# ESE 531: Digital Signal Processing

Lec 7: February 6, 2020
Sampling and Reconstruction



## Lecture Outline

- Sampling
  - Frequency Response of Sampled Signal
- Reconstruction
- Anti-aliasing Filter

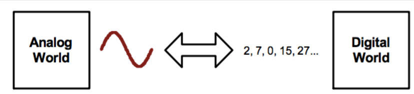


## Video Example


- https://www.youtube.com/watch?v=ByTsISFXUoY



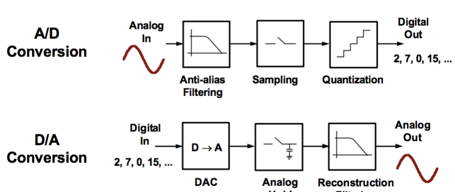
## The Data Conversion Problem




- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion



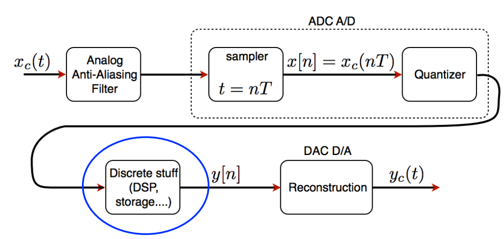
## Overview



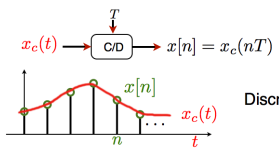

- We'll first look at these building blocks from a functional, "black box" perspective



## DSP System



## Ideal Sampling Model

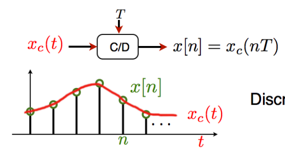



Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - $f_s$=1/T is the sampling frequency
  - $\Omega_s = 2\pi/T$

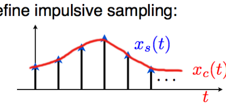


## Ideal Sampling Model




Discrete and Continuous

define impulsive sampling:




Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



## Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Dirac delta function, $\delta$ (t)
  - Infinitely high and thin, area of 1
  - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- Three signals. How are they related? In time? In frequency?



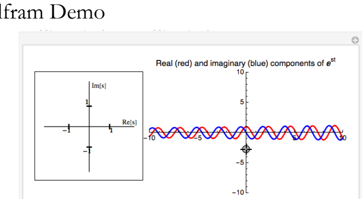
## Reminder: Laplace Transform

- The Laplace transform takes a function of time, t, and transforms it to a function of a complex variable, s.
- Because the transform is invertible, no information is lost and it is reasonable to think of a function f(t) and its Laplace transform F(s) as two views of the same phenomenon.
- Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.



## S-Plane

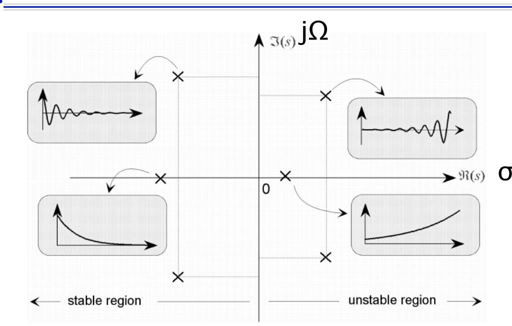
- $s = \sigma + j\Omega$
- Wolfram Demo



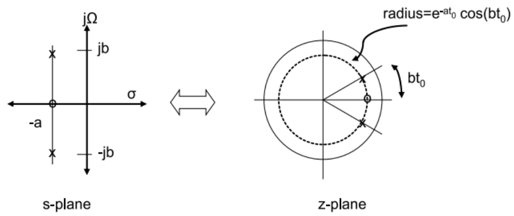

- http://pilot.cnxproject.org/content/collection/col10064/latest/module/m10060/latest



## S-plane and stability



## S-Plane Mapping to Z-Plane






## Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



## Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

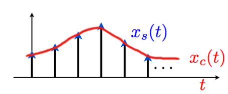
$x_s(t)$ :C.T $\qquad X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$

