University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

ESE531, Spring 2021 | Midterm | Thursday, March 5

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed.
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 79, Stdev: 12

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad (n_d \text{ an integer})$	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

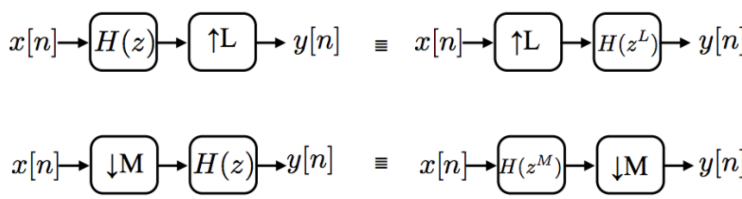
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

Interchange Identities:



1. (30 points) You are hired by a signal processing firm and you are hoping to impress them with the skills that you have acquired in this course. The firm asks you to design a discrete-time LTI system that has the property that if the input is given by

$$x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

the output is given by

$$y[n] = \left(\frac{1}{4}\right)^n u[n]$$

(a) Determine the transfer function, $H(z)$, and write the difference equation describing the LTI system.

(b) Draw the pole-zero diagram including the ROC indicated.

(c) Determine the impulse response, $h[n]$, of the LTI system.

(d) Is the system stable? Is the system causal? Explain your reasoning for both.

(a)

$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{1}{1-\frac{1}{4}z^{-1}} \div \left(\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right) \\
&= \frac{1-\frac{1}{3}z^{-1}}{12z^{-1}} \\
&= 12z - 4
\end{aligned}$$

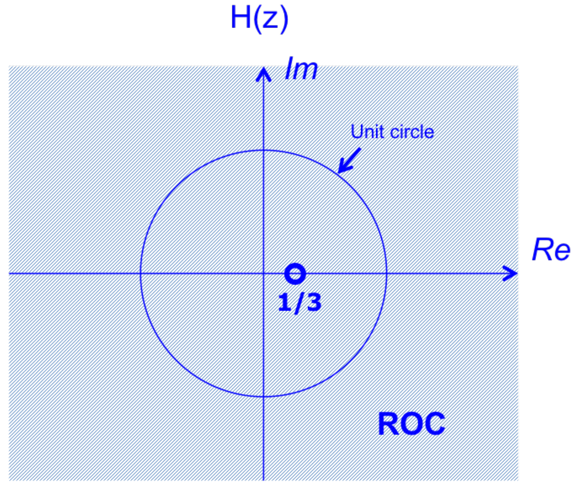
We can then derive the difference equation from:

$$\begin{aligned}
\frac{Y(z)}{X(z)} &= 12z - 4 \\
Y(z) &= 12zX(z) - 4X(z)
\end{aligned}$$

Take the inverse z-transform of both sides

$$y[n] = 12x[n+1] - 4x[n]$$

(b) There is one zero at $z = \frac{1}{3}$ and the ROC is the entire complex plane except $z = \infty$



(c) The impulse can be derived directly from the transfer function $H(z) = 12z - 4 \rightarrow h[-1] = 12$ and $h[0] = -4$. Therefore

$$h[n] = -4\delta[n] + 12\delta[n+1]$$

(d) The system is stable because the ROC of the transfer function includes the unit circle and the impulse response is absolutely infinitely summable since it's FIR. It is not causal because the current output depends on future inputs.

2. (25 points) A numerical operation is defined as

$$y[n] = x[n] - \frac{1}{2}x[n-1]$$

where $x[n]$ is the input and $y[n]$ is the output of the system.

(a) Show that this system is linear and time invariant.

(b) Find the impulse response, $h[n]$, of the system.

(c) Find the frequency response, $H(e^{j\omega})$, of the system.

(d) Find the impulse response of a system that could be cascaded with the system to recover the input; i.e., find $h_i[n]$, in the block diagram below.

$$x[n] \longrightarrow \boxed{h[n]} \xrightarrow{y[n]} \boxed{h_i[n]} \longrightarrow x[n]$$

(a) For linearity create a signal $x'[n] = \alpha x_1[n] + \beta x_2[n]$ where $y_1[n]$ and $y_2[n]$ are the outputs of $x_1[n]$ and $x_2[n]$ respectively.

$$\begin{aligned} y'[n] &= x'[n] - \frac{1}{2}x'[n-1] \\ &= \alpha x_1[n] + \beta x_2[n] - \frac{1}{2}(\alpha x_1[n-1] + \beta x_2[n-1]) \\ &= \alpha(x_1[n] - \frac{1}{2}x_1[n-1]) + \beta(x_2[n] - \frac{1}{2}x_2[n-1]) \\ &= \alpha y_1[n] + \beta y_2[n] \checkmark \end{aligned}$$

For time invariance create a signal $x'[n] = x[n-q]$ and show that $y'[n] = y[n-q]$

$$\begin{aligned} y'[n] &= x'[n] - \frac{1}{2}x'[n-1] \\ &= x[n-q] - \frac{1}{2}x[n-q-1] \\ &= y[n-q] \checkmark \end{aligned}$$

