ESE 531: Digital Signal Processing

Week 9

Lecture 15: March 14th, 2021

All-Pass Systems

Lecture Outline

- All Pass Systems

Review: Frequency Response of LTI System

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

- We can define a magnitude response…

$$\left|Y\left(e^{j\omega}\right)\right|=\left|H\left(e^{j\omega}\right)\right|\left|X\left(e^{j\omega}\right)\right|$$

- a phase response…

$$\angle Y\left(e^{j\omega}\right)=\angle H\left(e^{j\omega}\right)+\angle X\left(e^{j\omega}\right)$$

- and group delay

$$\operatorname{grd}[H(e^{j\omega})]=-\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

Review: Group Delay Math

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})} \qquad H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

- Look at each factor:

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r\cos(\omega - \theta)} \right)$$

$$\mathrm{grd}[1 - re^{j\theta} e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{\left| 1 - re^{j\theta} e^{-j\omega} \right|^2}$$

Stability and Causality

LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

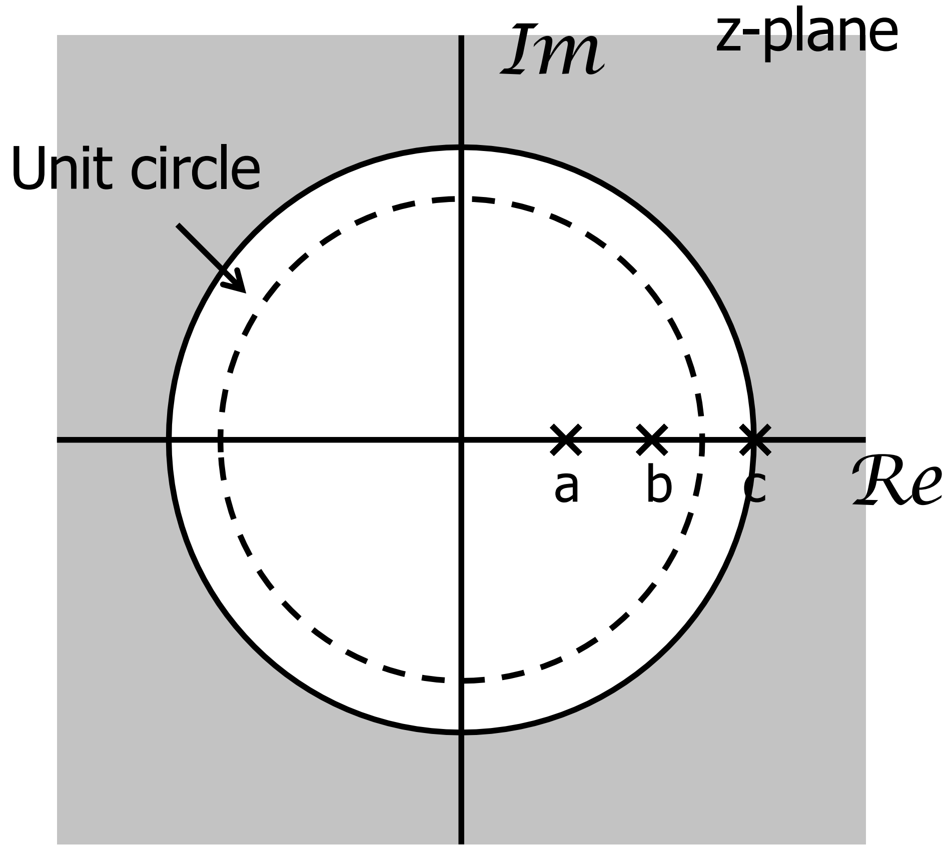
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

If stable and causal, all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

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All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \text{ all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

First Order All-Pass Filter (*a* real)

$$H(z) = \frac{z^{-1} - a^{*}}{1 - az^{-1}}$$

First Order All-Pass Filter (a real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$\left|H(e^{j\omega})\right| = \frac{\left|e^{-j\omega} - a^*\right|}{\left|1 - ae^{-j\omega}\right|}$$

$$= \frac{\left|e^{-j\omega}(1 - a^* e^{j\omega})\right|}{\left|1 - ae^{-j\omega}\right|}$$

$$= \frac{\left|1 - a^* e^{j\omega}\right|}{\left|1 - ae^{-j\omega}\right|} = 1$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\text{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

