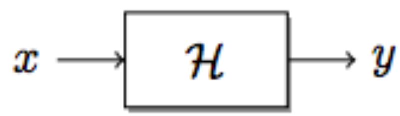
- The core "basis functions" (I.e eigenfunctions) of the DTFT are the complex sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \qquad -\infty \le n < \infty$$

Reminder: Frequency Response of LTI System

- We can use the DTFT to characterize an LTI system



$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

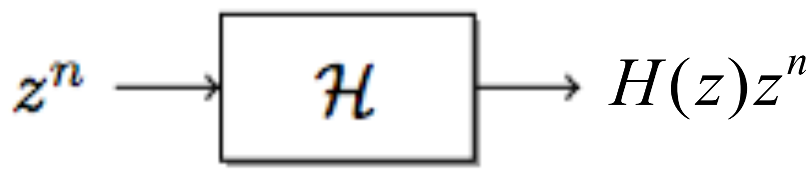
- and relate the DTFTs of the input and output

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \qquad H(\omega) = \sum_{m=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$Y(\omega) \;=\; X(\omega) H(\omega)$$

Complex Exponentials as Eigenfunctions

- Fact: A more general set of eigenfunctions of an LTI system are the complex exponentials z^n, $z \in C$

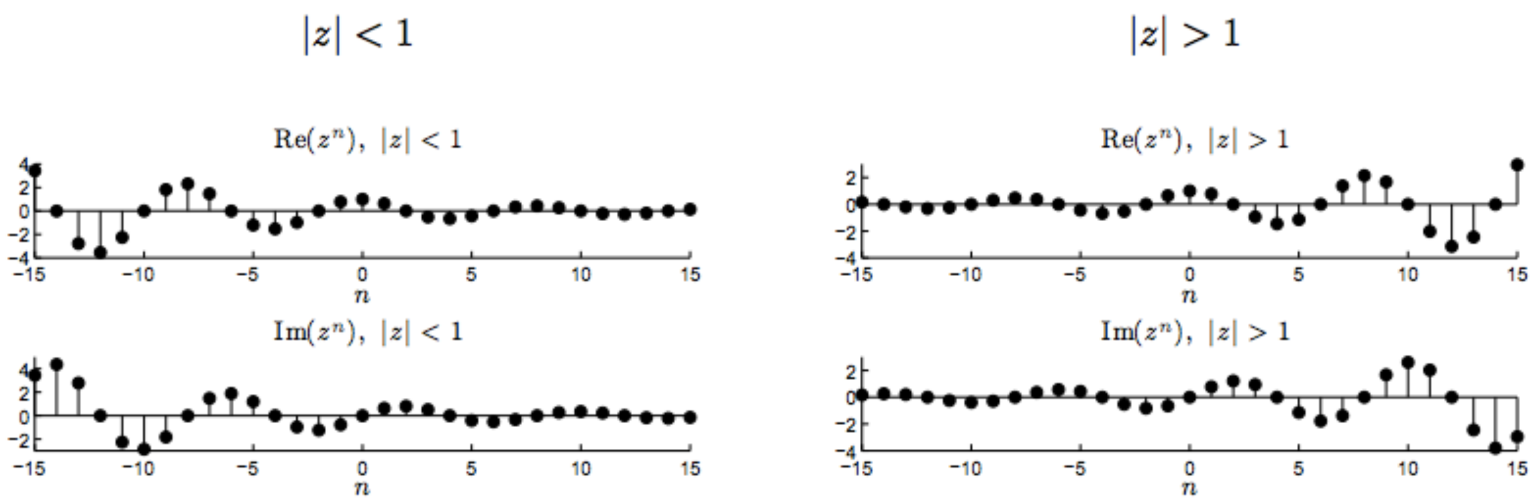


Reminder: Complex Exponentials

$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

$|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$

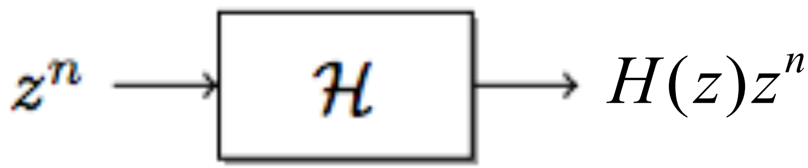
$e^{j\omega n}$ is a **complex sinusoid**



Bounded

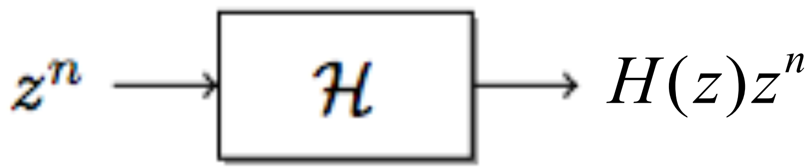
Unbounded

Proof: Complex Exponentials as Eigenfunctions



- Prove by computing the convolution with input $x[n] = z^n$

Proof: Complex Exponentials as Eigenfunctions



- Prove by computing the convolution with input $x[n] = z^n$

$$
\begin{aligned}
z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m}\, h[m] = \sum_{m=-\infty}^{\infty} z^n\, z^{-m}\, h[m] \\
&= \left(\sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \right) z^n \\
&= H(z) z^n \checkmark
\end{aligned}
$$

z-Transform

- Define the **forward z-transform** of x[n] as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