$x[n]$ :D.T $\qquad X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$

$$\boxed{X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}} \qquad \boxed{X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T}}$$



## Frequency Domain Analysis




Continuous

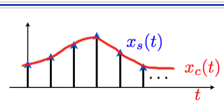
$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



## Frequency Domain Analysis




Continuous

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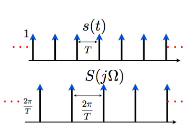
$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$s(t) \leftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

$\frac{2\pi}{T} = \Omega_s$






## Frequency Domain Analysis

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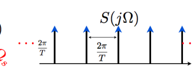




## Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T}$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T} k)$$

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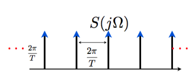




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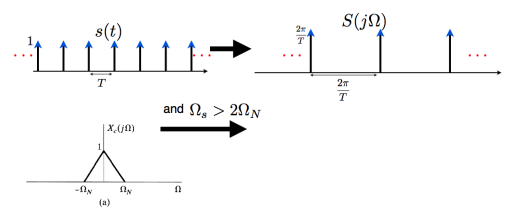




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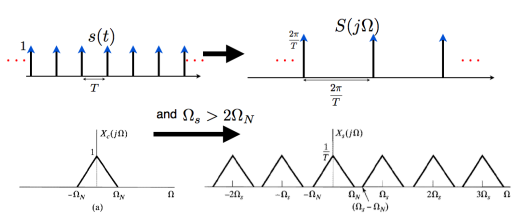
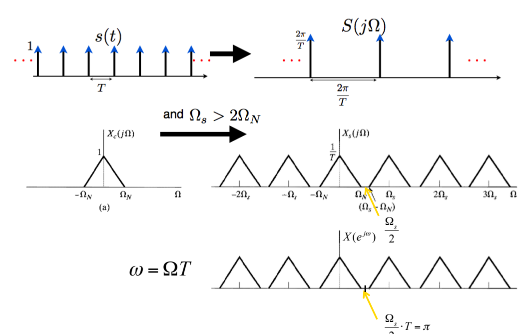
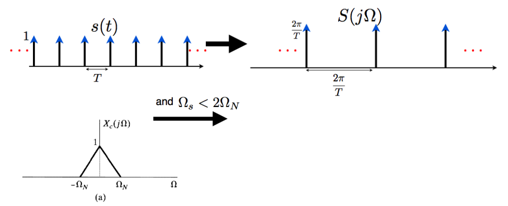





## Frequency Domain Analysis

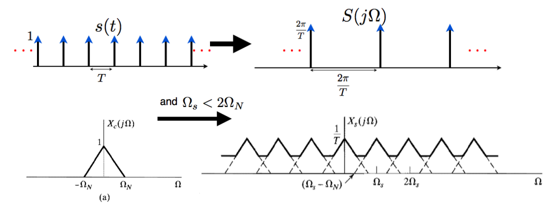


## Frequency Domain Analysis w/ Aliasing



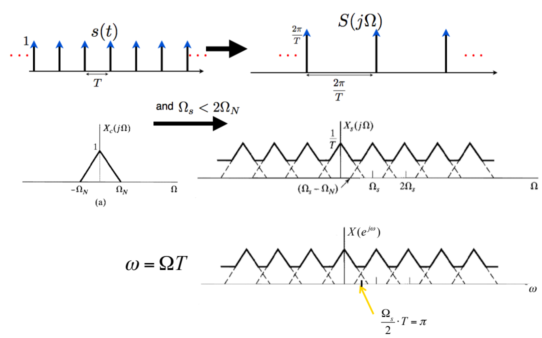



## Frequency Domain Analysis w/Aliasing






## Frequency Domain Analysis w/Aliasing



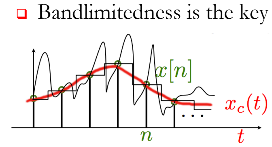



## Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

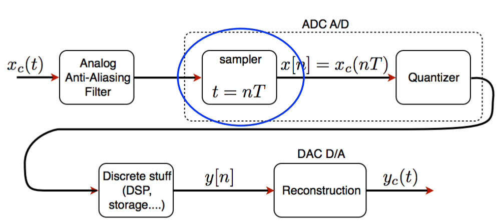
- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness




