ESE 531: Digital Signal Processing

Lec 9: February 13th, 2020

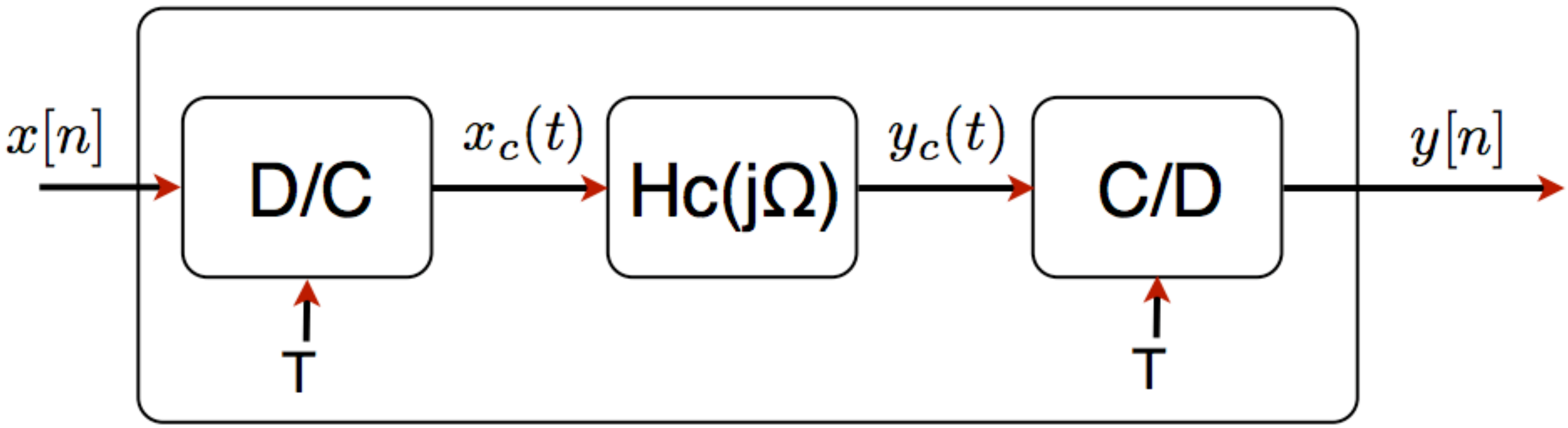
Downsampling/Upsampling and Practical Interpolation

Lecture Outline

- CT processing of DT signals
- Downsampling
- Upsampling
- Practical Interpolation (time permitting)

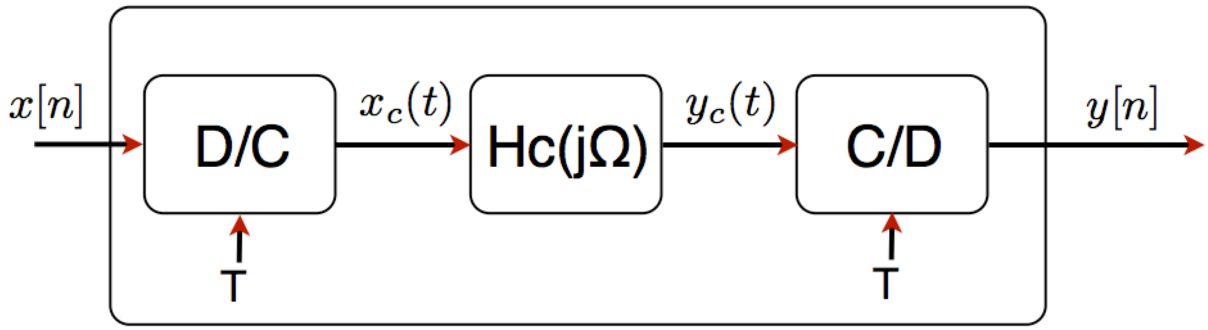
Continuous-Time Processing of Discrete-Time

- Useful to interpret DT systems with no simple interpretation in discrete time



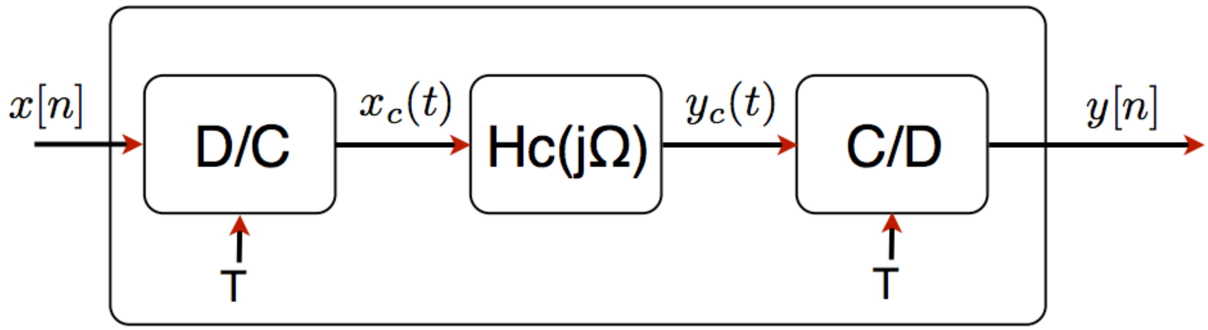
Is the effective $H(e^{j\omega})$ LTI?

Continuous-Time Processing of Discrete-Time



$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Continuous-Time Processing of Discrete-Time

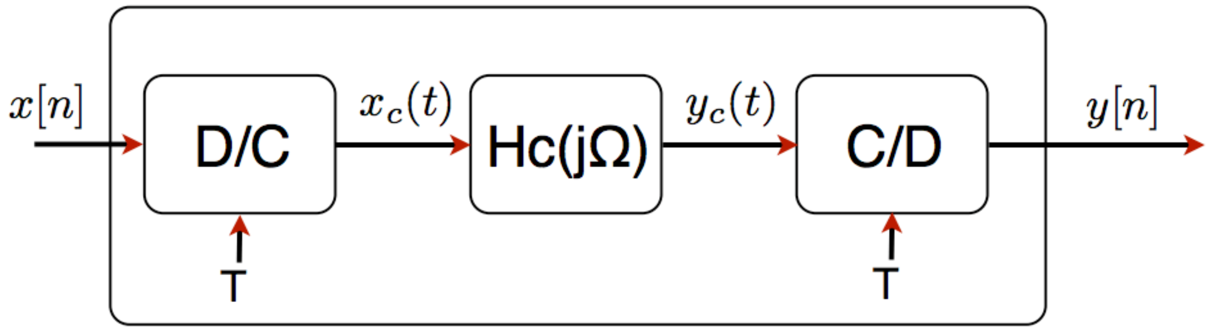


$$X_c(j\Omega) = \begin{cases} TX(e^{j\Omega T}) & |\Omega| < \pi / T \\ 0 & else \end{cases}$$

Also bandlimited

$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

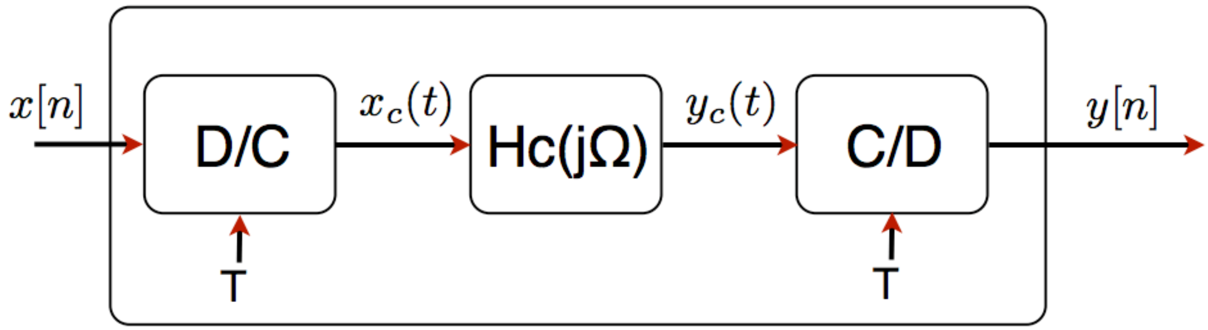
Continuous-Time Processing of Discrete-Time



$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$$

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left[j\left(\Omega - k\Omega_s\right)\right]\Big|_{\Omega=\omega/T}$$

Continuous-Time Processing of Discrete-Time



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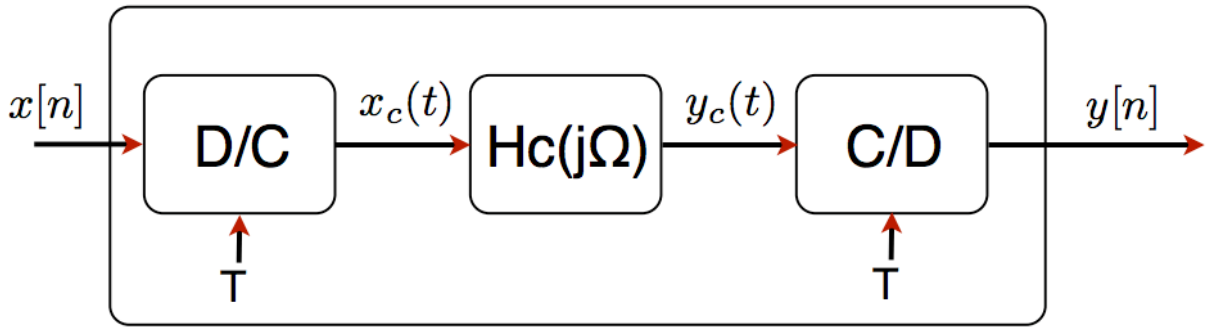

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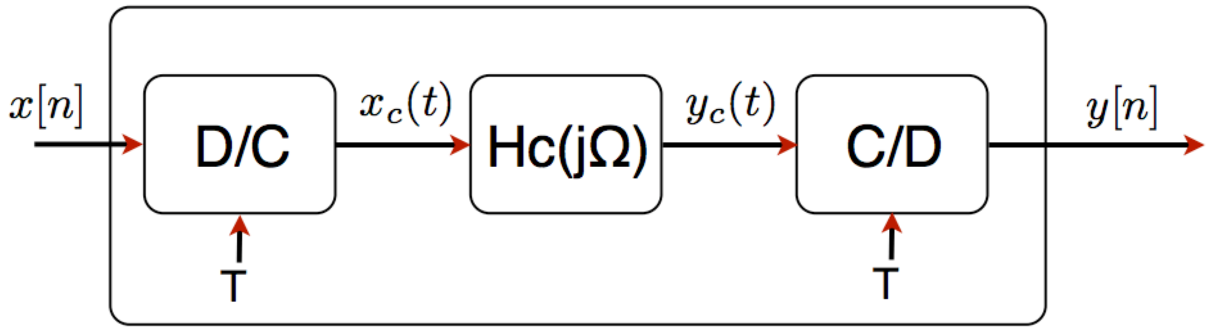

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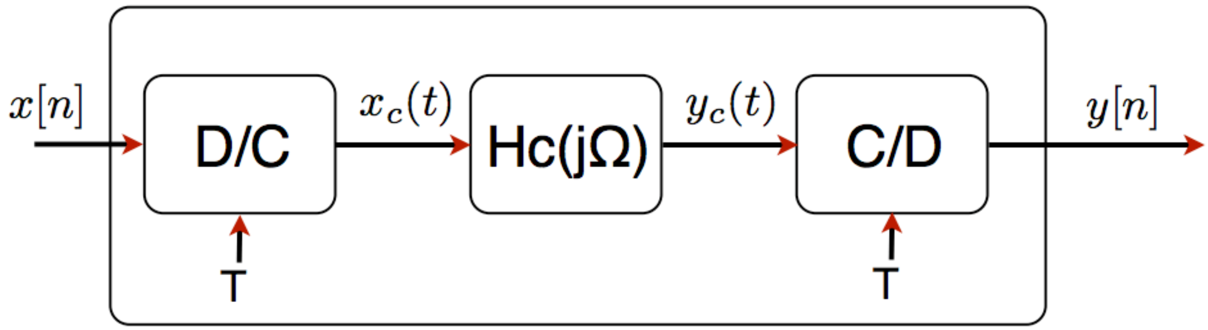

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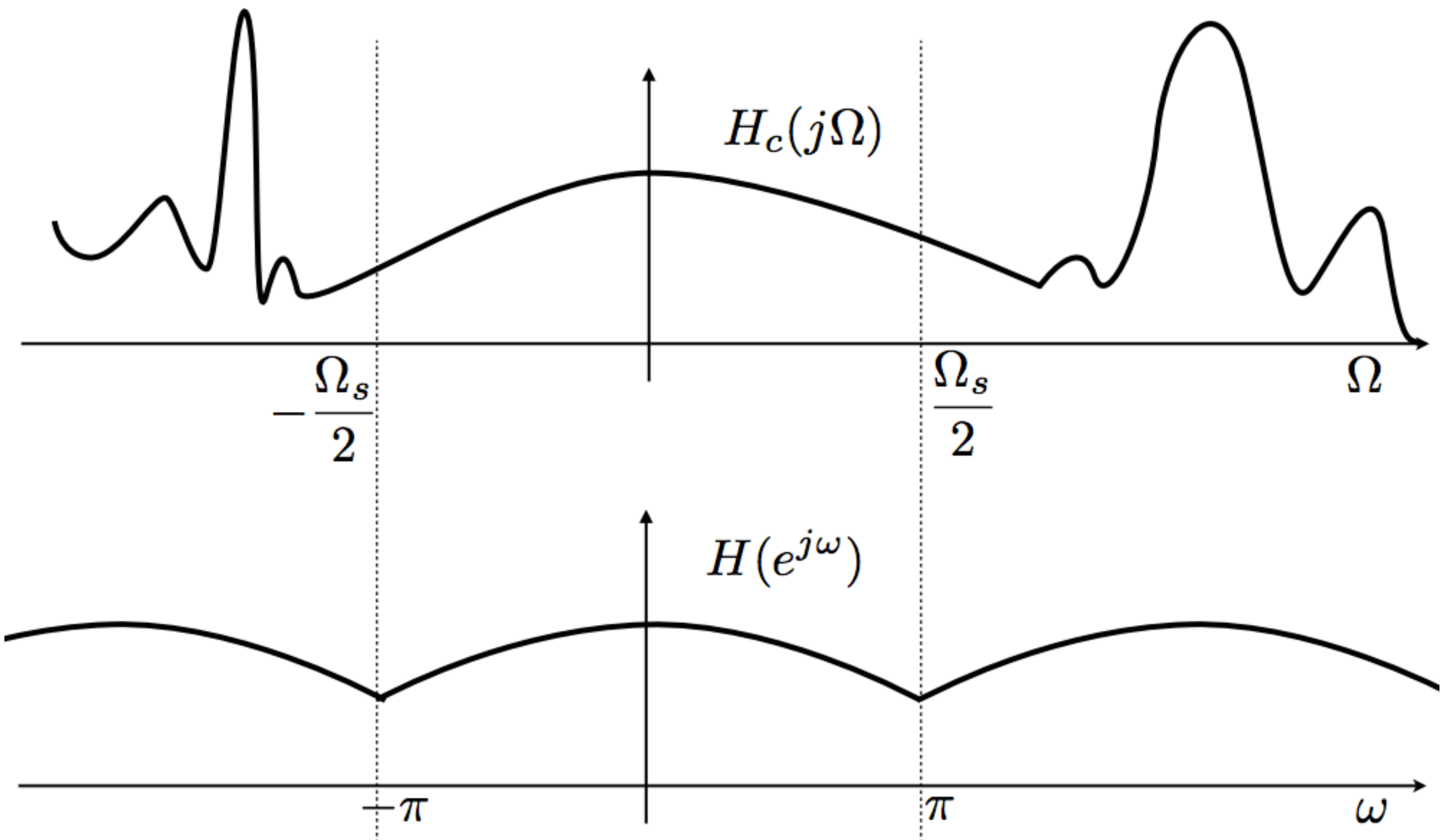
$$= \frac{1}{T}H_c(j\Omega)\Big|_{\Omega=\omega/T}(TX(e^{j\omega})) \qquad |\omega| < \pi$$

$$\boxed{H(e^{j\omega})}$$

Example



$H_c(j\Omega)$
$-\frac{\Omega_s}{2}$
$\frac{\Omega_s}{2}$
Ω
$H(e^{j\omega})$
$-\pi$
π
ω

