University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2020 Midterm Thursday, March 5

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

Name:

Grade:

Q1	
Q2	
Q3	
Q4	
Total	

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad (n_d$ an integer$)$	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

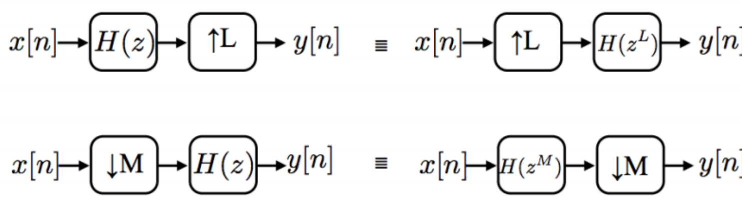
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

Interchange Identities:



1. (30 points) You are hired by a signal processing firm and you are hoping to impress them with the skills that you have acquired in this course. The firm asks you to design a discrete-time LTI system that has the property that if the input is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

the output is given by

$$y[n] = \left(\frac{1}{4}\right)^n u[n]$$

(a) Determine the transfer function, $H(z)$, and write the difference equation describing the LTI system.

(b) Draw the pole-zero diagram including the ROC indicated.

(c) Determine the impulse response, $h[n]$, of the LTI system.

(d) Is the system stable? Is the system causal? Explain your reasoning for both.

(You may continue problem 1 on this almost blank page.)

(This page intentionally left almost blank for pagination.)

2. (25 points) A numerical operation is defined as

$$y[n] = x[n] - \frac{1}{2}x[n-1]$$

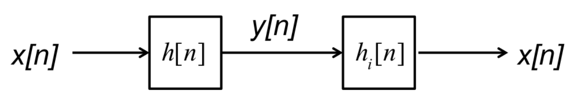
where $x[n]$ is the input and $y[n]$ is the output of the system.

(a) Show that this system is linear and time invariant.

(b) Find the impulse response, $h[n]$, of the system.

(c) Find the frequency response, $H(e^{j\omega})$, of the system.

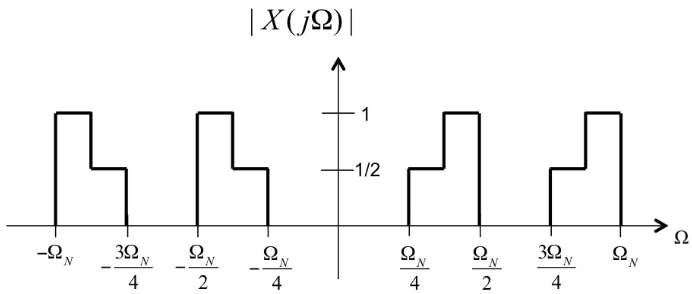
(d) Find the impulse response of a system that could be cascaded with the system to recover the input; i.e., find $h_i[n]$, in the block diagram below.

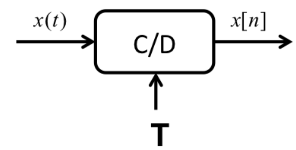


(You may continue problem 2 on this almost blank page.)

(This page intentionally left almost blank for pagination.)

3. (25 points) A continuous time signal $x(t)$ is sampled with sample period $T = \frac{2\pi}{\Omega_S}$, to produce a discrete time signal $x[n]$. The signal $x(t)$ is bandlimited, so that $X(j\Omega) = 0$ for all $|\Omega| > \Omega_N$. This process is described in the following figure as is the continuous-time Fourier transform of the signal $x(t)$. Assume the sampling frequency $\Omega_S = \Omega_N$.





(a) Sketch the resulting discrete-time Fourier transform $X(e^{j\omega})$ for $-\pi \le \omega \le \pi$. Make sure to label your axes and all relevant frequencies with fractional multiples of π.

(b) Does this sampling rate $\Omega_S = \Omega_N$ meet the Nyquist sampling theorem criterion that guarantees no aliasing? Explain your reasoning.

(c) Given $x[n]$, is it possible to perfectly recover $x(t)$? If so, design a system with only DT processing and a single D/C block (specify the period, T, for reconstruction) that does this. Carefully sketch the system and describe all operations that must occur in your system. You can utilize any ideal filters by sketching them, but label all axis points to fully specify them. If it is not possible to perfectly recover $x(t)$ from $x[n]$, carefully explain why this is the case.

