ESE 531: Digital Signal Processing

Lec 8: February 11th, 2020

DT/CT Processing of CT/DT Signals

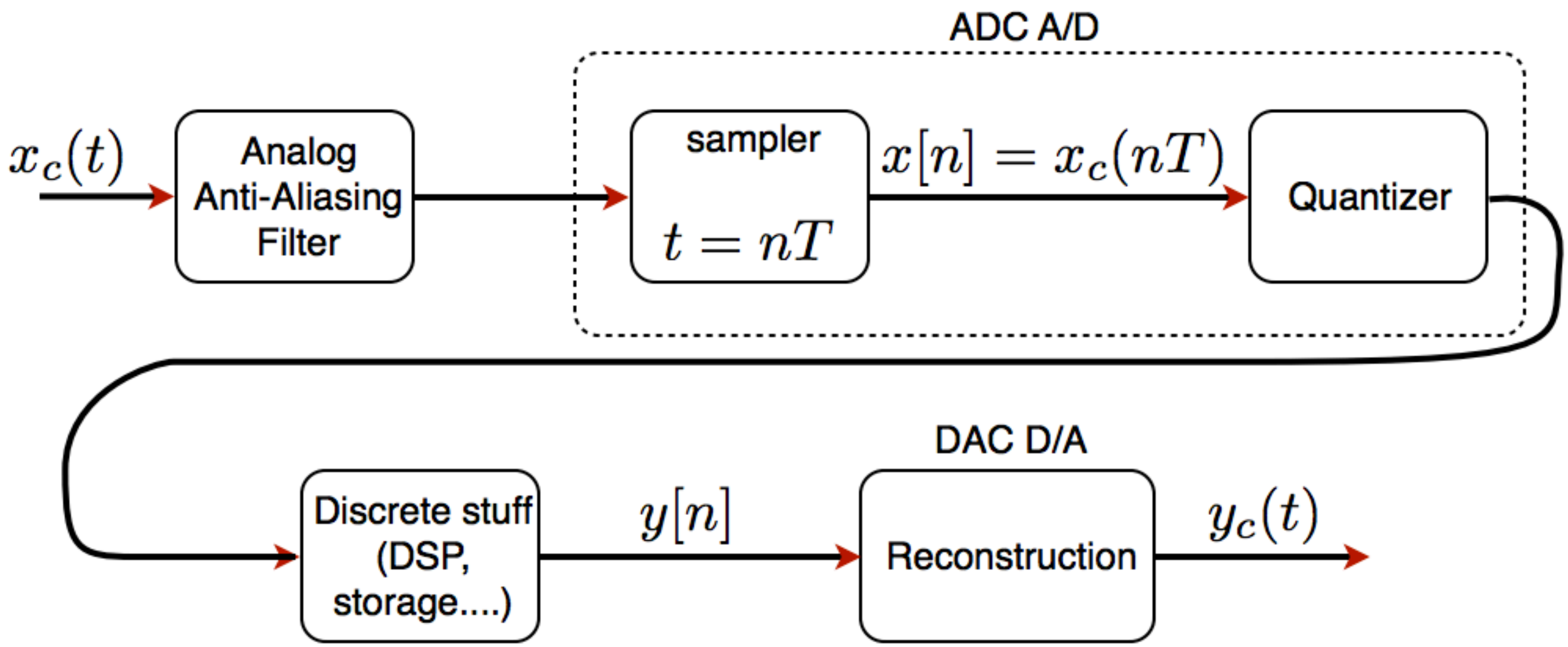
Lecture Outline

- Review
  - Ideal sampling
  - Frequency response of sampled signal
  - Reconstruction
  - Anti-aliasing filtering
- DT processing of CT signals
  - Impulse Invariance
- CT processing of DT signals (why??)

Last Time…

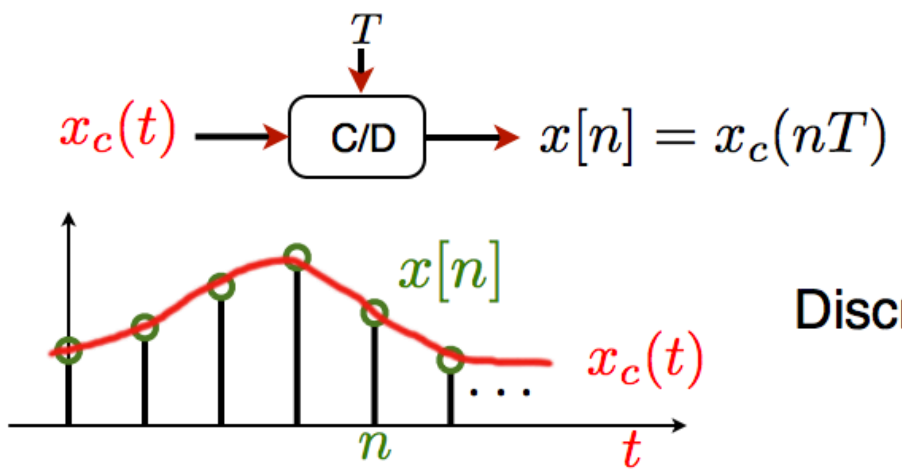
Sampling, Frequency Response of Sampled Signal, Reconstruction, Anti-aliasing filtering

DSP System



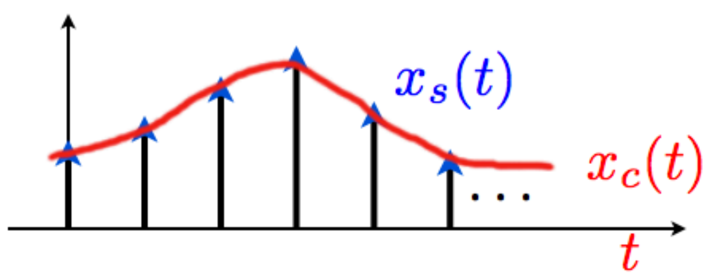
ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:

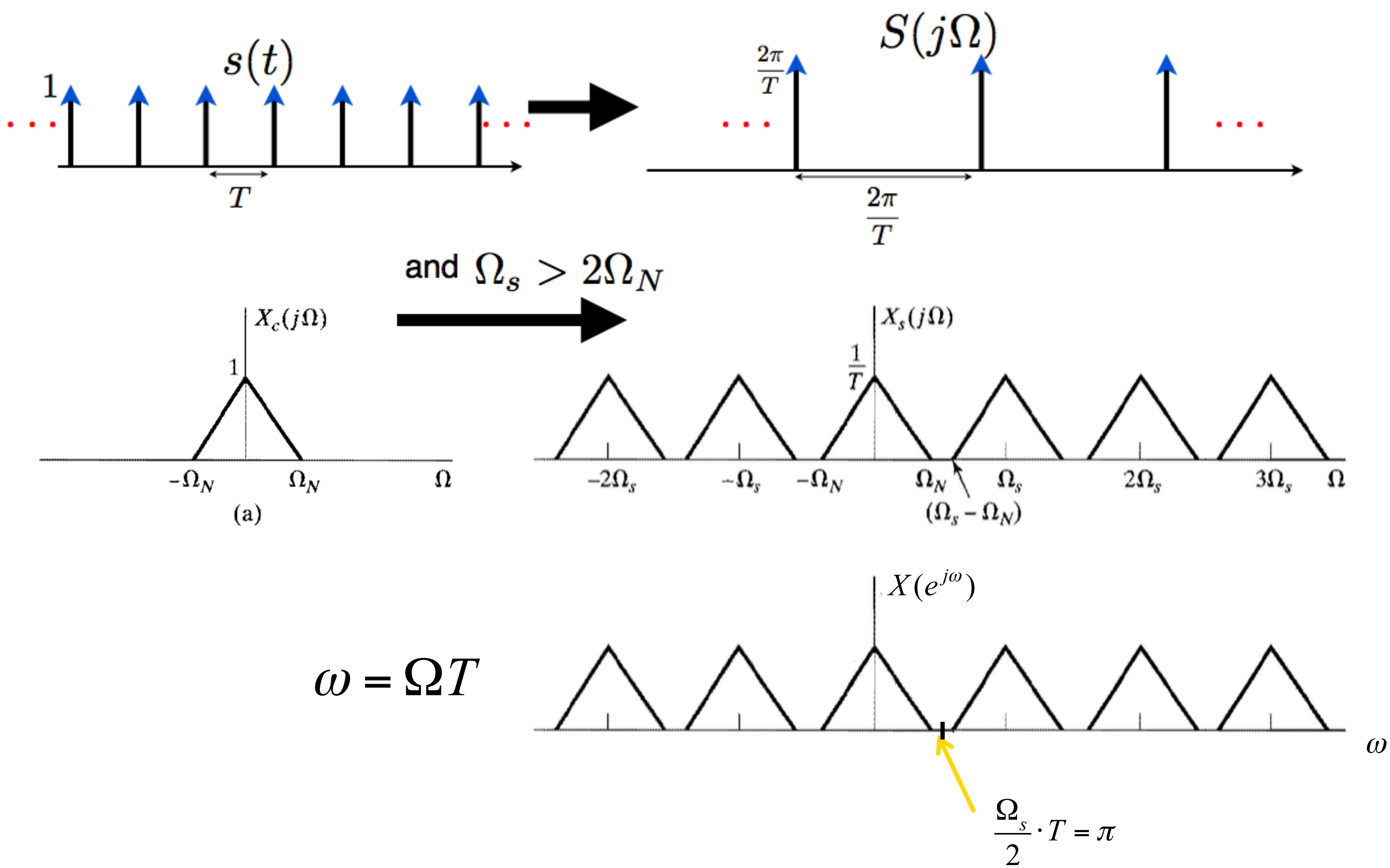


Continuous

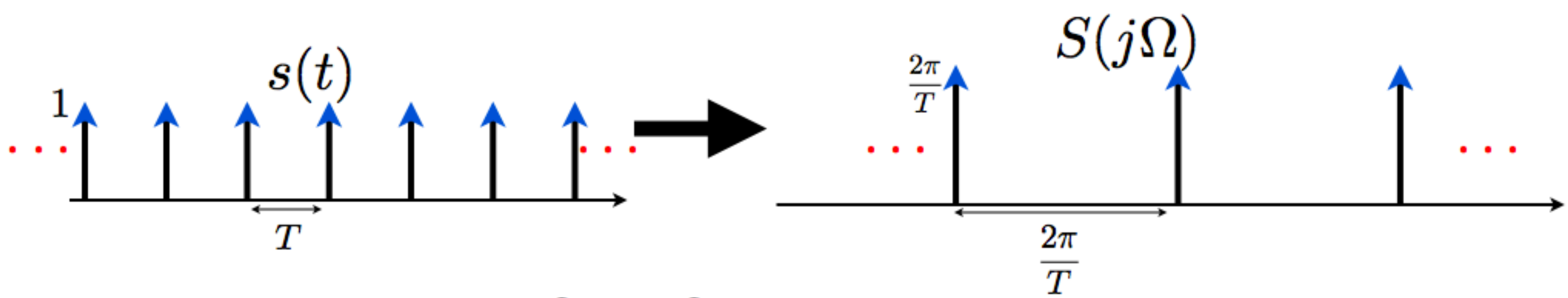
$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