Example: Non-integer Delay

- What is the time domain operation when Δ is non-integer? I.e $\Delta = 1/2$

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

$$\delta[n] \leftrightarrow 1$$

$$\delta[n - n_d] \leftrightarrow e^{-j\omega n_d}$$

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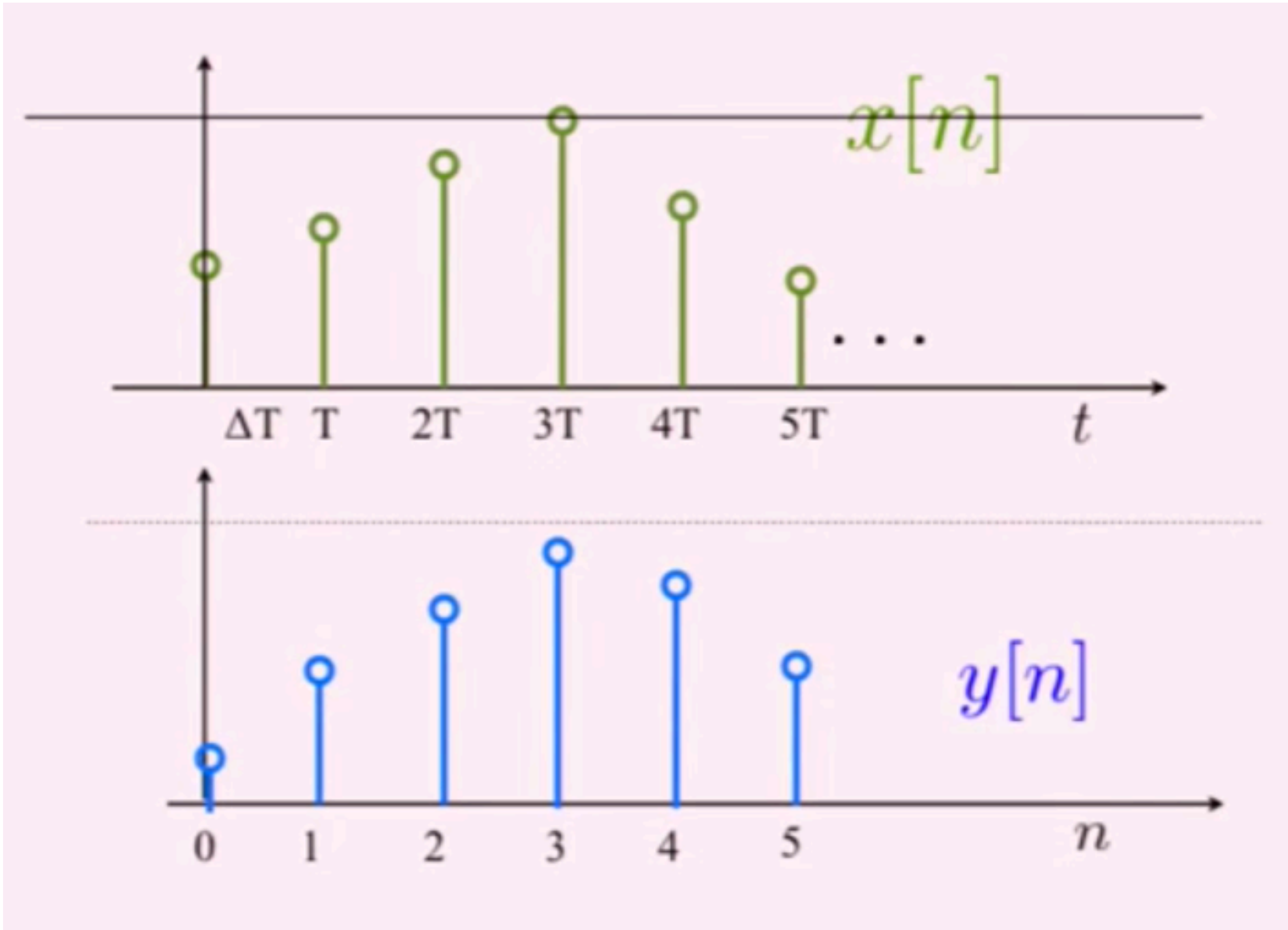

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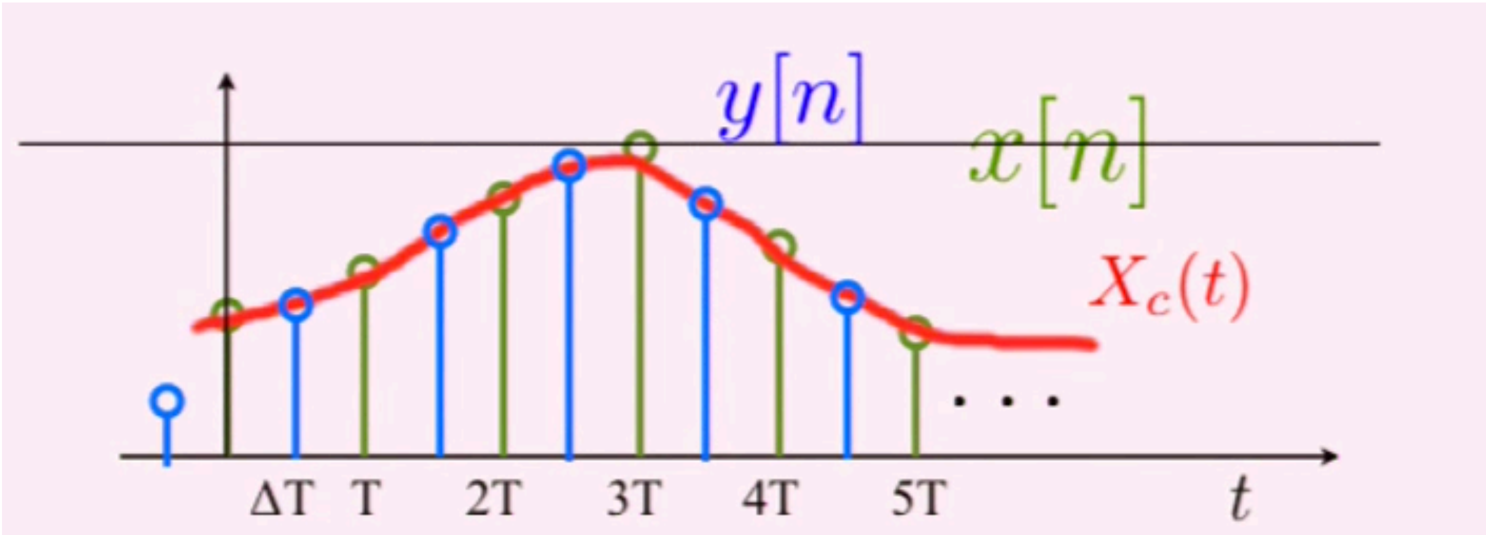

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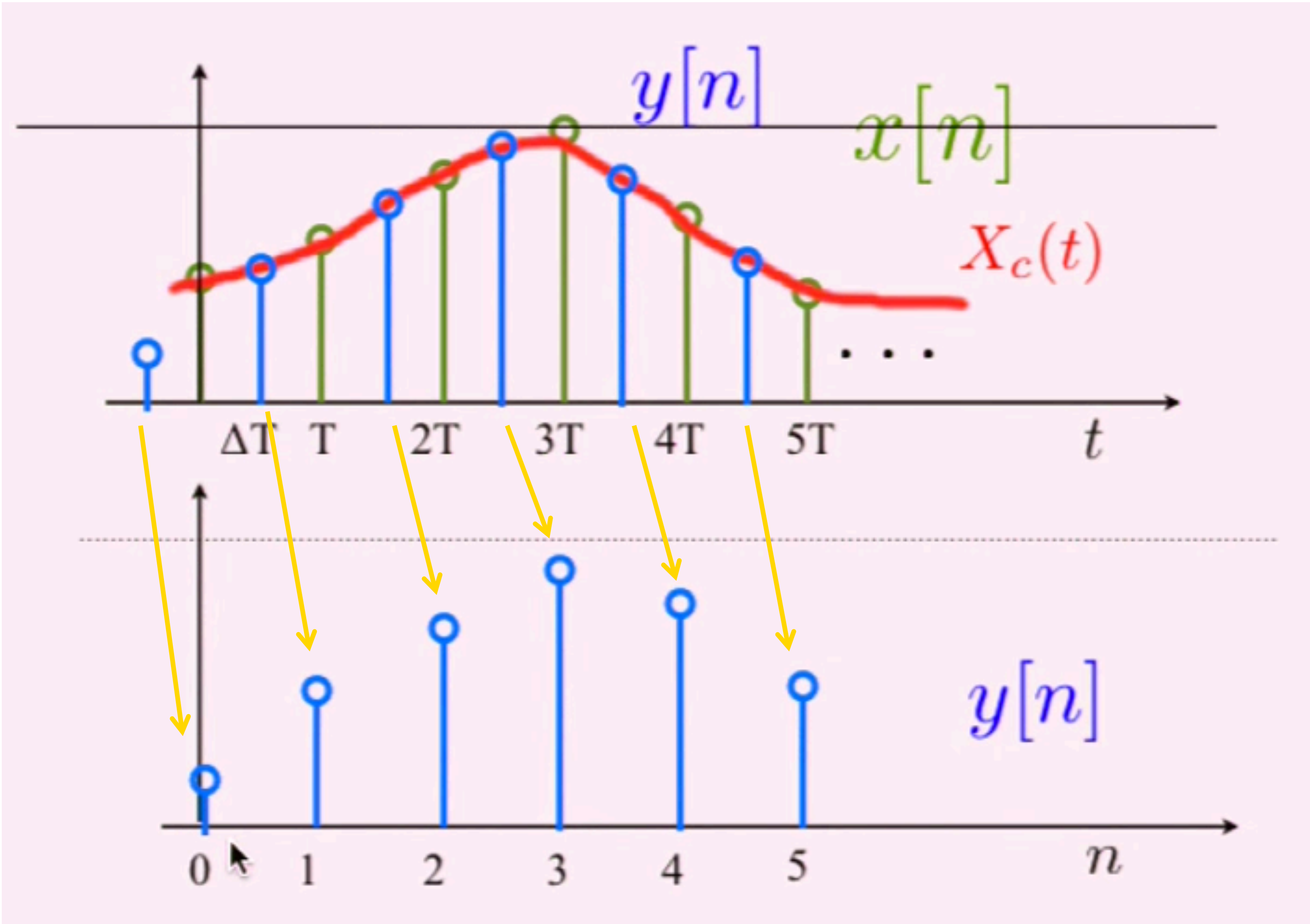

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Let: $H_c(j\Omega) = e^{-j\Omega\Delta T}$ <u>delay of ΔT in continuous time</u>

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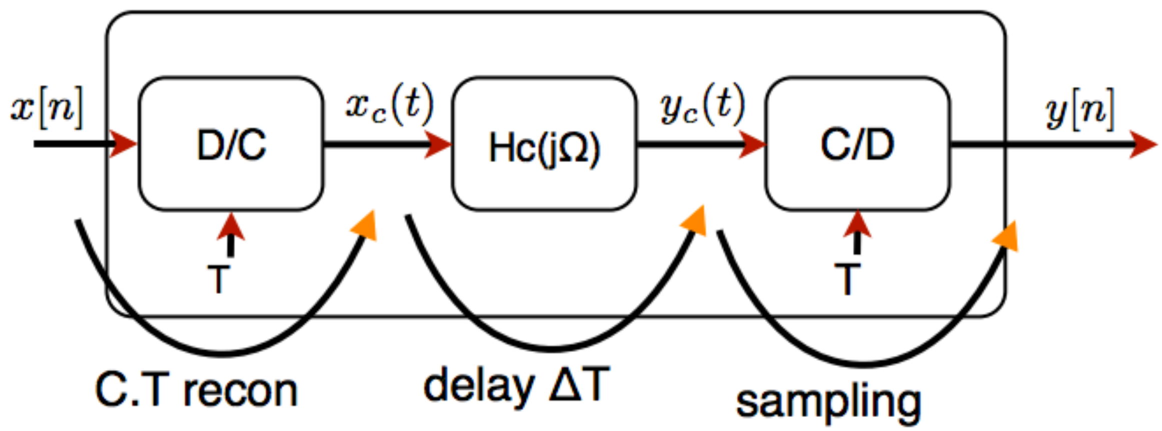
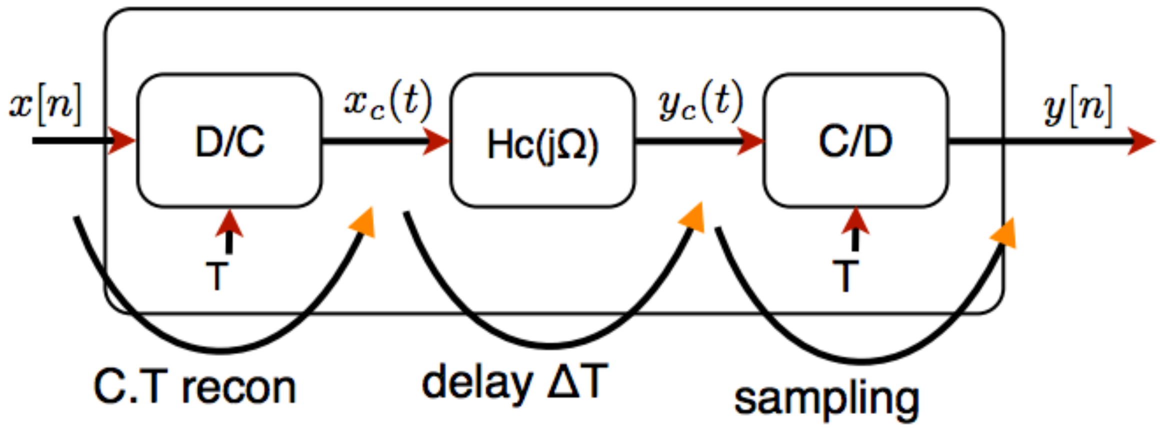


Example: Non-integer Delay

- The block diagram is for interpretation/analysis only



$$x_c(t) = \sum_k x[k]\,\mathrm{sinc}\left(\frac{t - kT}{T}\right)$$

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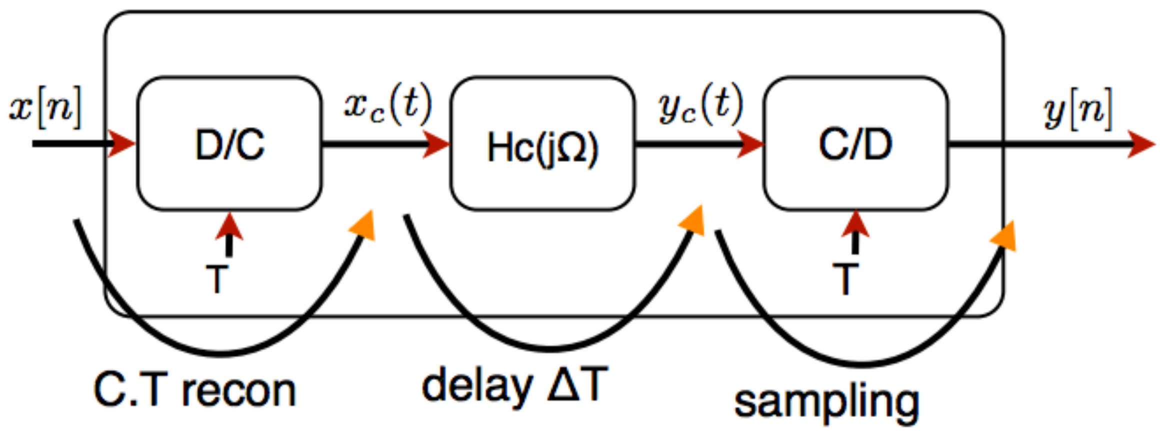

$$x_c(t) = \sum_k x[k]\,\mathrm{sinc}\left(\frac{t-kT}{T}\right) \qquad y_c(t) = x_c(t - T\Delta)$$