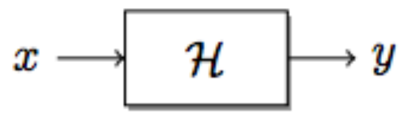
- The core "basis functions" of the z-transform are the complex exponentials z^n with arbitrary $z \in C$; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT $X(e^{j\omega})$ and the z-transform X(z); they are, in fact, intimately related

$$X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



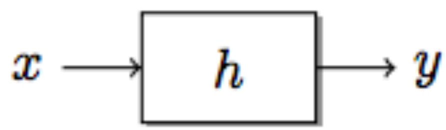
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \quad H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

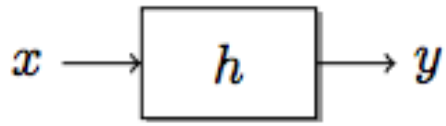
$$Y(z) = X(z)\, H(z)$$

Proof: Transfer Function of LTI Systems



- Compute z-transform of output by computing the convolution of impulse response with input $x[n] = z^n$

Proof: Transfer Function of LTI Systems



- Compute z-transform of output by computing the convolution of impulse response with input $x[n] = z^n$

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n]\, z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x[m]\, h[n-m] \right) z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{n=-\infty}^{\infty} h[n-m]\, z^{-n} \right) \quad (\text{let } r = n-m) \\
&= \sum_{m=-\infty}^{\infty} x[m] \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r-m} \right) = \left(\sum_{m=-\infty}^{\infty} x[m] z^{-m} \right) \left(\sum_{r=-\infty}^{\infty} h[r]\, z^{-r} \right) \\
&= X(z)\, H(z) \;\checkmark
\end{aligned}
$$

Z-transform

What are we missing?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Z-Transform

$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

$|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$

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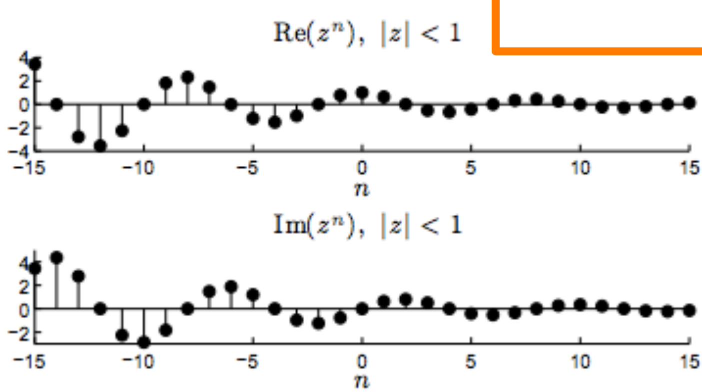
What are we missing?

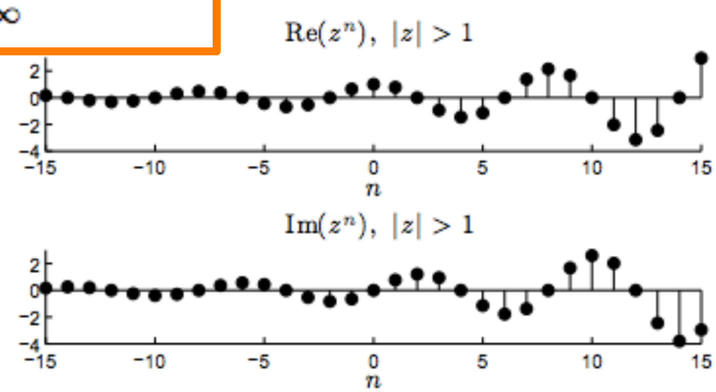
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$|z| < 1$



Bounded

$|z| > 1$



Unbounded

Region of Convergence (ROC)

Region of Convergence (ROC)

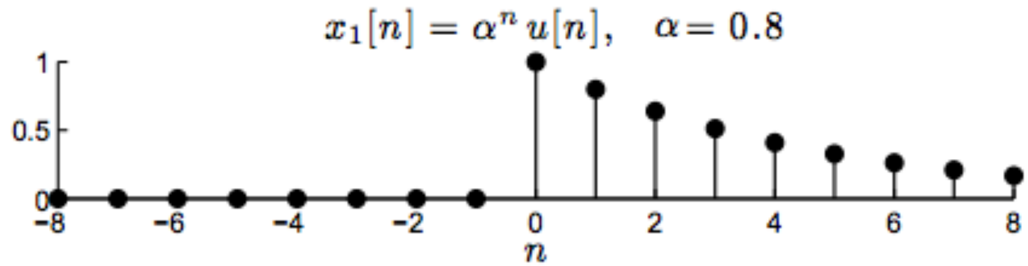
DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \; < \; \infty$$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence
- Example for $\alpha = 0.8$



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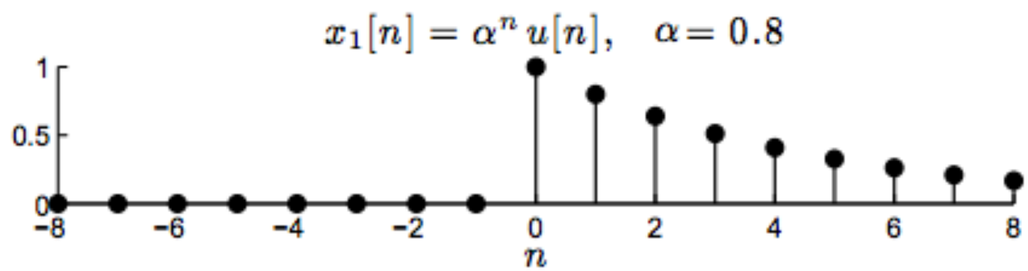