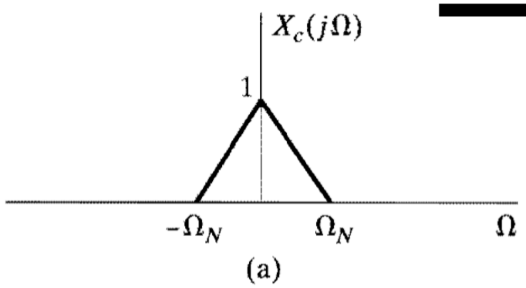
Frequency Domain Analysis

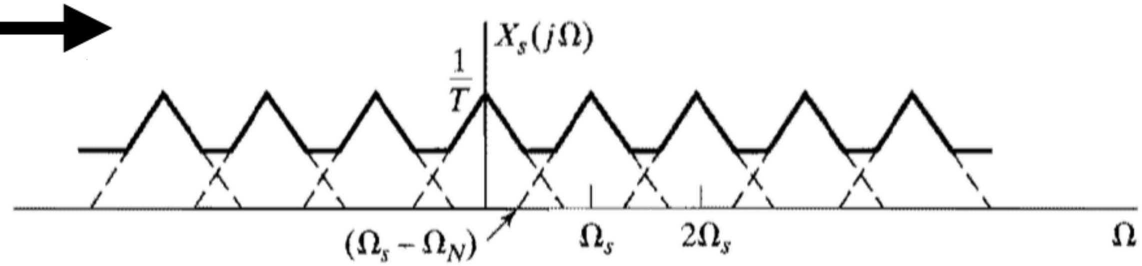


Frequency Domain Analysis

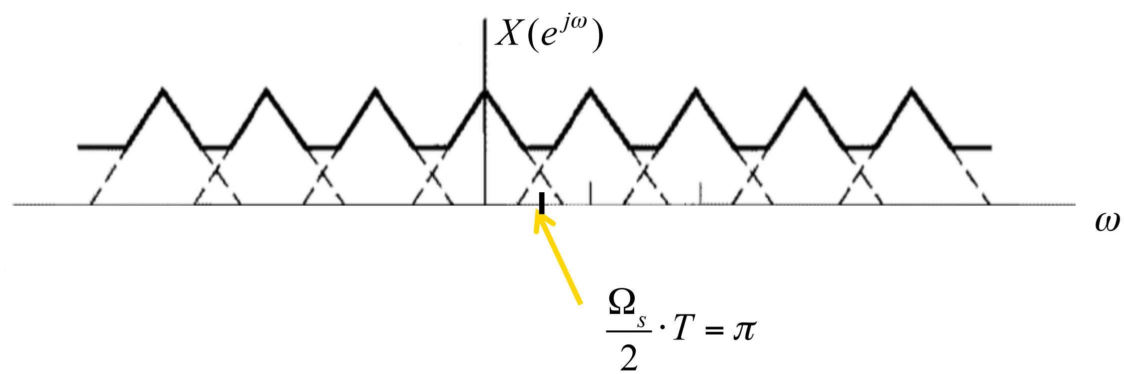


and $\Omega_s < 2\Omega_N$

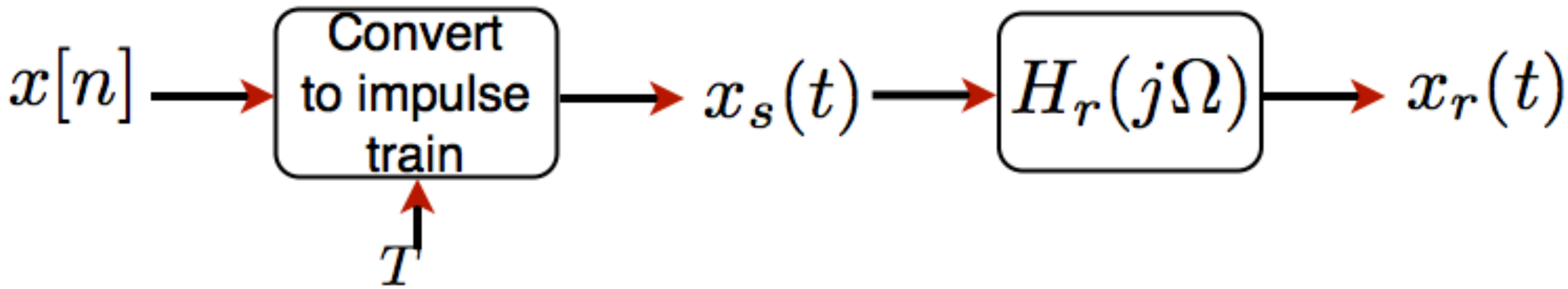


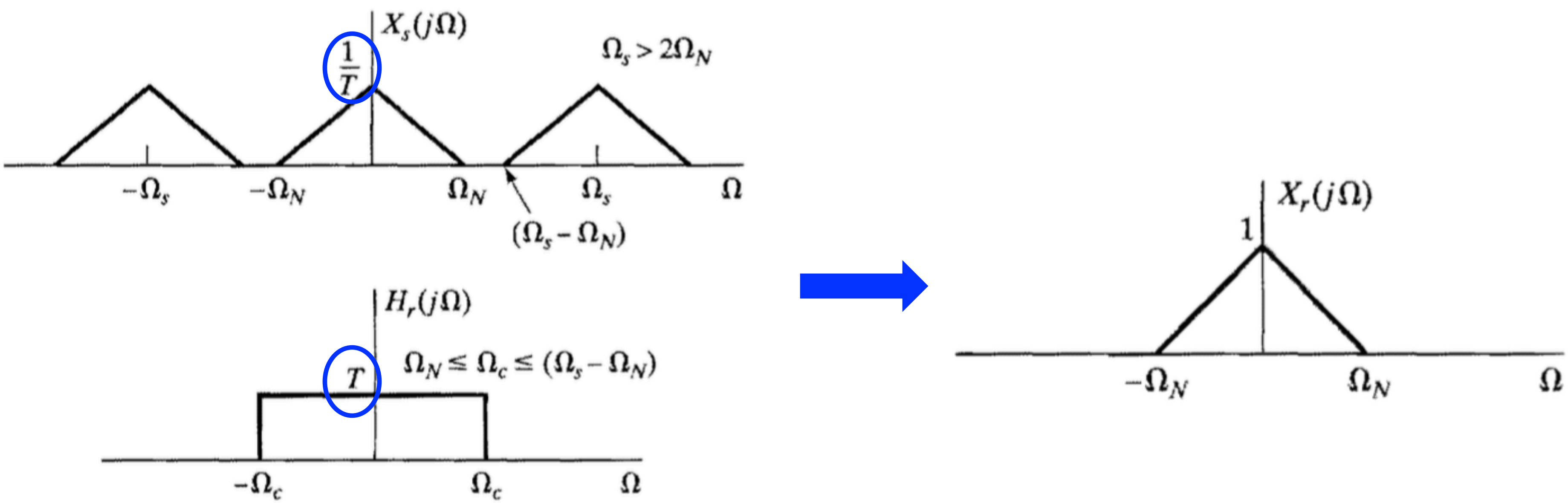


$$\omega = \Omega T$$



Reconstruction in Frequency Domain





Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period T=1/6000 s (f_s=6kHz)

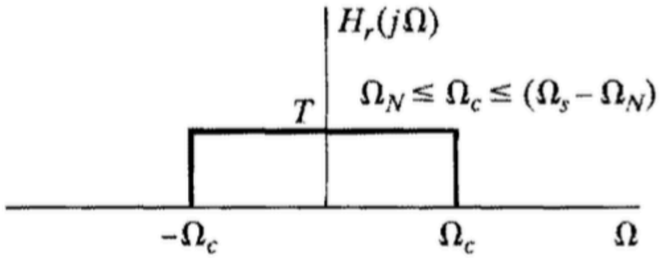
$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period $T=1/6000$ (f_s=6kHz)

Reconstruction in Time Domain

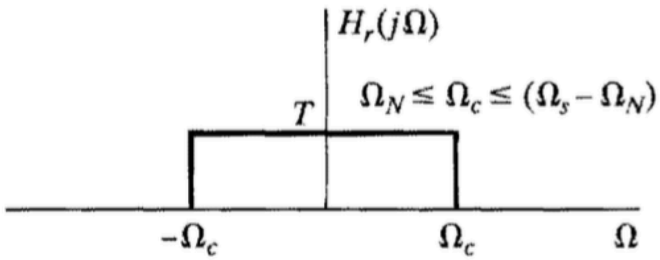
$\Omega_N = \Omega_C = \Omega_s/2$



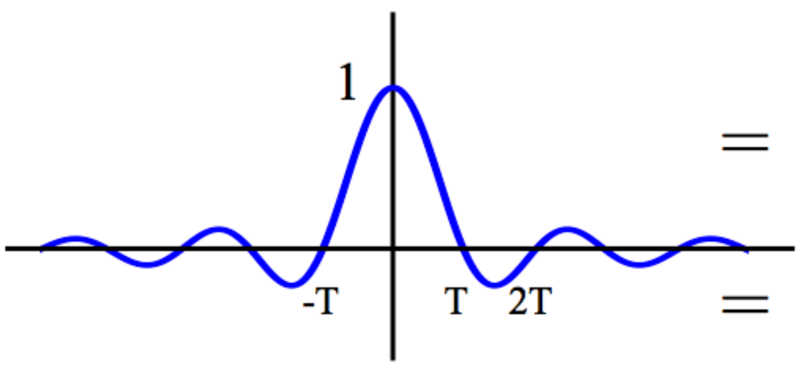
$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

$\Omega_N = \Omega_C = \Omega_s/2$



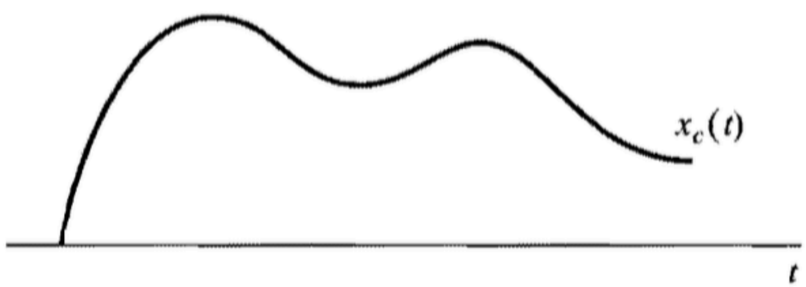
$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t}\, d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\
&= \mathrm{sinc}(\frac{t}{T})
\end{aligned}
$$

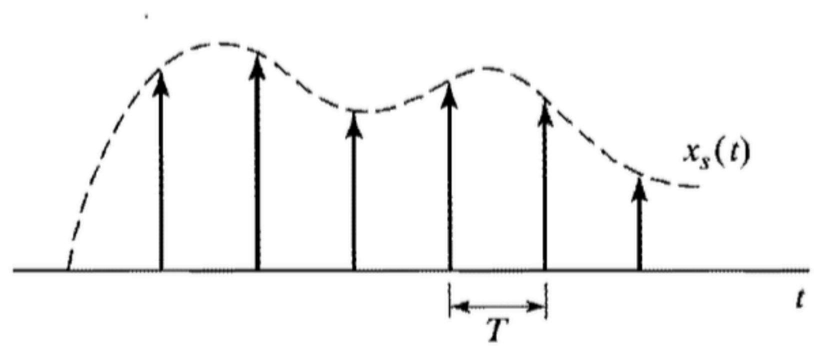


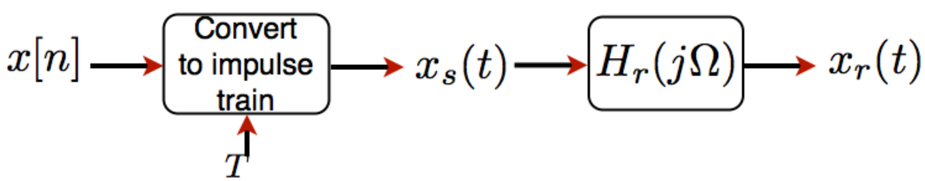
Reconstruction in Time Domain

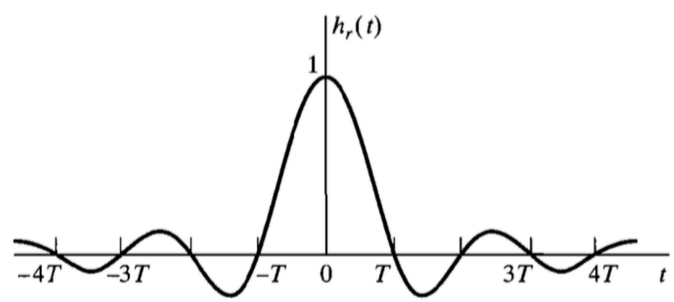
$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$

$$= \sum_n x[n] h_r(t - nT)$$









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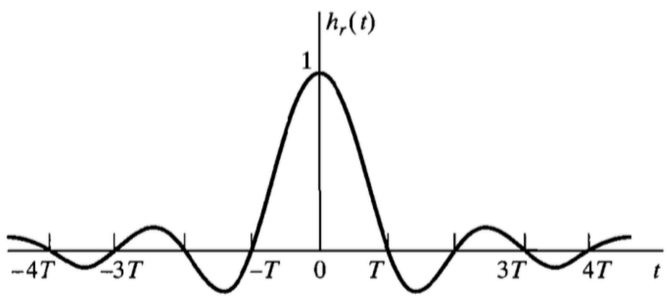

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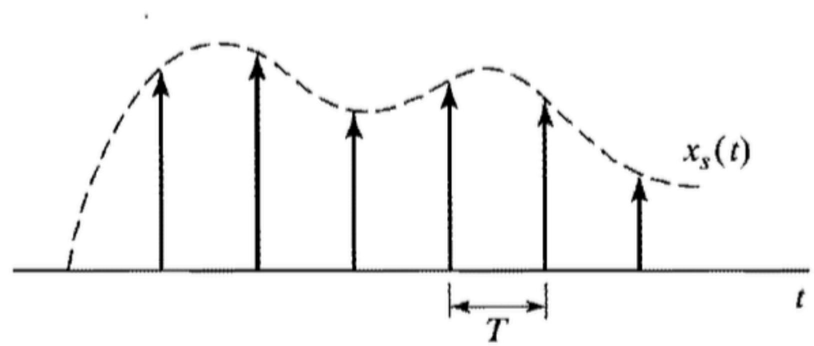

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Reconstruction in Time Domain

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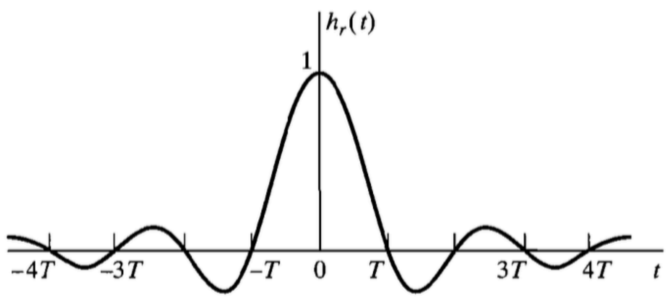

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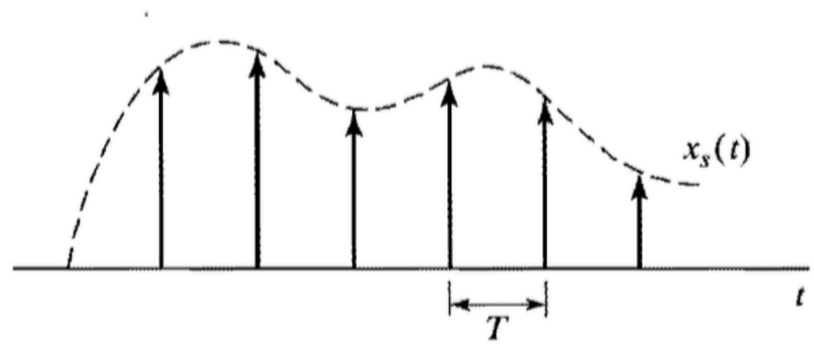

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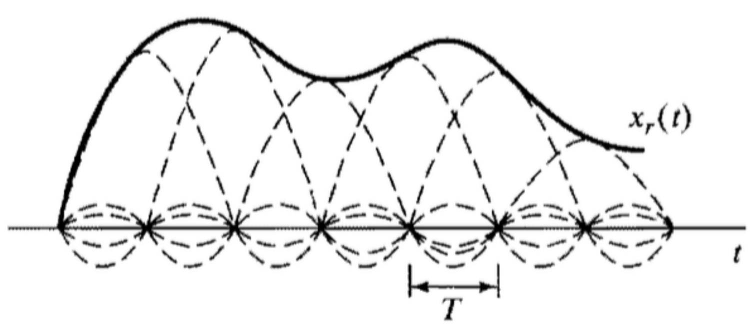
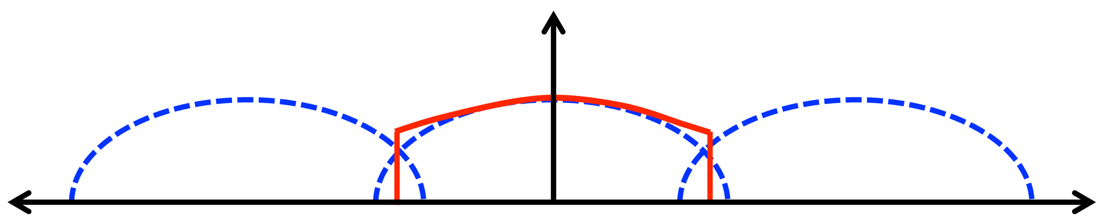