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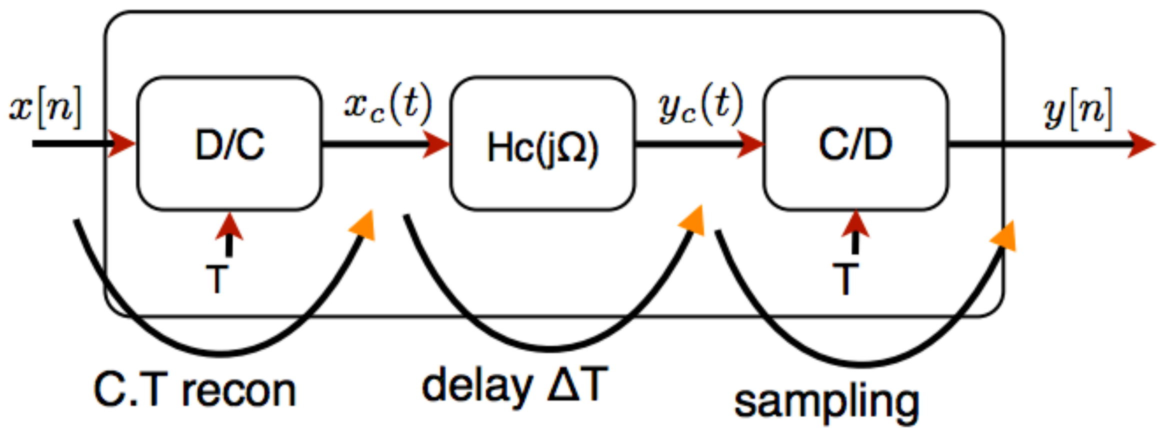

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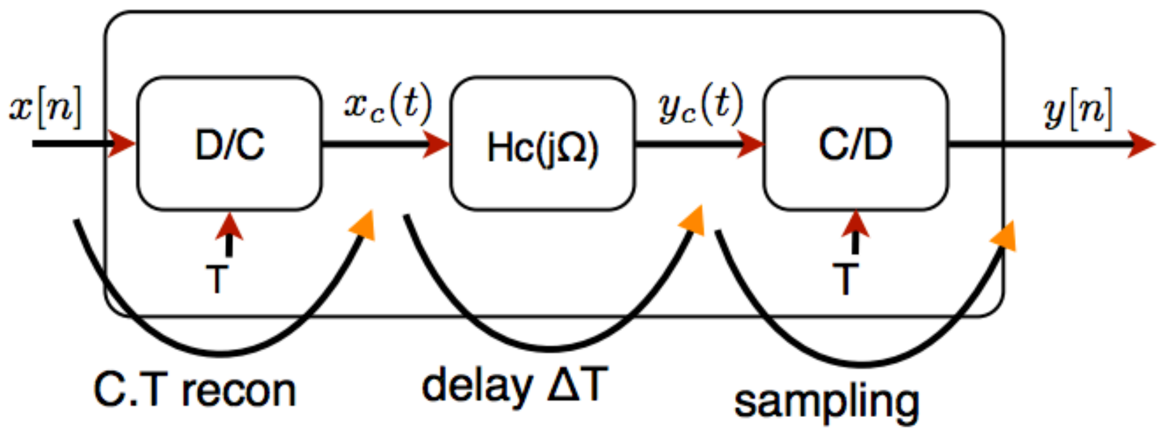

$$y[n] = y_c(nT) = x_c(nT - T \cdot \Delta)$$

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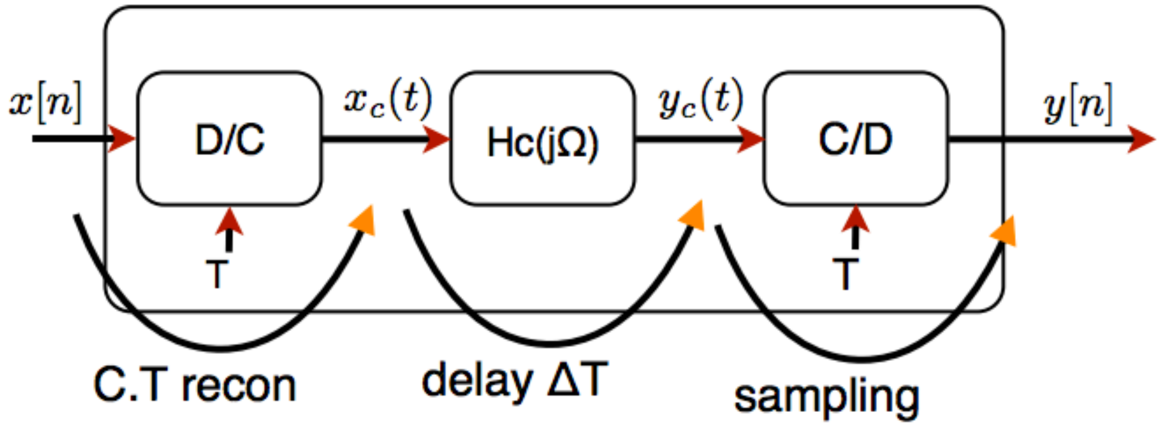

$$y[n] = y_c(nT) = x_c(nT - T\cdot\Delta) \qquad x_c(t) = \sum_k x[k]\operatorname{sinc}\left(\frac{t-kT}{T}\right)$$

$$x_c(nT - T\cdot\Delta) = \sum_k x[k]\operatorname{sinc}\left(\frac{nT - T\cdot\Delta - kT}{T}\right) = \sum_k x[k]\operatorname{sinc}\left(n - \Delta - k\right)$$

Example: Non-integer Delay

- Delay system has an impulse response of a sinc with a continuous time delay

$$y[n] = \sum_k x[k]\,\mathrm{sinc}(n - \Delta - k)$$
$$= x[n] * \mathrm{sinc}(n - \Delta)$$

$$\boxed{\Rightarrow h[n] = \mathrm{sinc}(n - \Delta)}$$

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$$\boxed{\begin{aligned} \delta[n] &\leftrightarrow 1 \\ \delta[n-n_d] &\leftrightarrow e^{-j\omega n_d} \end{aligned}}$$

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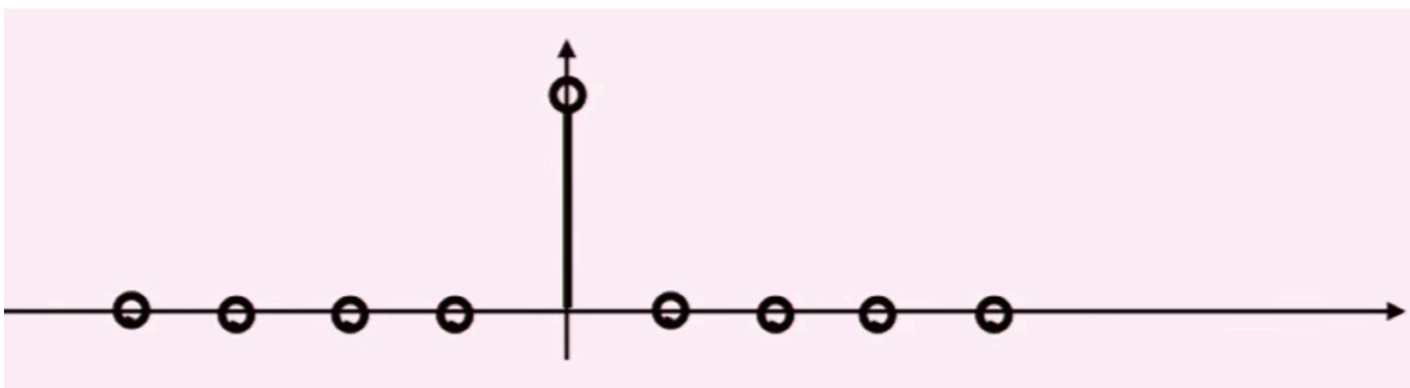

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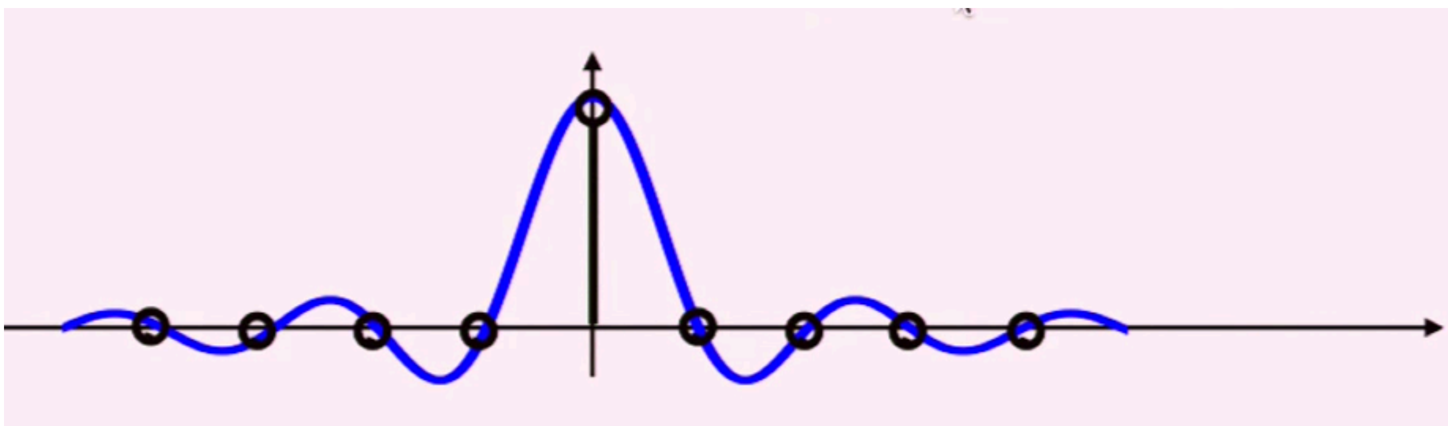

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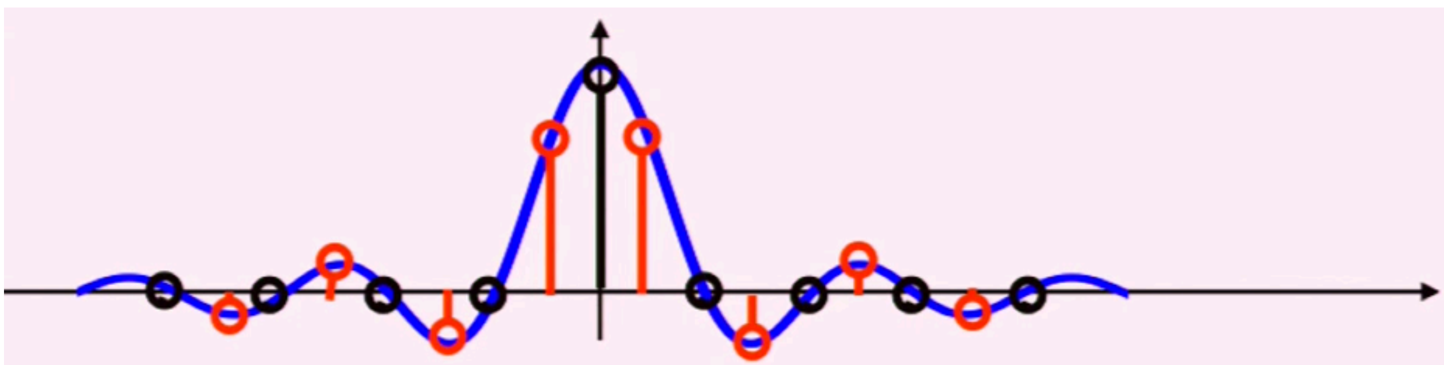

Downsampling

Downsampling

- Definition: Reducing the sampling rate by an integer number (M>1)

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$$

$$x[n] = x_c(nT)$$

$$x_d[n] = x[nM] = x_c(n\underbrace{MT}_{T'})$$

Downsampling

- Similar to C/D conversion
  - Need to worry about aliasing
  - Use anti-aliasing filter to mitigate effects

Downsampling

- Similar to C/D conversion
  - Need to worry about aliasing
  - Use anti-aliasing filter to mitigate effects
- If your discrete time signal is finely sampled (i.e oversampled) almost like a CT signal
  - Downsampling is just like sampling (C/D conversion)

Downsampling

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$$x[n] = x_c(nT)$$

$$x_d[n] = x[nM] = x_c(n\underbrace{MT}_{T'})$$

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

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- Want to relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$ not $X_c(j\Omega)$

- Separate sum into two sums—fine sum and coarse sum (i.e like counting minutes within hours)

Downsampling

The DTFT:

$$X(e^{j\omega}) = \frac{1}{T}\sum_k X_c\left(j\left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s}k\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

- k=rM+i
  - i = 0, 1, …, M-1
  - r = -∞, ..., ∞

Downsampling

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_k X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}k\right)\right)$$

$$= \frac{1}{M}\sum_{i=0}^{M-1} \underbrace{\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi}{MT}i - \frac{2\pi}{T}r\right)\right)}_{X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}i\right)}\right)}$$

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$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M} - \frac{2\pi}{M}i\right)}\right)$$