(b) The impulse response is simple the output when the input is $\delta[n]$. Therefore

$$h[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

(c) Using the fourier transform pairs and properties tables

$$H(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}$$

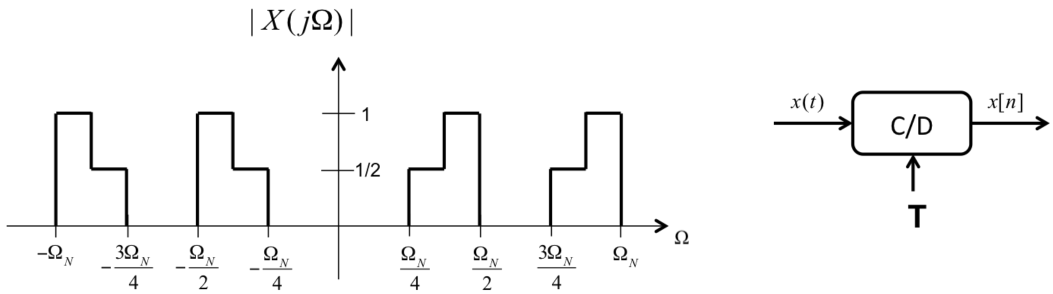
(d) We must find $h_i[n]$ such that $h_i[n] * h[n] = \delta[n]$. Taking the fourier transform of both sides

$$H_i(e^{j\omega}) \times H(e^{j\omega}) = 1$$
$$H_i(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

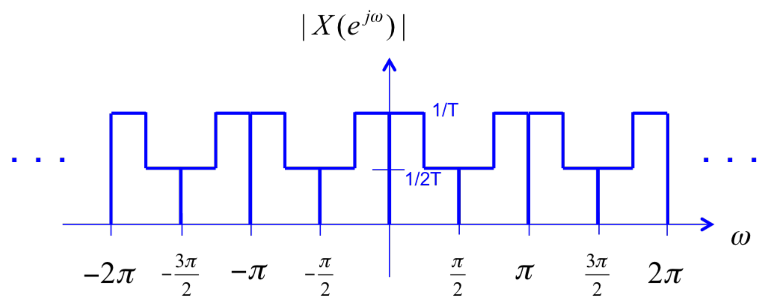
Taking the inverse fourier transform gives

$$h_i[n] = \left(\frac{1}{2}\right)^n u[n]$$

3. (25 points) A continuous time signal $x(t)$ is sampled with sample period $T = \frac{2\pi}{\Omega_S}$, to produce a discrete time signal $x[n]$. The signal $x(t)$ is bandlimited, so that $X(j\Omega) = 0$ for all $|\Omega| > \Omega_N$. This process is described in the following figure as is the continuous-time Fourier transform of the signal $x(t)$. Assume the sampling frequency $\Omega_S = \Omega_N$.



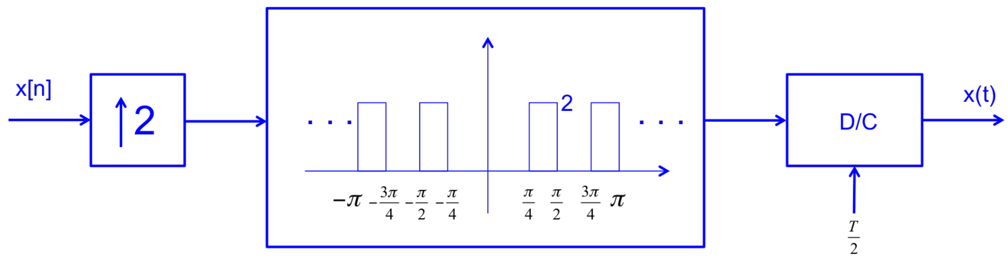
(a) Sketch the resulting discrete-time Fourier transform $X(e^{j\omega})$ for $-\pi \leq \omega \leq \pi$. Make sure to label your axes and all relevant frequencies with fractional multiples of π.



(b) Does this sampling rate $\Omega_S = \Omega_N$ meet the Nyquist sampling theorem criterion that guarantees no aliasing? Explain your reasoning.
No. Nyquist says we must sample at greater than twice the max frequency, i.e. $\Omega_s \geq 2\Omega_N$. This is sampling at half the Nyquist rate

(c) Given $x[n]$, is it possible to perfectly recover $x(t)$? If so, design a system with only DT processing and a single D/C block (specify the period, T, for reconstruction) that does this. Carefully sketch the system and describe all operations that must occur in your system. You can utilize any ideal filters by sketching them, but label all axis points to fully specify them. If it is not possible to perfectly recover $x(t)$ from $x[n]$, carefully explain why this is the case.

One possible system is below:



4. (20 points) Figure 1 below shows a system in which two continuous-time signals are multiplied and a discrete-time signal is then obtained from the product by sampling the product at the Nyquist rate; i.e., $y_1[n]$ is samples of $y_c(t)$ taken at the Nyquist rate. The signal $x_1(t)$ is bandlimited to 25 kHz ($X_1(j\Omega) = 0$ for $|\Omega| \geq 5\pi \times 10^4$), and $x_2(t)$ is bandlimited to 2.5 kHz ($X_2(j\Omega) = 0$ for $|\Omega| \geq 0.5\pi \times 10^4$).