The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

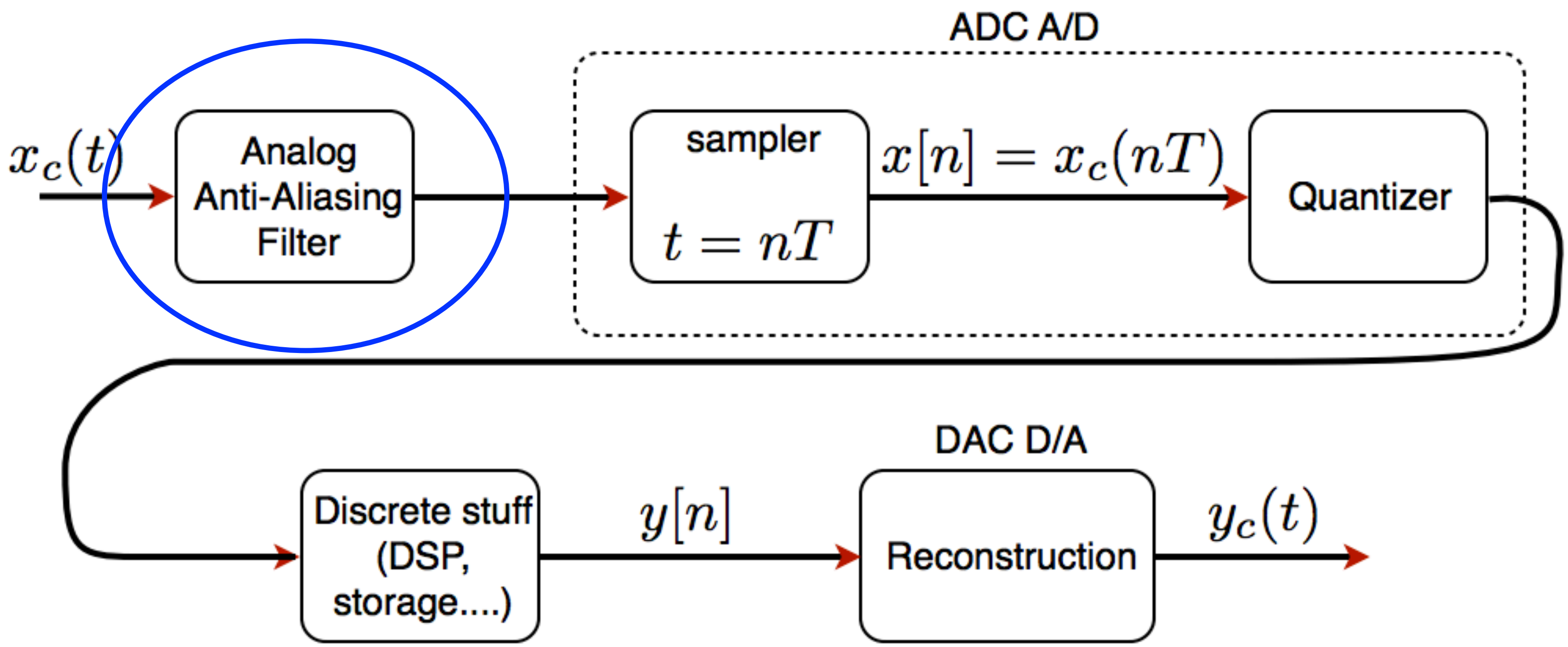
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$



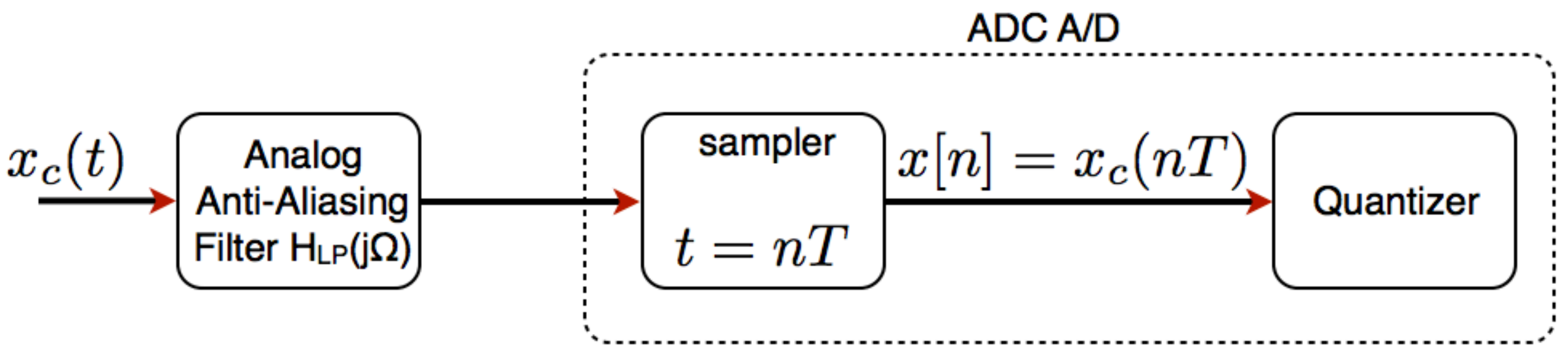
$$X_r(j\Omega) = \begin{cases} T X_s(j\Omega) & \text{if} \quad |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

DSP System

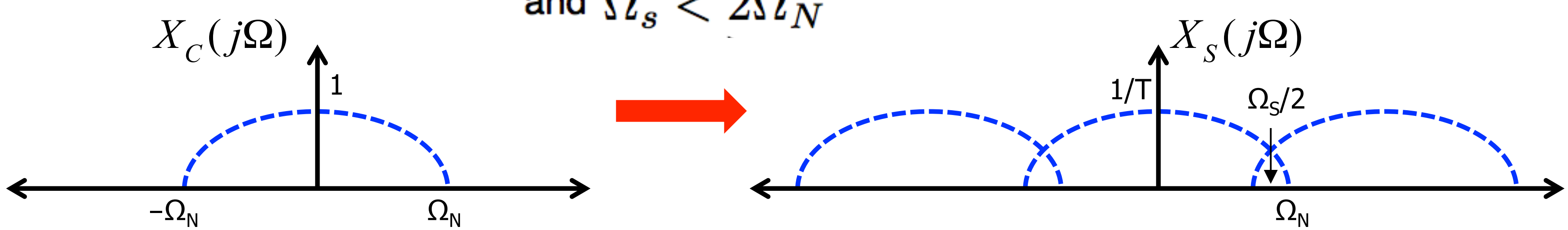


ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

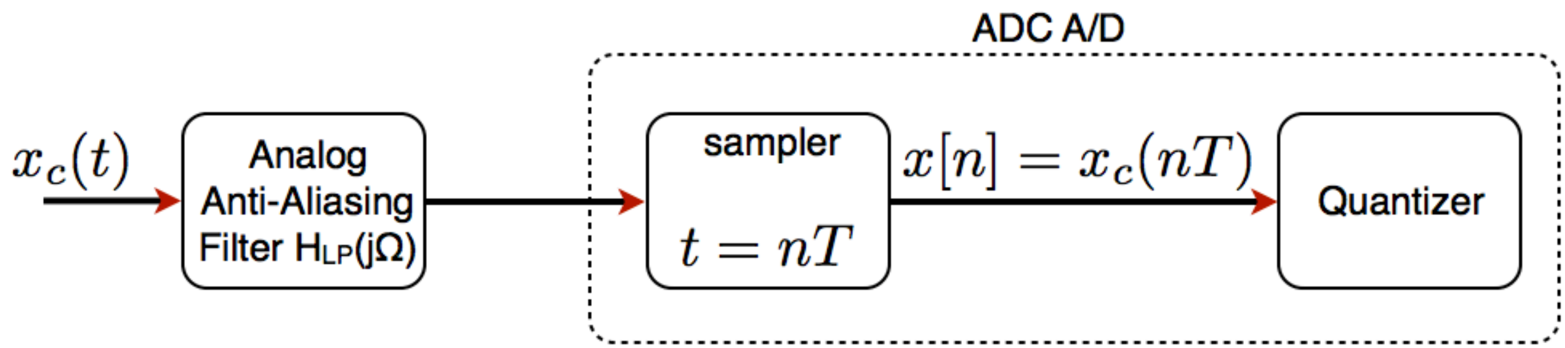
Anti-Aliasing Filter



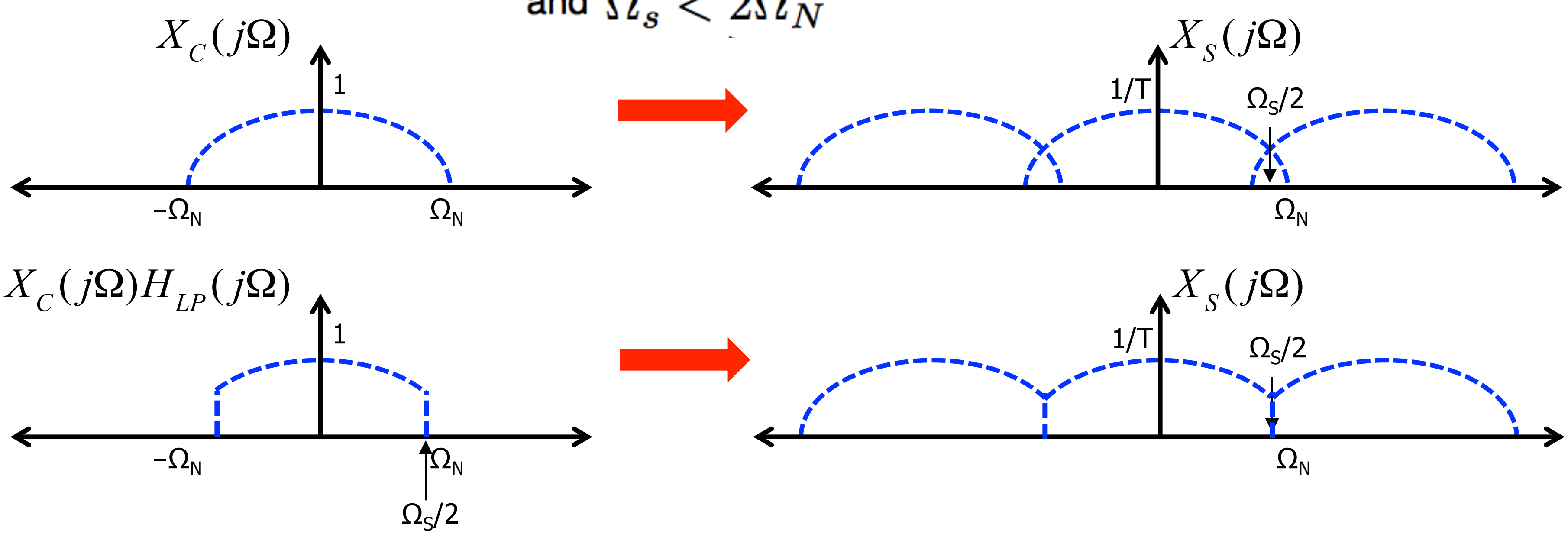
and $\Omega_s < 2\Omega_N$



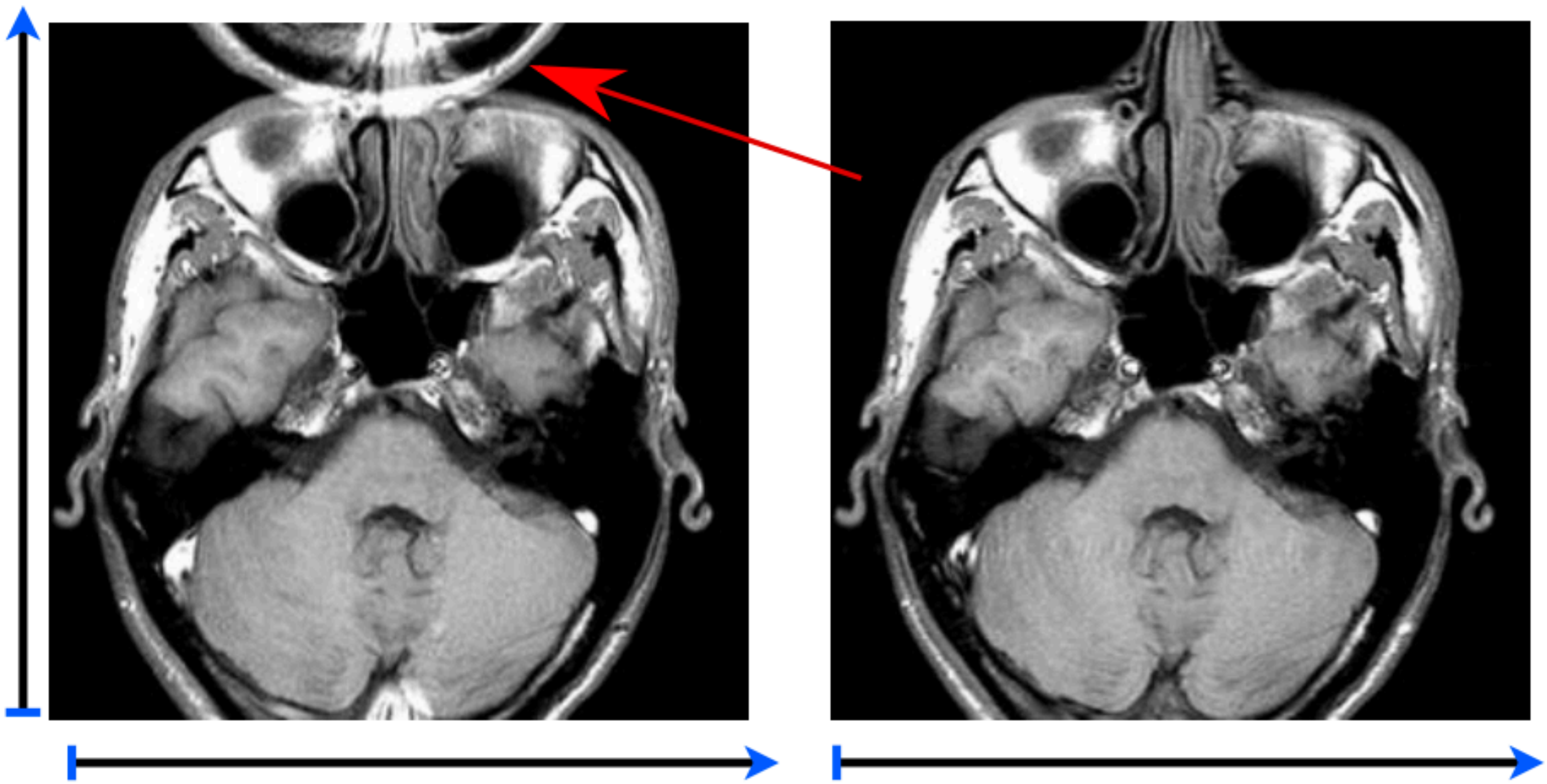
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



