

ESE 531: Digital Signal Processing

Lec 7: February 6, 2020

Sampling and Reconstruction



Lecture Outline

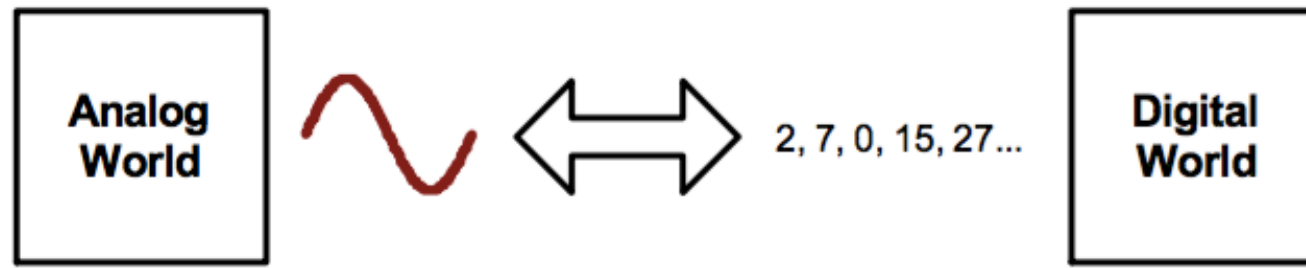
- ❑ Sampling
 - Frequency Response of Sampled Signal
- ❑ Reconstruction
- ❑ Anti-aliasing Filter

Video Example



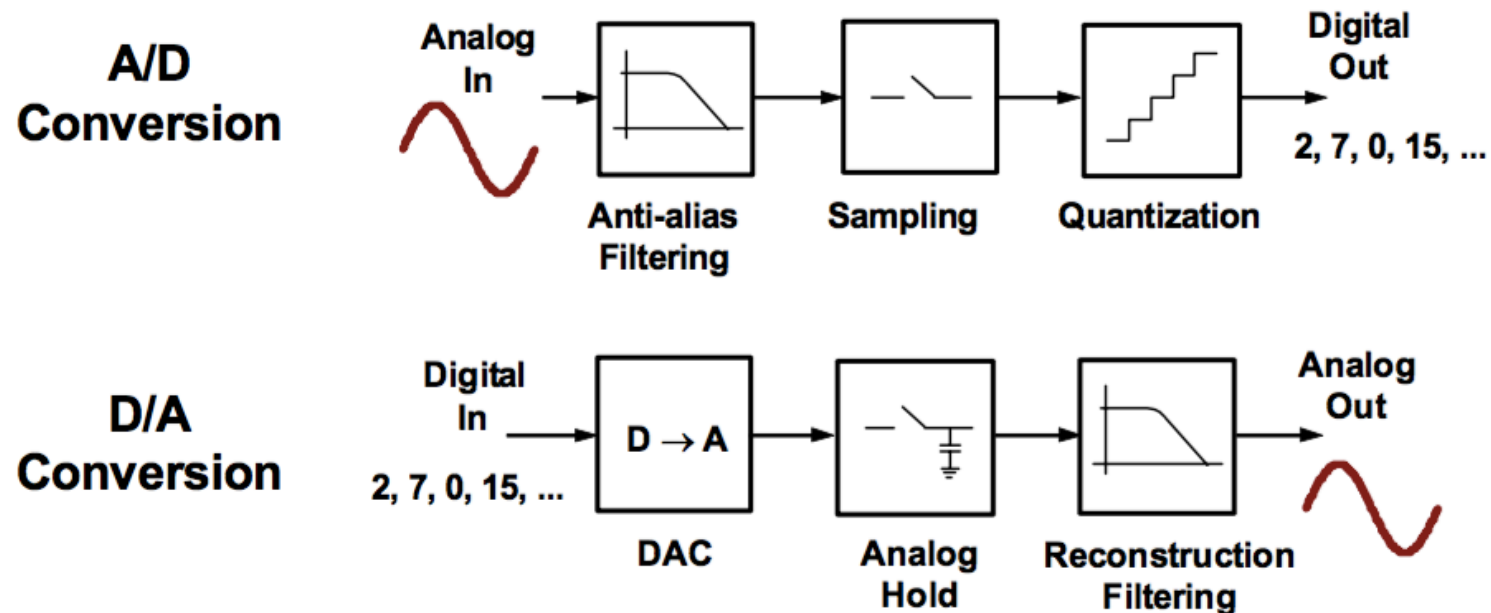
❑ <https://www.youtube.com/watch?v=ByTsISFXUoY>

The Data Conversion Problem



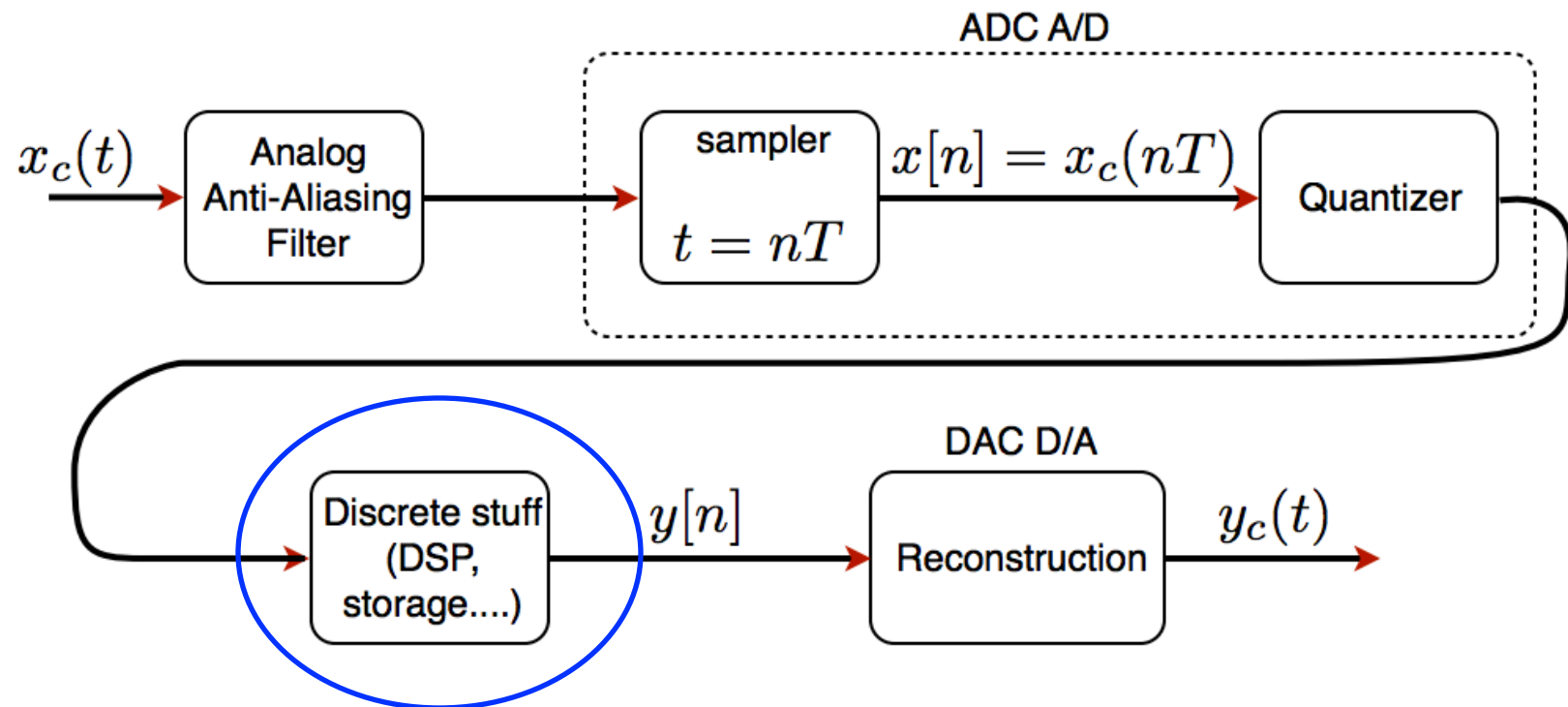
- ❑ Real world signals
 - Continuous time, continuous amplitude
- ❑ Digital abstraction
 - Discrete time, discrete amplitude
- ❑ Two problems
 - How to go discretize in time and amplitude
 - A/D conversion
 - How to "undescretize" in time and amplitude
 - D/A conversion

Overview

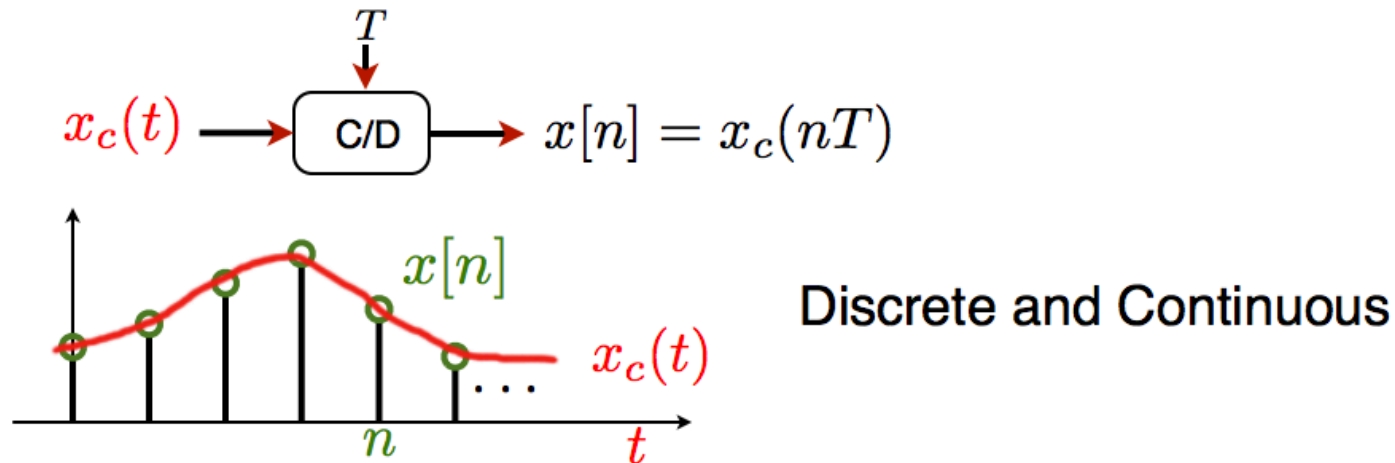


- ❑ We'll first look at these building blocks from a functional, "black box" perspective

DSP System

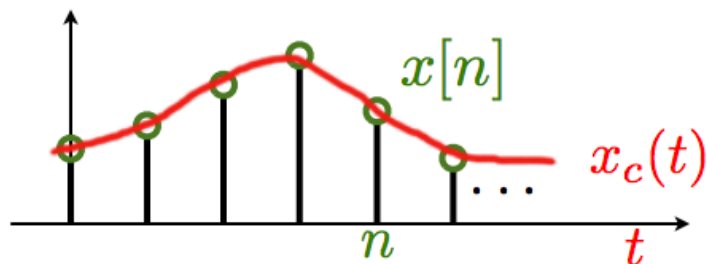
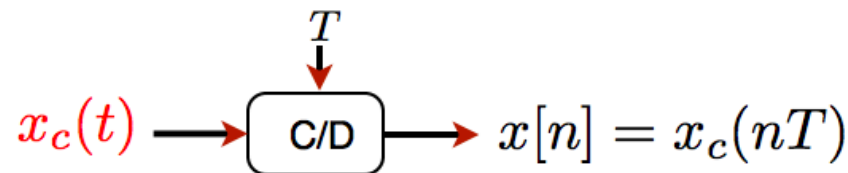


Ideal Sampling Model



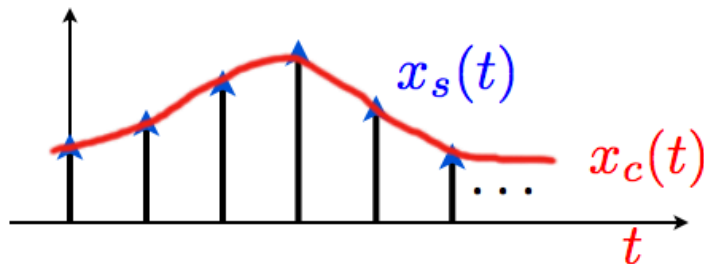
- ❑ Ideal continuous-to-discrete time (C/D) converter
 - T is the sampling period
 - $f_s = 1/T$ is the sampling frequency
 - $\Omega_s = 2\pi/T$

Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- ❑ Dirac delta function, $\delta(t)$
 - Infinitely high and thin, area of 1
 - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- ❑ Three signals. How are they related? In time? In frequency?