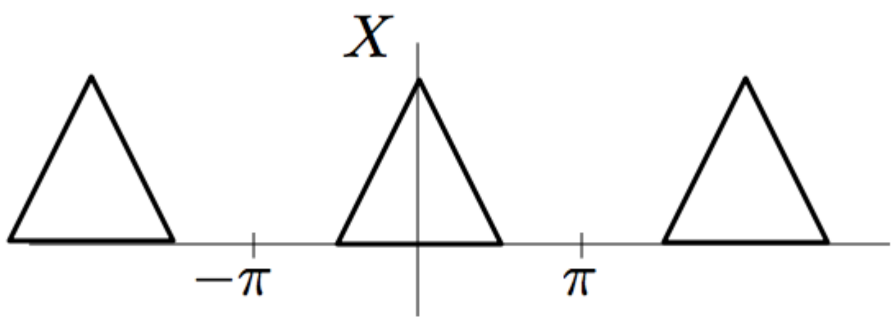
scale by 1/M

stretch by M

replicate

Example: M=2

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



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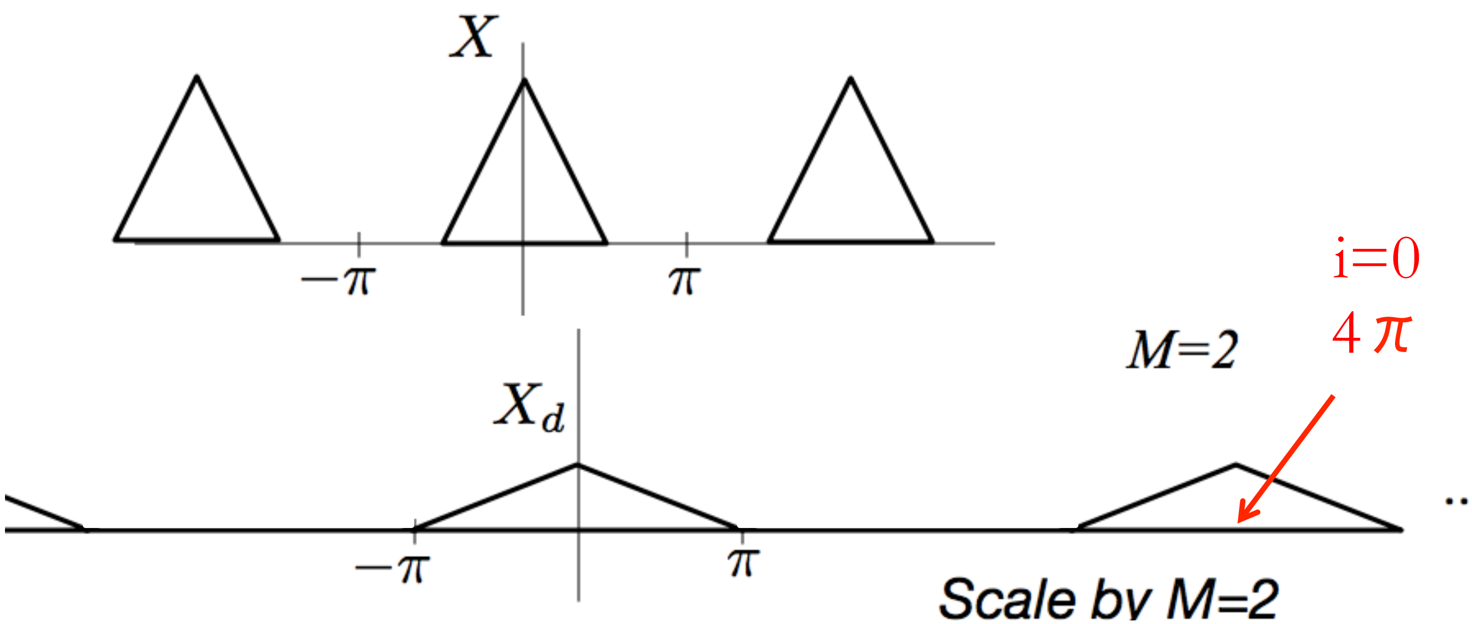

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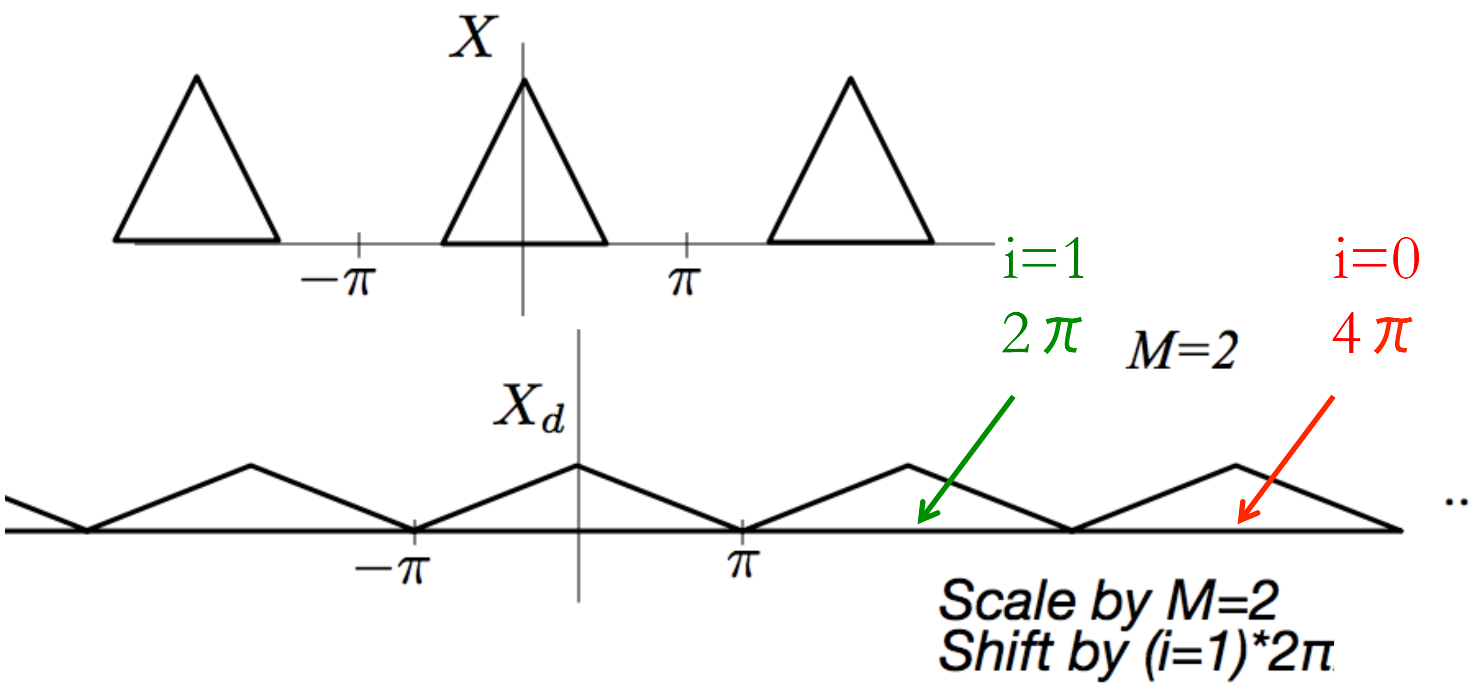

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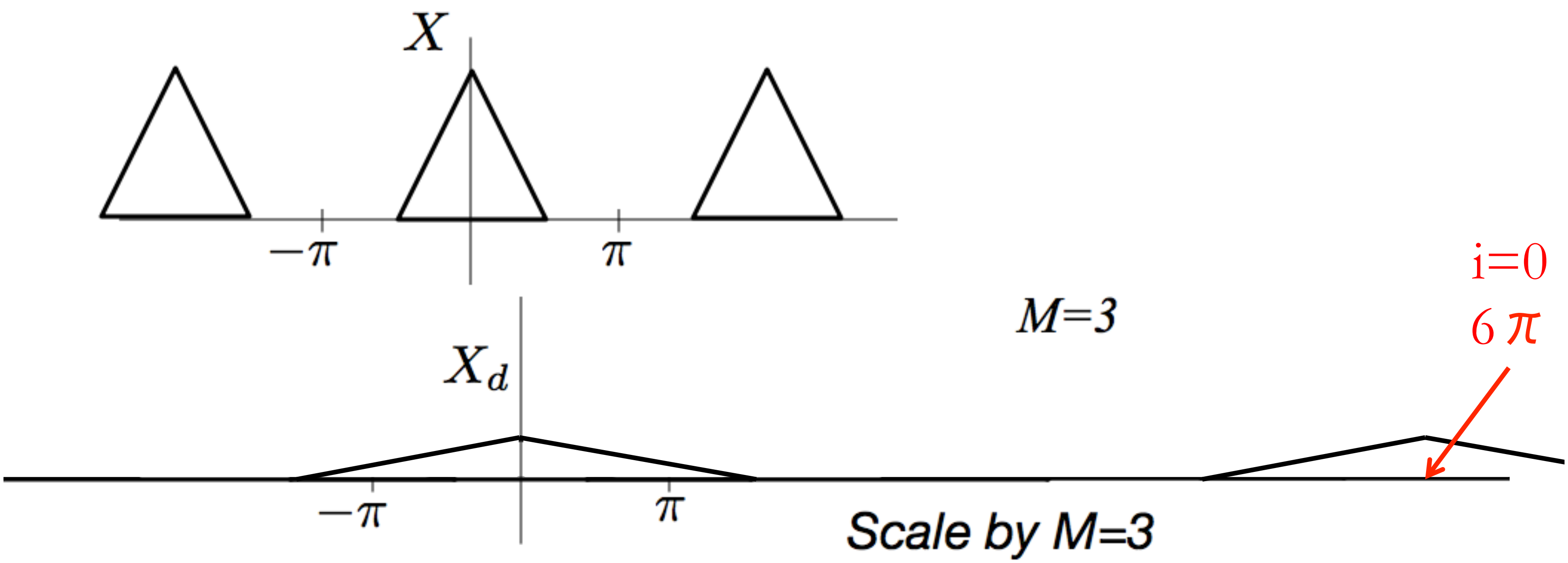

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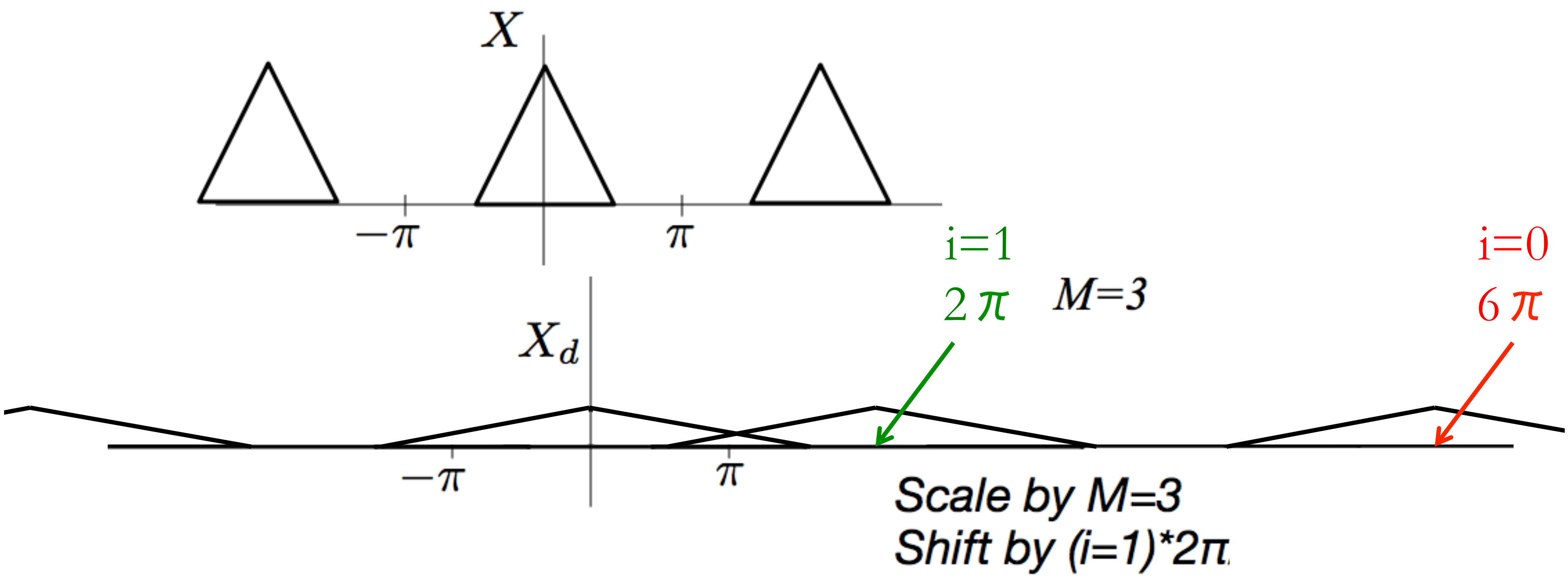

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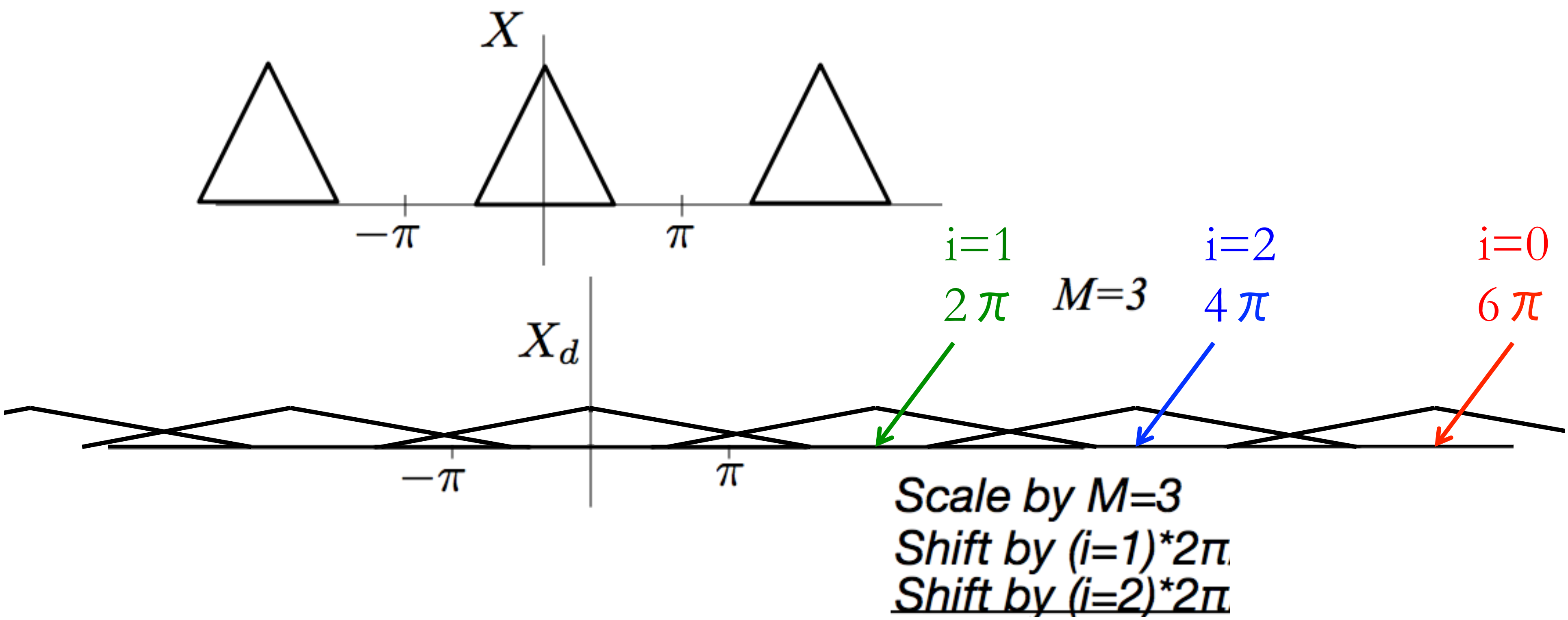

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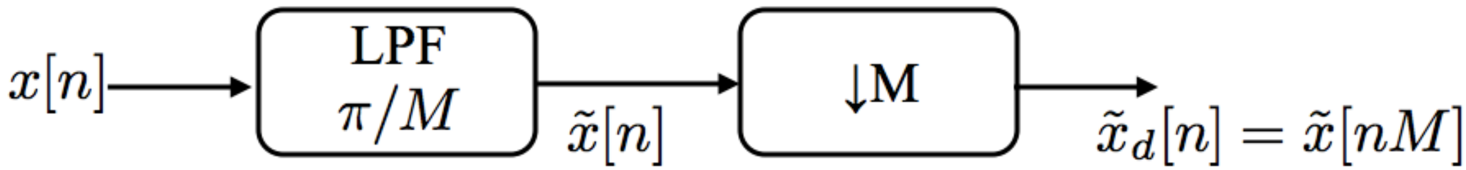