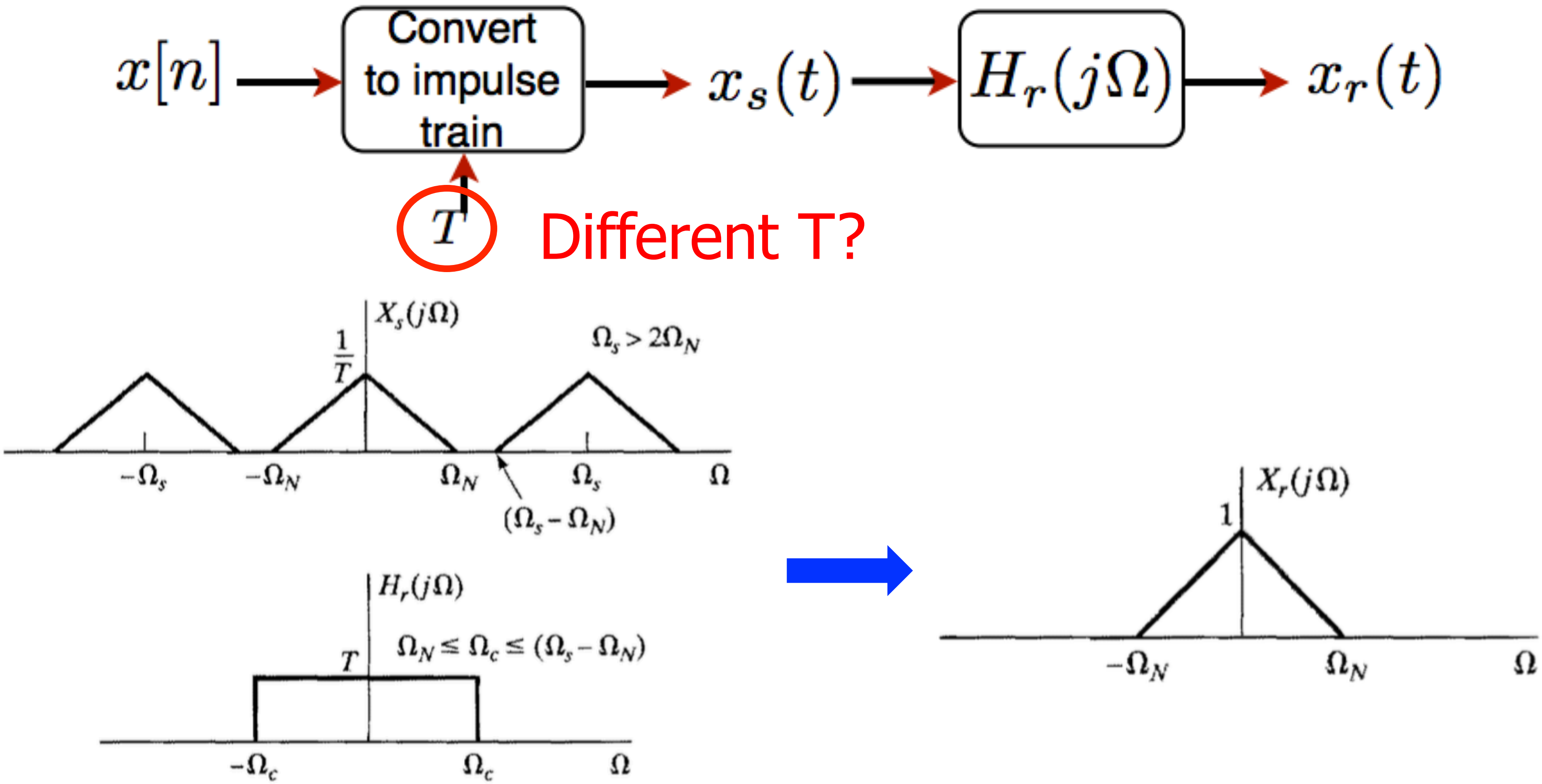
MRI aliasing example



MRI anti-aliasing example

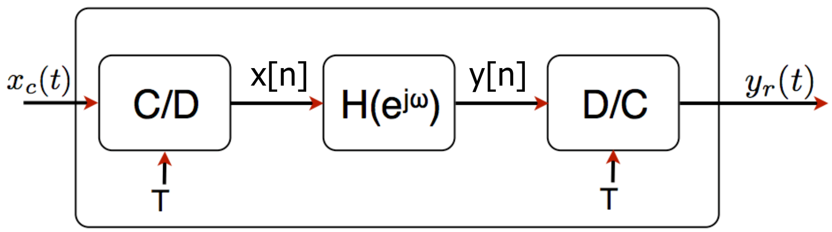
MRI anti-aliasing example

Reconstruction in Frequency Domain



Signal Processing

- Use theory of sampling (C/D) and reconstruction (D/C) to implement signal processing
- Two cases:
  - Discrete-time processing of continuous-time signals



Signal Processing

- Use theory of sampling (C/D) and reconstruction (D/C) to implement signal processing
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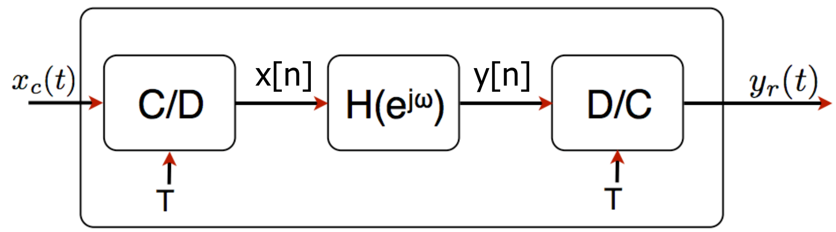
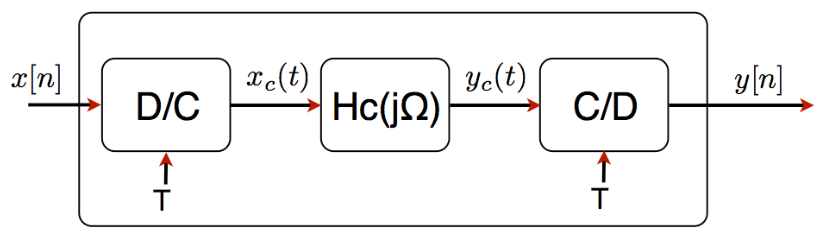
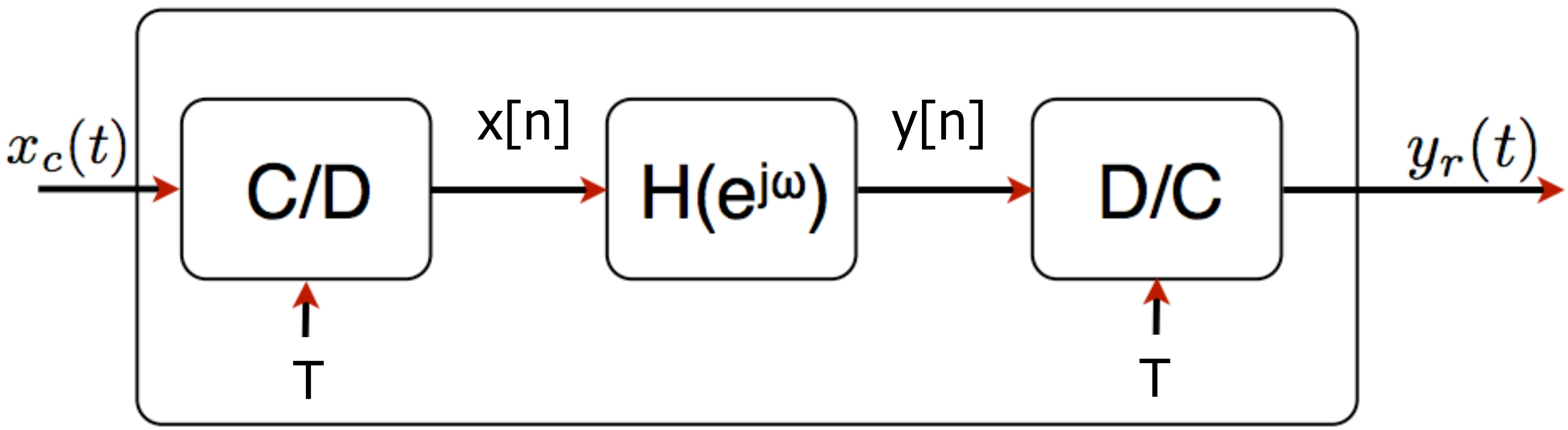


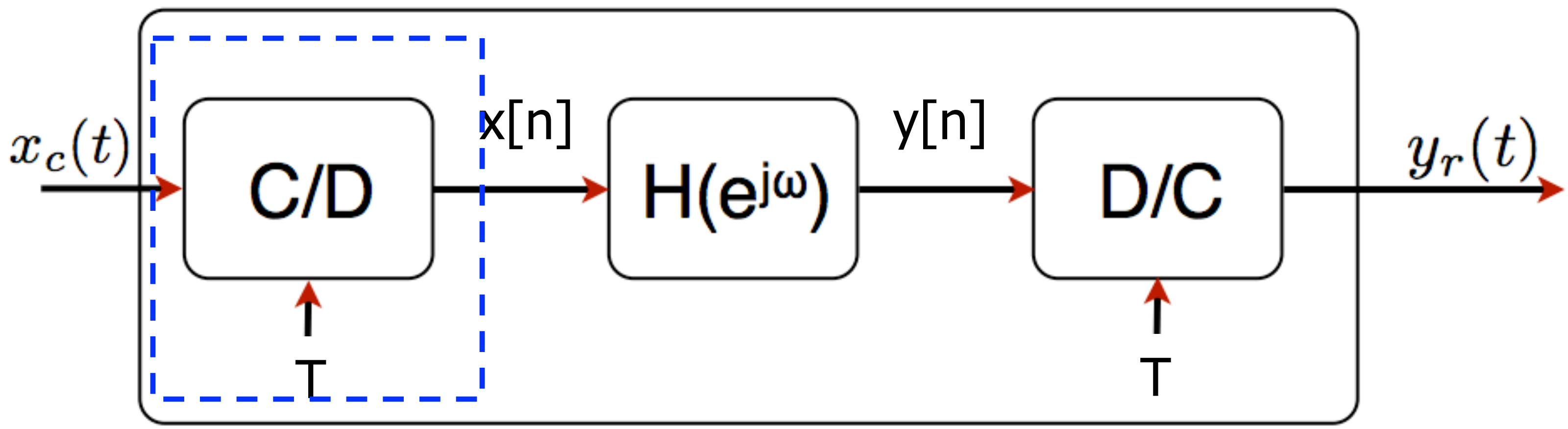
  - Continuous-time processing of discrete-time signals



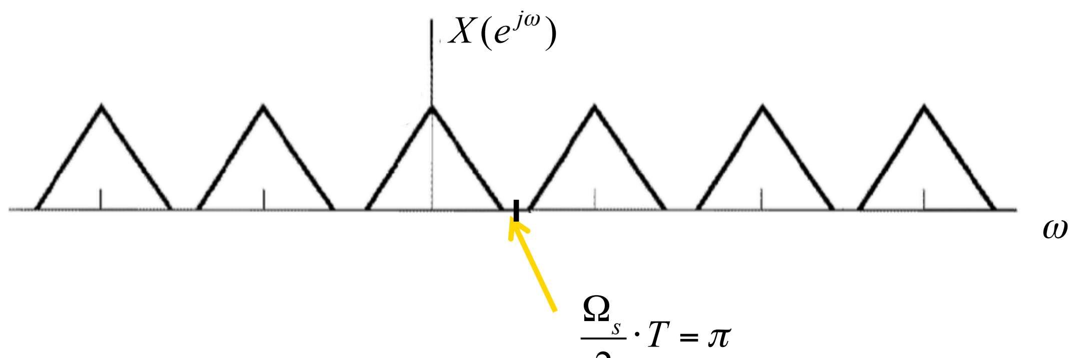
Discrete-Time Processing of Continuous Time



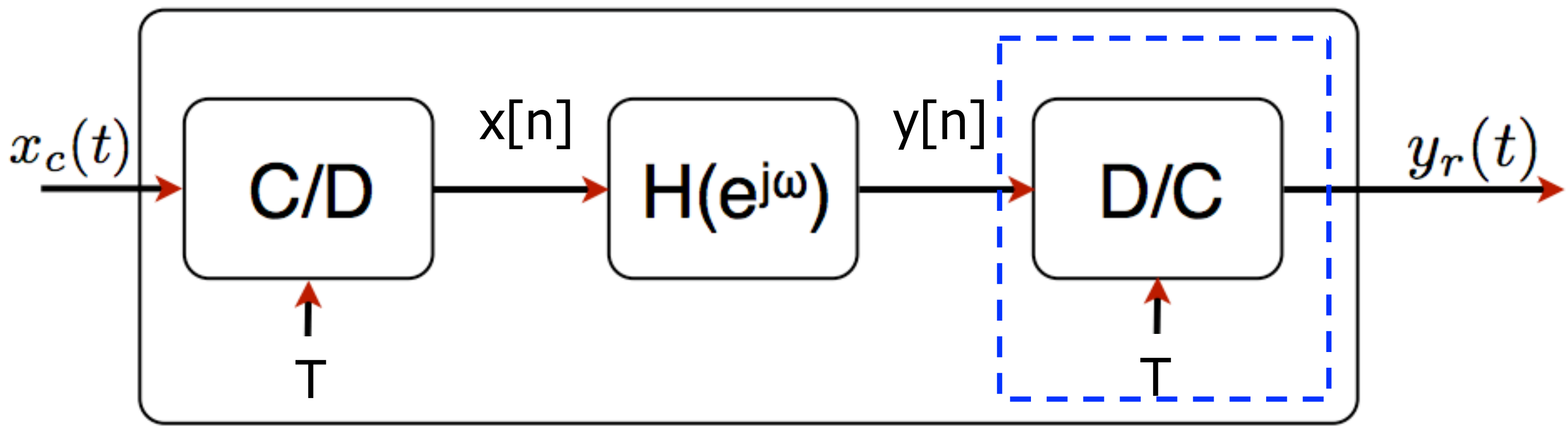
Discrete-Time Processing of Continuous Time

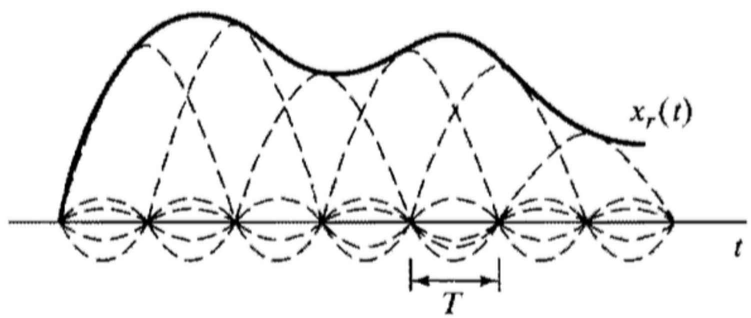


$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$



Discrete-Time Processing of Continuous Time

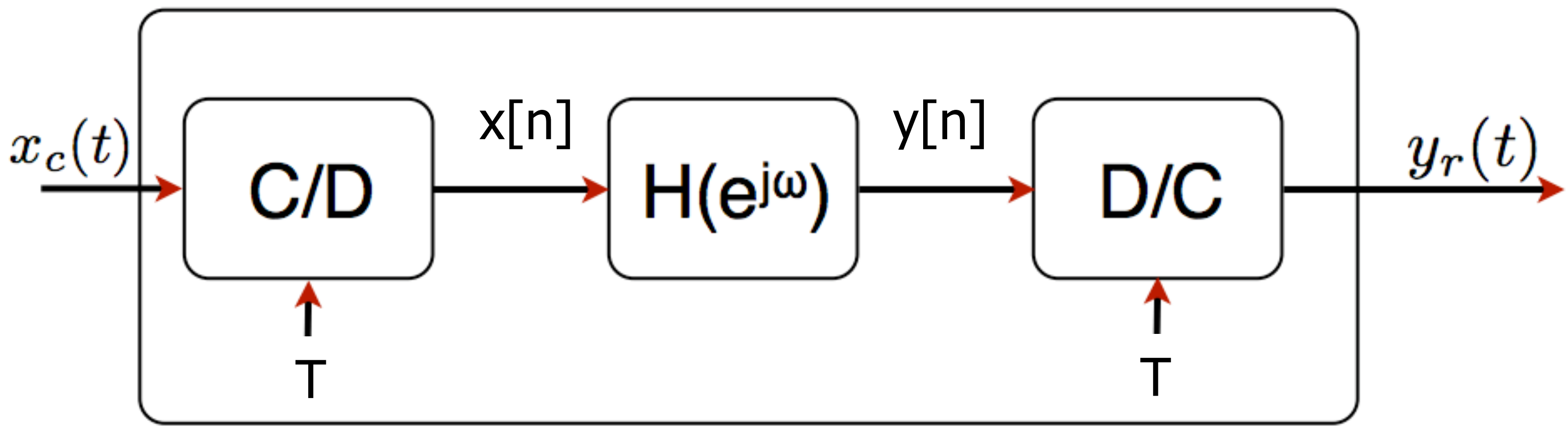




Sum of scaled shifted sincs

$$y_r(t) = \sum_{n=-\infty}^{\infty} y[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