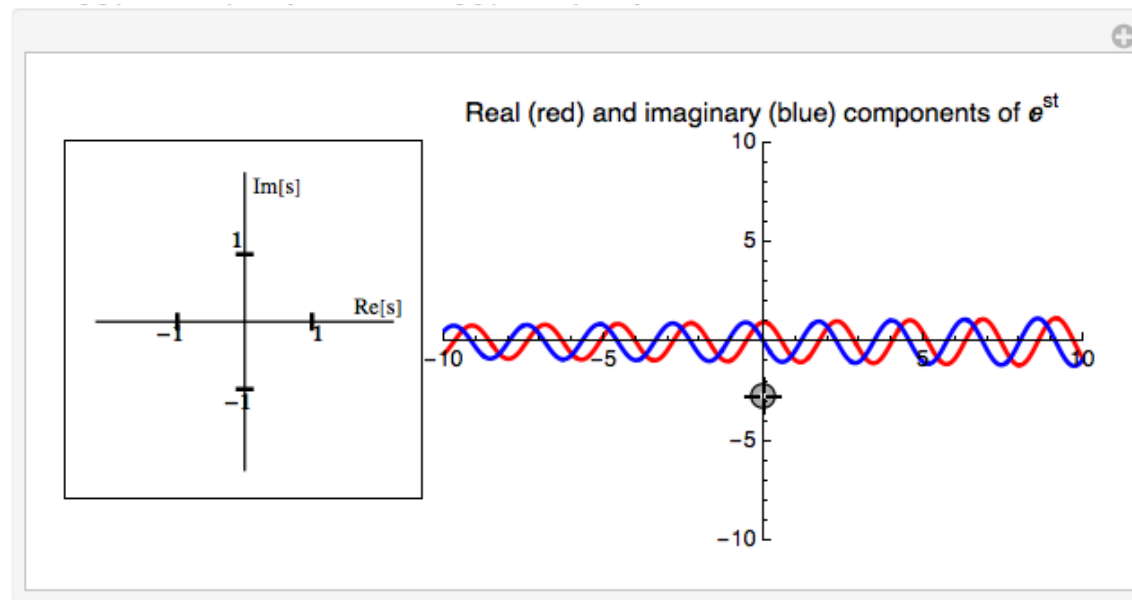
Reminder: Laplace Transform

- ❑ The Laplace transform takes a function of time, t , and transforms it to a function of a complex variable, s .
- ❑ Because the transform is invertible, no information is lost and it is reasonable to think of a function $f(t)$ and its Laplace transform $F(s)$ as two views of the same phenomenon.
- ❑ Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

S-Plane

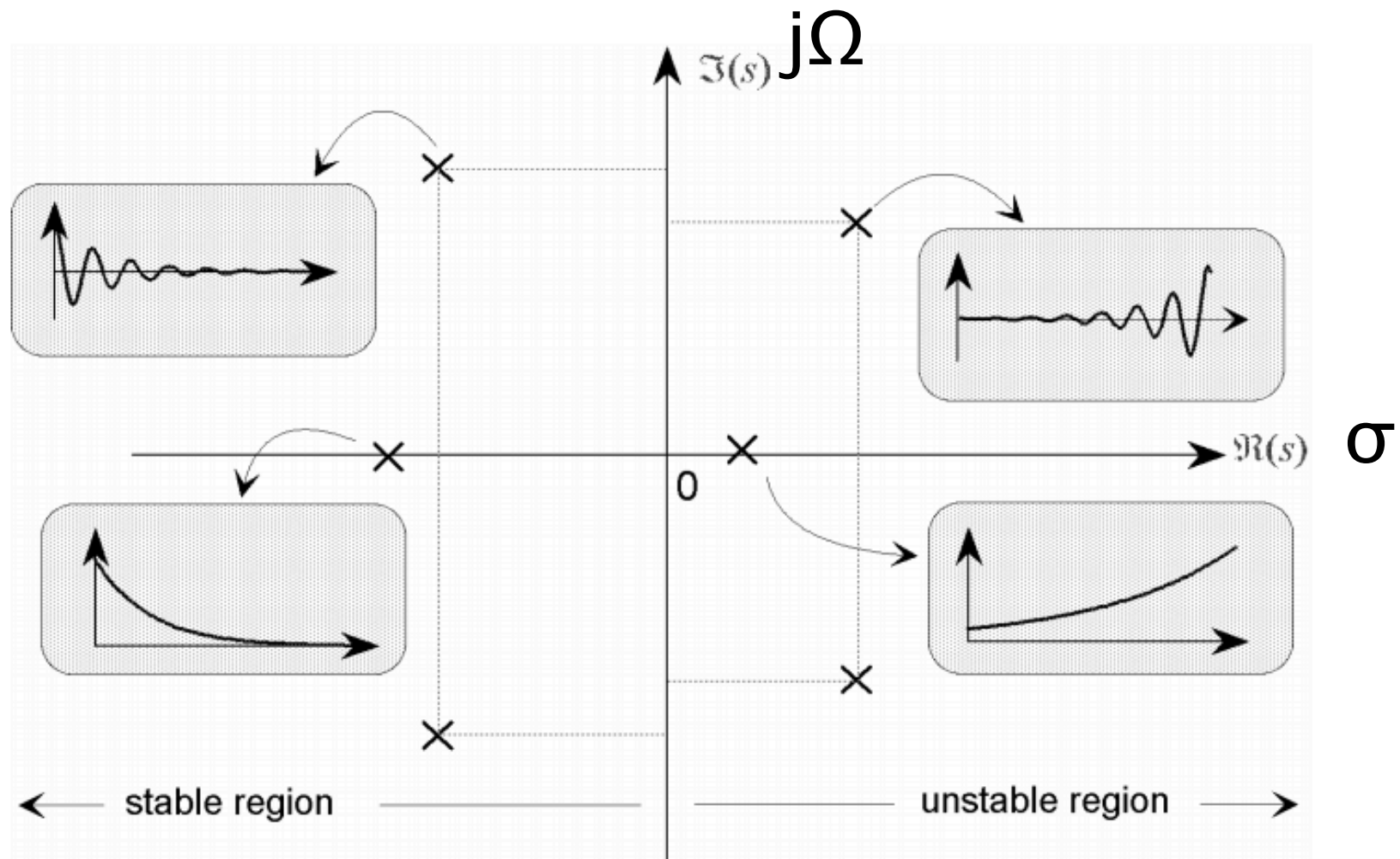
□ $s = \sigma + j\Omega$

□ Wolfram Demo

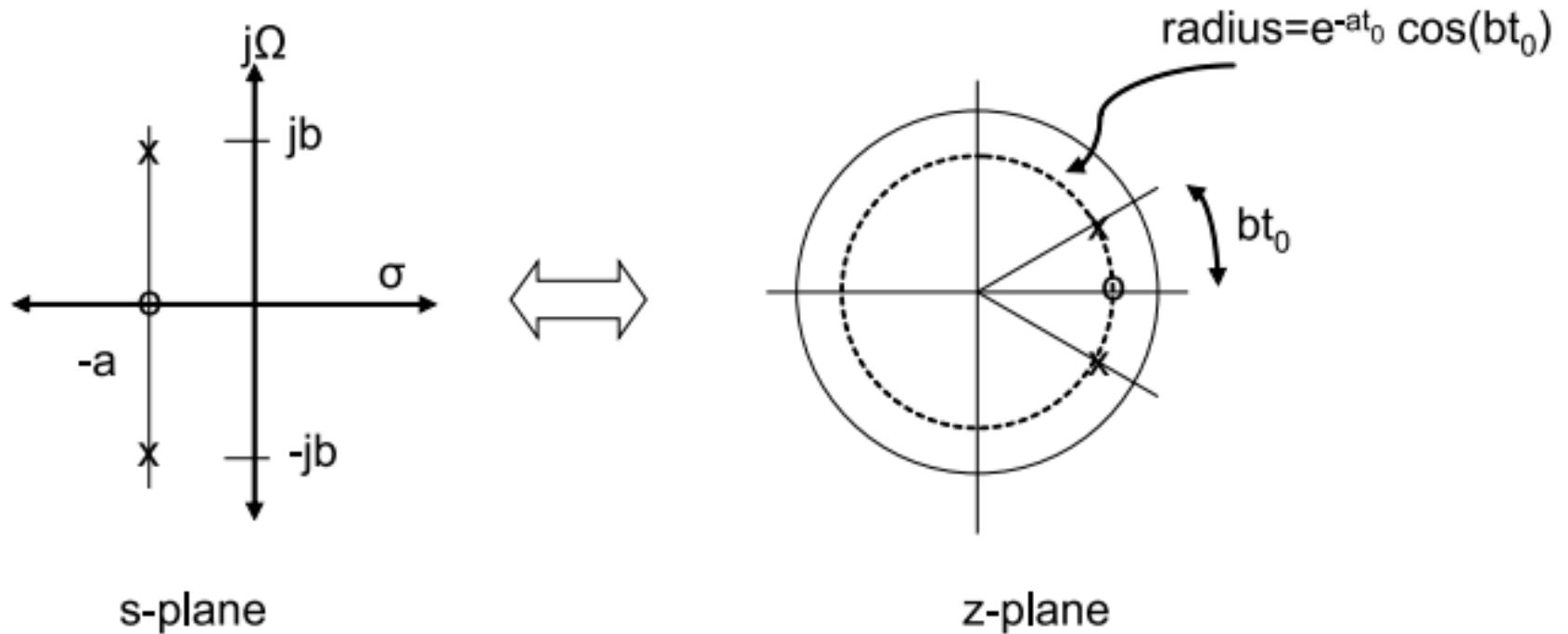


□ <http://pilot.cnxproject.org/content/collection/col10064/latest/module/m10060/latest>

S-plane and stability



S-Plane Mapping to Z-Plane





Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT)$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

- How is $x[n]$ related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) \quad : \text{C.T.}$$

$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

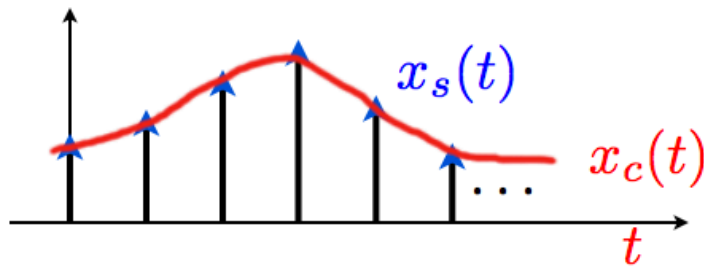
$$x[n] \quad : \text{D.T.}$$

$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}$$

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}$$

Frequency Domain Analysis



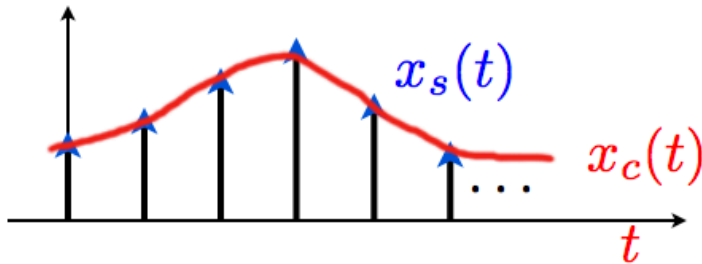
Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis



Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t - T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$s(t) \leftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

Ω_s



Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$



Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

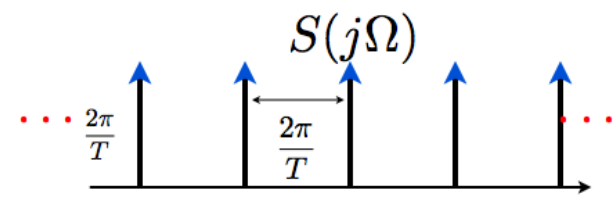
$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$



Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

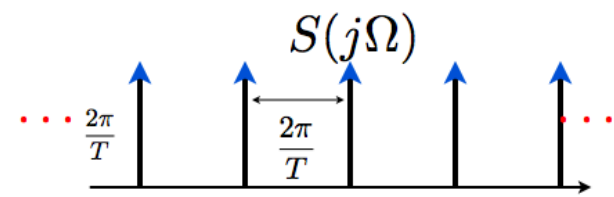
$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$


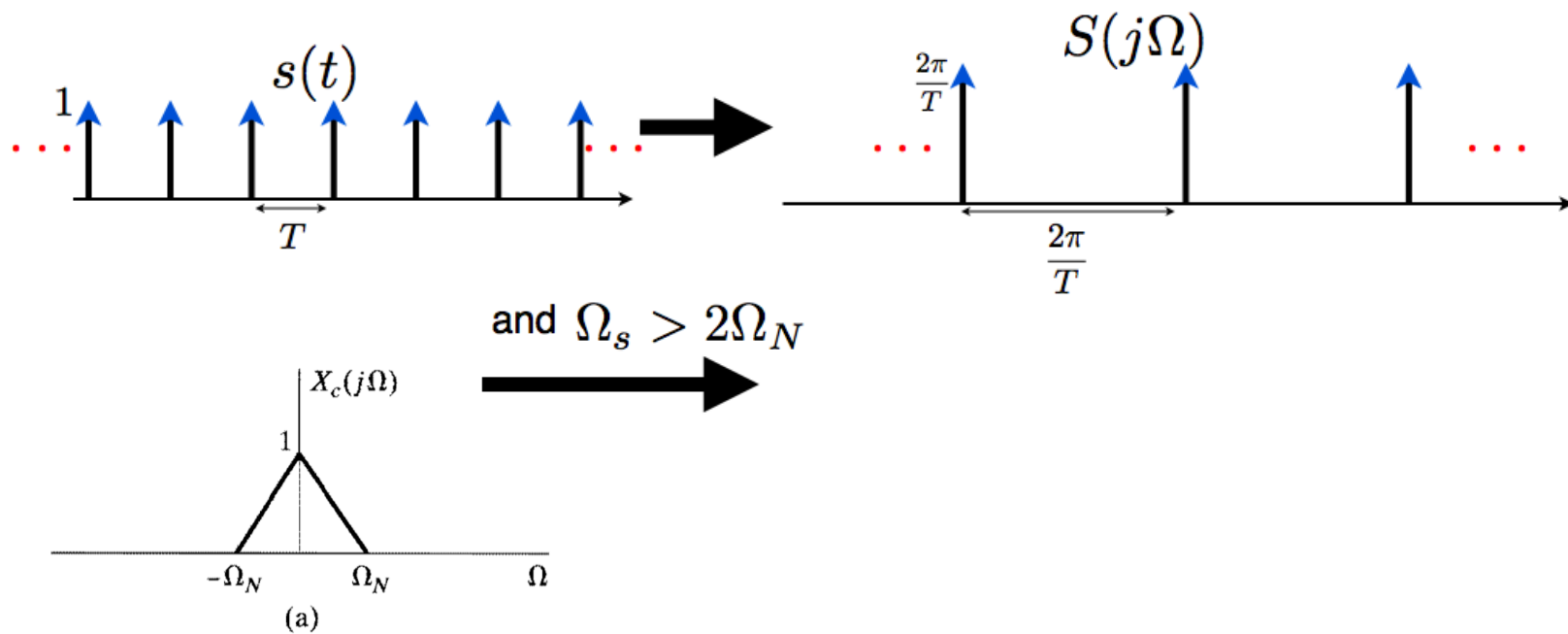
Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

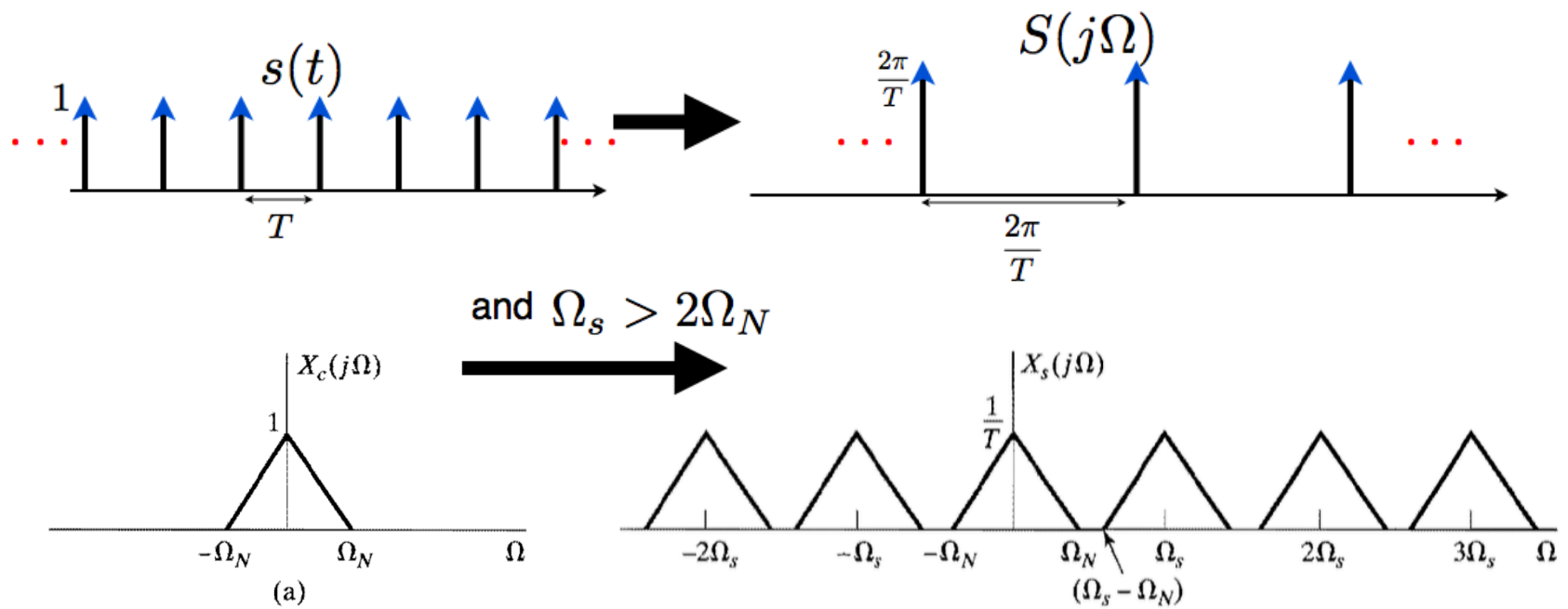
$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$