Mulitiple signals go through the samples, but only one is bandlimited within our sampling band



## DSP System



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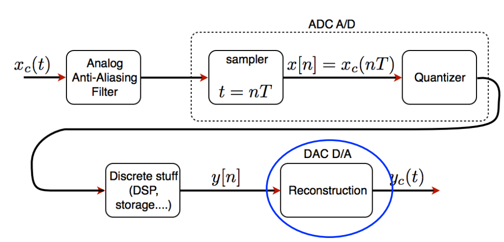
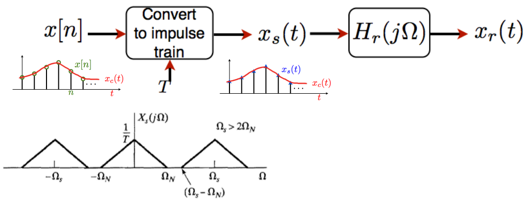


## Reconstruction in Frequency Domain



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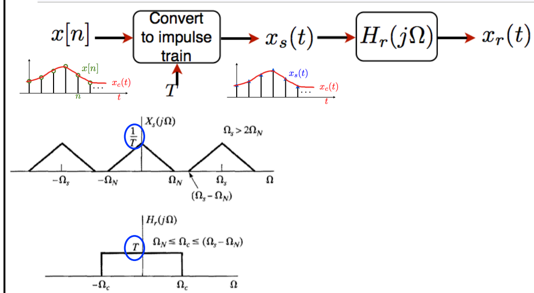
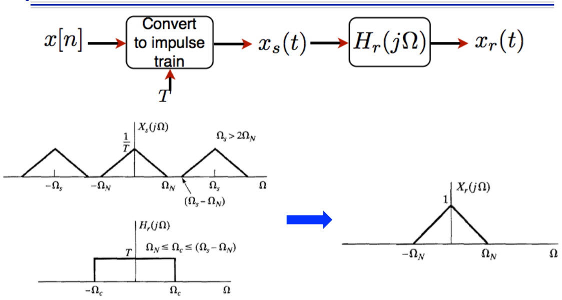





## Reconstruction in Frequency Domain






## Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period T=1/6000 s ($f_s$=6kHz)

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad | \quad \Omega_s = \frac{2\pi}{T}$$

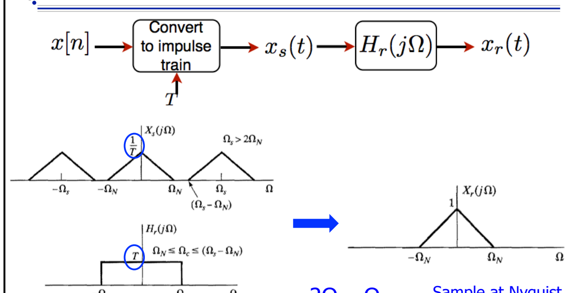


## Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period T=1/6000 ($f_s$=6kHz)



## Reconstruction in Frequency Domain



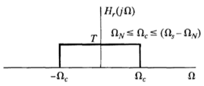

$2\Omega_N=\Omega_s$

$\Omega_N=\Omega_C=\Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

## Reconstruction in Time Domain

$\Omega_N=\Omega_C=\Omega_s/2$

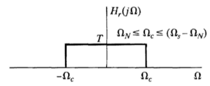



$$h_r(t) = \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} Te^{j\Omega t}d\Omega$$