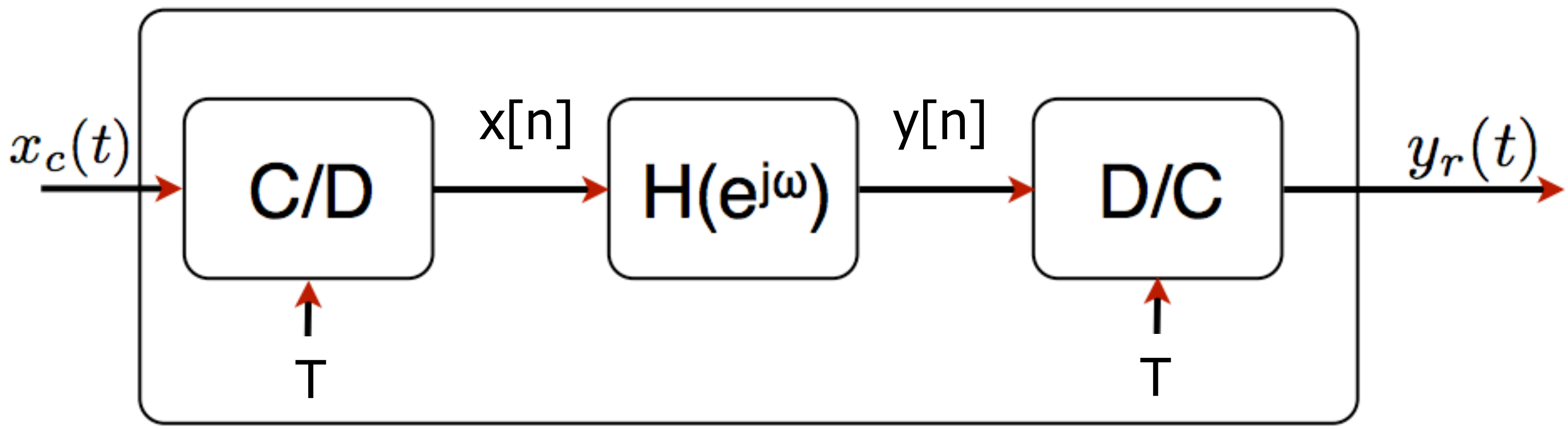
Discrete-Time Processing of Continuous Time



$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- If h[n]/H($e^{j\omega}$) is LTI
  - Is the whole system from $x_c(t) \rightarrow y_c(t)$ LTI?

Discrete-Time Processing of Continuous Time



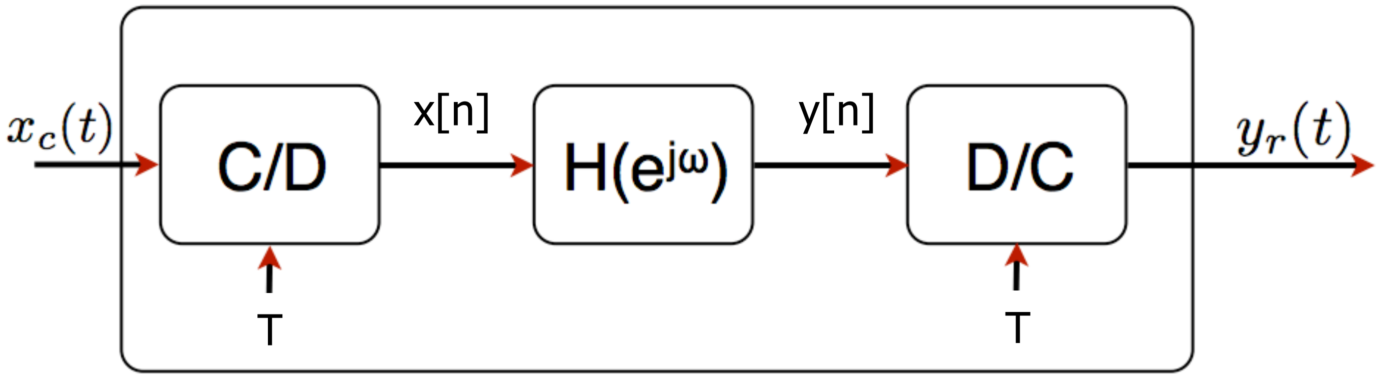
$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right] \qquad y_r(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- If $x_c(t)$ is bandlimited by $\Omega_s/T = \pi/T$, then,

$$\frac{Y_r(j\Omega)}{X_c(j\Omega)} = H_{eff}(j\Omega) = \begin{cases} H(e^{j\omega})\big|_{\omega=\Omega T} & |\Omega| < \Omega_s/T \\ 0 & else \end{cases}$$

Example 1

- Consider the following system



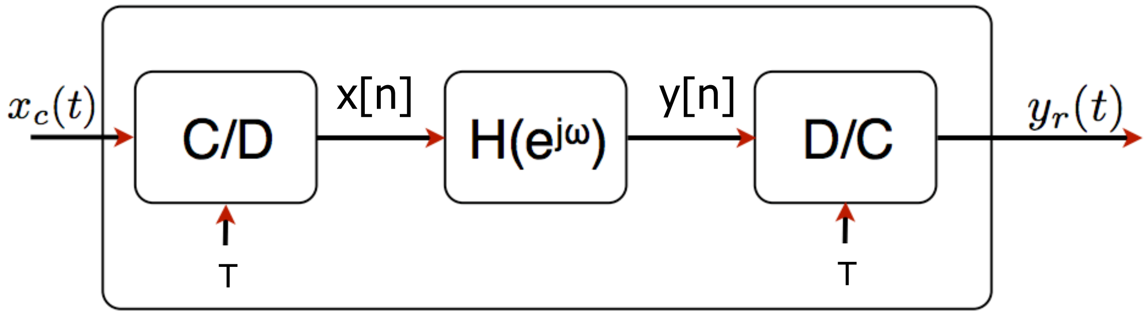
- Where

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- What is the effective frequency response of the system? What happens to a signal bandlimited by Ω_N?

Example 2

- DT implementation of an ideal CT bandlimited differentiator



- The ideal CT differentiator is defined by

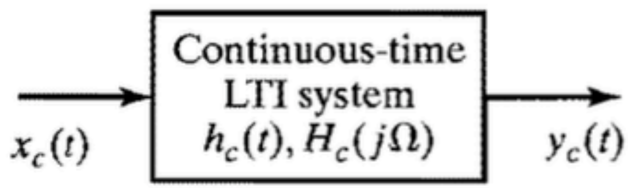
$$y_C(t) = \frac{d}{dt}[x_C(t)]$$

- With corresponding

$$H_C(j\Omega) = j\Omega$$

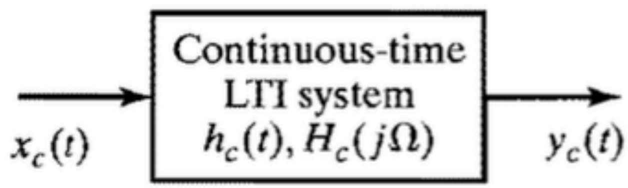
Impulse Invariance

- Want to implement continuous-time system…

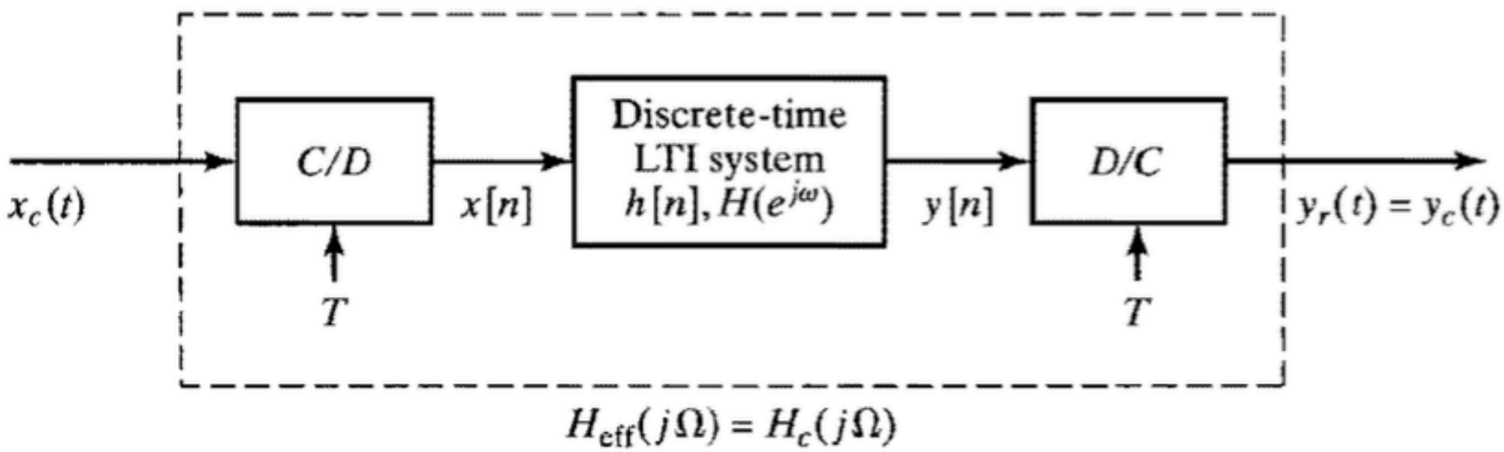


Impulse Invariance

- Want to implement continuous-time system…



- …in discrete-time



Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\Omega)\Big|_{\Omega=\frac{\omega}{T}}, \qquad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi / T$$

Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

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$$\boxed{h[n] = Th_c(nT)}$$

Impulse Invariance

- Let,

$$h[n] = h_c(nT)$$

Impulse Invariance

- Let,

$$h[n] = h_c(nT)$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right]$$

Impulse Invariance

- Let,

$$h[n] = h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

- If sampling at Nyquist Rate then

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

$$H(e^{j\omega}) = \frac{1}{T} H_c\left(j\frac{\omega}{T} \right), \qquad |\omega| < \pi$$

Impulse Invariance

- Let,

$$h[n] = T h_c(nT)$$

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

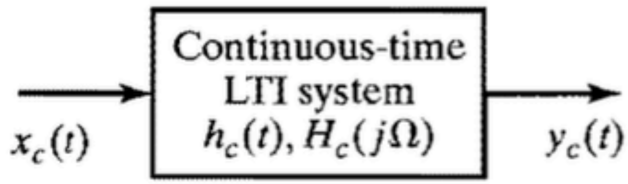
- If sampling at Nyquist Rate then

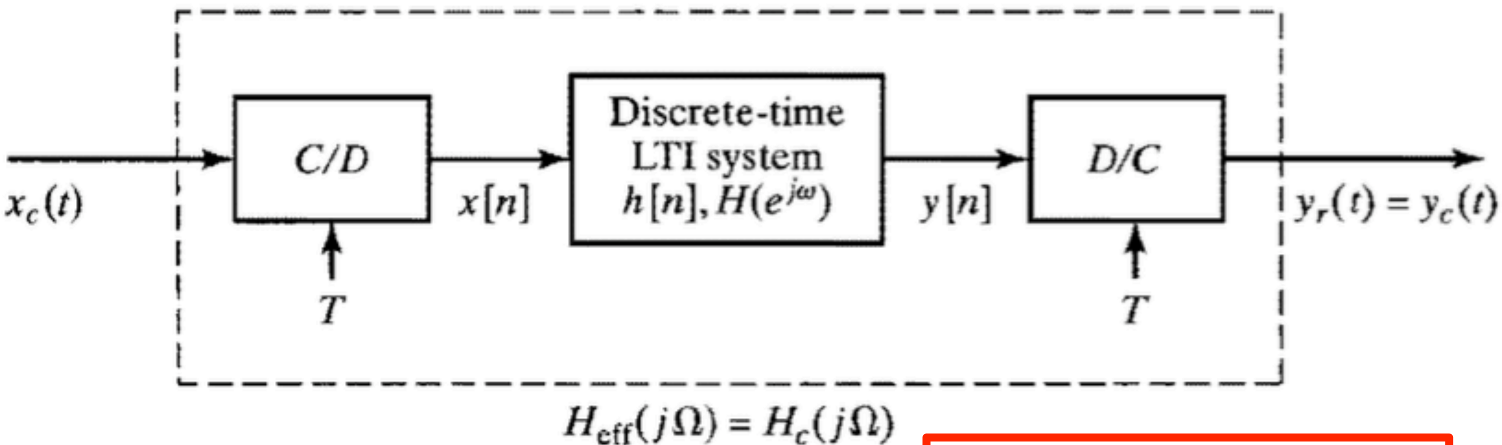
$$H(e^{j\omega}) = \frac{T}{T} \sum_{k=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} - \frac{2\pi k}{T} \right) \right]$$

$$H(e^{j\omega}) = \frac{T}{T} H_c\left(j\frac{\omega}{T} \right), \qquad |\omega| < \pi$$