4. (20 points) Figure 1 below shows a system in which two continuous-time signals are multiplied and a discrete-time signal is then obtained from the product by sampling the product at the Nyquist rate; i.e., $y_1[n]$ is samples of $y_c(t)$ taken at the Nyquist rate. The signal $x_1(t)$ is bandlimited to 25 kHz ($X_1(j\Omega) = 0$ for $|\Omega| \geq 5\pi \times 10^4$), and $x_2(t)$ is bandlimited to 2.5 kHz ($X_2(j\Omega) = 0$ for $|\Omega| \geq 0.5\pi \times 10^4$).

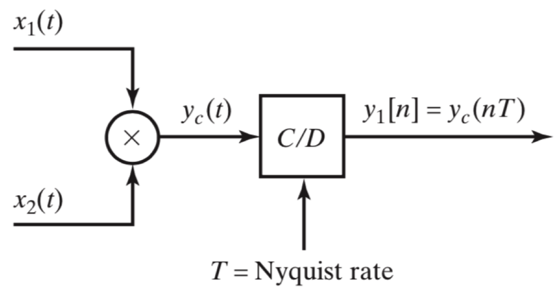


Figure 1

(a) Find the Nyquist sampling period T.

In some situations (digital transmission, for example), the continuous-time signals have already been sampled at their individual Nyquist rates, and the multiplication is to be carried out in the discrete-time domain, perhaps with some additional processing before and after multiplication, as indicated in Figure 2 below. Each of the systems A and B either is an identity or can be implemented using one or more of the modules shown in Figure 3.

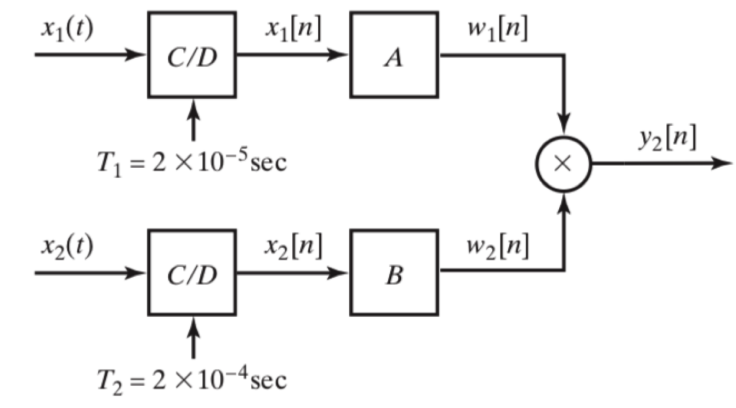


Figure 2

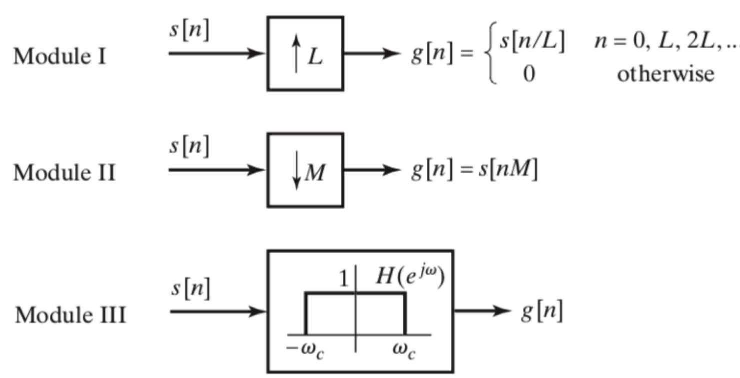


Figure 3

(b) For each of the systems A and B, either specify that the system is an identity system or specify an appropriate interconnection of one or more of the modules shown in Figure 3. Also, specify all relevant parameters L, M, and ω_c. The systems A and B should be constructed such that $y_2[n]$ is proportional to $y_1[n]$, i.e., $y_2[n] = ky_1[n] = ky_c(nT) = kx_1(nT) \times x_2(nT)$, and these samples are at the Nyquist rate ($y_2[n]$ does not represent oversampling or undersampling of $y_c(t)$).

(You may continue problem 4 on this almost blank page.)

(This page intentionally left almost blank for pagination.)