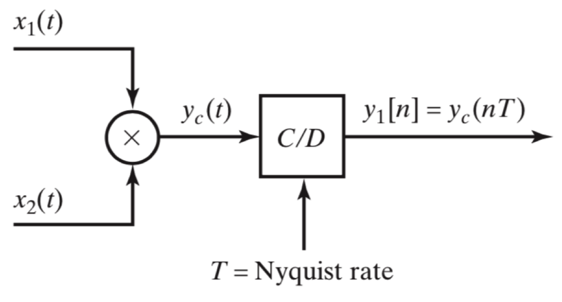


Figure 1

(a) Find the Nyquist sampling period T.

Since $y_c(t) = x_1(t)x_2(t)$, $Y_c(j\Omega) = \frac{1}{2\pi}(X_1(j\Omega) * X_2(j\Omega))$, and so $Y_c(j\Omega) = 0$ for $|\Omega| \geq 11\pi \times 10^4$. (i.e. The bandwidth of the convolution of the two signals is twice the sum of the bandwidths of the individual signals). Hence the Nyquist rate is $T = 1/55000s$ or $f_s = 55kHz$.

In some situations (digital transmission, for example), the continuous-time signals have already been sampled at their individual Nyquist rates, and the multiplication is to be carried out in the discrete-time domain, perhaps with some additional processing before and after multiplication, as indicated in Figure 2 below. Each of the systems A and B either is an identity or can be implemented using one or more of the modules shown in Figure 3.

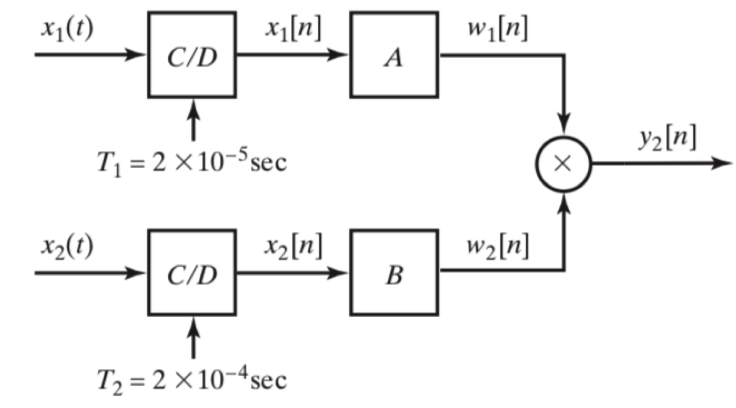


Figure 2

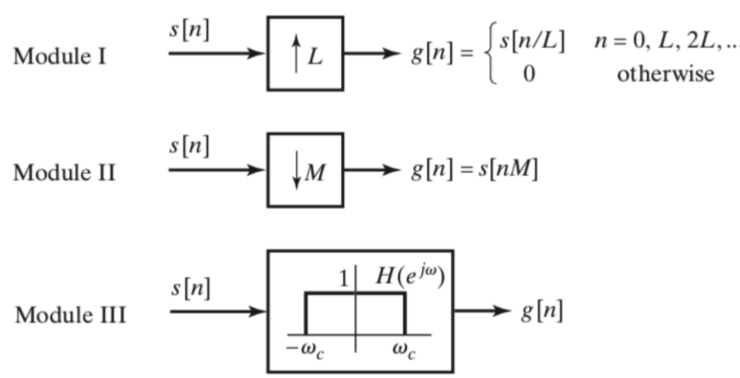
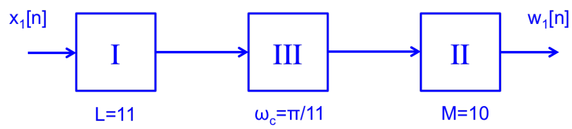


Figure 3

(b) For each of the systems A and B, either specify that the system is an identity system or specify an appropriate interconnection of one or more of the modules shown in Figure 3. Also, specify all relevant parameters L, M, and ω_c. The systems A and B should be constructed such that $y_2[n]$ is proportional to $y_1[n]$, i.e., $y_2[n] = ky_1[n] = ky_c(nT) = kx_1(nT) \times x_2(nT)$, and these samples are at the Nyquist rate ($y_2[n]$ does not represent oversampling or undersampling of $y_c(t)$).

We need to choose systems A and B such that $w_1[n] = ax_1(nT)$ and $w_2[n] = bx_2(nT)$. For system A must resample such that

$$T = T_1 \frac{M}{L} \rightarrow \frac{T}{T_1} = \frac{1/55000}{2 \times 10^{-5}} = \frac{10}{11} = \frac{M}{L}$$



For system B must resample such that

$$T = T_2 \frac{M}{L} \rightarrow \frac{T}{T_2} = \frac{1/55000}{2 \times 10^{-4}} = \frac{1}{11} = \frac{M}{L}$$