Impulse Invariance

- Want to implement continuous-time system in discrete-time

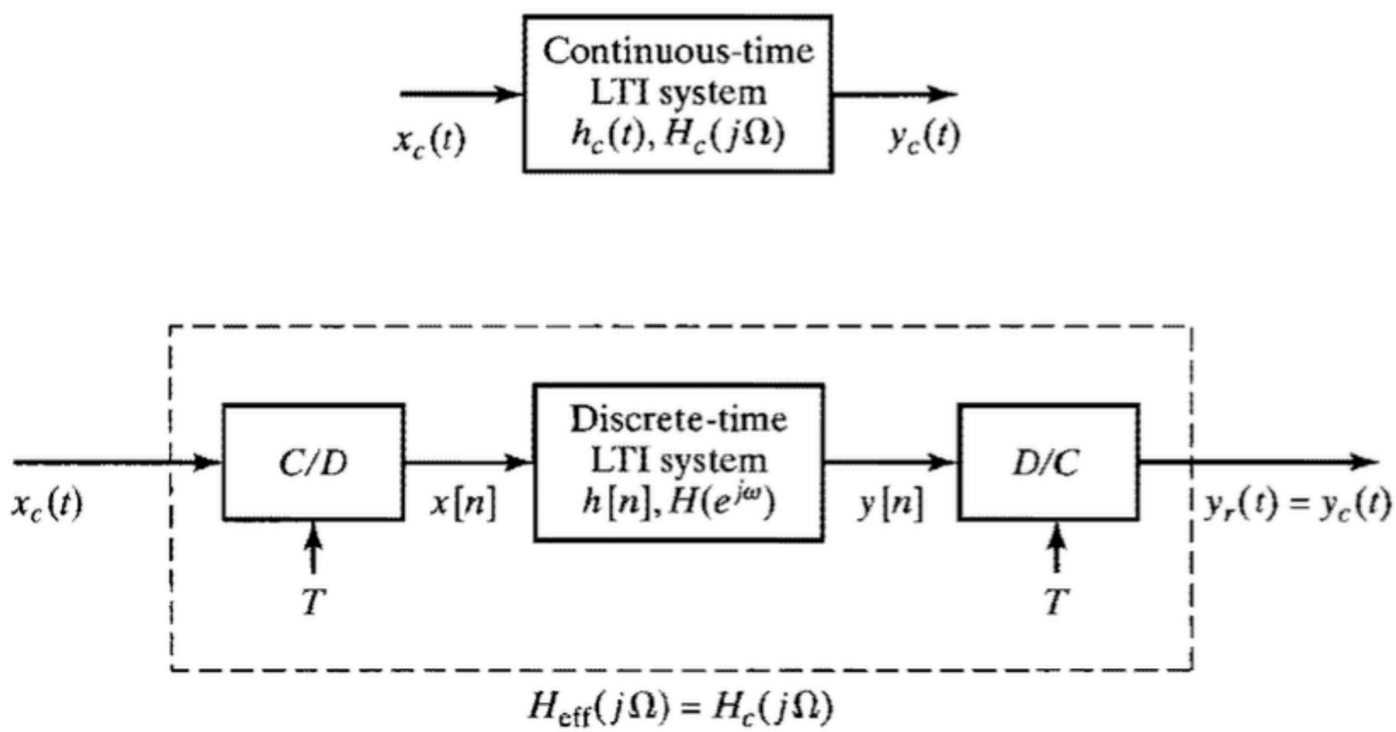




$$h[n] = Th_c(nT)$$

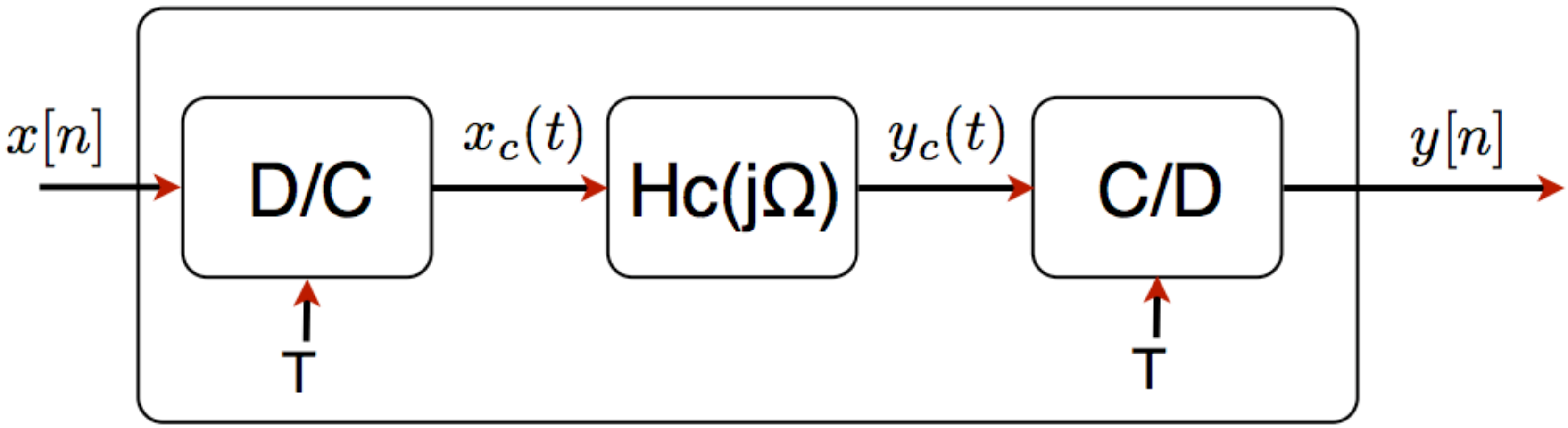
Example 3: DT Lowpass Filter

- We wish to implement a lowpass filter with cutoff frequency Ω_c on continuous time signal in discrete time with the following system



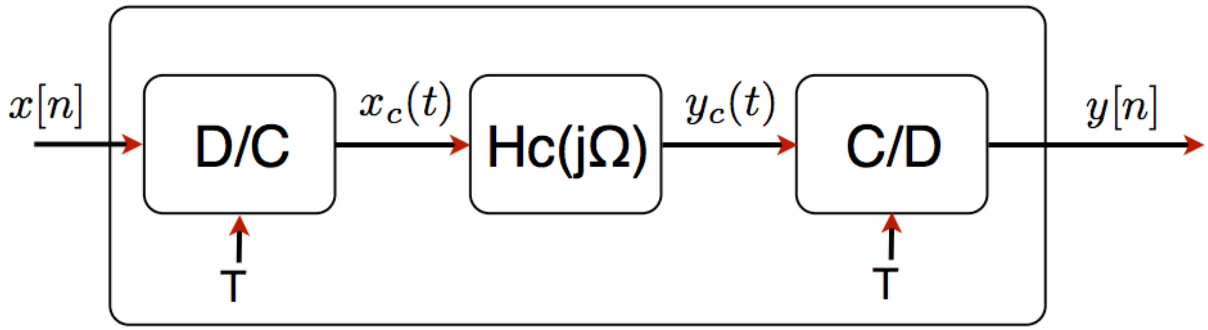
Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time



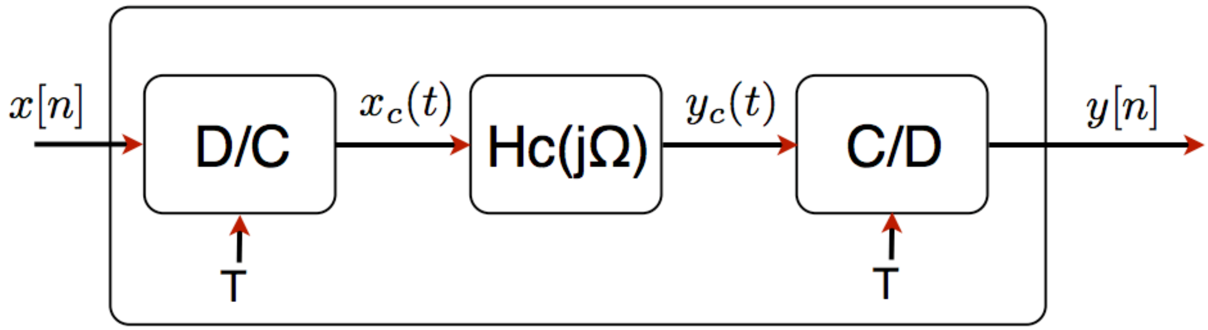
Is the effective $H(e^{j\omega})$ LTI?

Continuous-Time Processing of Discrete-Time



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Continuous-Time Processing of Discrete-Time



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Also bandlimited

$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

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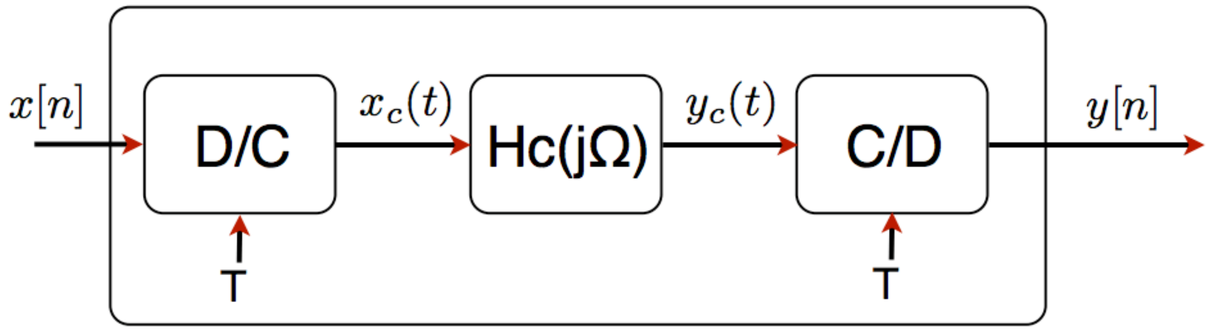

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left[j\left(\Omega - k\Omega_s\right)\right]\Big|_{\Omega=\omega/T}$$

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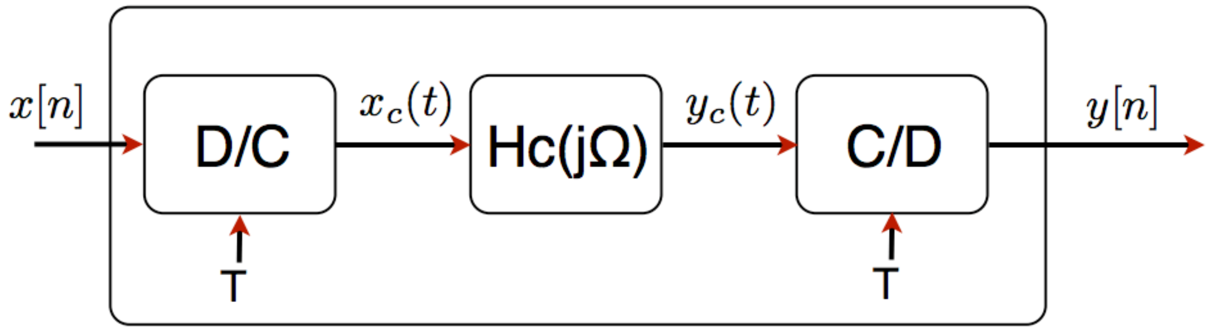

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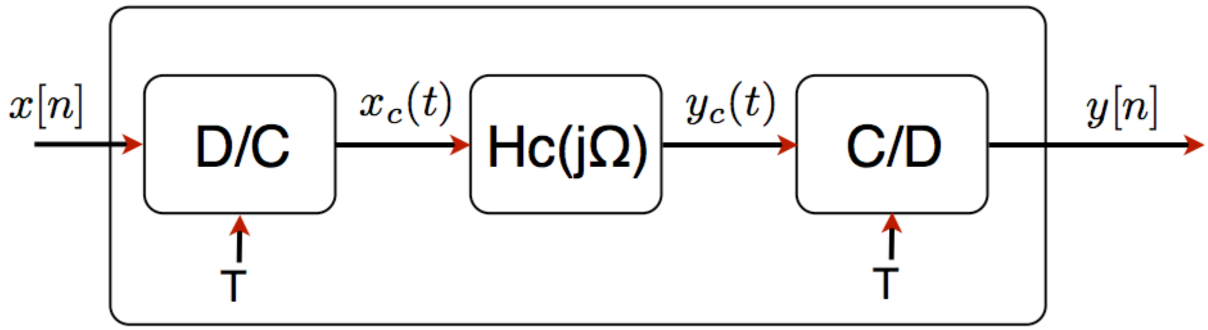

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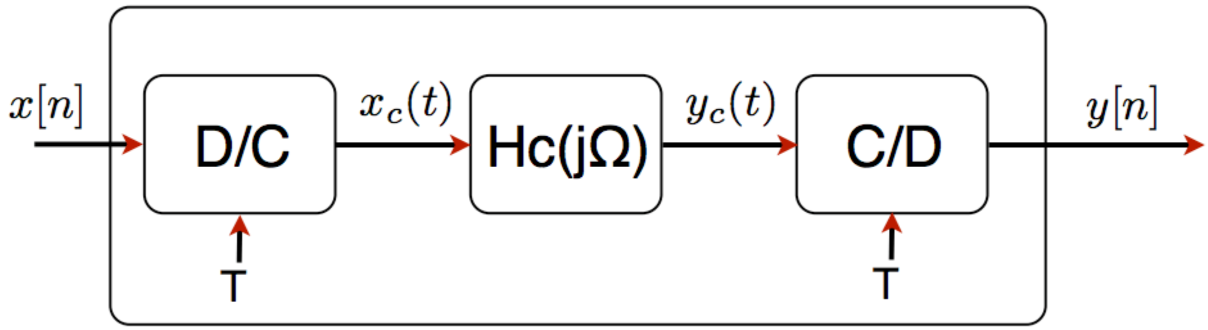

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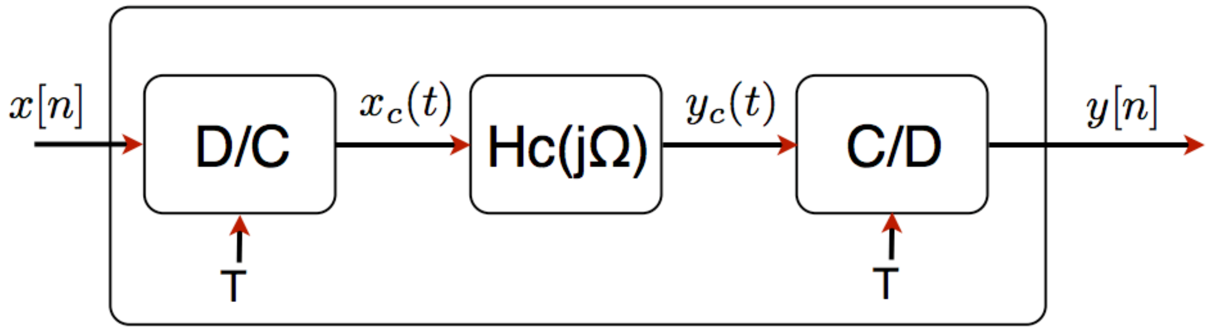