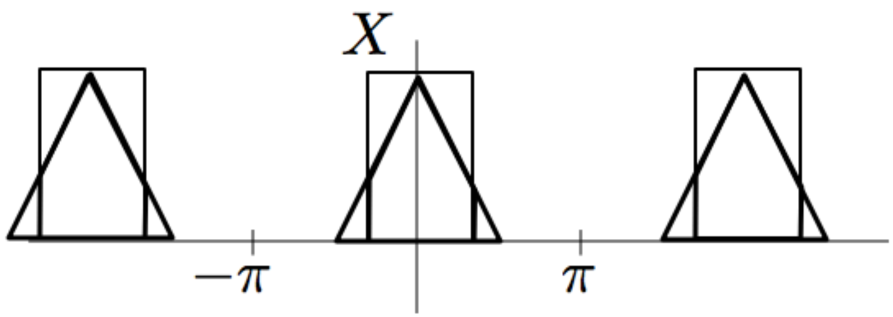

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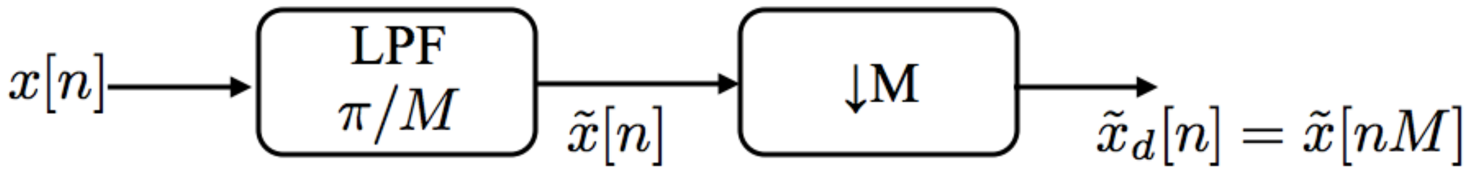

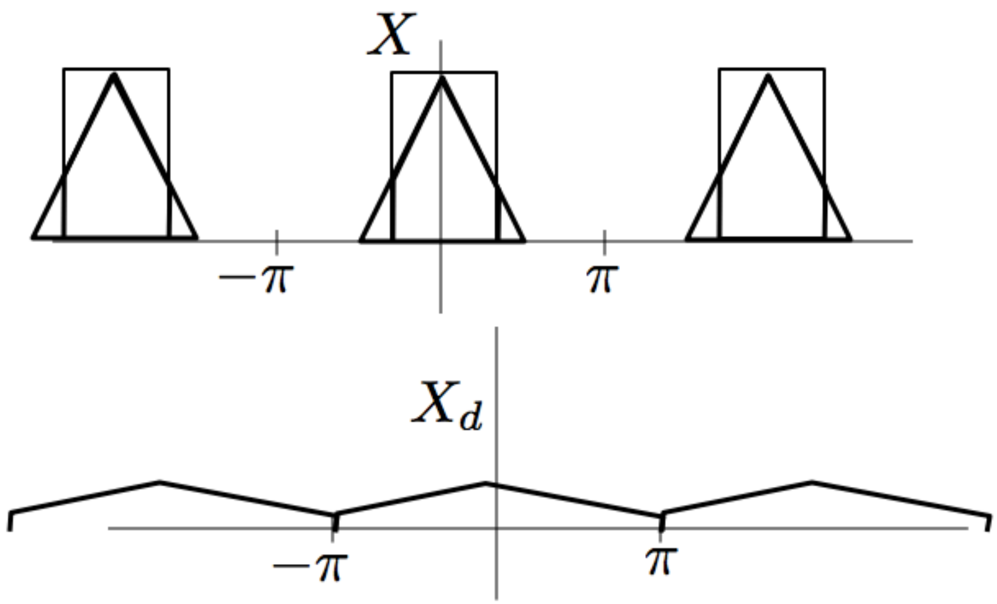

M=3

Upsampling

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

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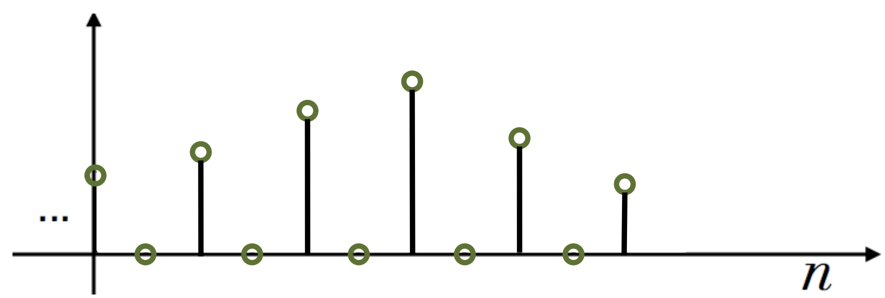
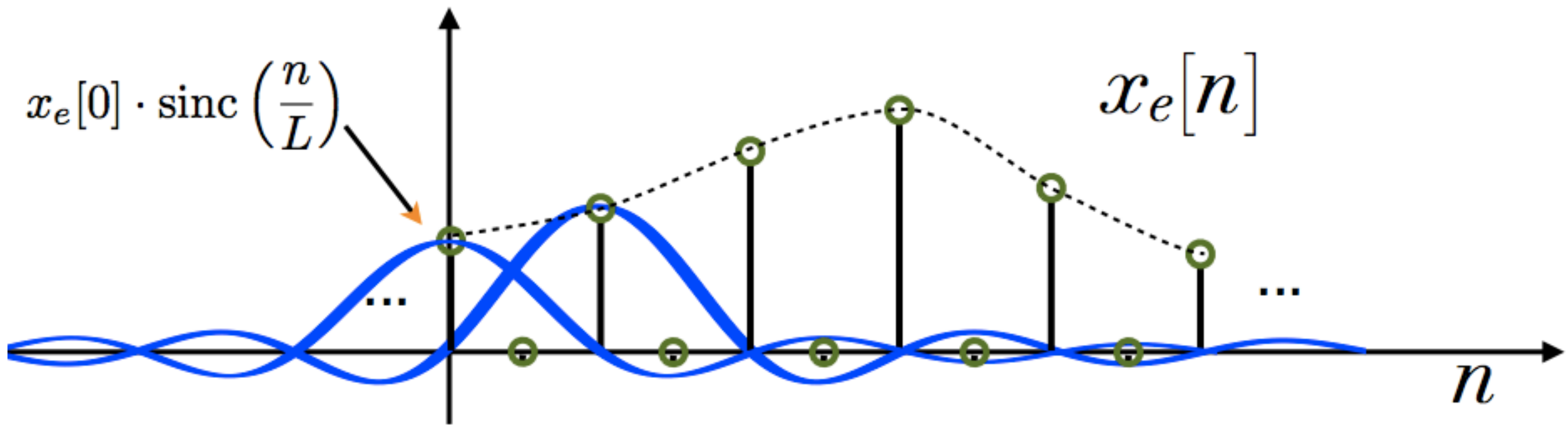
Obtain $x_i[n]$ from $x[n]$ in two steps:

(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

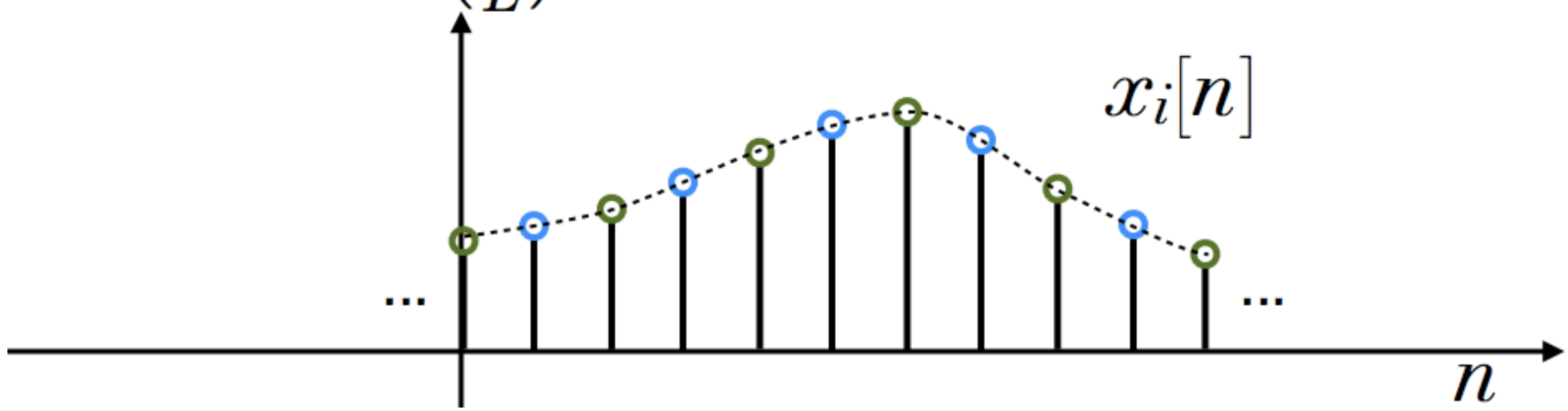


Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$$x_i[n] = x_e[n] * \text{sinc}\left(\frac{n}{L}\right)$$



Upsampling

- Much like D/C converter
- Upsample by A LOT → almost continuous

- Intuition:
  - Recall our D/C model: $x[n] \rightarrow x_s(t) \rightarrow x_c(t)$
  - Approximate "$x_s(t)$" by placing zeros between samples
  - Convolve with a sinc to obtain "$x_c(t)$"

Upsampling

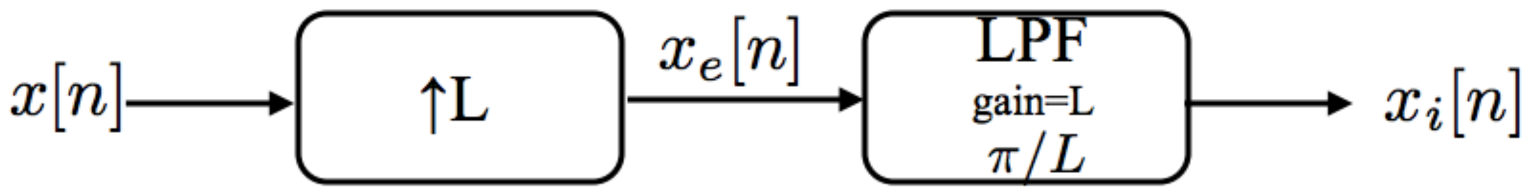
$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

$$x_i[n] = x_e[n] * \mathrm{sinc}(n/L)$$

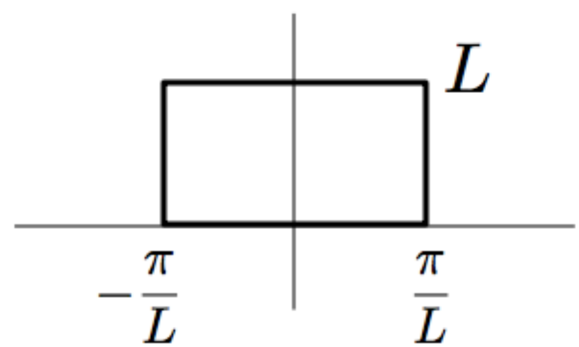
$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\mathrm{sinc}(\frac{n-kL}{L})$$

Frequency Domain Interpretation

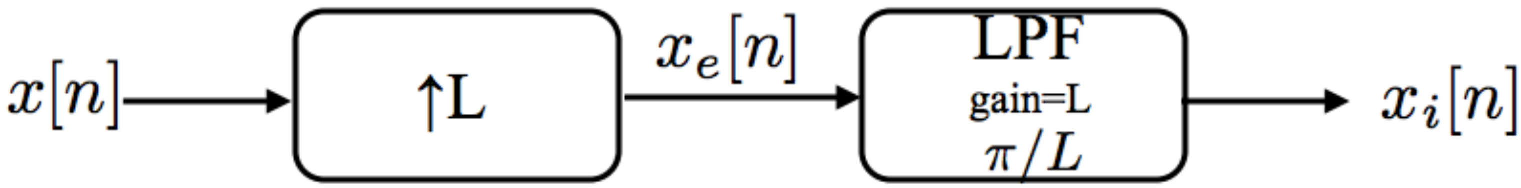
$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$



$\text{sinc}(n/L)$ **DTFT ⇒**

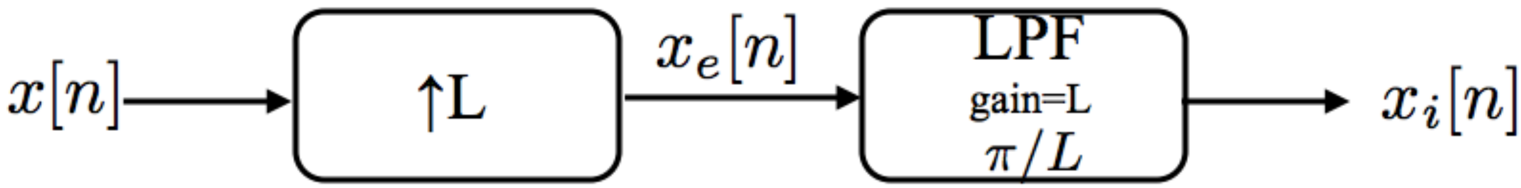


Frequency Domain Interpretation



$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for n=mL} \\ \text{(integer m)}}} e^{-j\omega n}$$

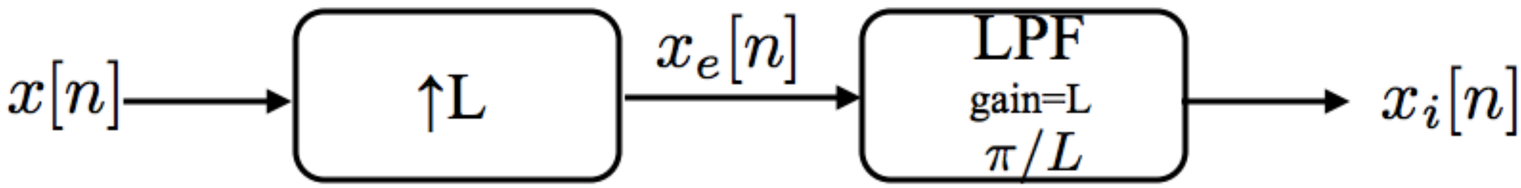
Frequency Domain Interpretation



$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for n=mL} \\ \text{(integer m)}}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]} e^{-j\omega mL}$$