- The **forward z-transform** of $x_1[n]$

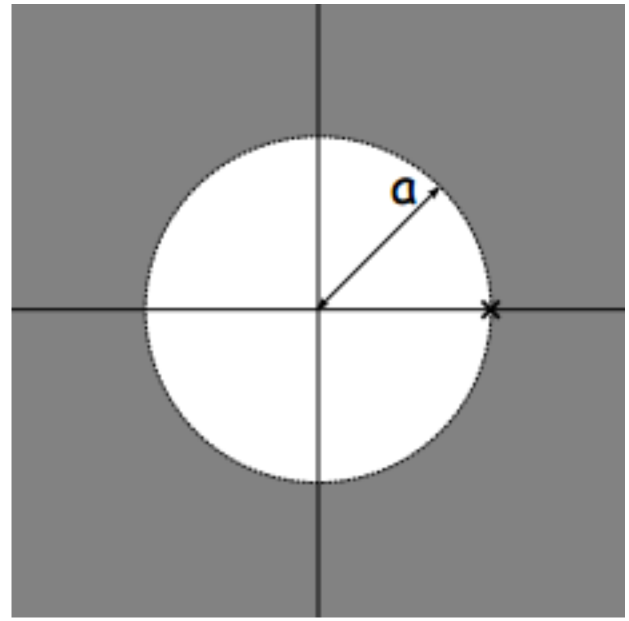
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} = \sum_{n=0}^{\infty} \alpha^n\, z^{-n} = \sum_{n=0}^{\infty} (\alpha\, z^{-1})^n = \frac{1}{1-\alpha\, z^{-1}} = \frac{z}{z-\alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha\, z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)

$$ROC = \{z : \ |z| > |a|\}$$



- The **forward** z**-transform** of $x_1[n]$

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- **Important:** We can apply the geometric sum formula only when $|\alpha\, z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- What is the DTFT of $x_1[n]=a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad -\infty \le n < \infty$$

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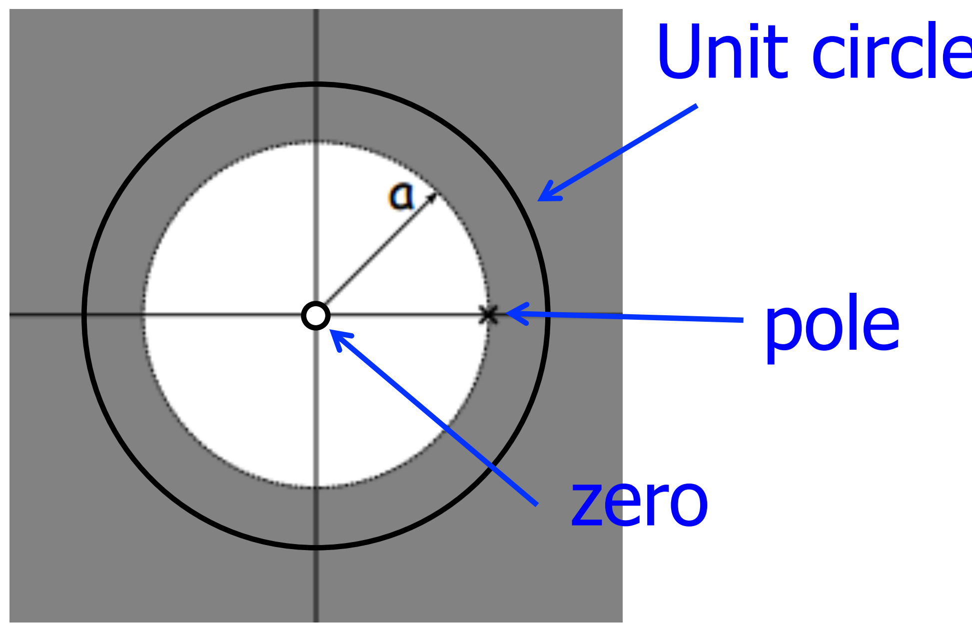
$$X_1(z) = \frac{z}{z-a}$$

$$ROC = \{z: \quad |z| > |a|\}$$



ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

- Hint:

$$x_1[n] = a^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Left-sided sequence

ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The z-transform without ROC does not uniquely define a sequence!

ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

- Hint:

$$x_1[n] = a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \quad |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \quad |z| < |a|\}$$

ROC Example 5

- What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \ |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \ |z| < |a|\}$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n+M-1]$$

finite length sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n+M-1]$$

$$X_6(z) = \frac{1-a^M z^{-M}}{1-az^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} \left(1 - a e^{j2\pi k/M} z^{-1}\right)$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

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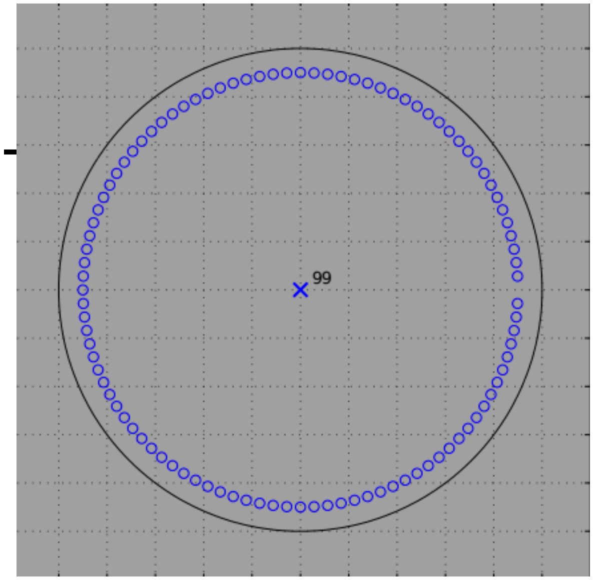
$$x_6[n] = a^n u[n] u[-n + M -$$



$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} (1 - a e^{j2\pi k/M} z^{-1})$$

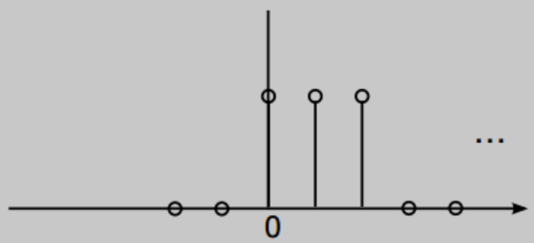
M=100

Properties of ROC

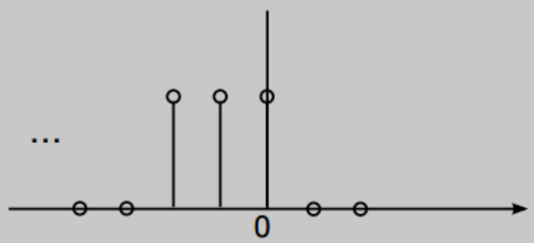
- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1,2
- For left-sided: inwards from inner most pole to zero
  - Example 3
- For two-sided, ROC is a ring - or do not exist
  - Examples 4,5

Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly z=0, z=∞ (Example 6)



$X(z) = 1 + z^{-1} + z^{-2}$ ROC excludes $z = 0$



$X(z) = 1 + z^{1} + z^{2}$ ROC excludes $z = \infty$

Formal Properties of the ROC

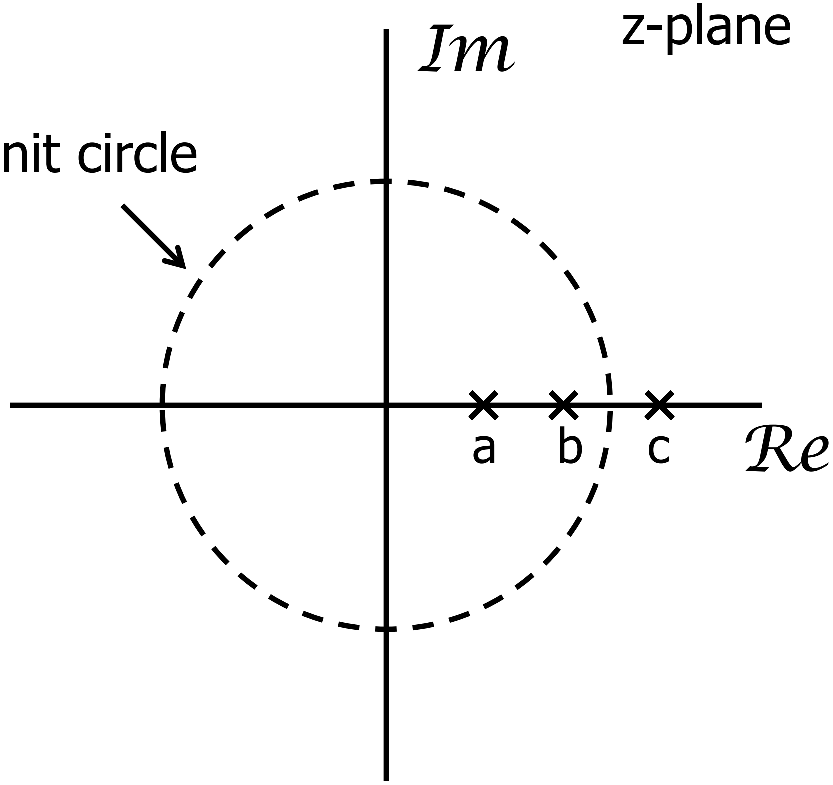
- PROPERTY 1:
  - The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.
- PROPERTY 2:
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
- PROPERTY 3:
  - The ROC cannot contain any poles.
- PROPERTY 4:
  - If x[n] is *a finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

Formal Properties of the ROC

- PROPERTY 5:
  - If x[n] is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in X(z) to (and possibly including) $z = \infty$.
- PROPERTY 6:
  - If x[n] is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in X(z) to (and possibly including) z=0.
- PROPERTY 7:
  - A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.
- PROPERTY 8:
  - The ROC must be a connected region.

