ESE 531: Digital Signal Processing

Lec 6: February 4, 2020
Inverse z-Transform

Lecture Outline

- z-Transform
  - Tie up loose ends
  - Regions of convergence properties
  - Transform properties
- Inverse z-transform
  - Inspection
  - Partial fraction
  - Power series expansion
- z-transform of difference equations

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z-Transform

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z-Transform

- Define the **forward z-transform** of x[n] as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

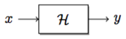
- The core "basis functions" of the z-transform are the complex exponentials z^n with arbitrary z ∈ C; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT X(ω) and the z-transform X(z); they are, in fact, intimately related

$$X_{\text{DTFT}}(\omega) = X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \quad H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

$$Y(z) = X(z)\, H(z)$$

Region of Convergence (ROC)

Region of Convergence (ROC)

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

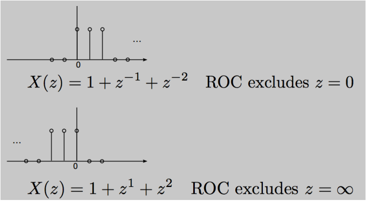
$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \;<\; \infty$$

Summary: Properties of ROC

- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1,2
- For left-sided: inwards from inner most pole to zero
  - Example 3
- For two-sided, ROC is a ring - or does not exist
  - Examples 4,5

Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly z=0, z=∞
  - Example 6



$X(z) = 1 + z^{-1} + z^{-2}$ ROC excludes $z = 0$

$X(z) = 1 + z^{1} + z^{2}$ ROC excludes $z = \infty$

Formal Properties of the ROC

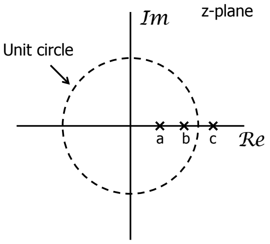
- PROPERTY 1:
  - The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.
- PROPERTY 2:
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
- PROPERTY 3:
  - The ROC cannot contain any poles.
- PROPERTY 4:
  - If x[n] is *a finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z-plane, except possibly z = 0 or z = ∞.

Formal Properties of the ROC

- PROPERTY 5:
  - If x[n] is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in X(z) to (and possibly including) z = ∞.
- PROPERTY 6:
  - If x[n] is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in X(z) to (and possibly including) z=0.
- PROPERTY *7:*
  - A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.
- PROPERTY 8:
  - The ROC must be a connected region.

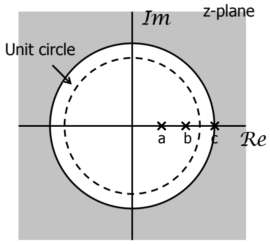
Example: ROC from Pole-Zero Plot

- How many possible ROCs?



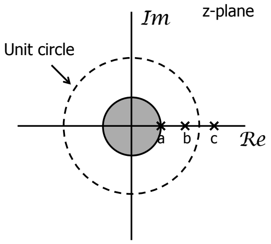
Example: ROC from Pole-Zero Plot

ROC 1: right-sided



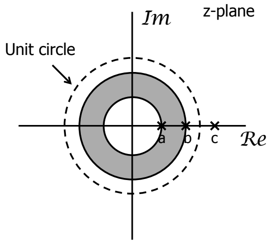
Example: ROC from Pole-Zero Plot

ROC 2: left-sided



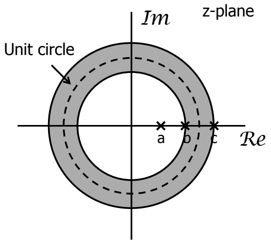
Example: ROC from Pole-Zero Plot

ROC 3: two-sided



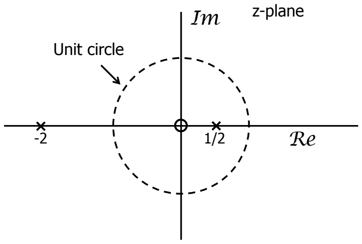
Example: ROC from Pole-Zero Plot

ROC 4: two-sided



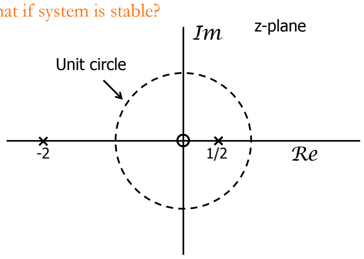
Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?



Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is stable?



Region of Convergence (ROC)

DEFINITION

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$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \ < \ \infty$$

BIBO Stability Revisited

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

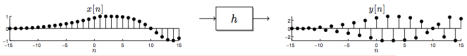
DEFINITION

An LTI system is **bounded-input bounded-output** (

input x always produces a bounded output y

bounded x $\longrightarrow \boxed{h} \longrightarrow$ bounded y

- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$

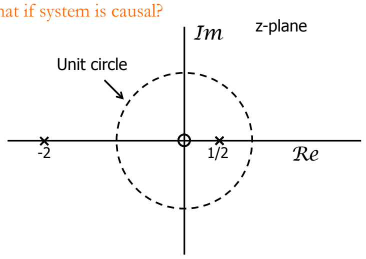


- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 \ = \ \sum_{n=-\infty}^{\infty} |h[n]| \ < \ \infty$$

Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is causal?



Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \leftrightarrow X(z)$$

$$x[n-n_d] \leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \leftrightarrow X(z)$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

Properties of z-Transform

- Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

- Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

ROC_Y at least $\text{ROC}_X \wedge \text{ROC}_H$

Inverse z-Transform

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Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

- There is a similar formula for the inverse z-transform using a contour integral

$$x[n] = \oint_C X(z)\, z^n\, \frac{dz}{j2\pi z}$$

- Contour integrals are fun but beyond the scope of this course!

Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion

Z-Transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Z-Transform Properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Partial Fraction Expansion

- Let

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- M zeros and N poles at nonzero locations

Partial Fraction Expansion

$$X(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{z^N\sum_{k=0}^{M} b_k z^{M-k}}{z^M\sum_{k=0}^{N} a_k z^{N-k}}$$

- Factored numerator/denominator

$$X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

Partial Fraction Expansion

- If M<N and the poles are 1st order

$$X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}=\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- where

$$A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)},\qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)},\qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles $\quad A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1=(1-\frac{1}{4}z^{-1})X(z)\Big|_{z=1/4}=\frac{(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}\Bigg|_{z=1/4}=-1$$

$$A_2=(1-\frac{1}{2}z^{-1})X(z)\Big|_{z=1/2}=\frac{(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}\Bigg|_{z=1/2}=2$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)},\qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z: \frac{1}{2} < |z|\right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z: \frac{1}{2} < |z|\right\}$$

5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z: \frac{1}{2} < |z|\right\}$$

5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

Partial Fraction Expansion

- If M<N and the poles are 1st order

$$X(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} = \sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- where

$$A_k = (1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Partial Fraction Expansion

- If M≥N and the poles are 1st order

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- Where B_k is found by long division

$$A_k = (1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC = \left\{z: 1 < |z|\right\}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC=\{z:\ 1<|z|\}$$

$$=\frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}, \qquad ROC=\{z:\ 1<|z|\}$$

$$\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\overline{\big)\,z^{-2}+2z^{-1}+1}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}, \qquad ROC=\{z:\ 1<|z|\}$$

$$\begin{array}{r l} & \qquad\qquad\qquad\quad 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\big) & \overline{z^{-2}+2z^{-1}+1} \\ & \underline{z^{-2}-3z^{-1}+2} \\ & \qquad\quad 5z^{-1}-1 \end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=2-\frac{9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}}, \qquad ROC=\{z:\ 1<|z|\}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=2-\frac{9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}}, \qquad ROC=\{z:\ 1<|z|\}$$

$$x[n]=2\delta[n]-9\left(\frac{1}{2}\right)^{n}u[n]+8u[n]$$

Power Series Expansion

- Expansion of the z-transform definition

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

$$=\cdots+x[-2]z^{2}+x[-1]z+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

Example: Finite-Length Sequence

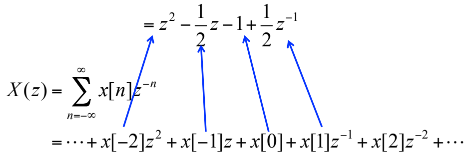
- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$



$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

4. $\delta[n-m]$ $\quad z^{-m}$

Reminder: Difference Equations

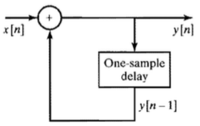
- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$



$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right)x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

Difference Equation to z-Transform

$$y[n]=-\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k]+\sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right)x[n-m]$$

$$Y(z)=-\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)z^{-k}Y(z)+\sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right)z^{-m}X(z)$$

Difference Equation to z-Transform

$$y[n]=-\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k]+\sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right)x[n-m]$$

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$$\boxed{H(z)=\frac{\sum_{m=0}^{M}\left(b_k\right)z^{-k}}{\sum_{k=0}^{N}\left(a_k\right)z^{-k}}}$$

Example: 1st-Order System

$$y[n]=ay[n-1]+x[n]$$

$$\boxed{\begin{gathered}\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m x[n-m]\\[1em] H(z)=\frac{\sum_{m=0}^{M}\left(b_k\right)z^{-k}}{\sum_{k=0}^{N}\left(a_k\right)z^{-k}}\end{gathered}}$$

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$$y[n]=ay[n-1]+x[n]$$

$$H(z)=\frac{\overset{b_0}{1}}{\underset{a_0}{1}-\underset{a_1}{a}z^{-1}}$$

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Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\overset{b_0}{1}}{\underset{a_0}{1} - \underset{a_1}{a}z^{-1}}$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

$$h[n] = a^n u[n]$$

Why right sided?

Big Ideas

- z-Transform
  - Draw pole-zero plots
  - Must specify region of convergence (ROC)
    - ROC properties
- z-Transform properties
  - Similar to DTFT
- Inverse z-transform
  - Avoid it!
  - Inspection, properties, partial fractions, power series
- Difference equations easy to transform

Admin

- HW 2 due Sunday 2/9 at midnight