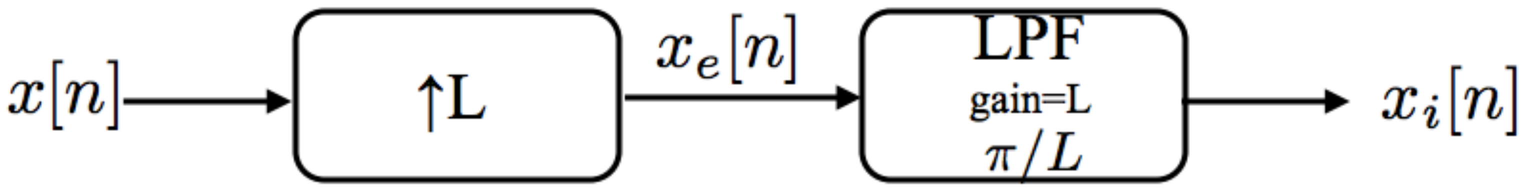
Frequency Domain Interpretation



$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL}$$

Frequency Domain Interpretation

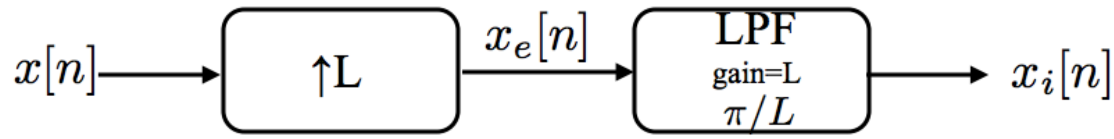


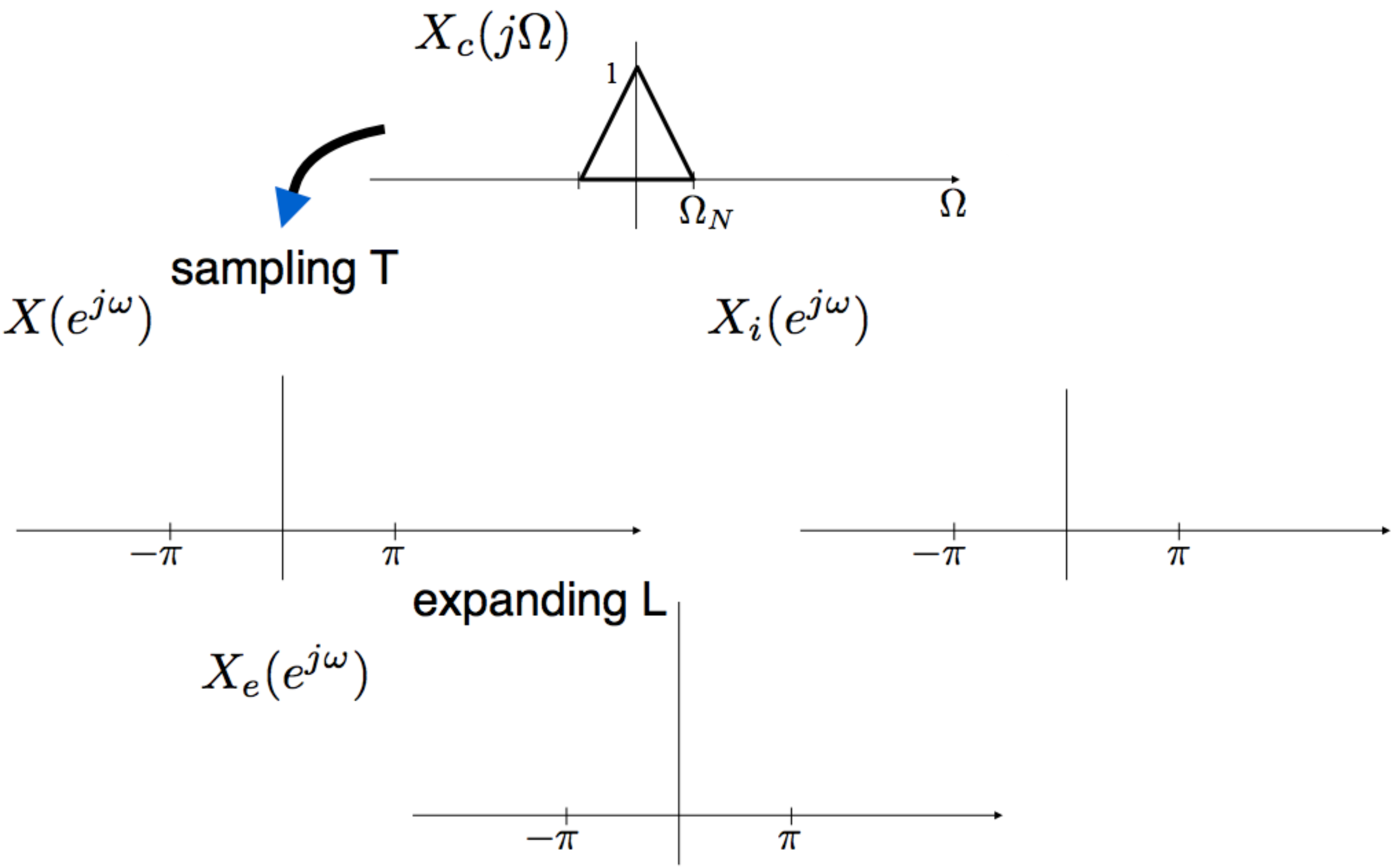
$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\neq 0 \text{ only for n=mL (integer m)}} e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} \quad = X(e^{j\omega L})$$

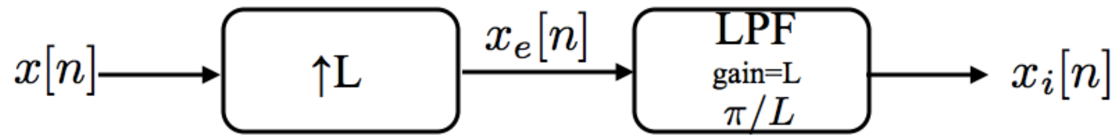
Shrink DTFT by a factor of L!

Example

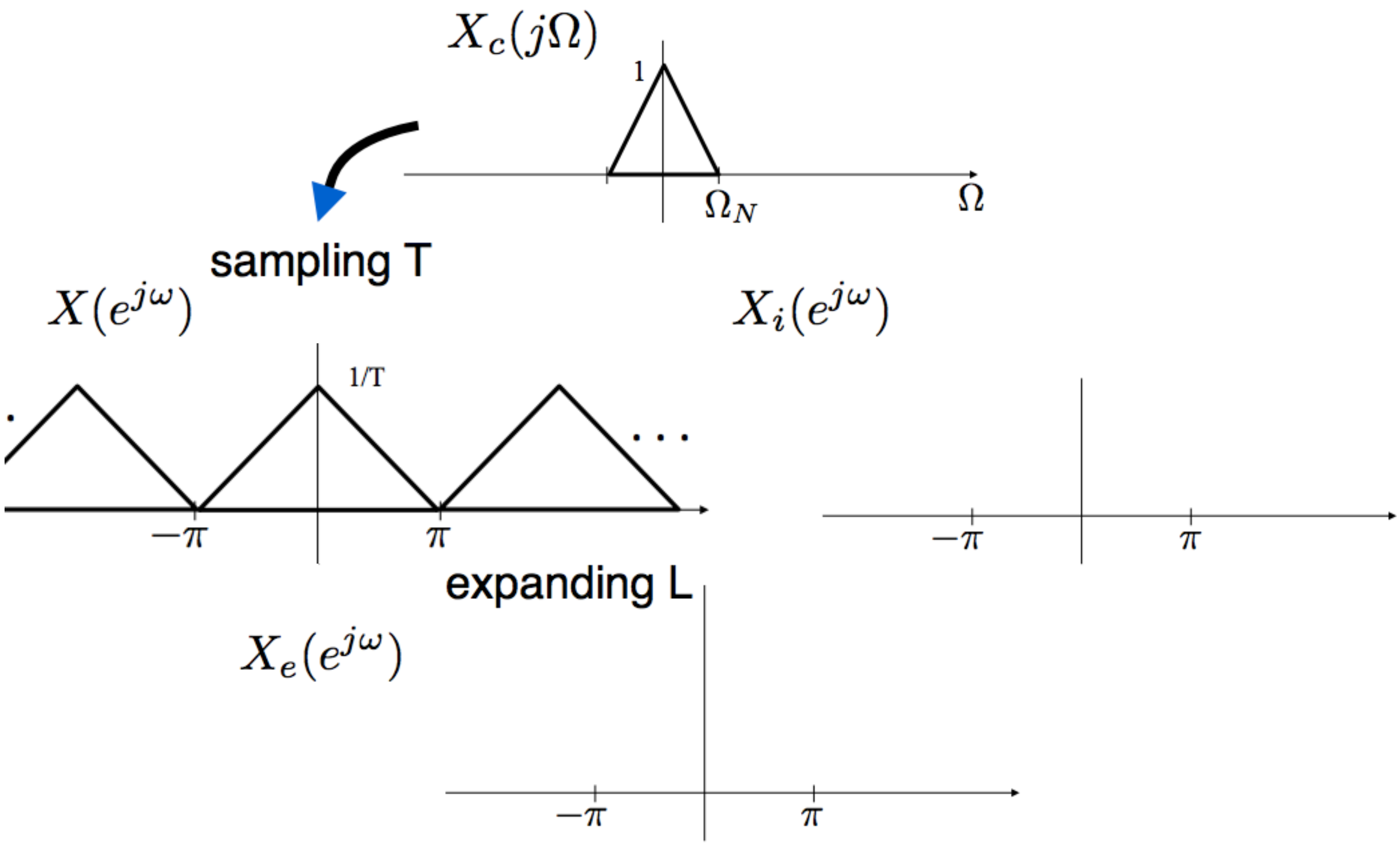

$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

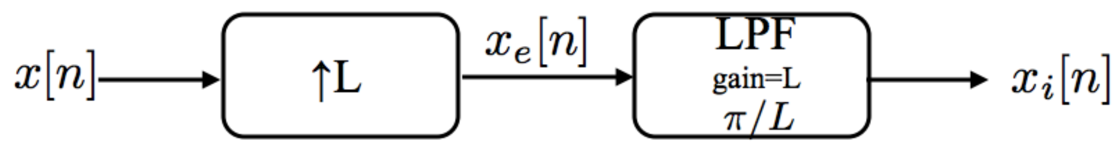


$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$

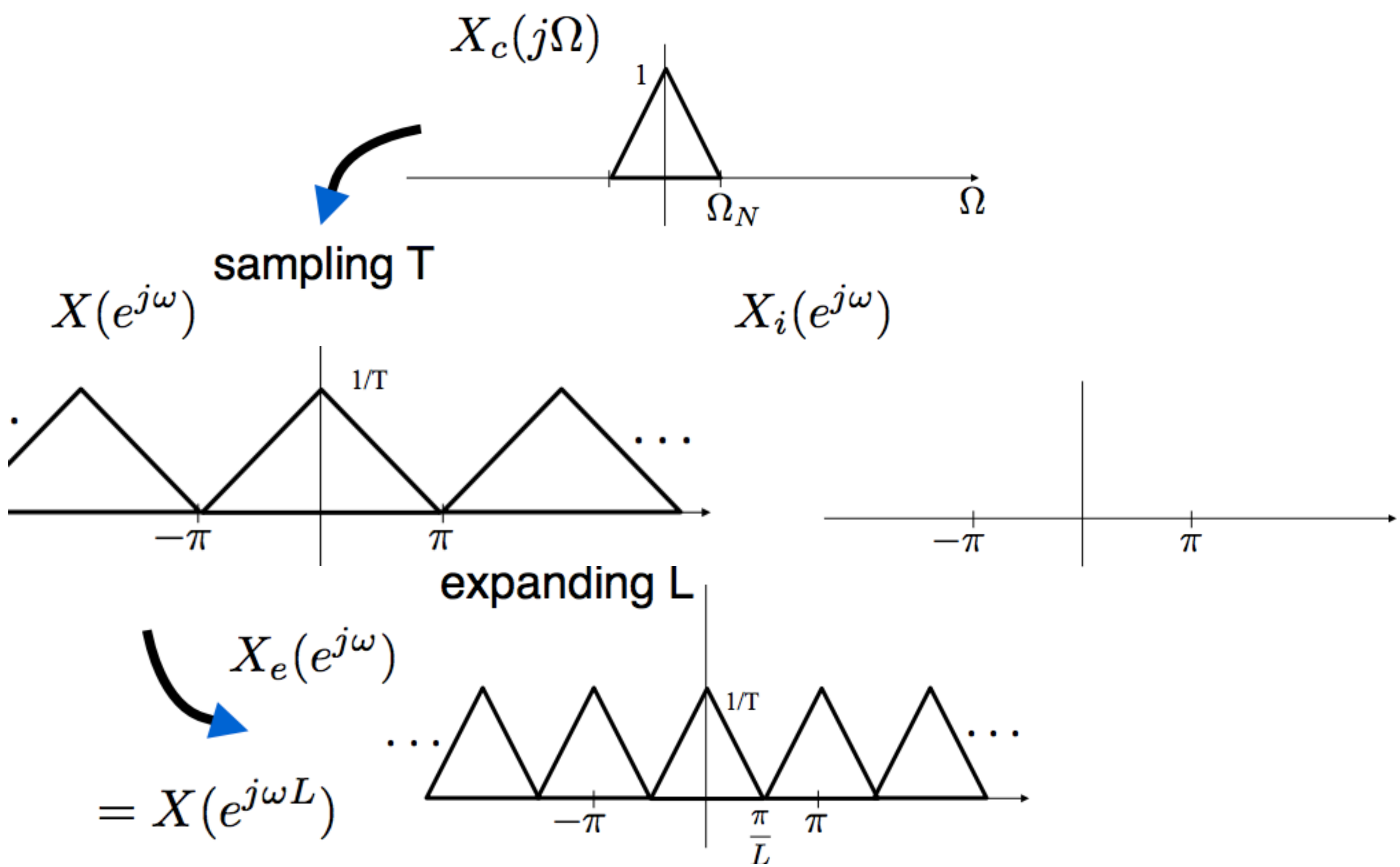


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$-\pi$
π

Example

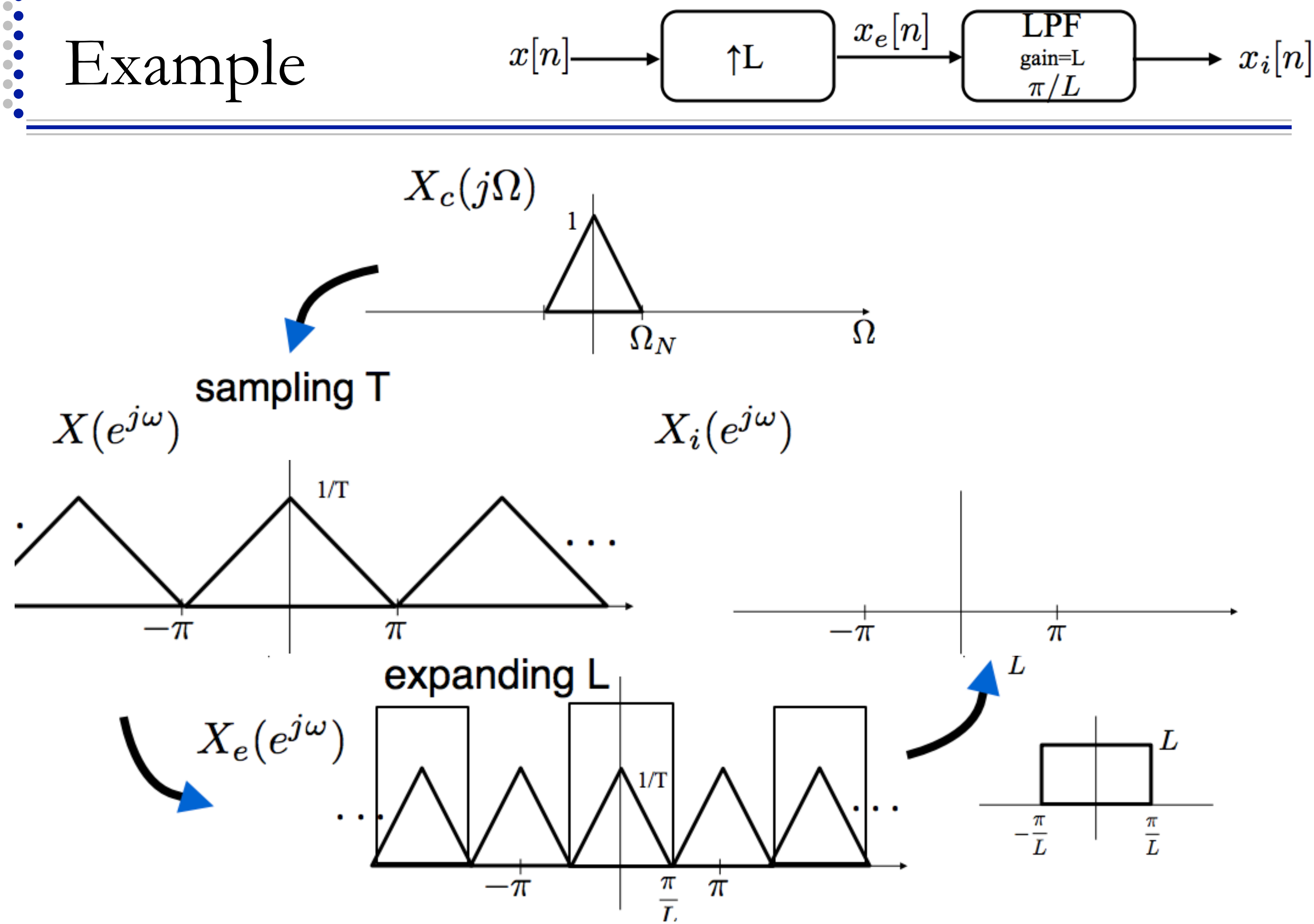


$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$



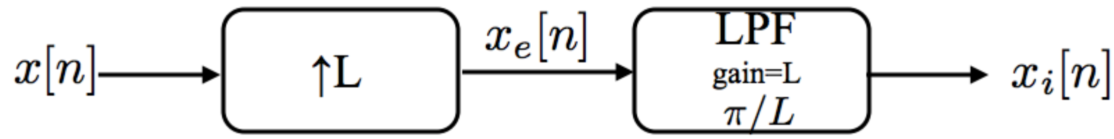
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
$= X(e^{j\omega L})$
1/T
$-\pi$
$\frac{\pi}{L}$
π