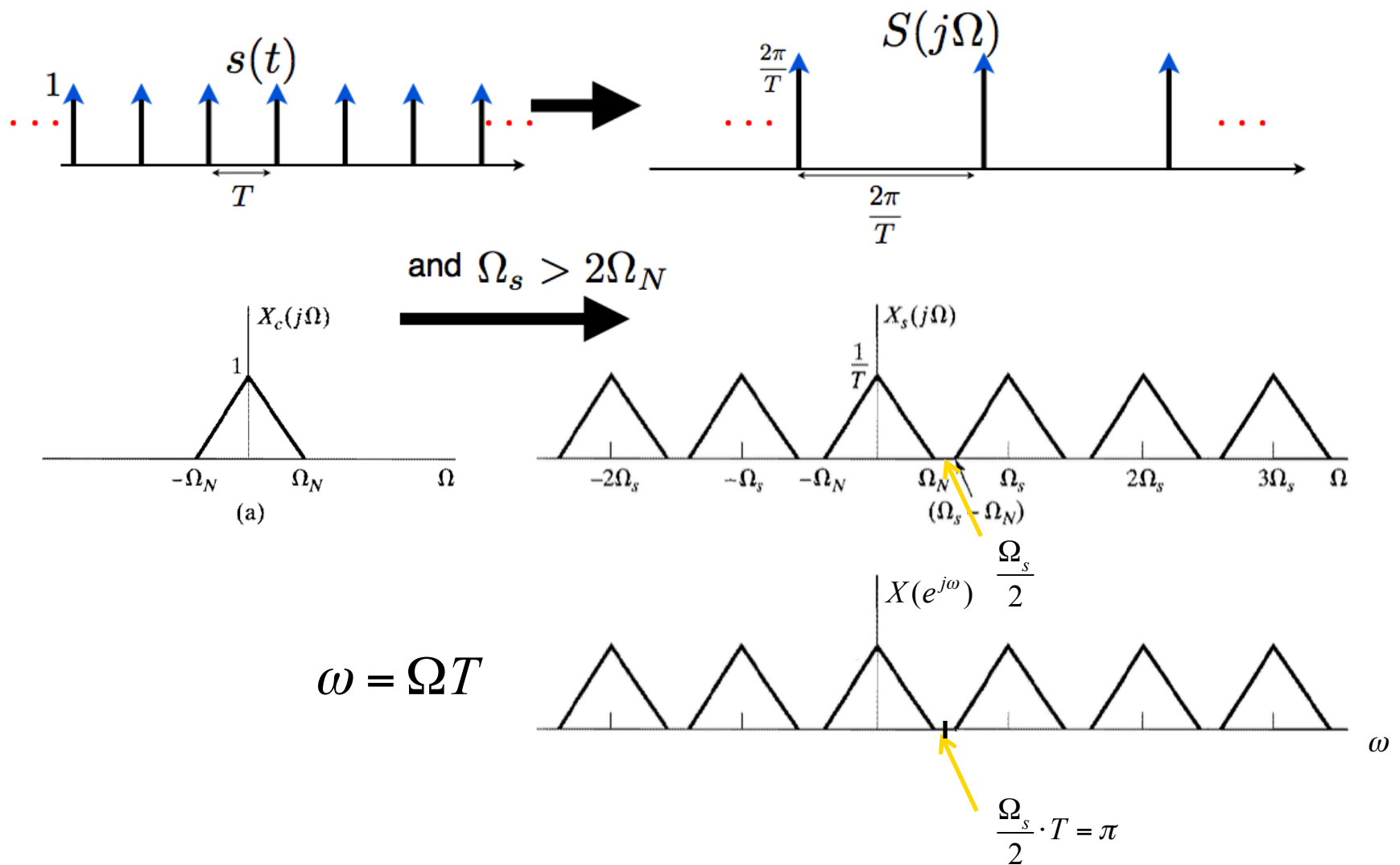
Frequency Domain Analysis



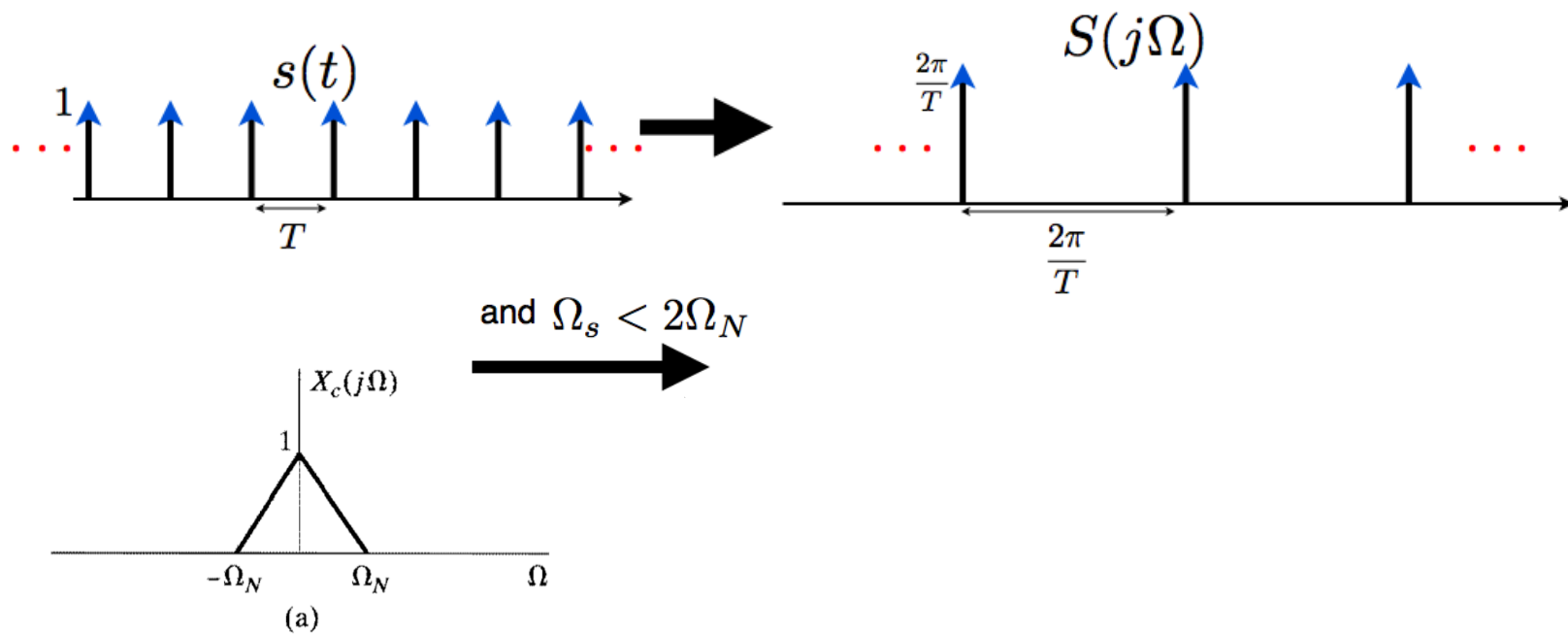
Frequency Domain Analysis



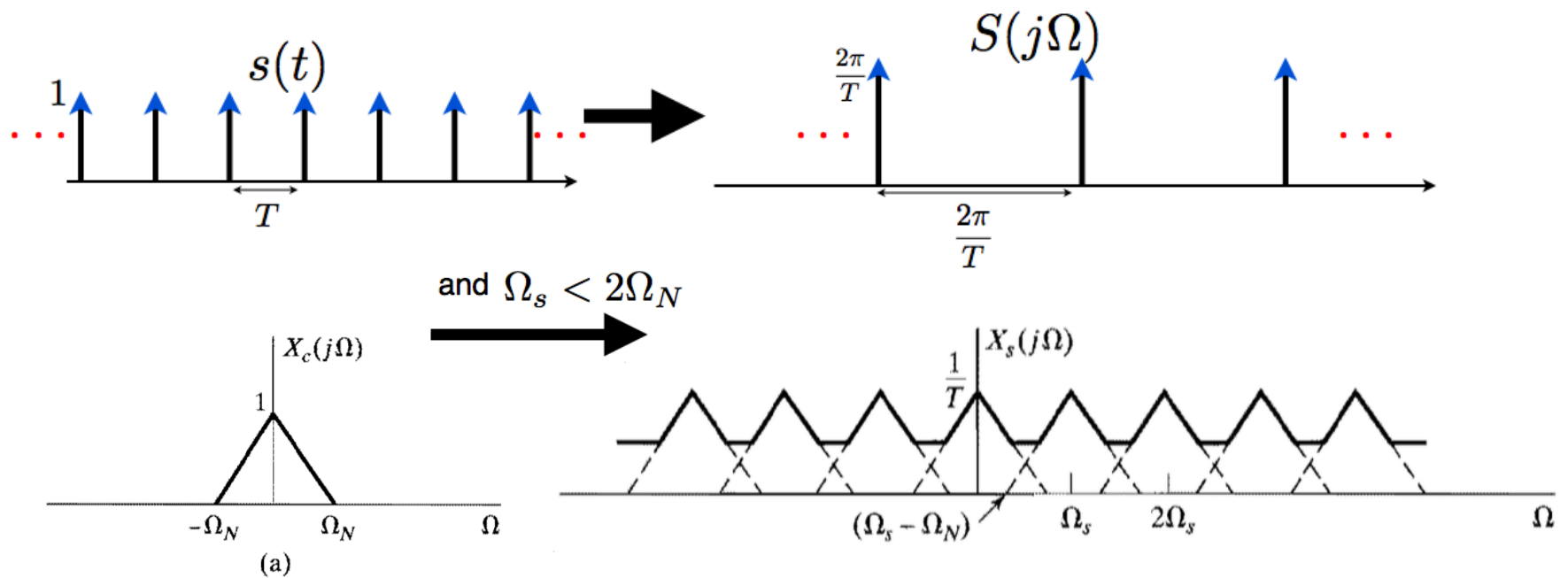
Frequency Domain Analysis



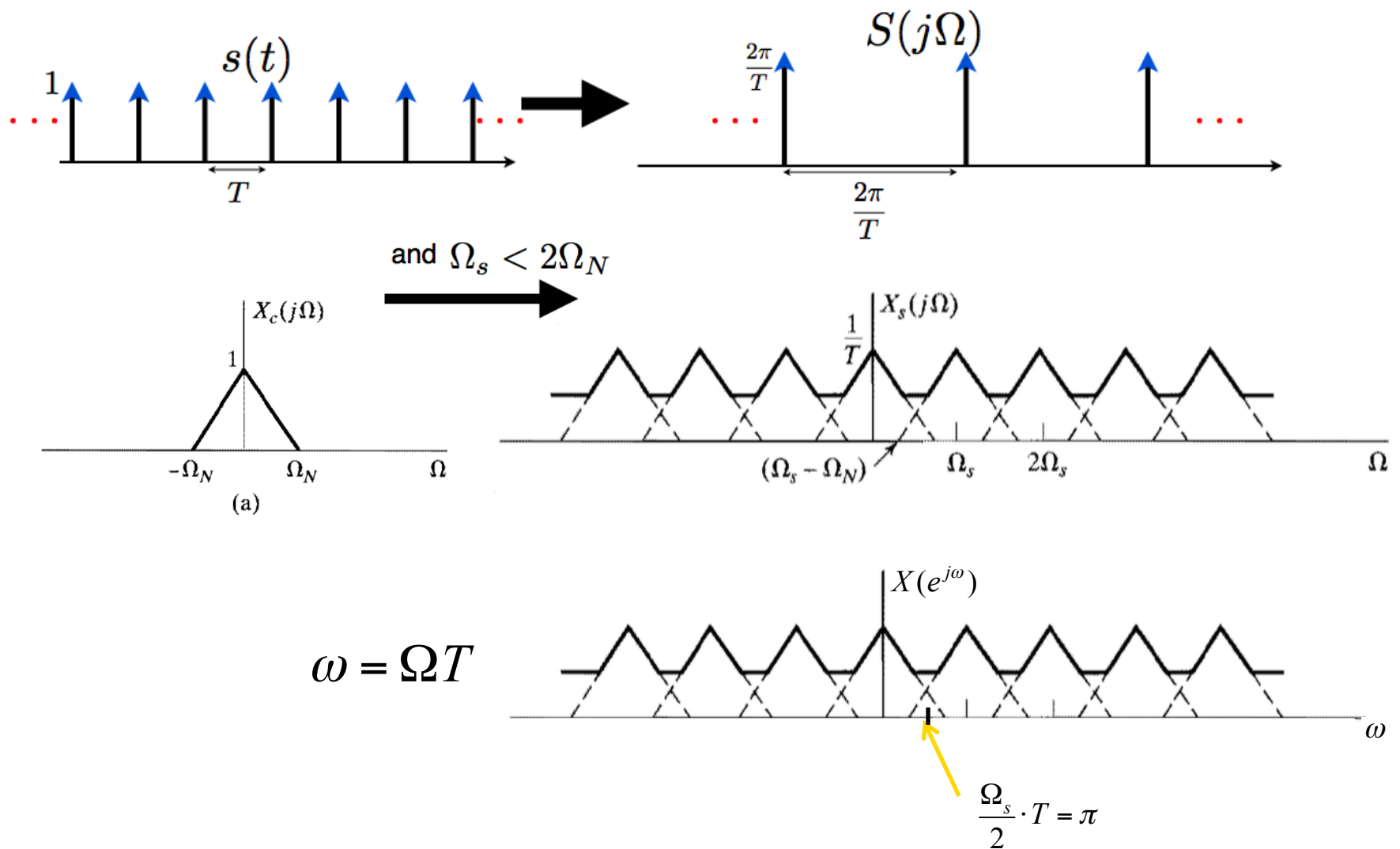
Frequency Domain Analysis w/ Aliasing



Frequency Domain Analysis w/ Aliasing



Frequency Domain Analysis w/ Aliasing