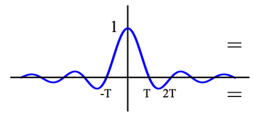


## Reconstruction in Time Domain

$\Omega_N=\Omega_C=\Omega_s/2$



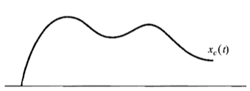

$$\begin{aligned} h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} Te^{j\Omega t}d\Omega \\ &= \frac{T}{2\pi}\frac{1}{jt}e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\ &= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t}-e^{-j\frac{\Omega_s}{2}t}}{2j} \\ &= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\ &= \text{sinc}(\frac{t}{T}) \end{aligned}$$

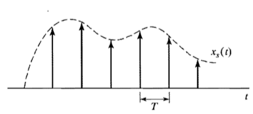


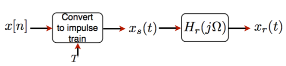


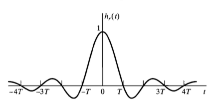

## Reconstruction in Time Domain

$$\begin{aligned} x_r(t) = x_s(t) * h_r(t) &= \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t) \\ &= \sum_n x[n]h_r(t-nT) \end{aligned}$$



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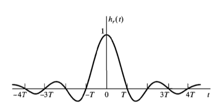


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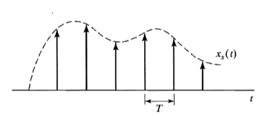


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## Reconstruction in Time Domain

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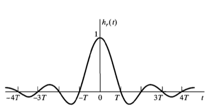


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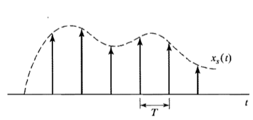


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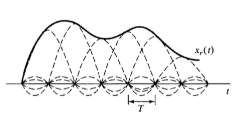
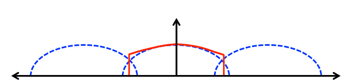



The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth



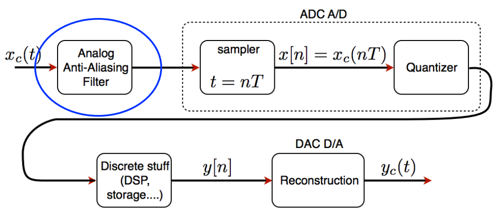
## Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$



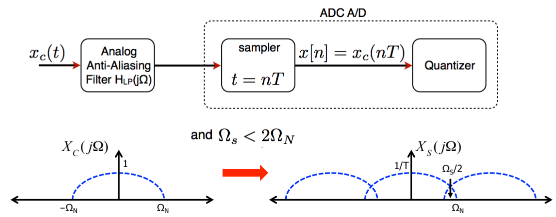
$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if } \quad |\Omega| \le \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

## DSP System



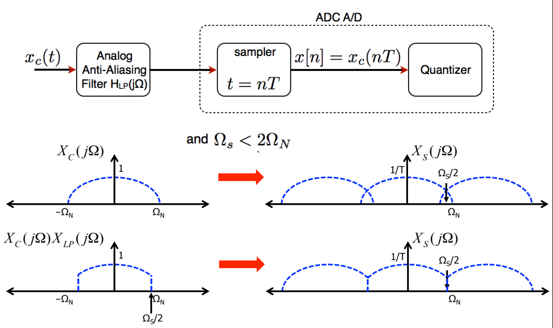



## Anti-Aliasing Filter



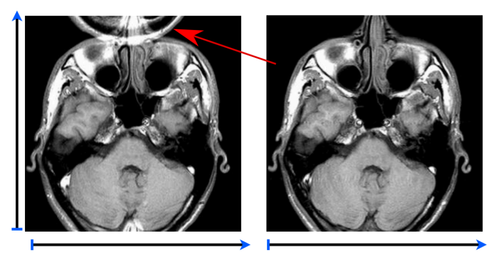



## Anti-Aliasing Filter






## MRI aliasing example





## MRI anti-aliasing example



## MRI anti-aliasing example

## Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited



## Admin

- HW 2 due Sunday at midnight
- HW 3 posted Sunday

- Check your submission!