Example

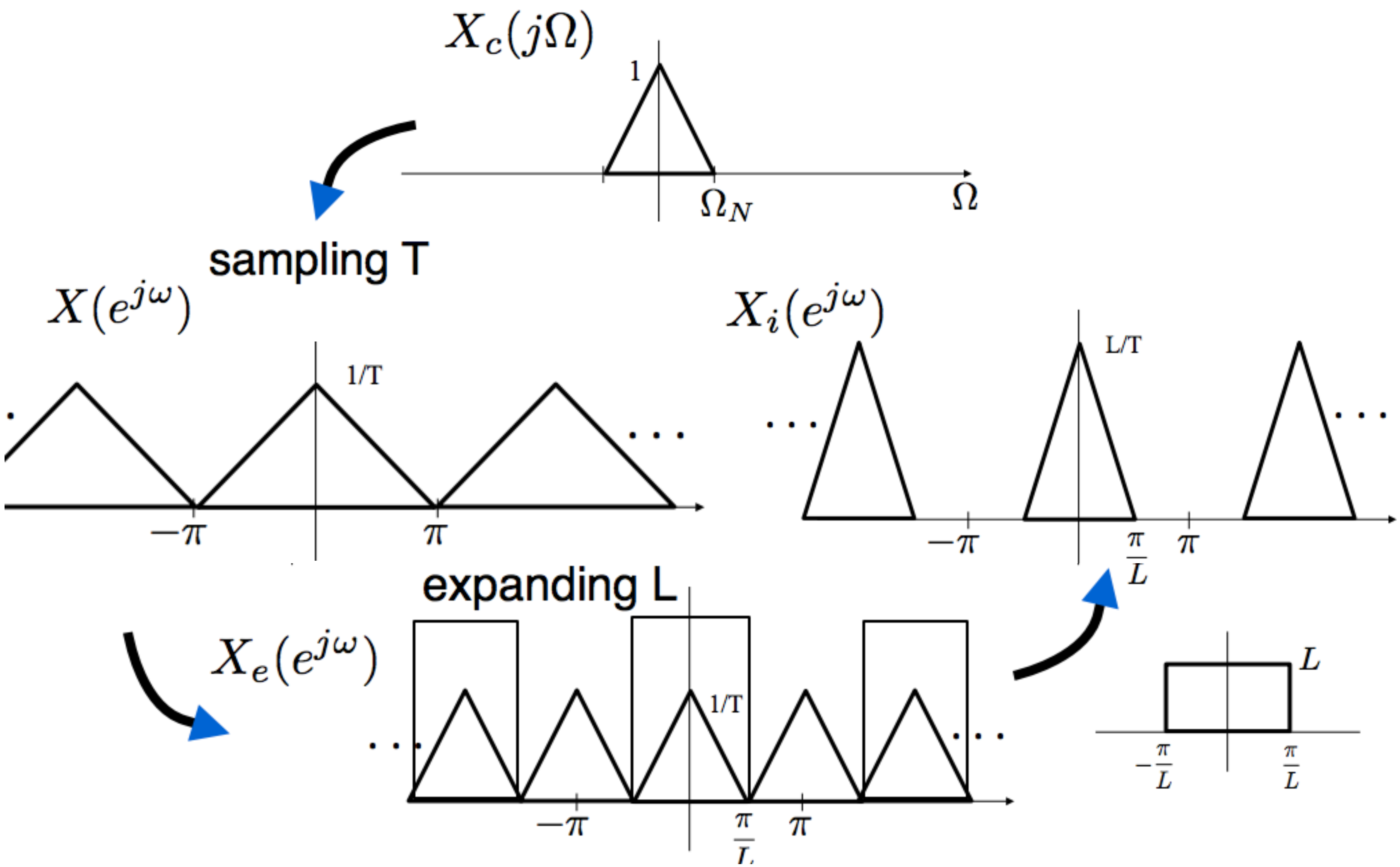


$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
$-\pi$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Example

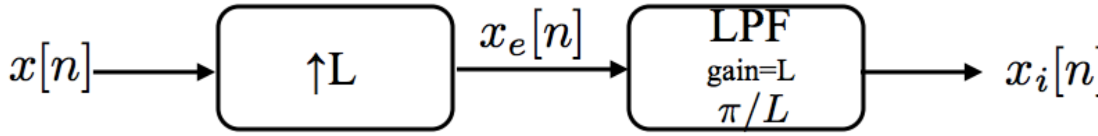


$x[n]$
$\uparrow L$
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$

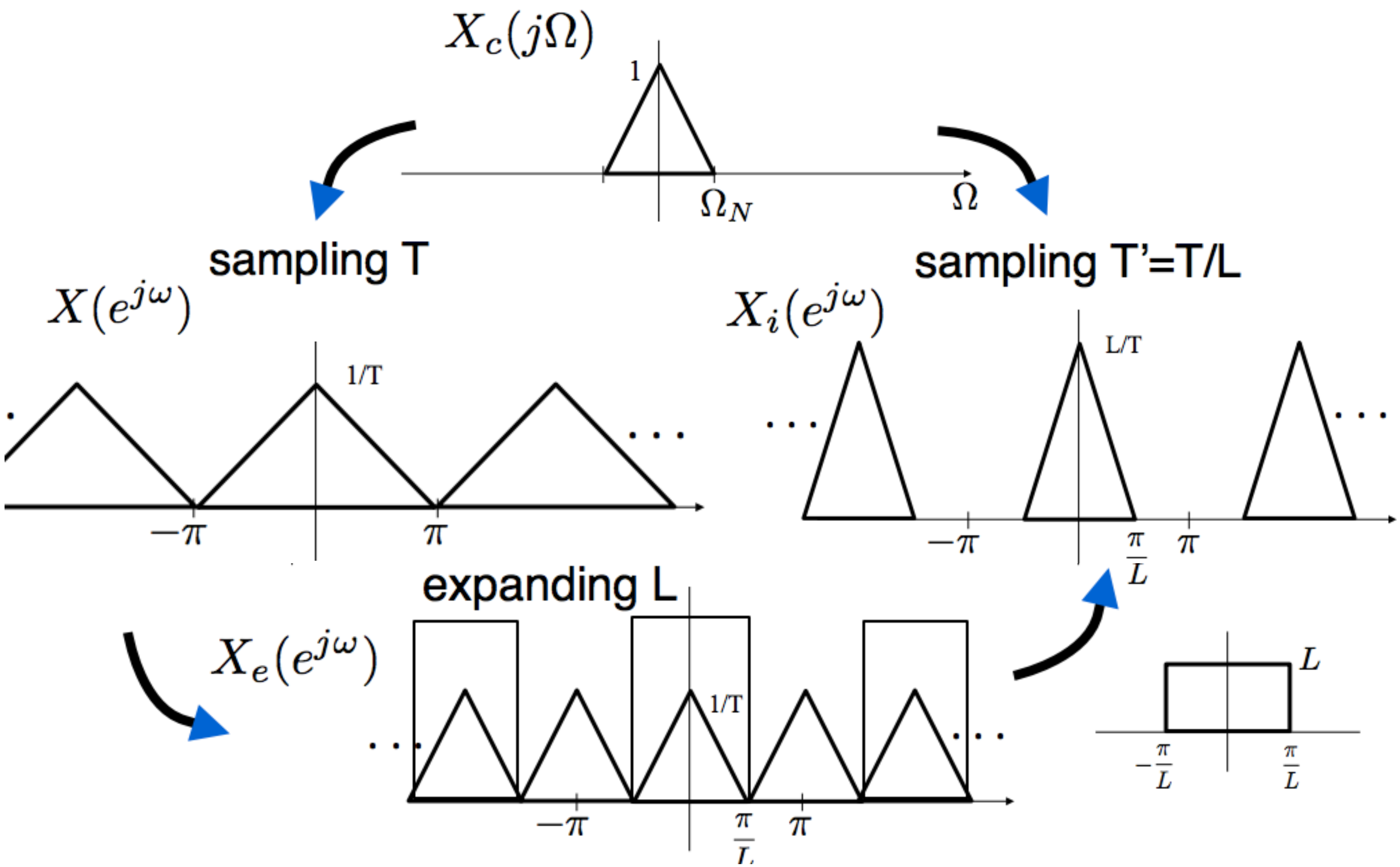


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Example



$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$

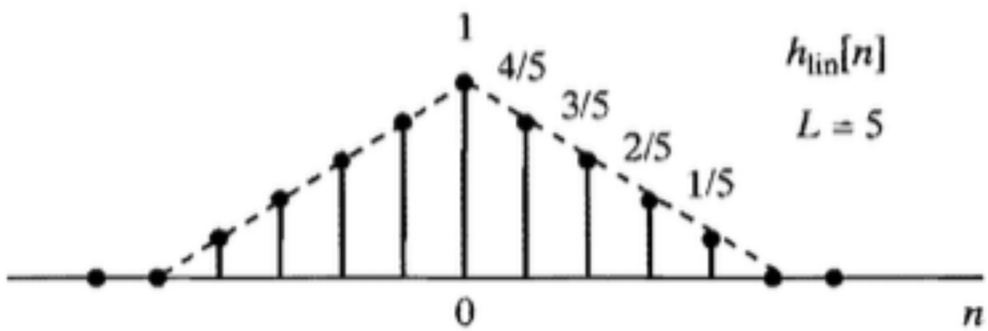


$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
$X(e^{j\omega})$
1/T
$-\pi$
π
sampling T'=T/L
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Practical Interpolation

- Interpolate with simple, practical filters
  - Linear interpolation – samples between original samples fall on a straight line connecting the samples
    - Convolve with triangle instead of sinc

$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise,} \end{cases}$$



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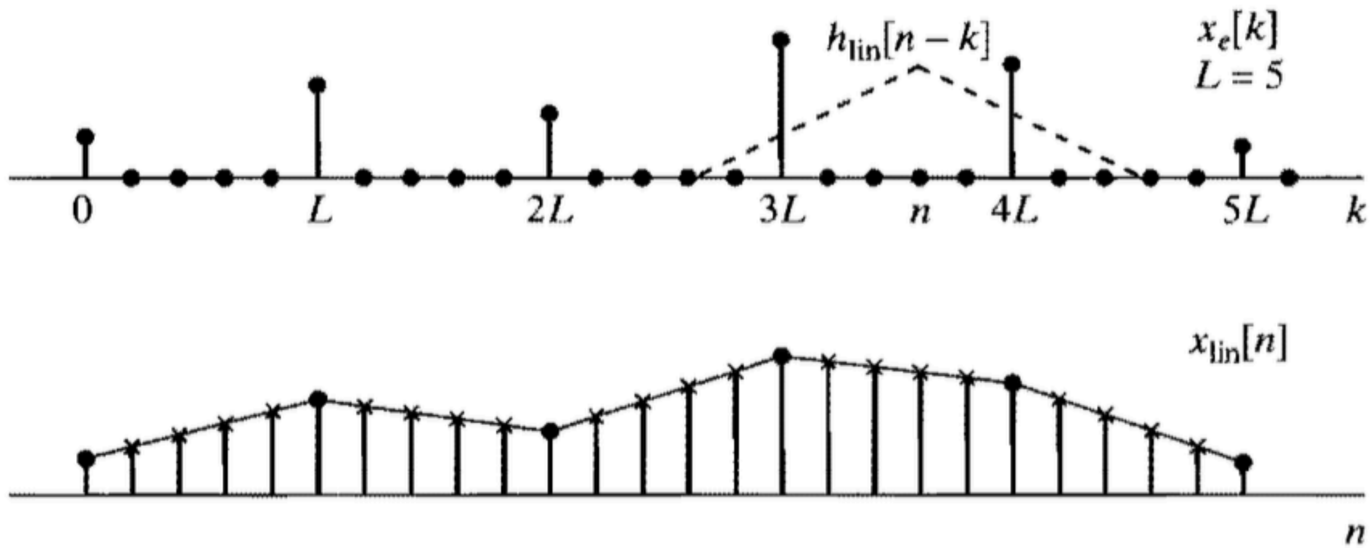
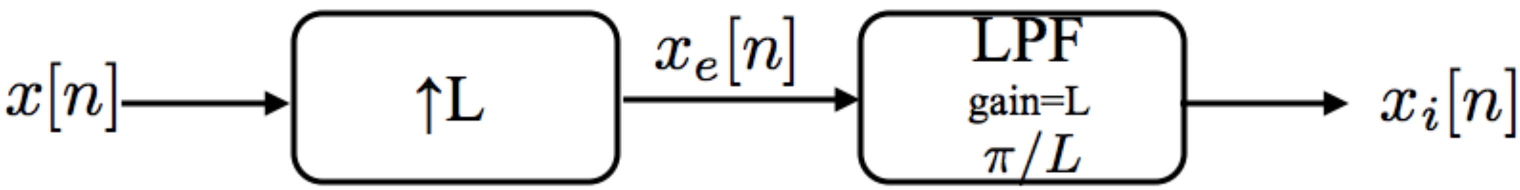
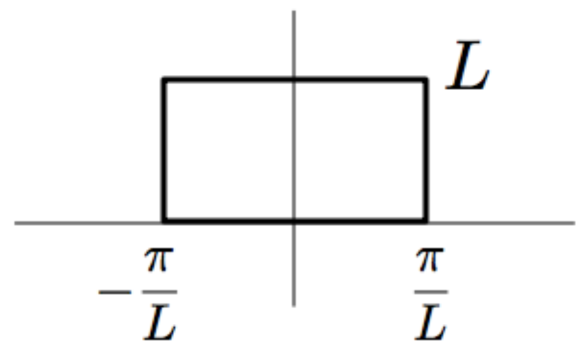


Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

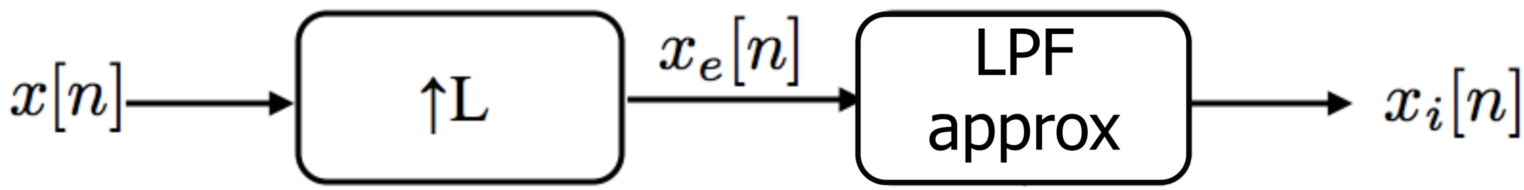


$\text{sinc}(n/L)$ **DTFT ⇒**

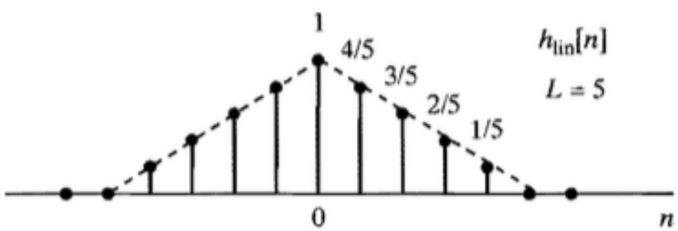


Linear Interpolation -- Frequency Domain

$$x_i[n] = x_e[n] * h_{lin}[n]$$



$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \le L, \\ 0, & \text{otherwise}, \end{cases}$$