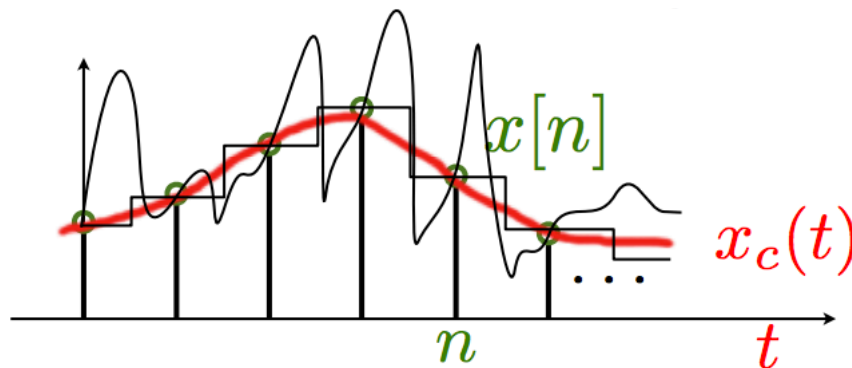


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

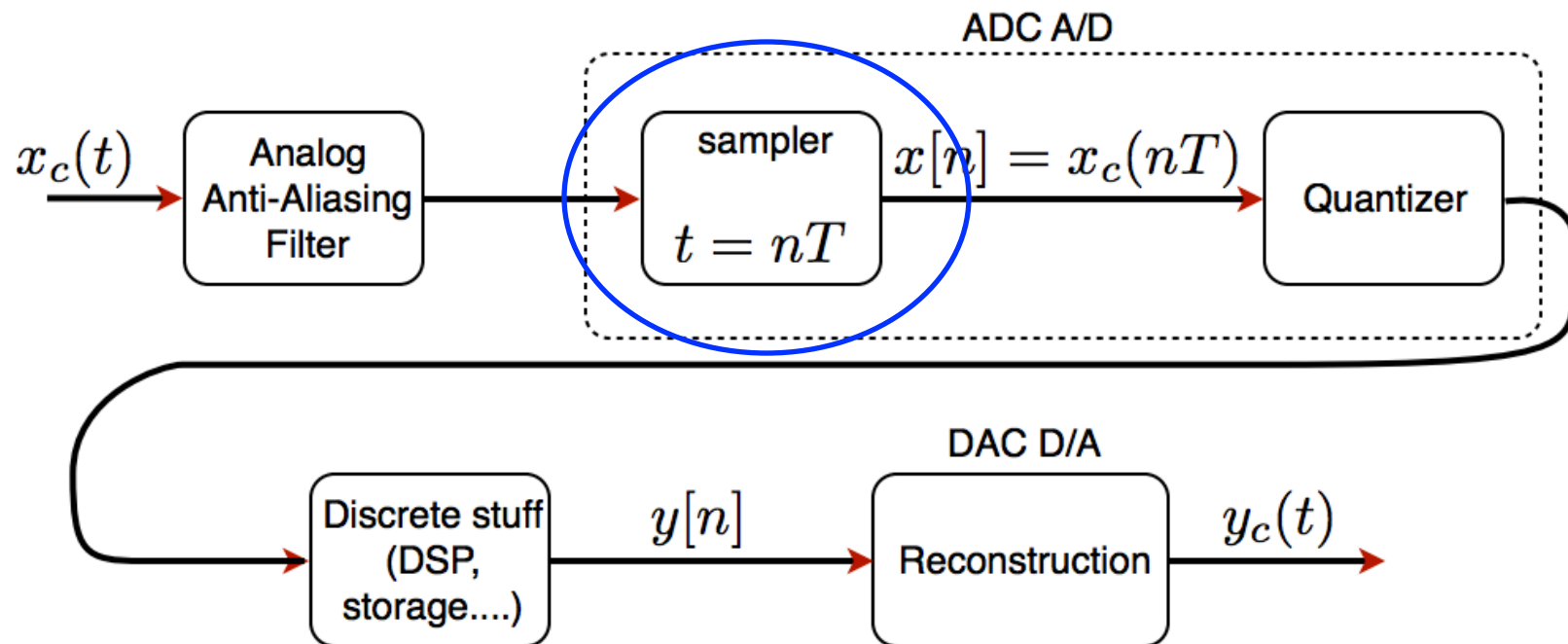
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$
- Bandlimitedness is the key to uniqueness

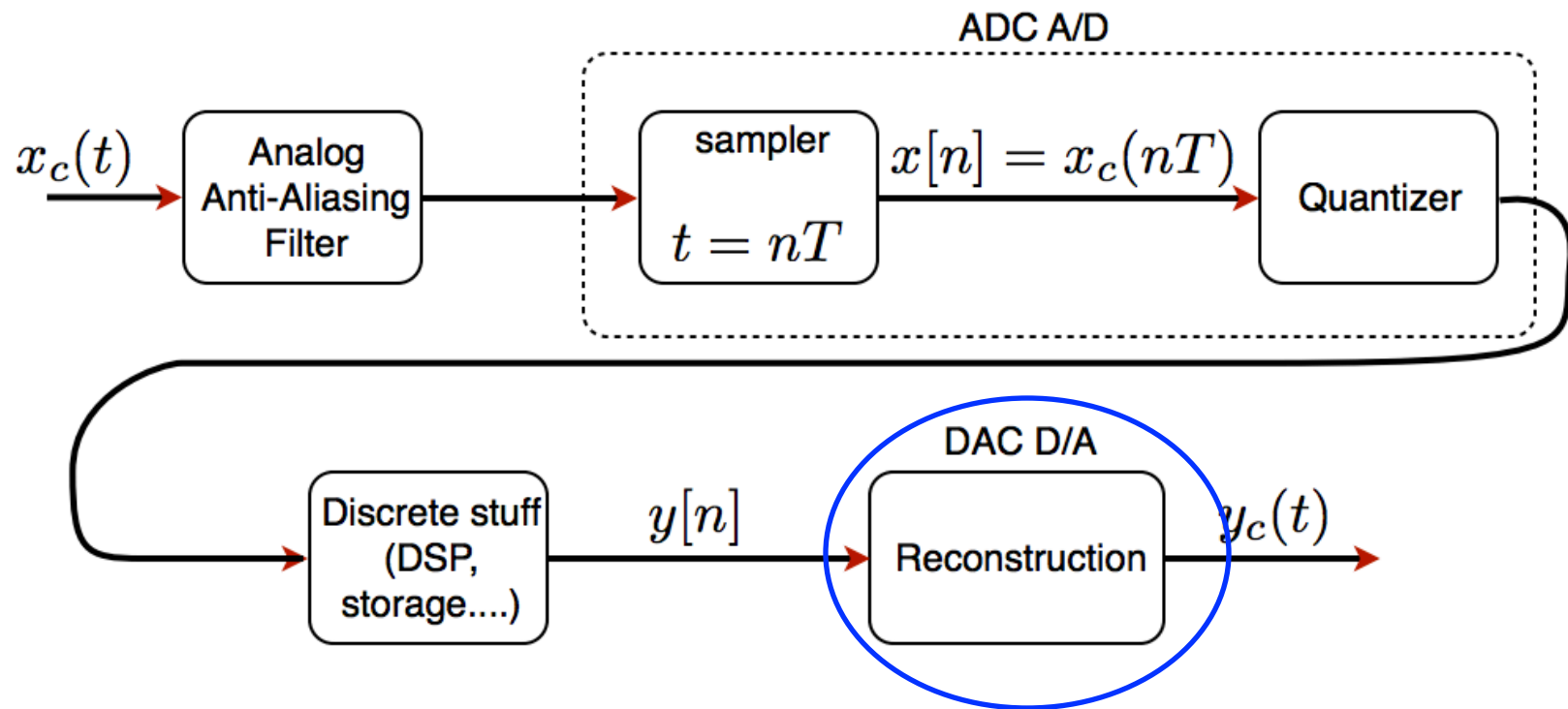


Multiple signals go through the samples, but only one is bandlimited within our sampling band