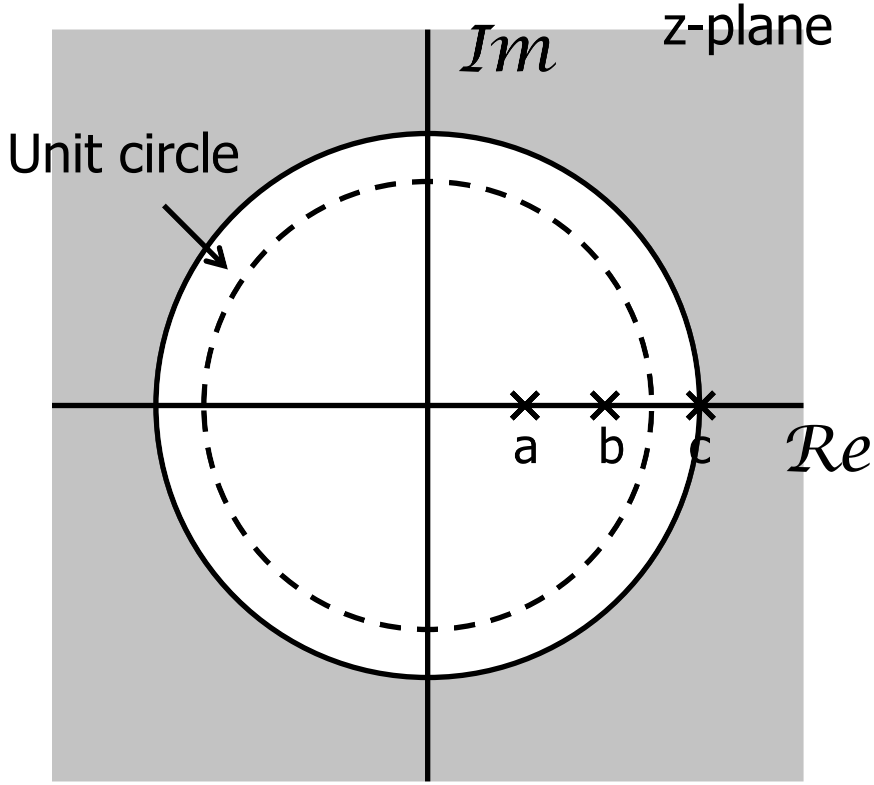
Example: ROC from Pole-Zero Plot

- How many possible ROCs?