Linear Interpolation -- Frequency Domain

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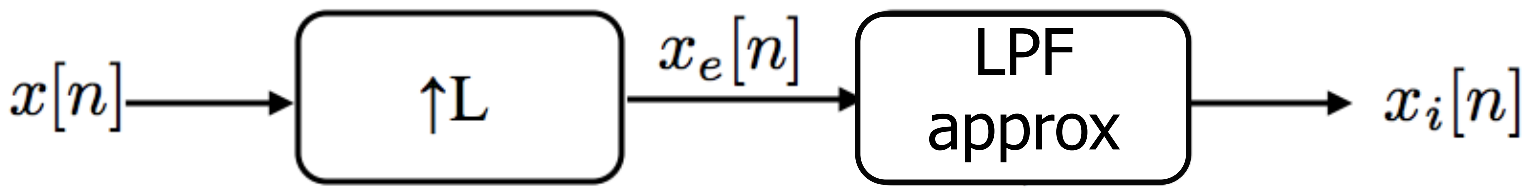

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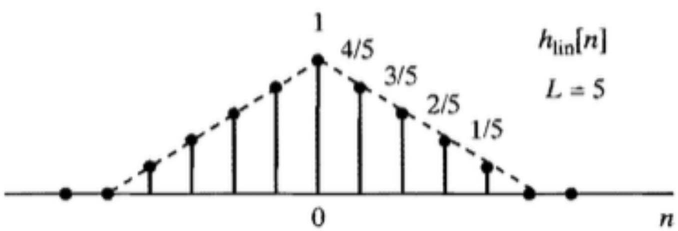
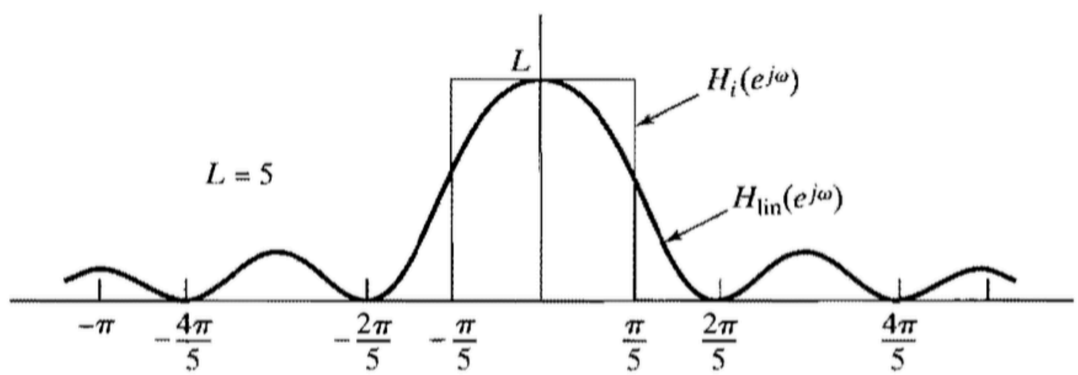


DTFT ⇒



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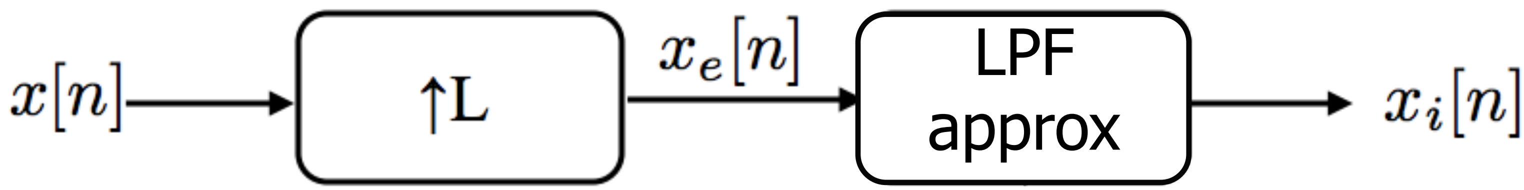

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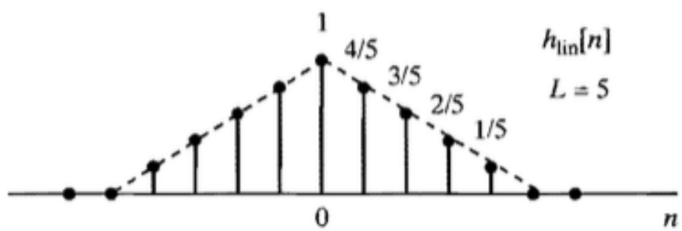
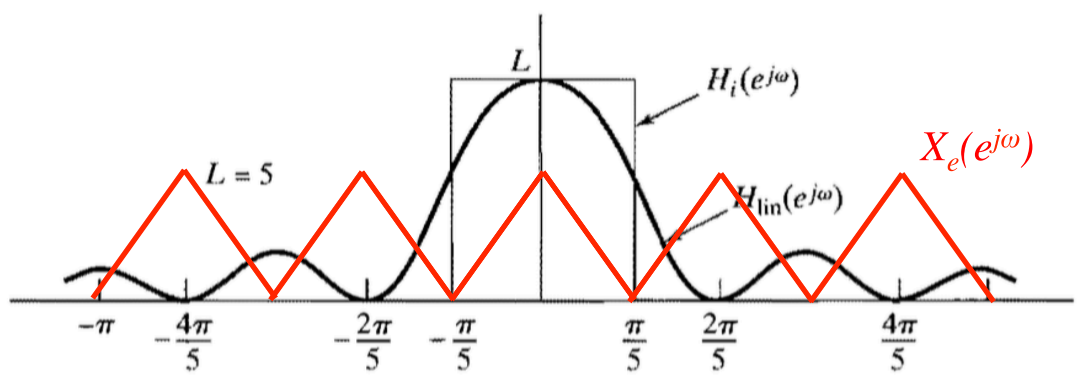


DTFT ⇒



Linear Interpolation -- Frequency Domain

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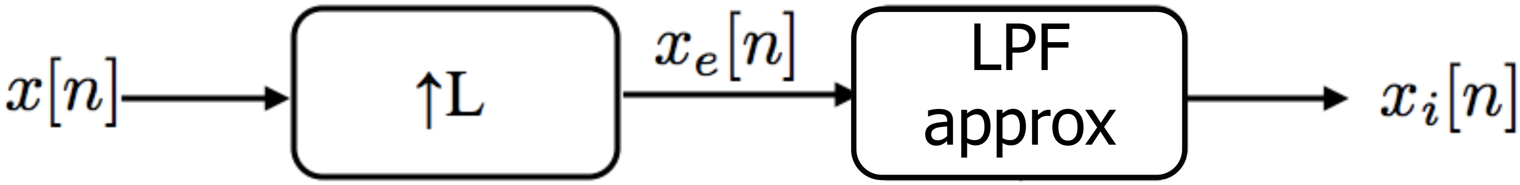

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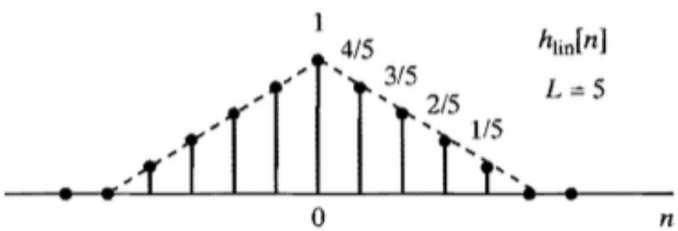
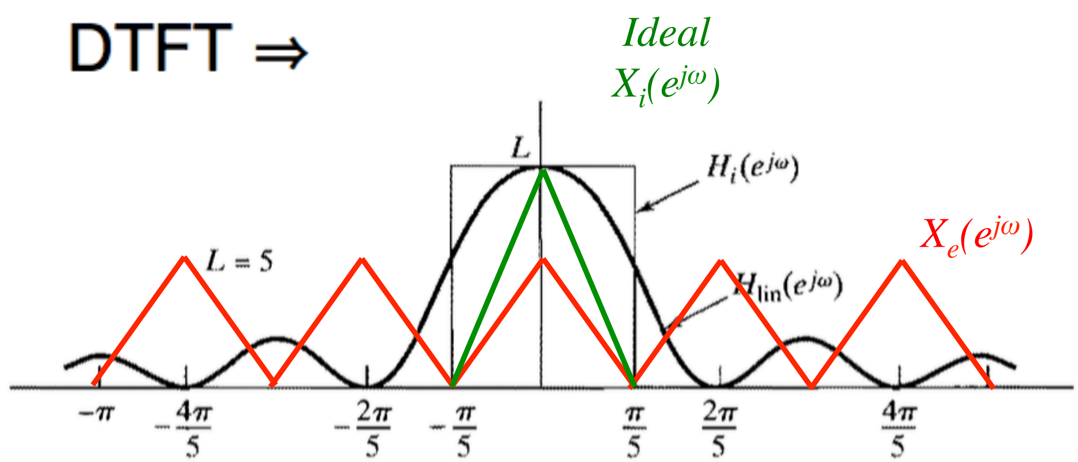


DTFT ⇒



Linear Interpolation -- Frequency Domain

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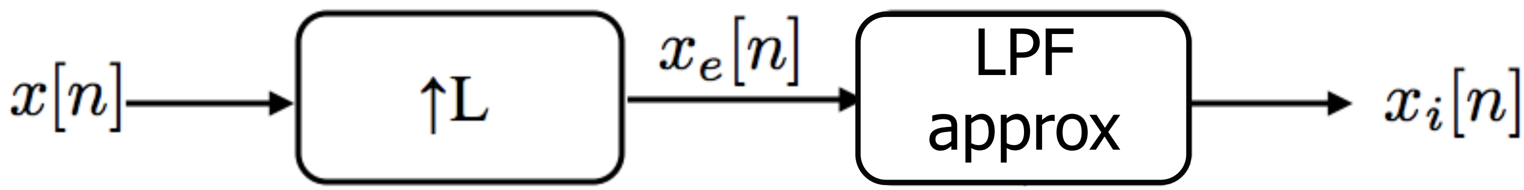

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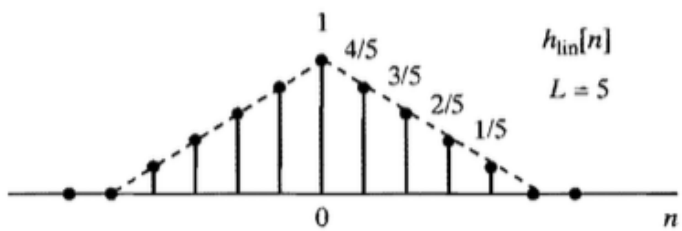
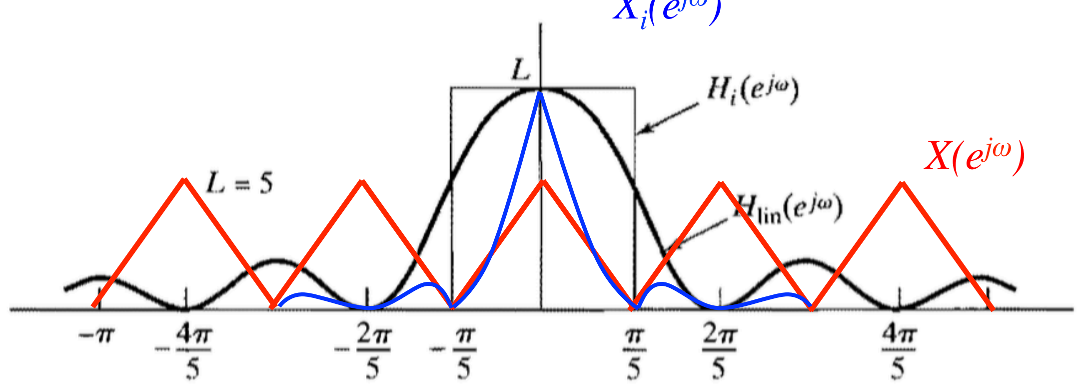


DTFT ⇒



Linear Interpolation -- Frequency Domain

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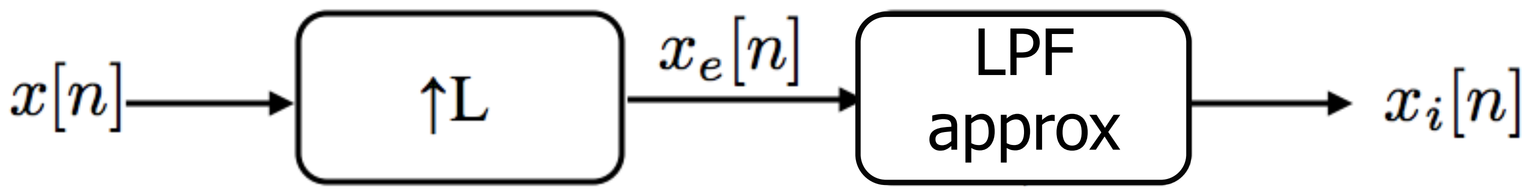

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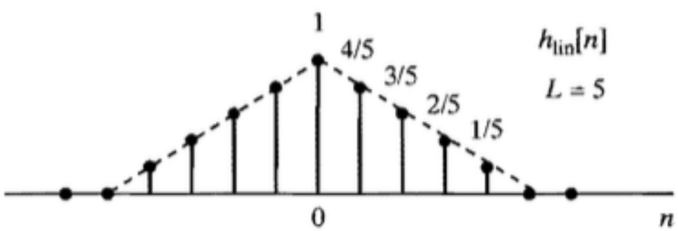
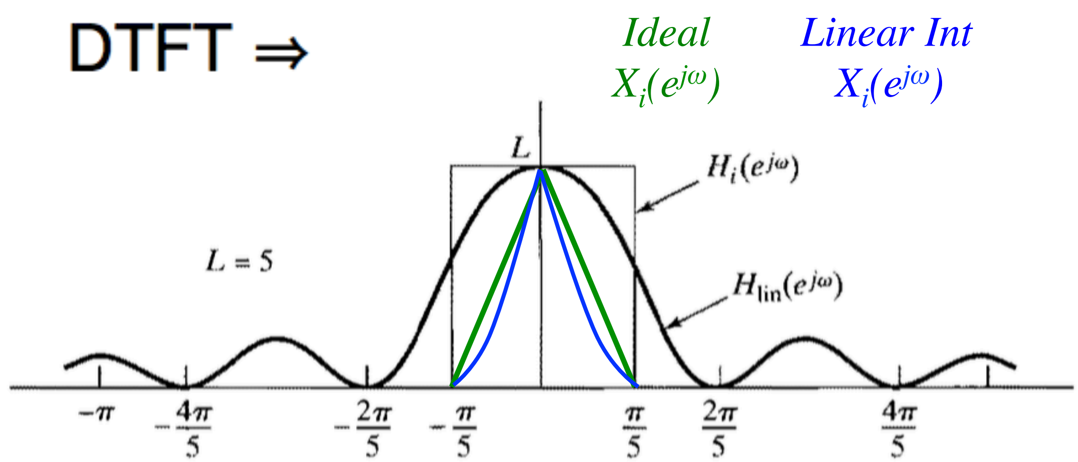


DTFT ⇒



Big Ideas

- CT processing of DT signals
  - Allows for interpretation of DT systems
- Downsampling
  - Like a C/D converter
- Upsampling
  - Like a D/C converter
- Practical Interpolation
  - Linear interpolation
  - Approximate sinc function with triangle

Admin

- HW 4 due Sunday