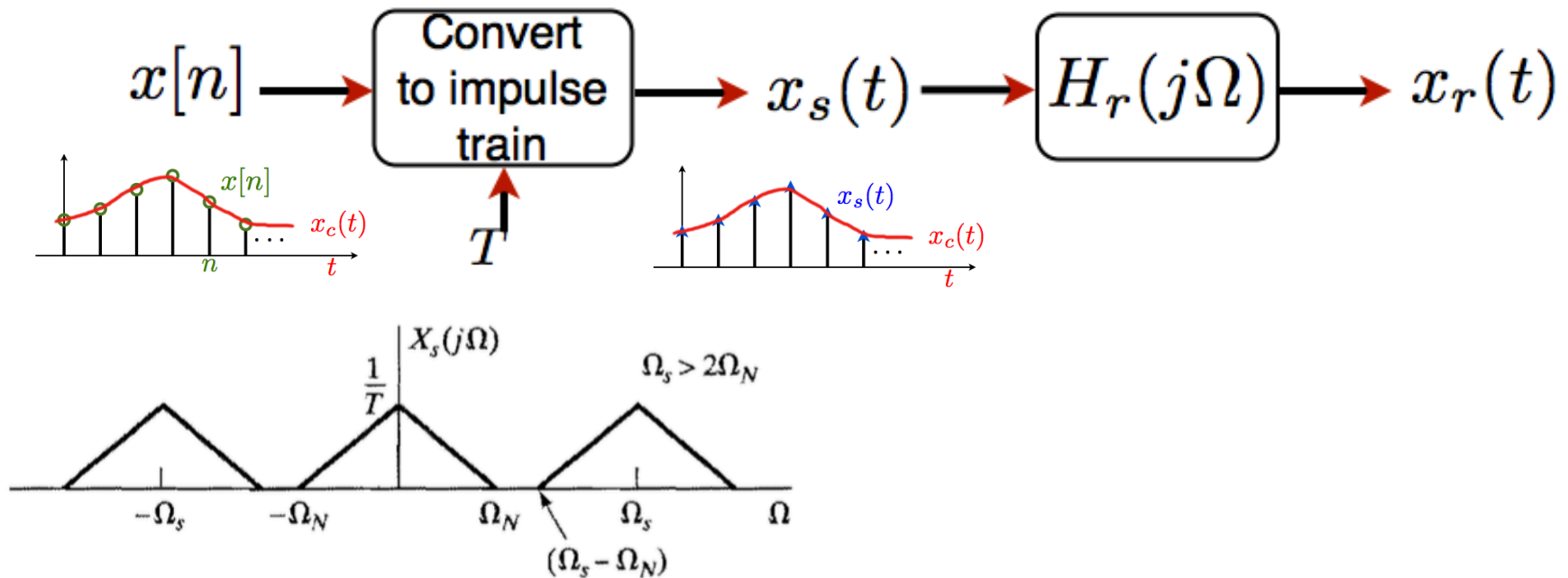
DSP System



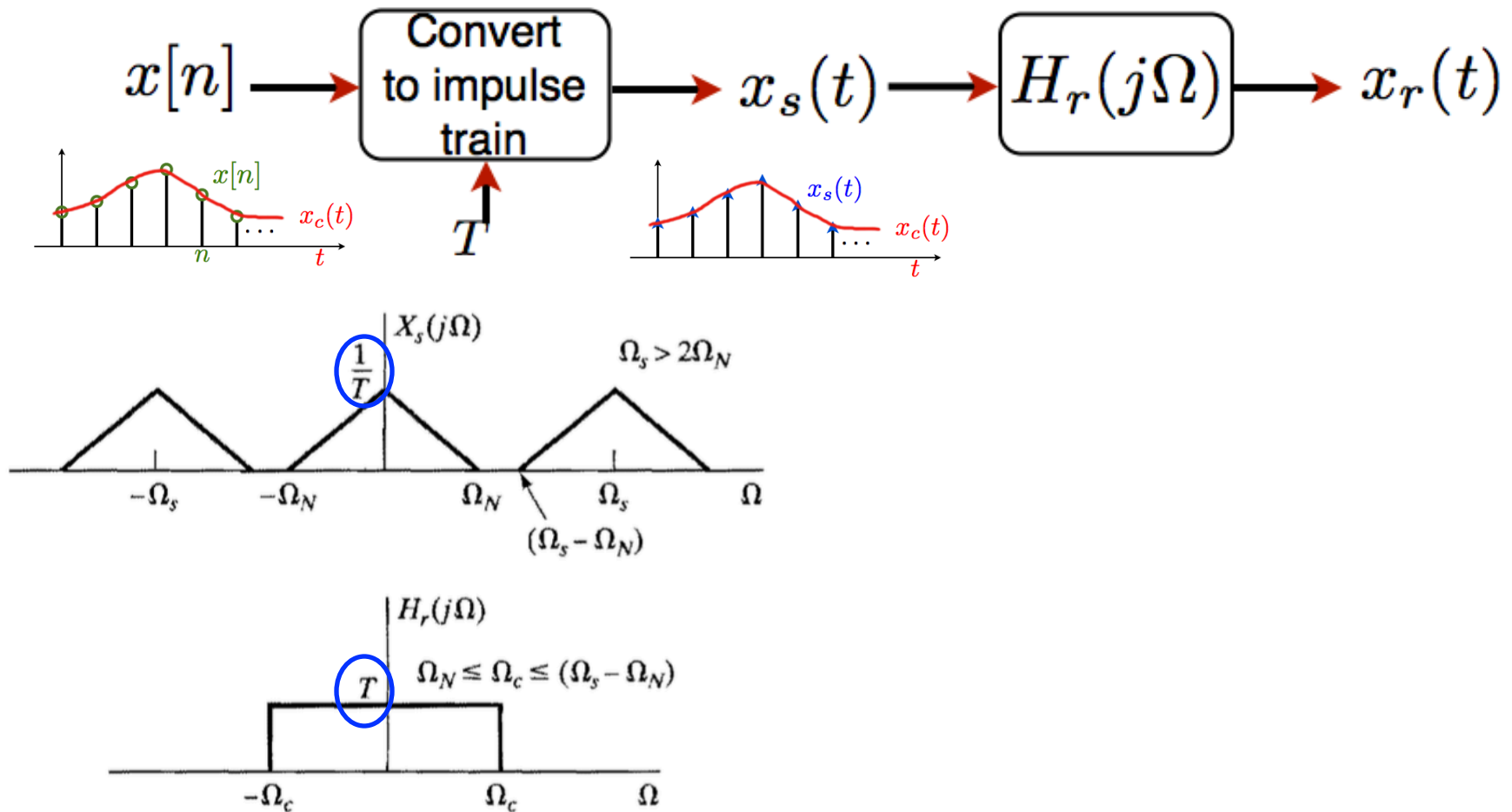
DSP System



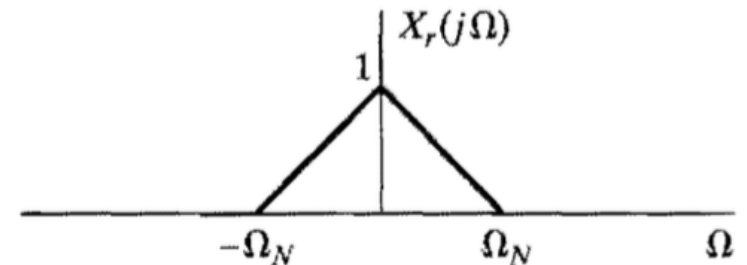
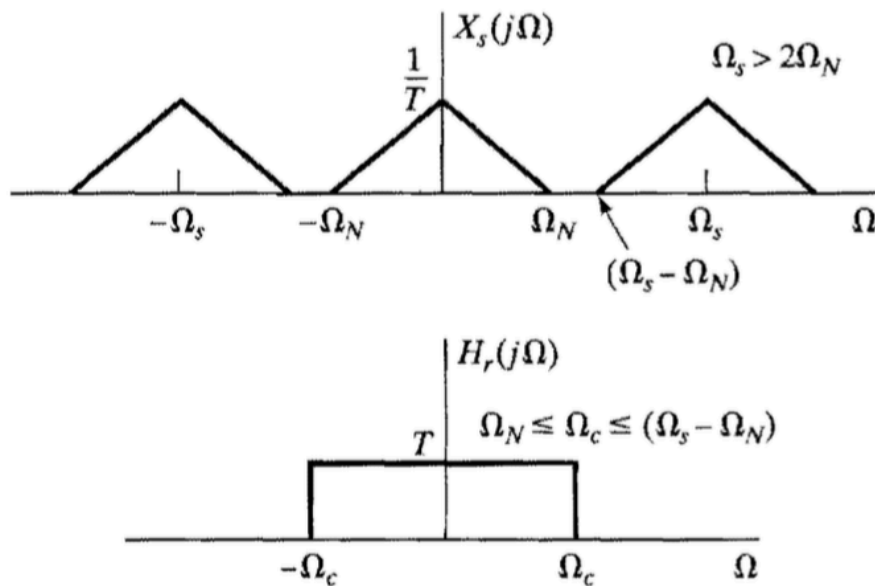
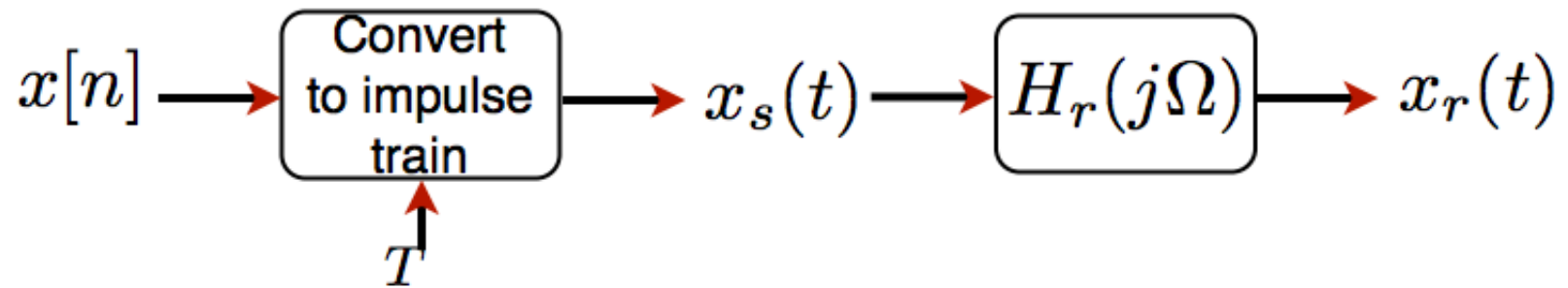
Reconstruction in Frequency Domain



Reconstruction in Frequency Domain



Reconstruction in Frequency Domain





Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t) = \cos(4000 \pi t)$ with sampling period $T = 1/6000$ s ($f_s = 6$ kHz)

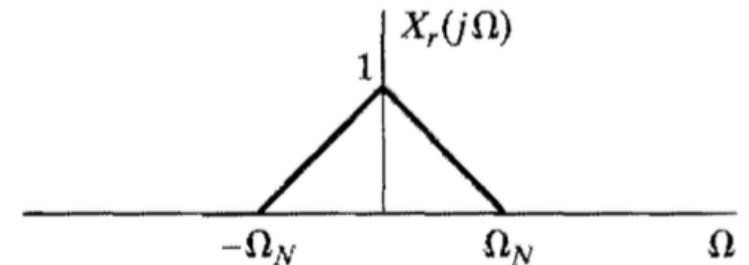
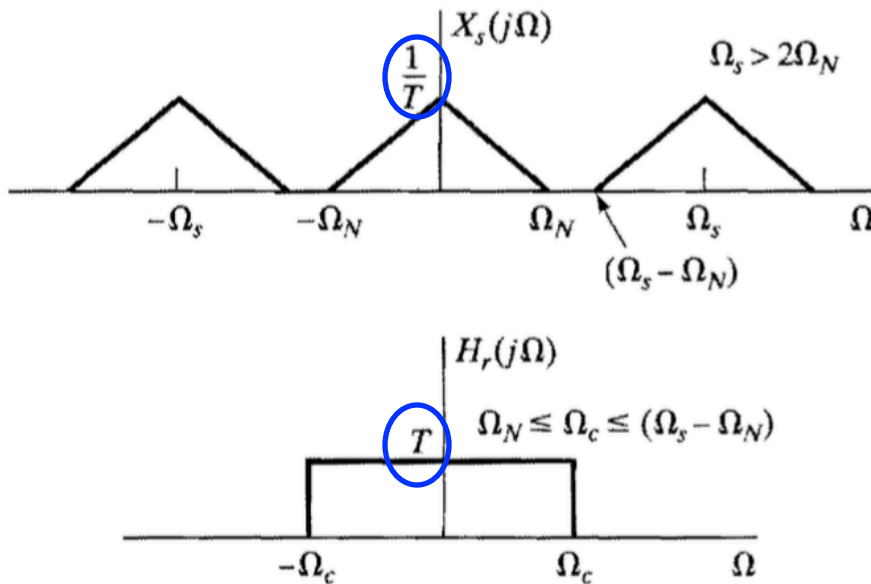
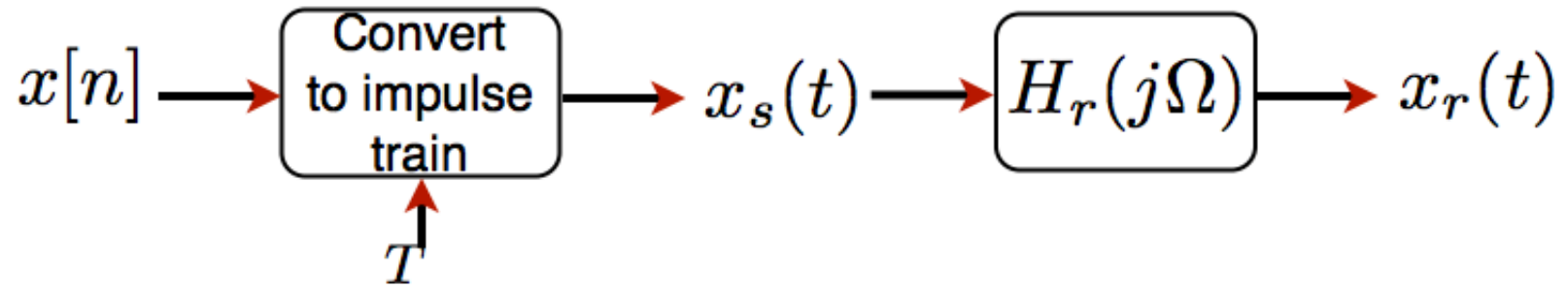
$$\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T} \end{aligned}$$



Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t) = \cos(16000 \pi t)$ with sampling period $T = 1/6000$ ($f_s = 6\text{kHz}$)

Reconstruction in Frequency Domain



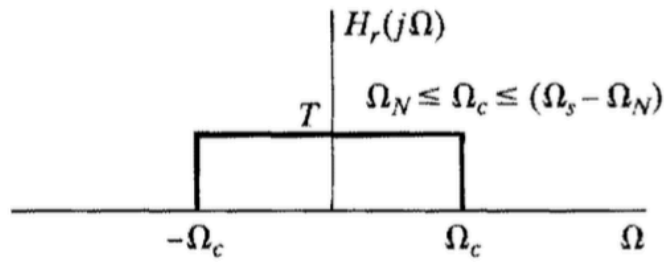
$$2\Omega_N = \Omega_s$$

$$\Omega_N = \Omega_c = \Omega_s/2$$

Sample at Nyquist
and filter at signal
bandwidth

Reconstruction in Time Domain

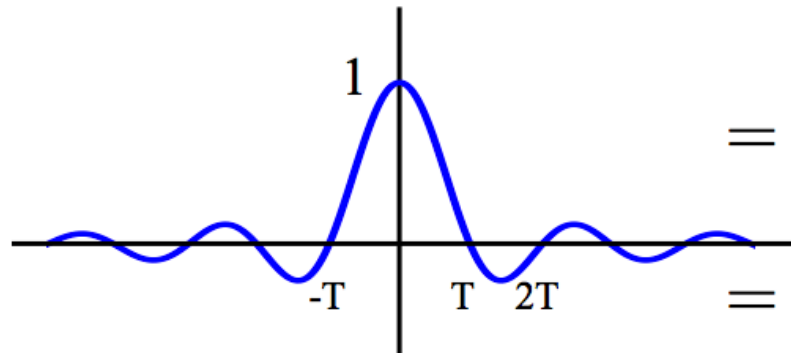
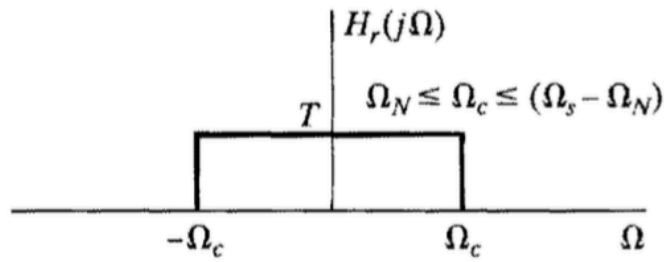
$$\Omega_N = \Omega_C = \Omega_s/2$$