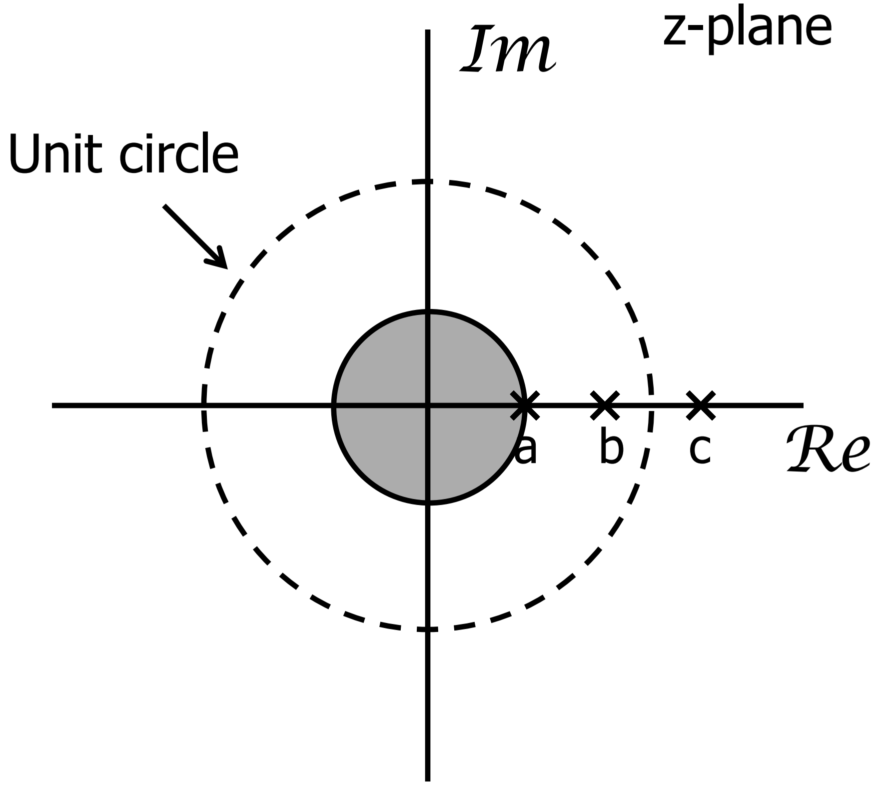
Example: ROC from Pole-Zero Plot

ROC 1: right-sided



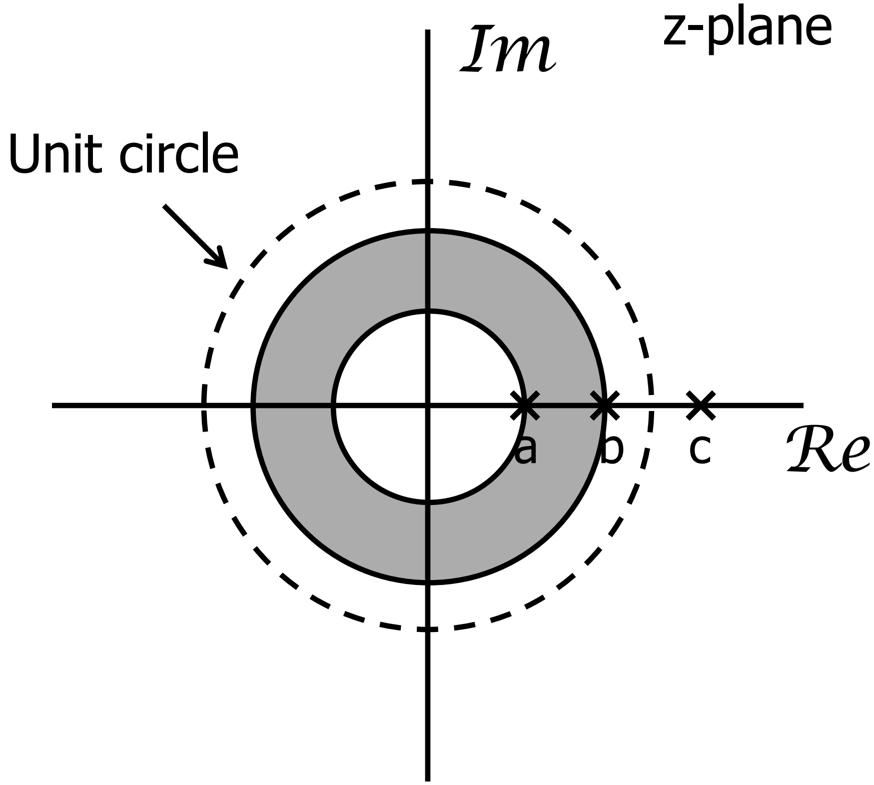
Example: ROC from Pole-Zero Plot

ROC 2: left-sided



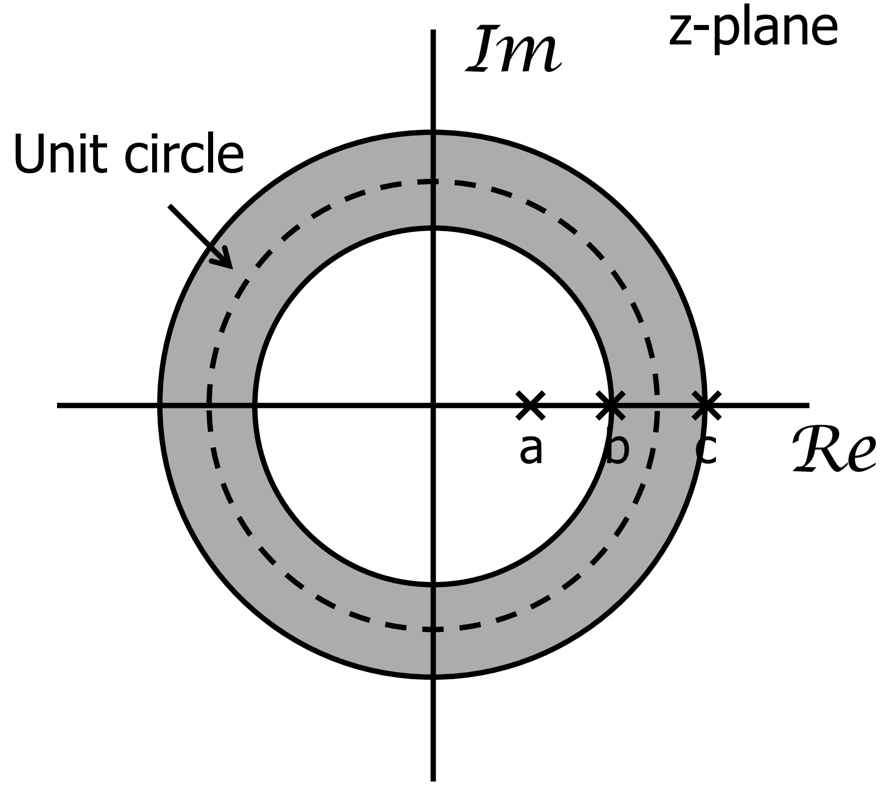