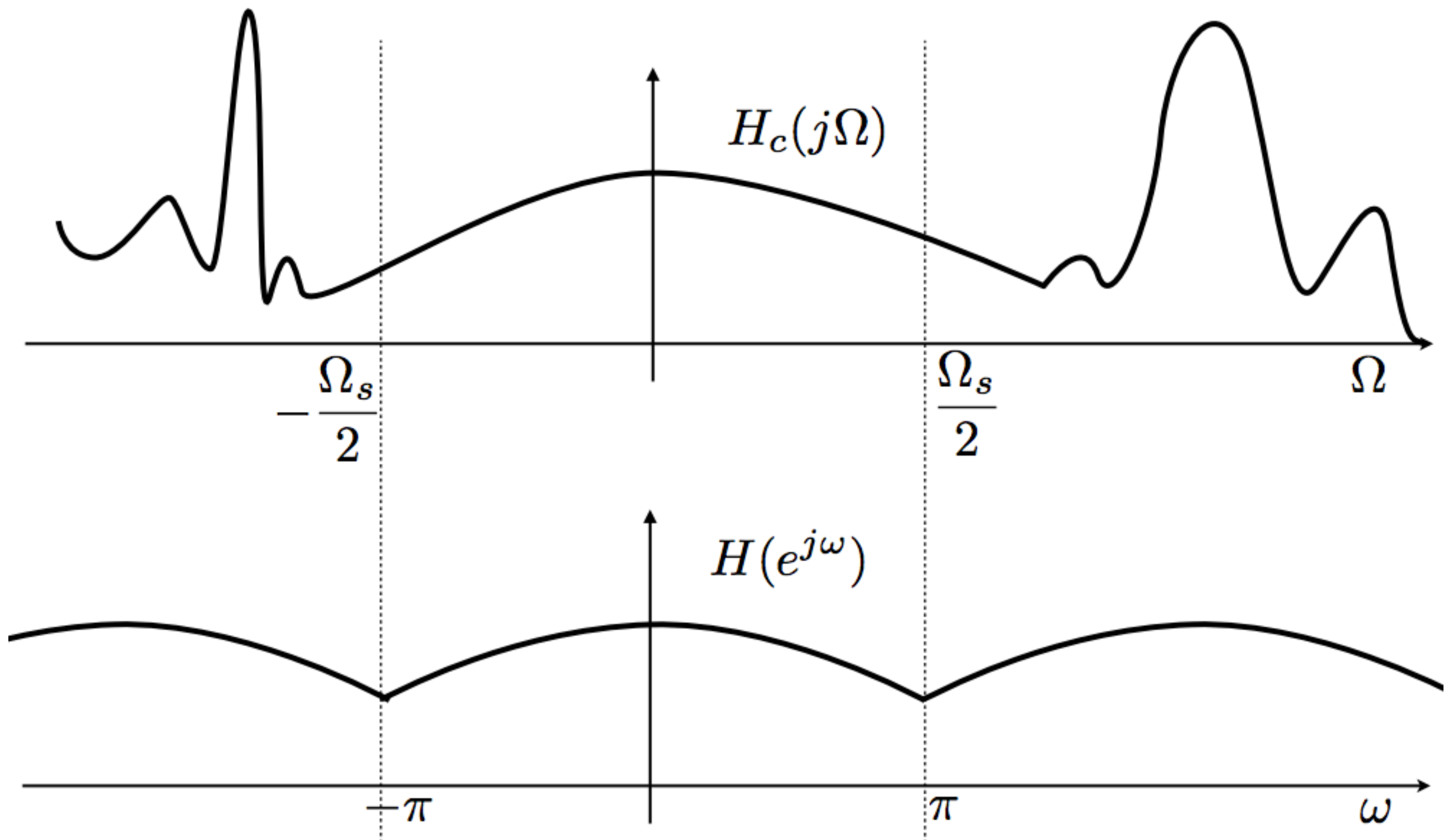
$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad Y(e^{j\omega}) = \frac{1}{T}Y_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$Y(e^{j\omega}) = \frac{1}{T}H_c(j\Omega)\Big|_{\Omega=\omega/T} X_c(j\Omega)\Big|_{\Omega=\omega/T}$$

$$= \frac{1}{T}H_c(j\Omega)\Big|_{\Omega=\omega/T}(TX(e^{j\omega})) \qquad |\omega| < \pi$$

$$H(e^{j\omega})$$

Example



$H_c(j\Omega)$
$-\frac{\Omega_s}{2}$
$\frac{\Omega_s}{2}$
Ω
$H(e^{j\omega})$
$-\pi$
π
ω

Example: Non-integer Delay

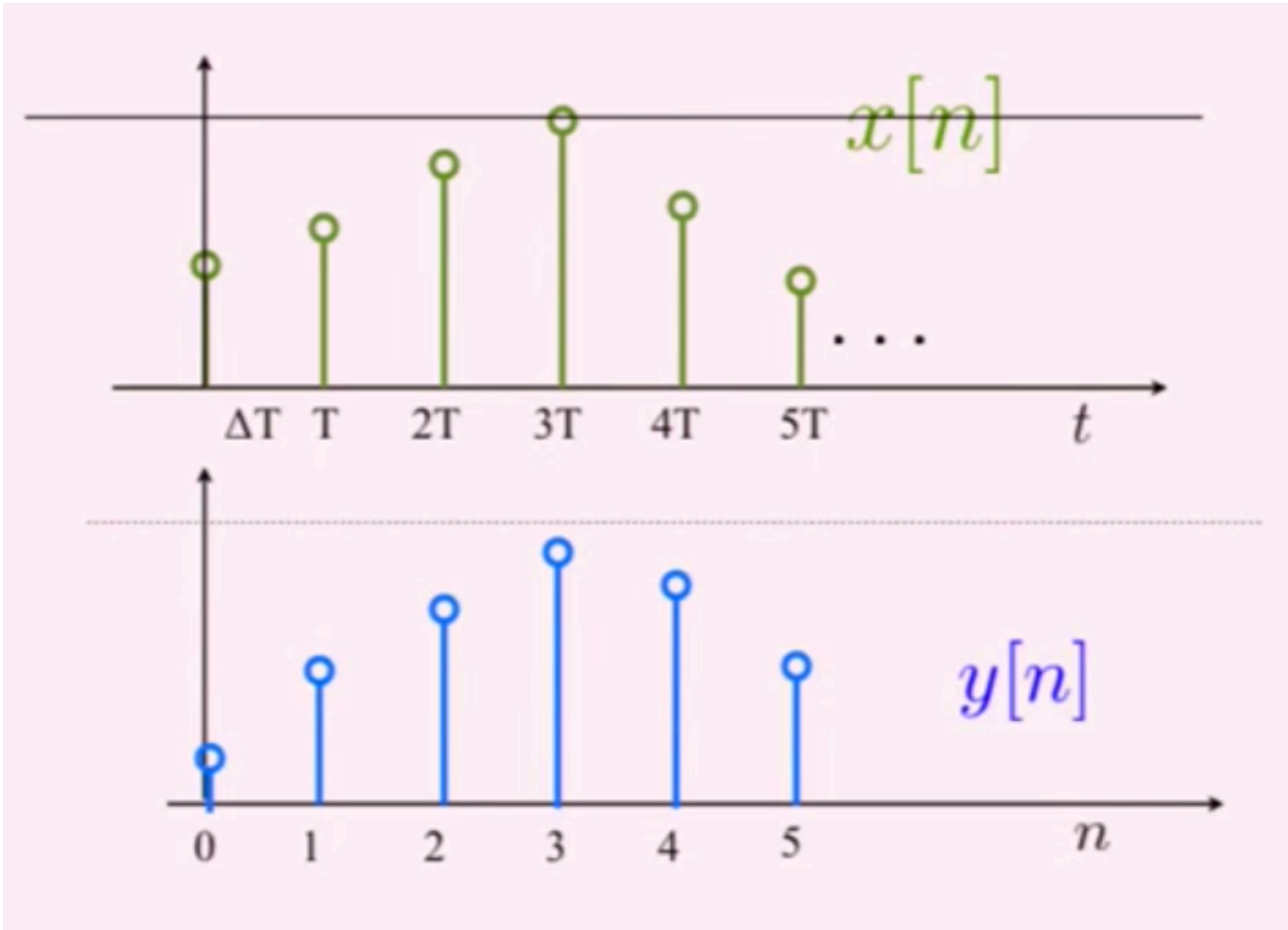
- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

$$\delta[n] \leftrightarrow 1$$

$$\delta[n - n_d] \leftrightarrow e^{-j\omega n_d}$$

Example: Non-integer Delay



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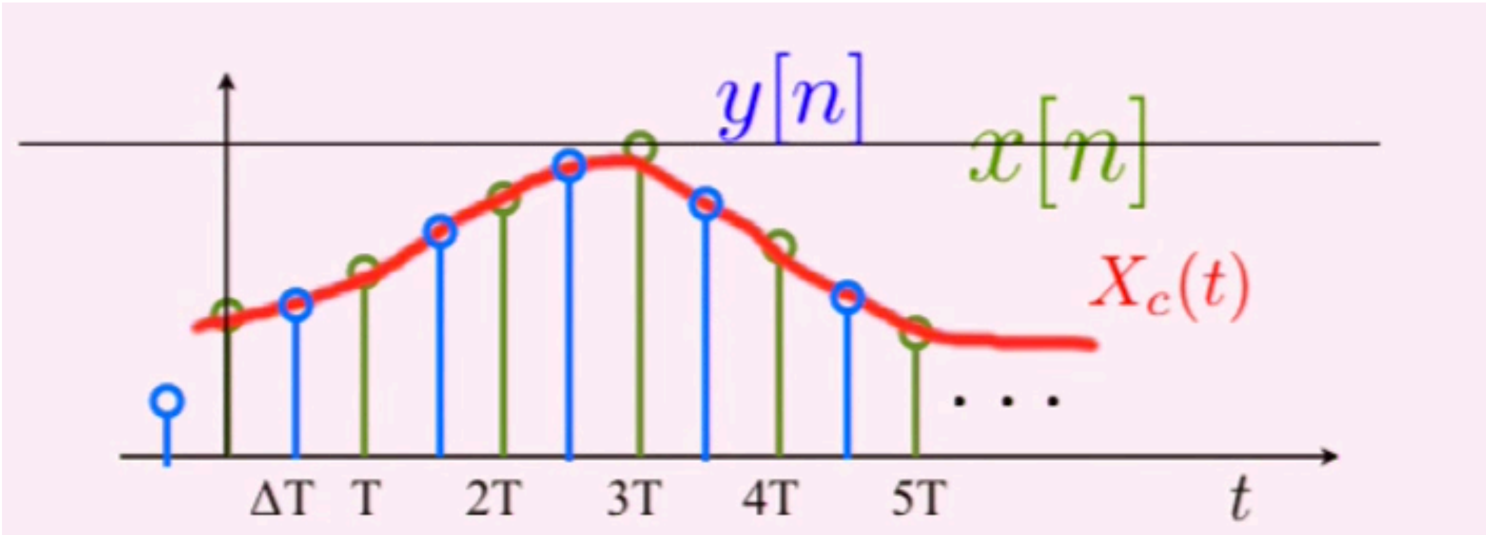

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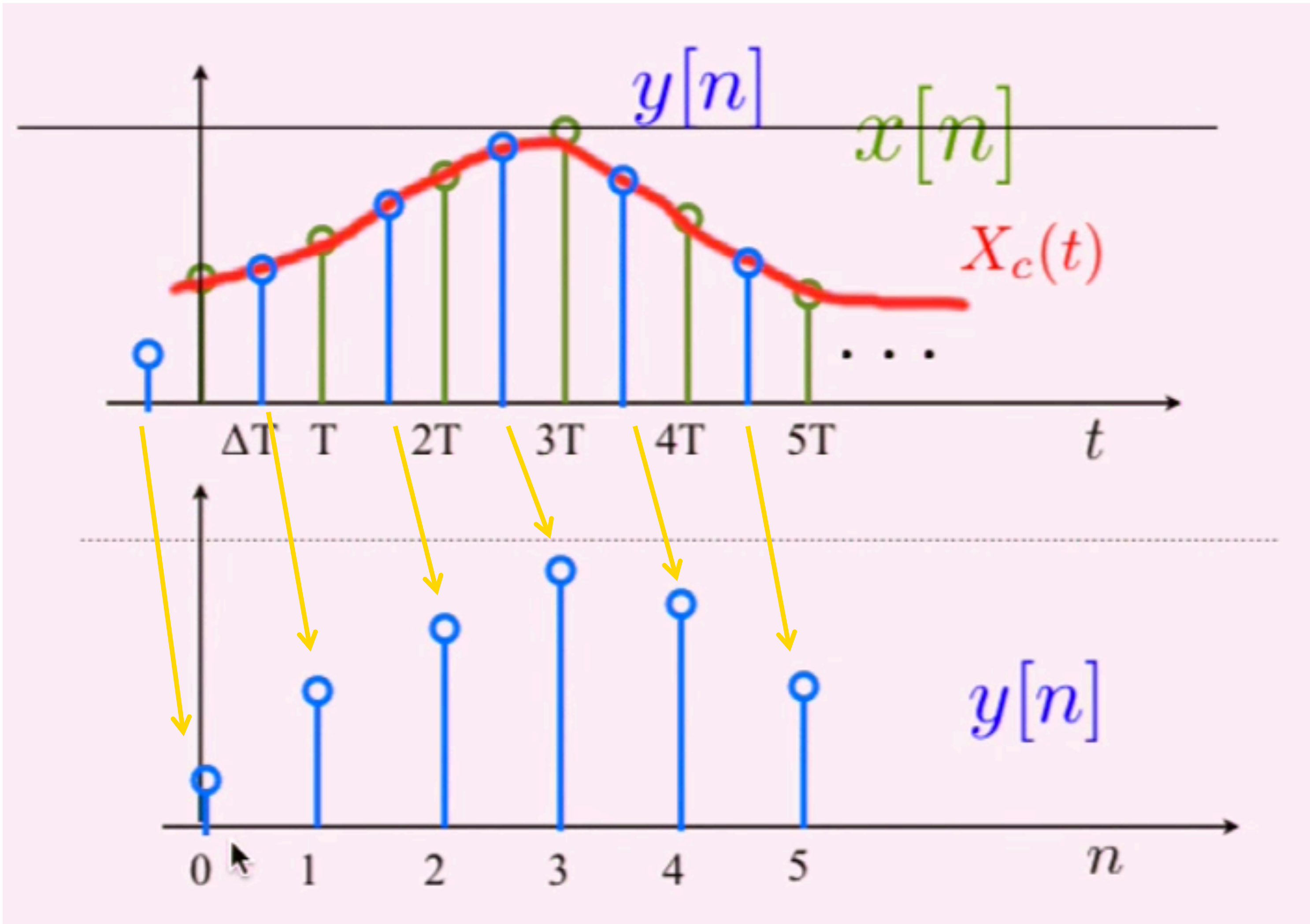

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Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ <u>delay of ΔT in continuous time</u>

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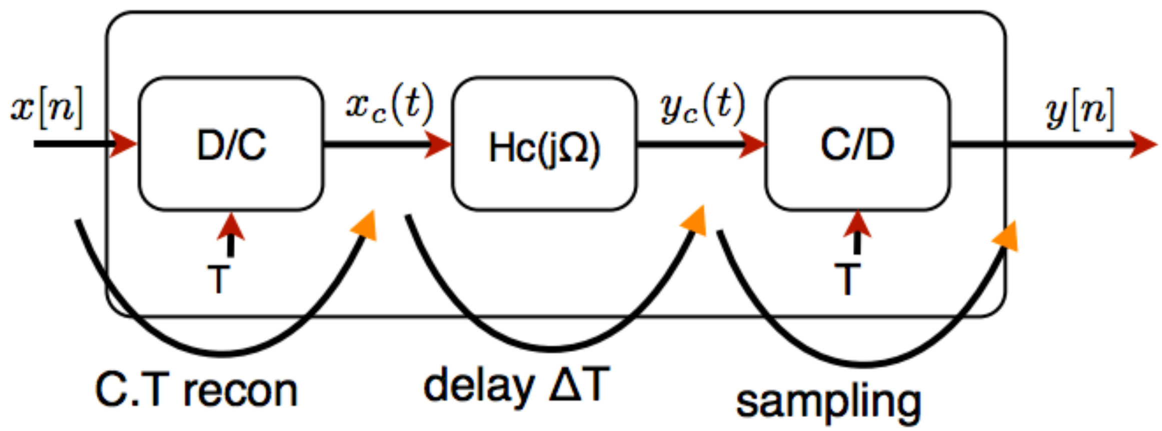
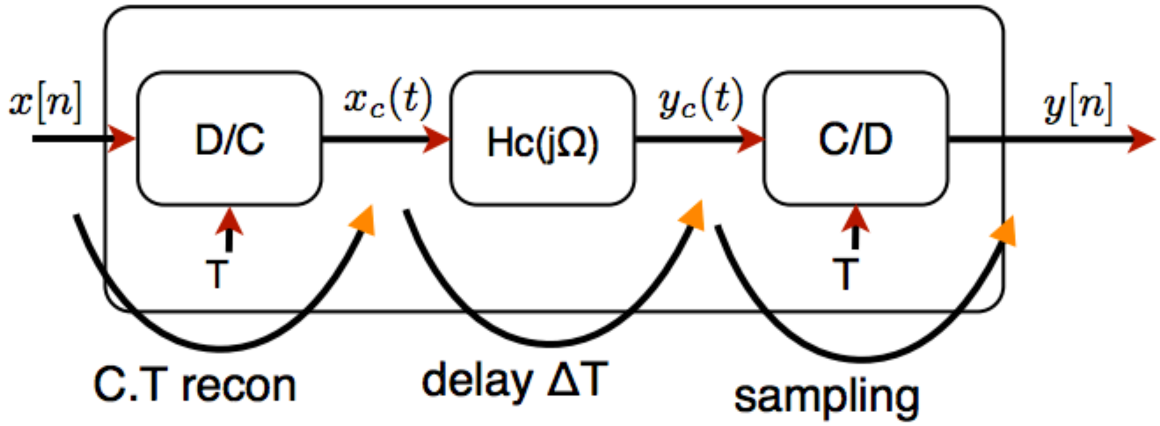


Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



$$x_c(t) = \sum_k x[k] \operatorname{sinc}\left(\frac{t - kT}{T}\right)$$

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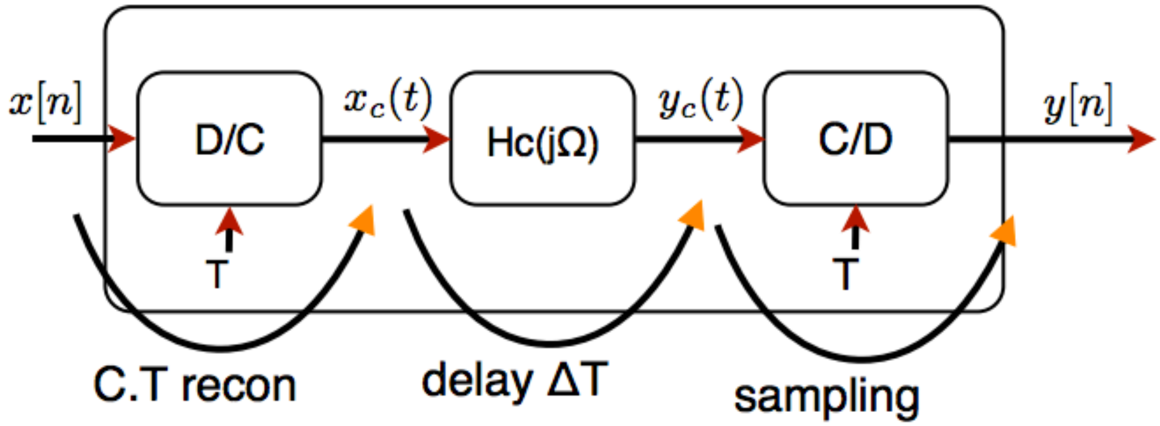

$$x_c(t) = \sum_k x[k]\,\mathrm{sinc}\left(\frac{t-kT}{T}\right) \qquad y_c(t) = x_c(t - T\Delta)$$

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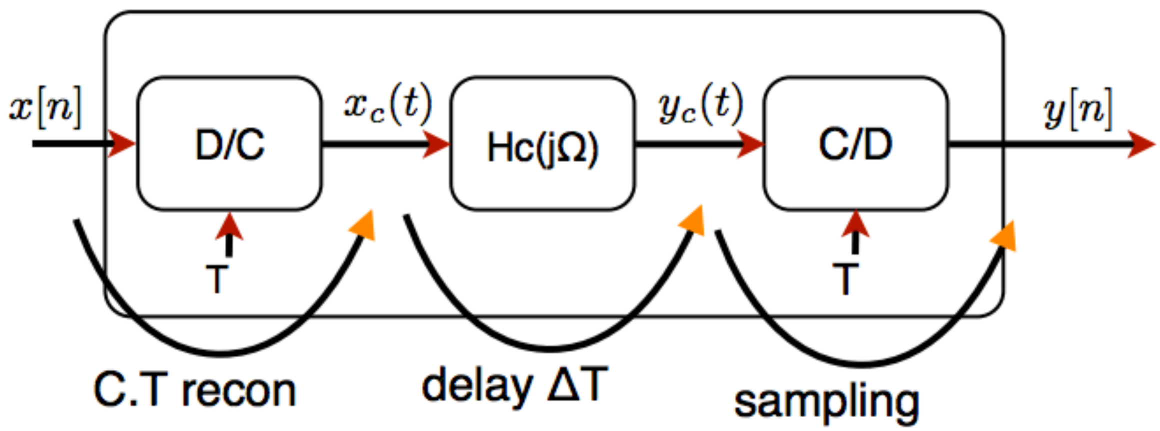

$$x_c(t) = \sum_k x[k]\text{sinc}\left(\frac{t-kT}{T}\right)$$

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$$y[n] = y_c(nT)$$

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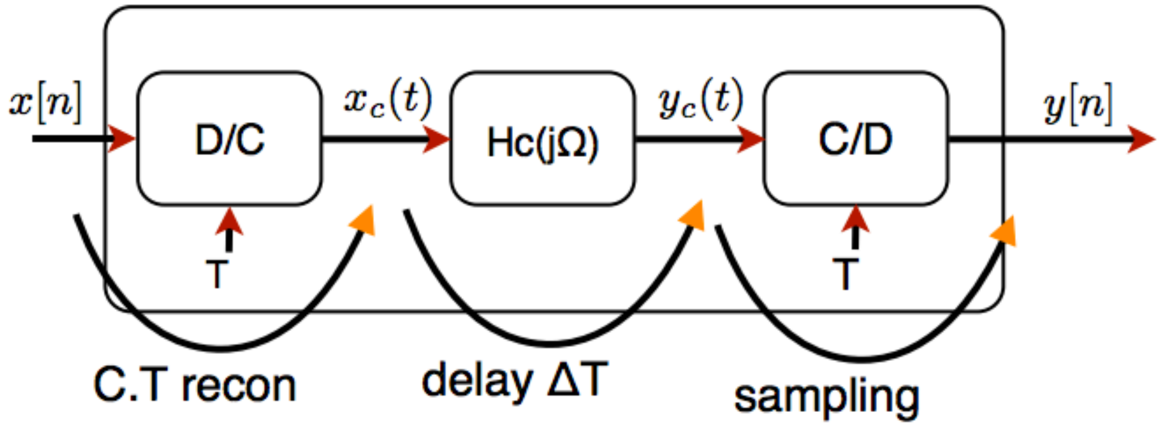

$$y[n] = y_c(nT) = x_c(nT - T \cdot \Delta)$$

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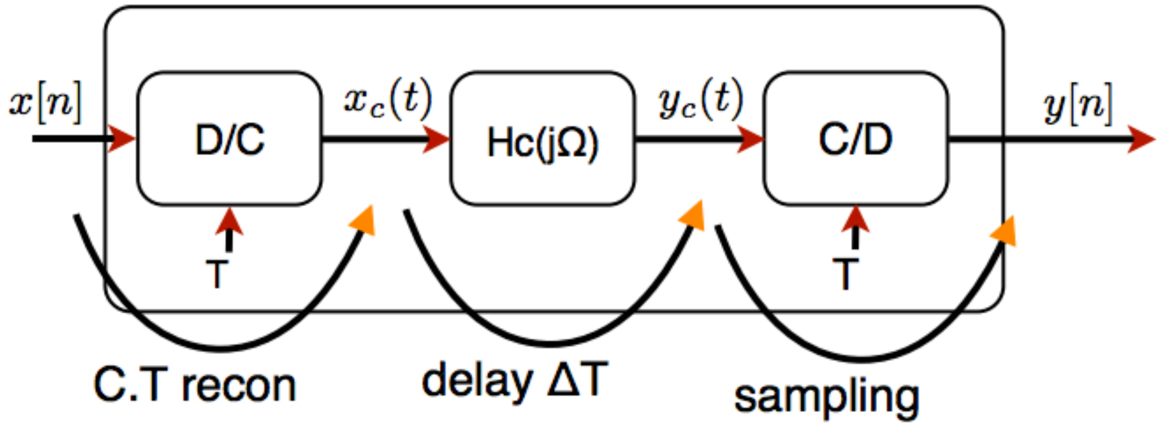

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$$x_c(nT - T \cdot \Delta) = \sum_k x[k] \operatorname{sinc}\left(\frac{nT - T \cdot \Delta - kT}{T}\right) = \sum_k x[k] \operatorname{sinc}(n - \Delta - k)$$

Example: Non-integer Delay

- Delay system has an impulse response of a sinc with a continuous time delay

$$y[n] = \sum_k x[k]\,\mathrm{sinc}\left(n - \Delta - k\right)$$
$$= x[n] * \mathrm{sinc}\left(n - \Delta\right)$$

$$\Rightarrow h[n] = \mathrm{sinc}(n - \Delta)$$

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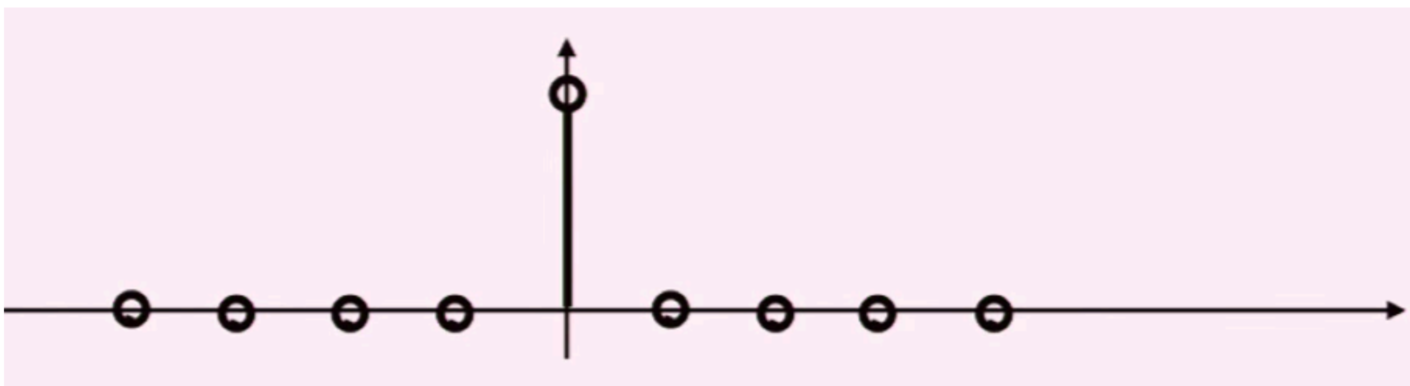

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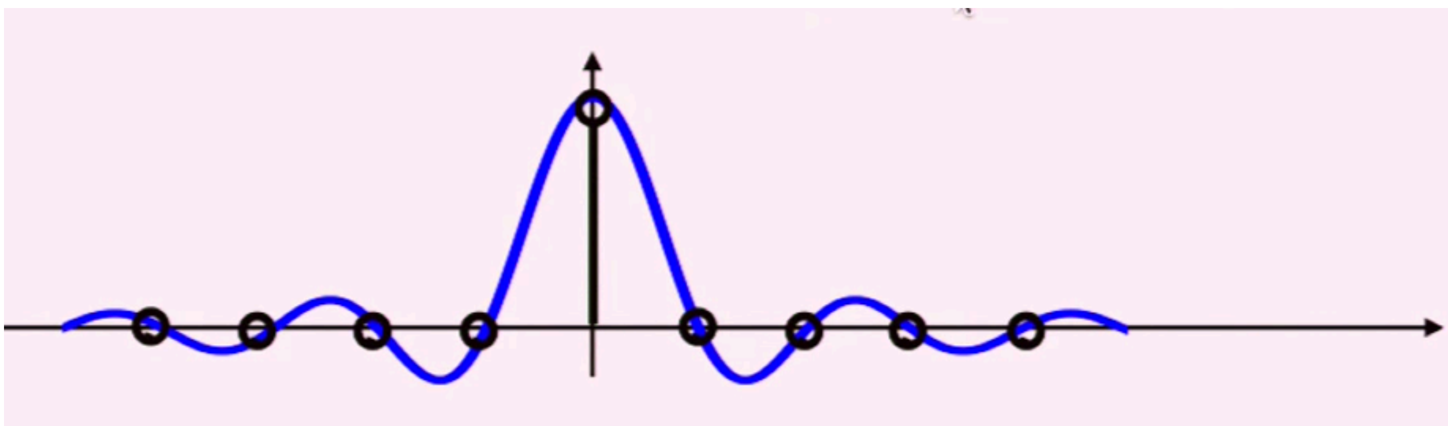

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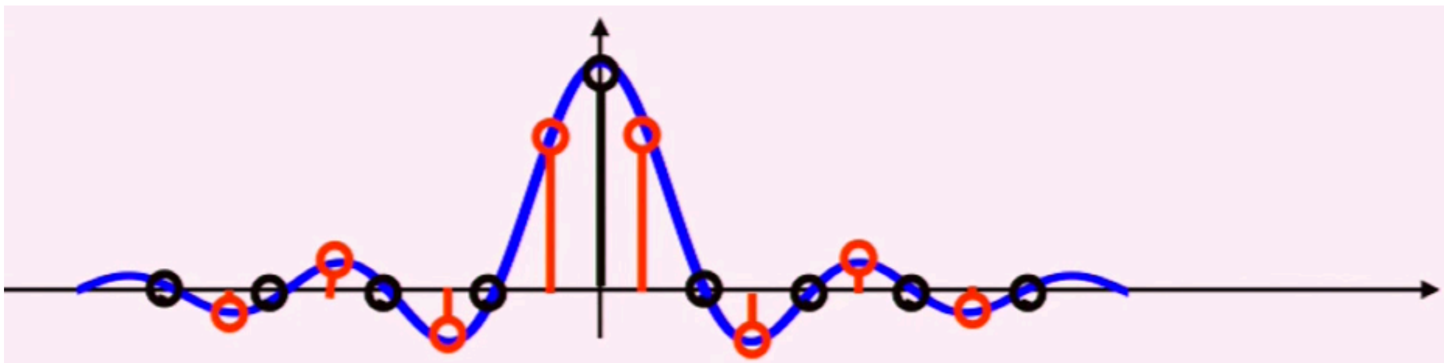

Big Ideas

- Sampling and reconstruction
  - Rely on bandlimitedness for unique reconstruction
- CT processing of DT
  - Effectively LTI if no aliasing
- DT processing of CT
  - Always LTI
  - Useful for interpretation
- Changing the sampling rates next time
  - Upsampling, downsampling

Admin

- HW 3 due Sunday