$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

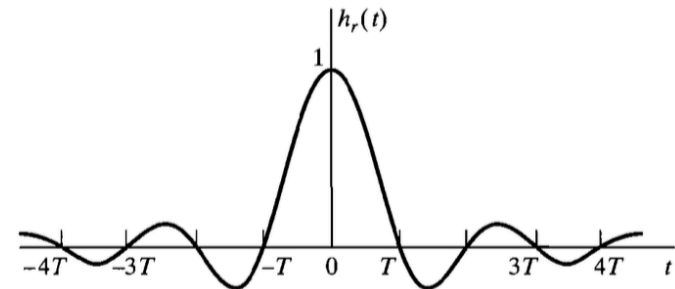
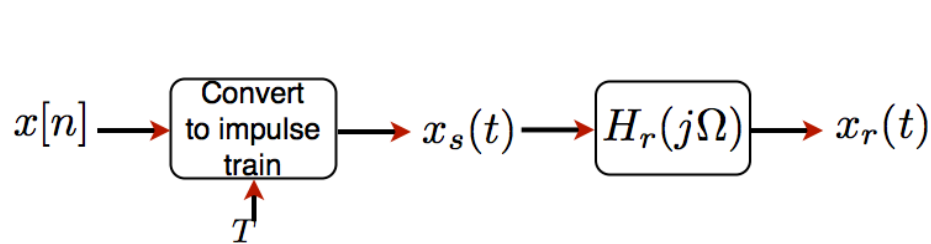
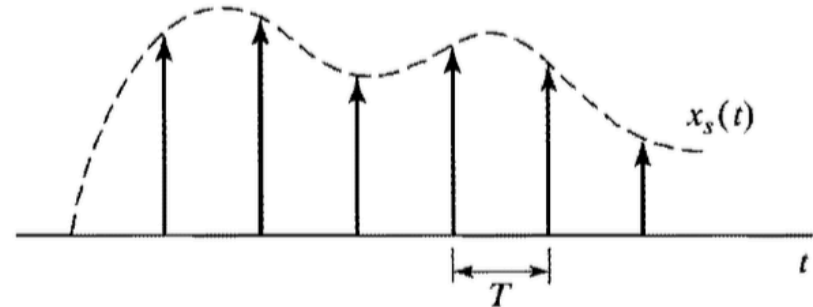
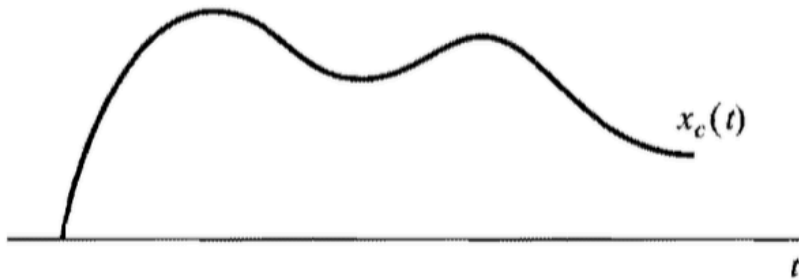
$$\Omega_N = \Omega_C = \Omega_s/2$$



$$\begin{aligned} h_r(t) &= \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega \\ &= \frac{T}{2\pi} \frac{1}{jt} e^{j\Omega t} \Big|_{-\Omega_s/2}^{\Omega_s/2} \\ &= \frac{T}{\pi t} \frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\ &= \frac{T}{\pi t} \sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t} \sin\left(\frac{\pi}{T}t\right) \\ &= \text{sinc}\left(\frac{t}{T}\right) \end{aligned}$$

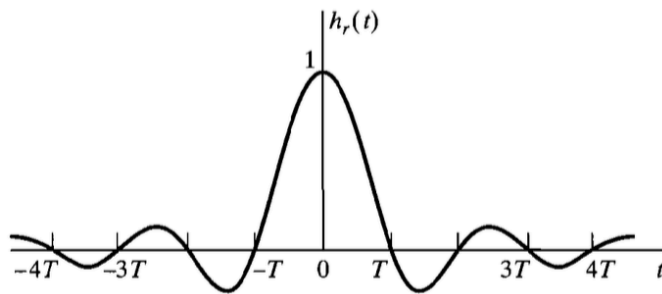
Reconstruction in Time Domain

$$\begin{aligned}
 x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) \\
 &= \sum_n x[n] h_r(t - nT)
 \end{aligned}$$

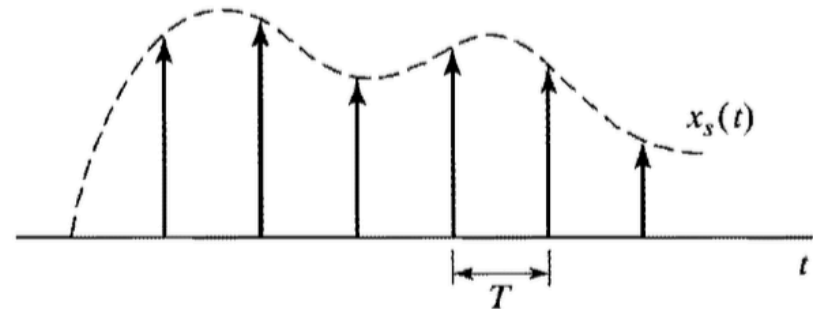


Reconstruction in Time Domain

$$\begin{aligned}x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n] \delta(t - nT) \right) * h_r(t) \\ &= \sum_n x[n] h_r(t - nT)\end{aligned}$$



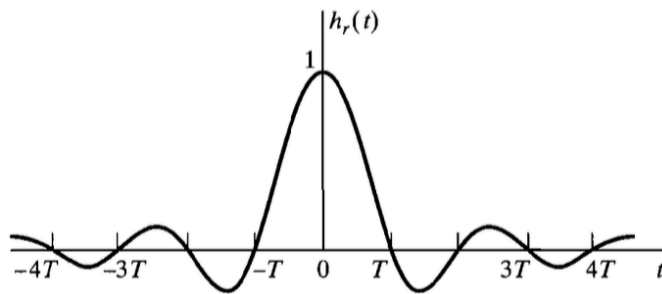
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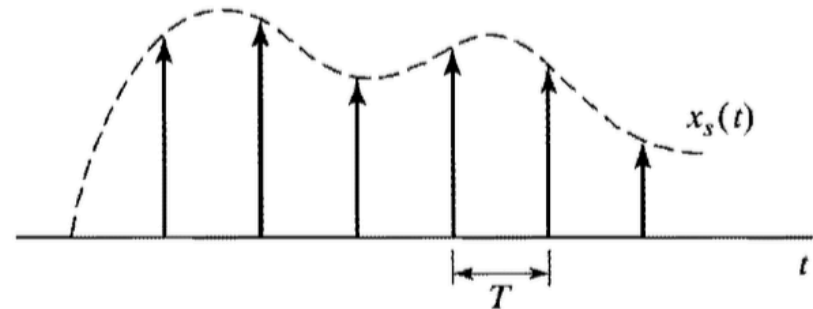
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Reconstruction in Time Domain

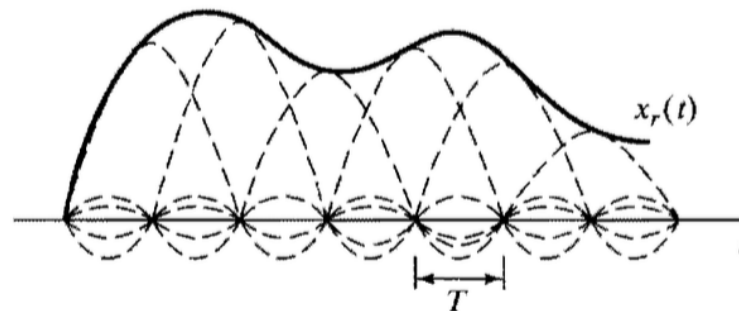
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 \end{aligned}$$



*



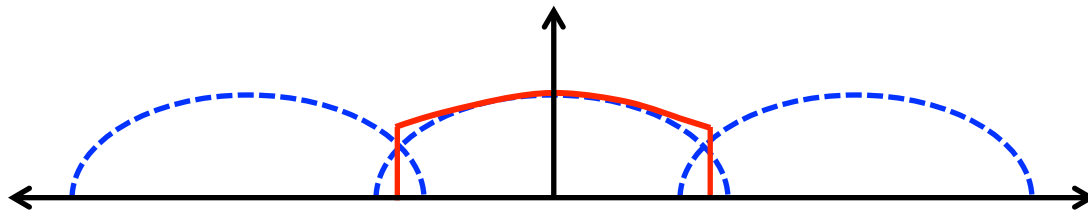
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The sum of “sincs”
gives $x_r(t) \rightarrow$ unique
signal that is
bandlimited by
sampling bandwidth

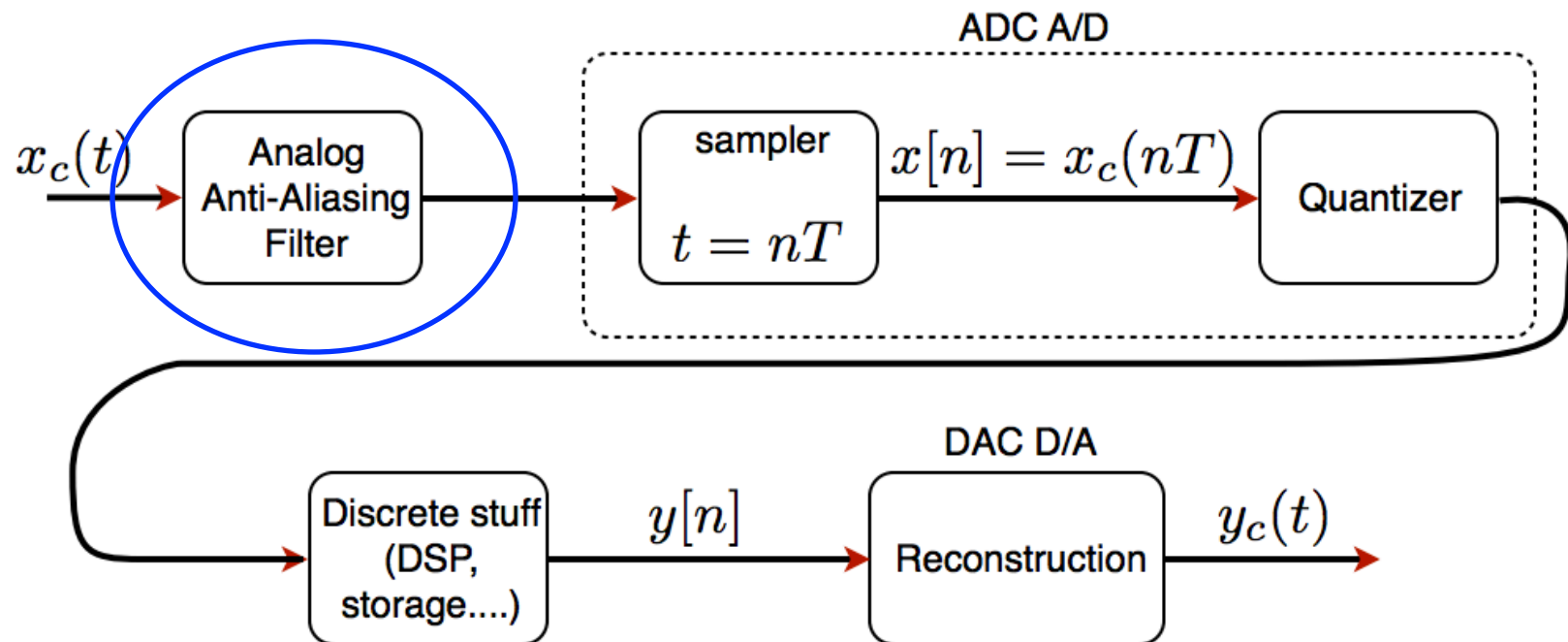
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$