Example: ROC from Pole-Zero Plot

ROC 3: two-sided



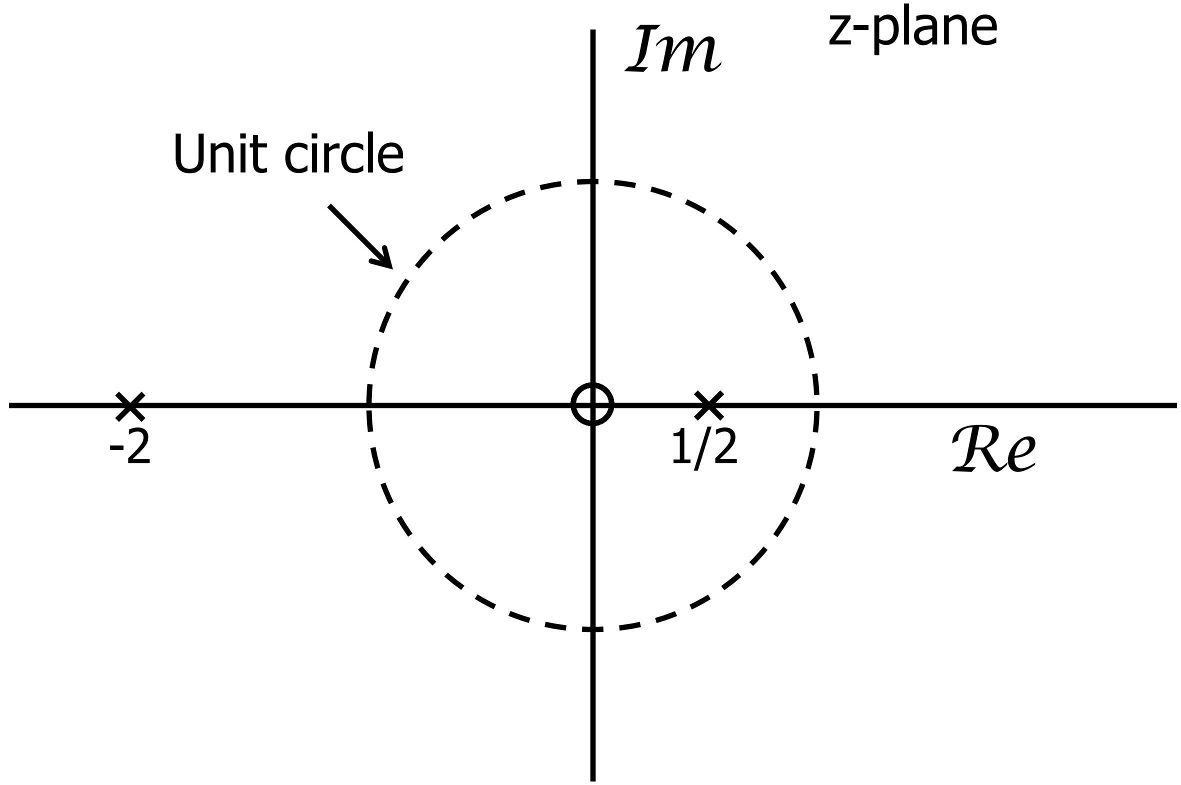
Example: ROC from Pole-Zero Plot

ROC 4: two-sided



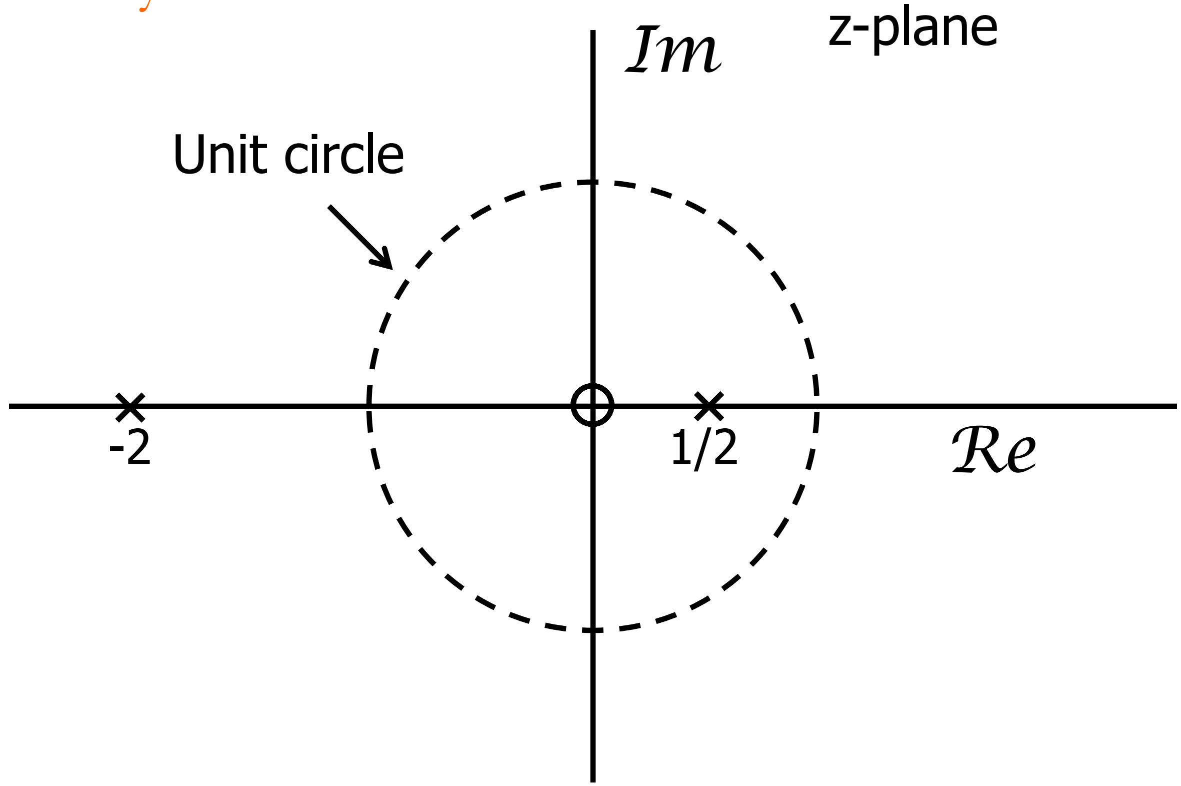
Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?



Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is stable?



BIBO Stability Revisited

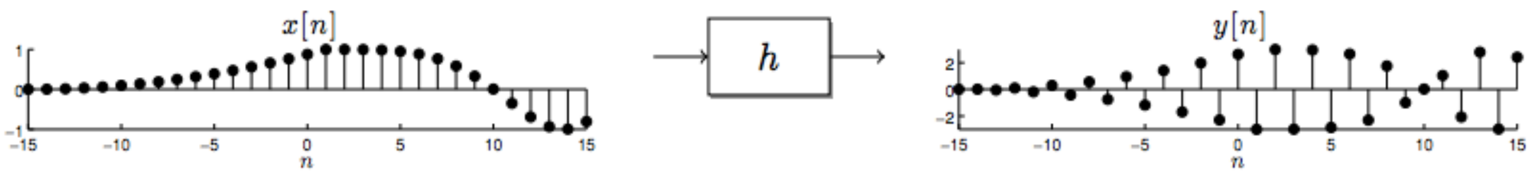
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output (**
input x always produces a bounded output y

bounded x → h → bounded y

- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$

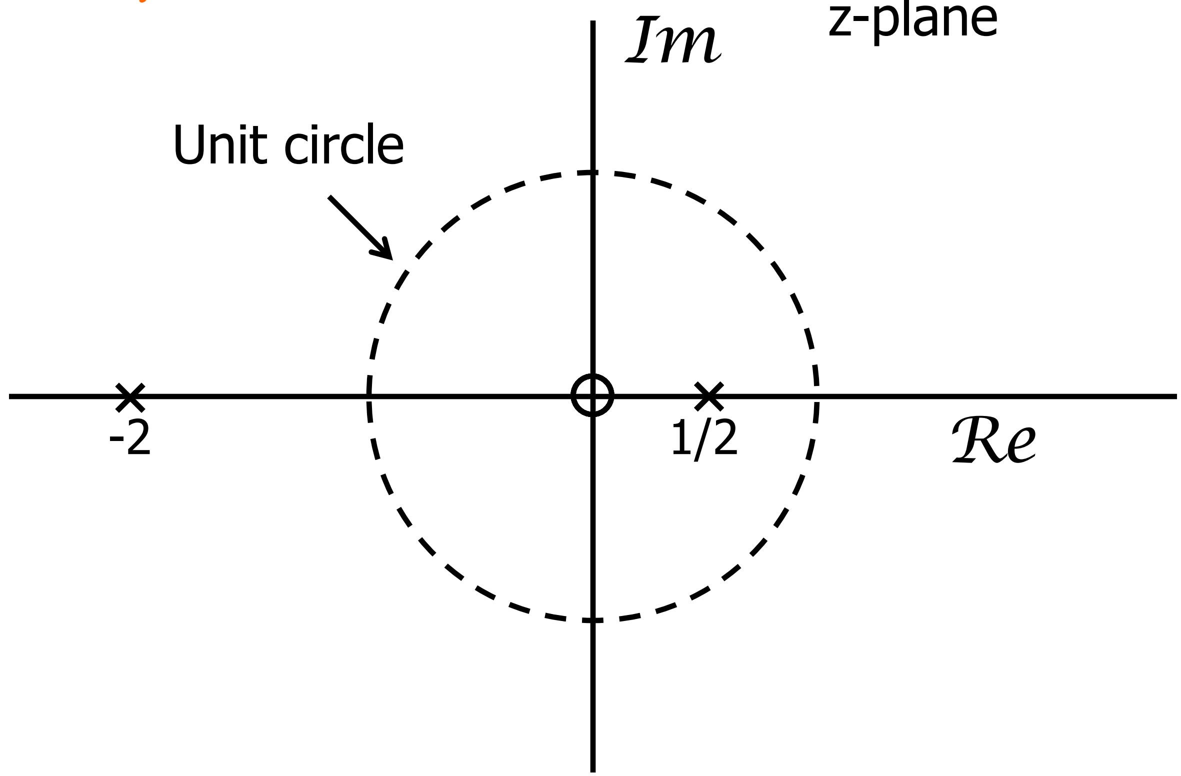


- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is causal?



Properties of z-Transform

- Linearity:

$$ax_1[n]+bx_2[n] \leftrightarrow aX_1(z)+bX_2(z)$$

- Time shifting:

$$x[n] \leftrightarrow X(z)$$

$$x[n-n_d] \leftrightarrow z^{-n_d}X(z)$$

- Multiplication by exponential sequence

$$x[n] \leftrightarrow X(z)$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

Properties of z-Transform

- Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

- Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

ROC_Y at least $ROC_X \wedge ROC_H$

Big Ideas

- z-Transform
  - Uses complex exponential eigenfunctions to represent discrete time sequence
    - DTFT is z-Transform where $z=e^{j\omega}$, $|z|=1$
  - Draw pole-zero plots
  - Must specify region of convergence (ROC)

- z-Transform properties
  - Similar to DTFT

Admin

- HW 1 due Sunday at midnight
  - Submit **single pdf** in Canvas
  - Don’t need to submit .m file, just the code in your pdf
- HW 2 posted Sunday
- Advice
  - Want to try and develop intuition and toolbox
  - Practice problems at end of chapter with answers in the text