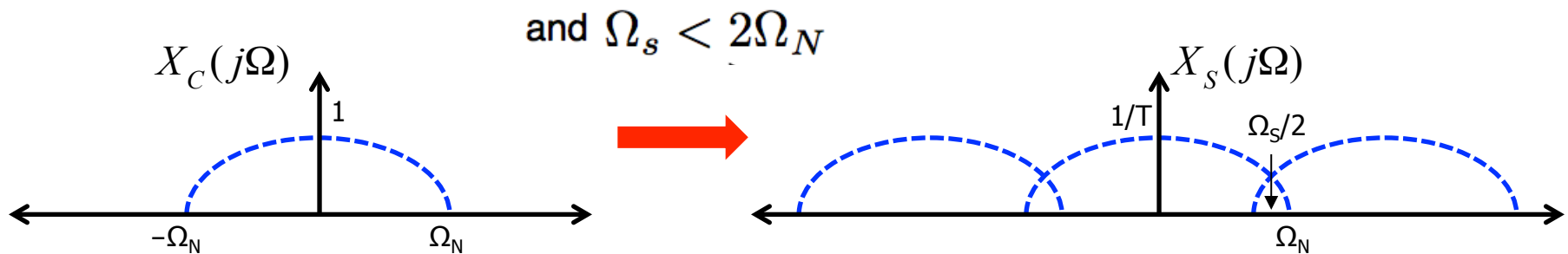
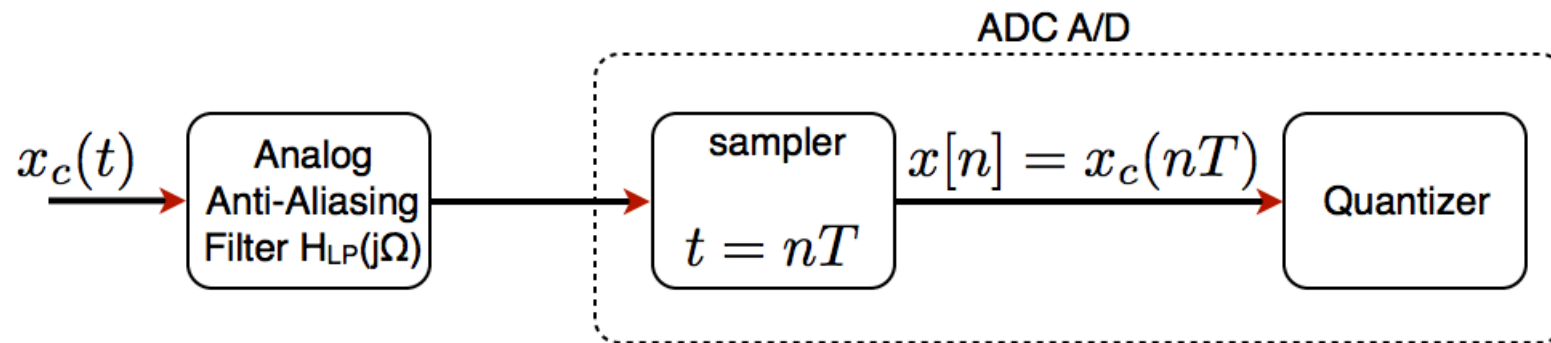


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

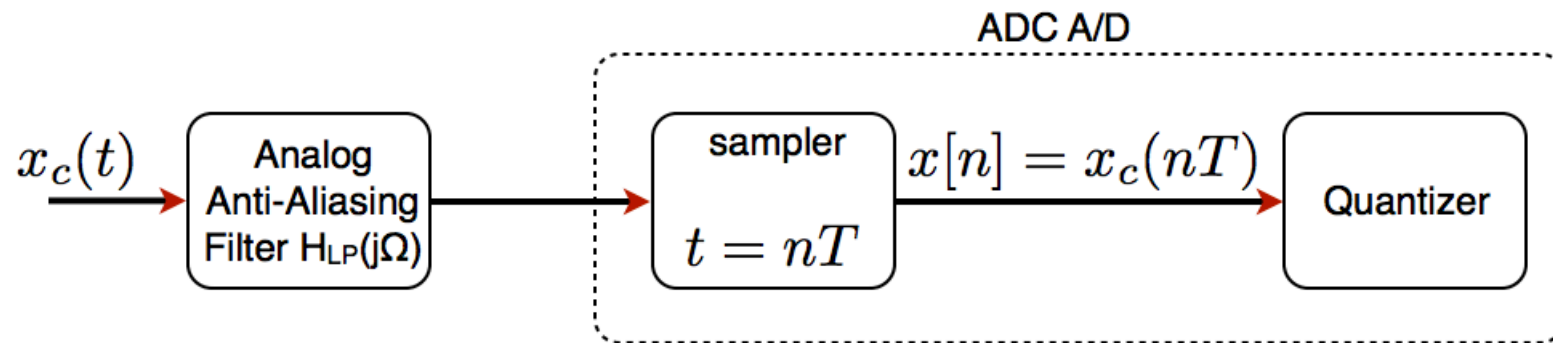
DSP System



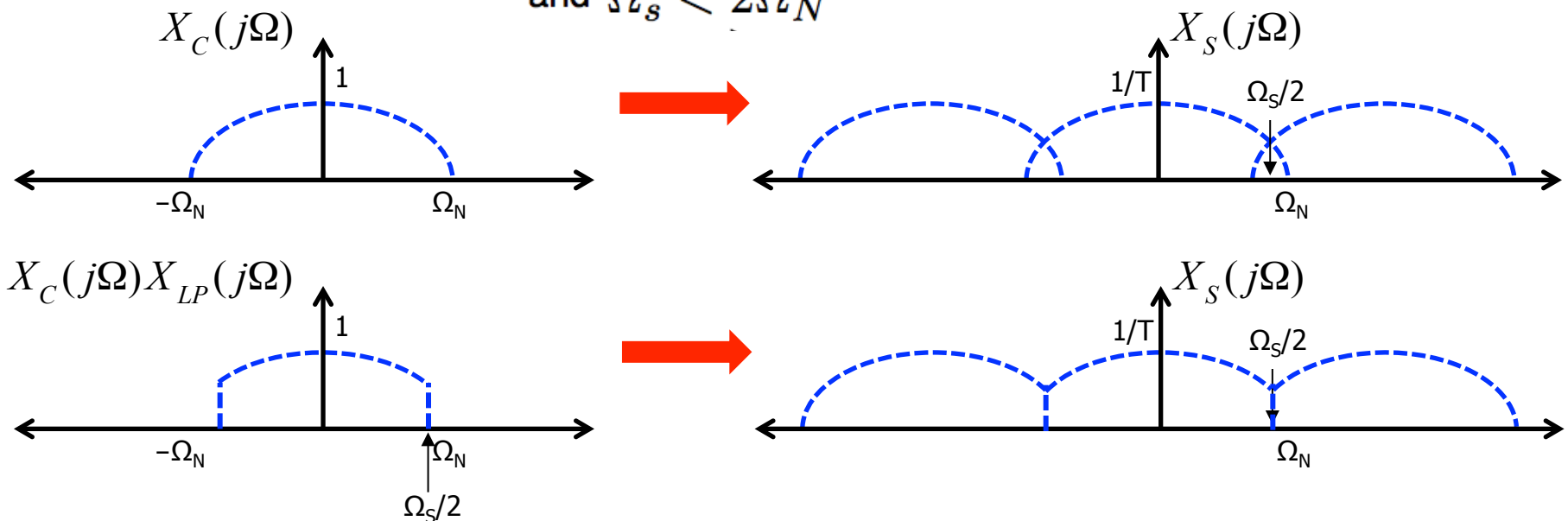
Anti-Aliasing Filter



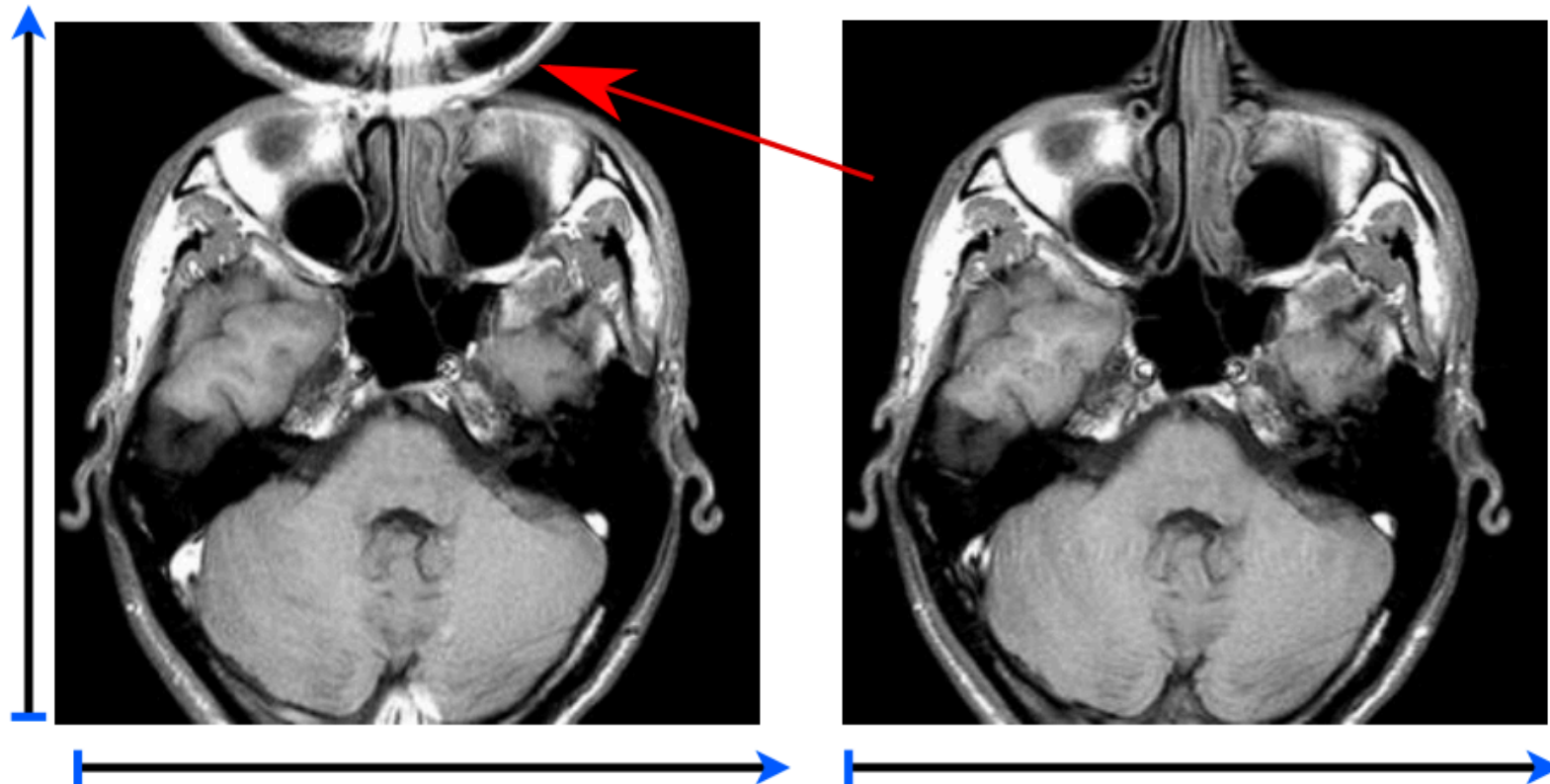
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



MRI aliasing example



MRI anti-aliasing example



MRI anti-aliasing example





Big Ideas

- ❑ Sampling
 - Ideal sampling modeled as impulsive sampling
 - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- ❑ Frequency Response of Sampled Signal
 - Sampled signal is period replicated input CT signal
- ❑ Reconstruction
 - Low pass/reconstruction filter results in sum of sincs
- ❑ Anti-aliasing filtering
 - Force input signal to be bandlimited



Admin

- ❑ HW 2 due Sunday at midnight
- ❑ HW 3 posted Sunday
- ❑ Check your submission!