General All-Pass Filter

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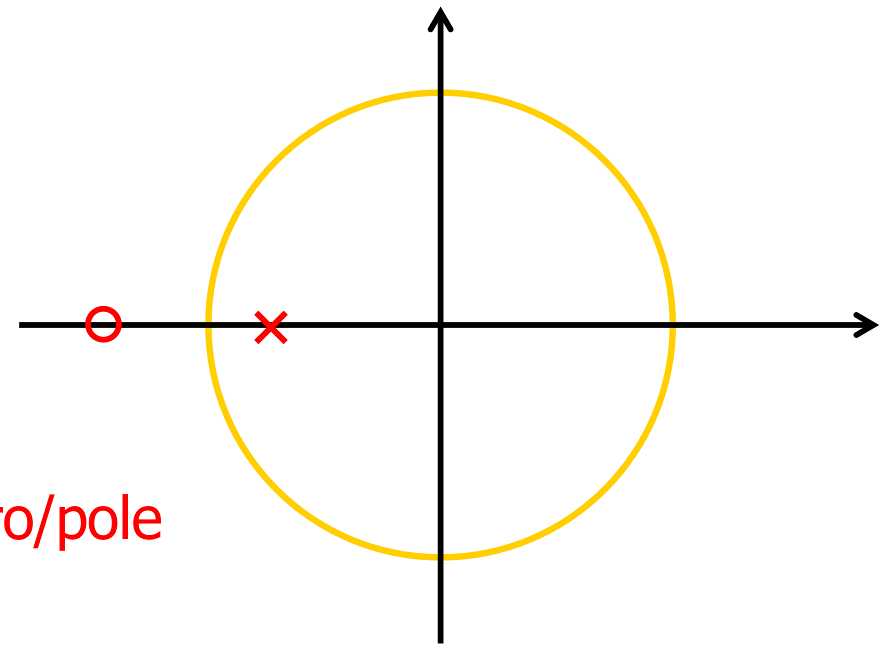
$$e_k = 0.8e^{j\pi/4}$$

Real zero/pole



General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

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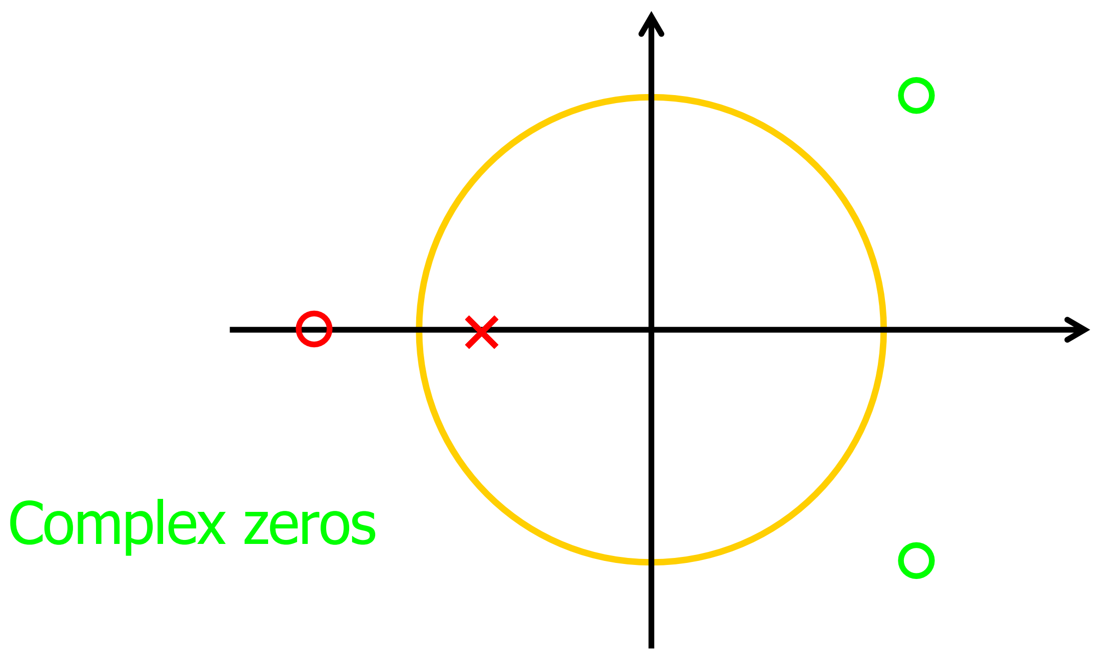

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

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- Example:

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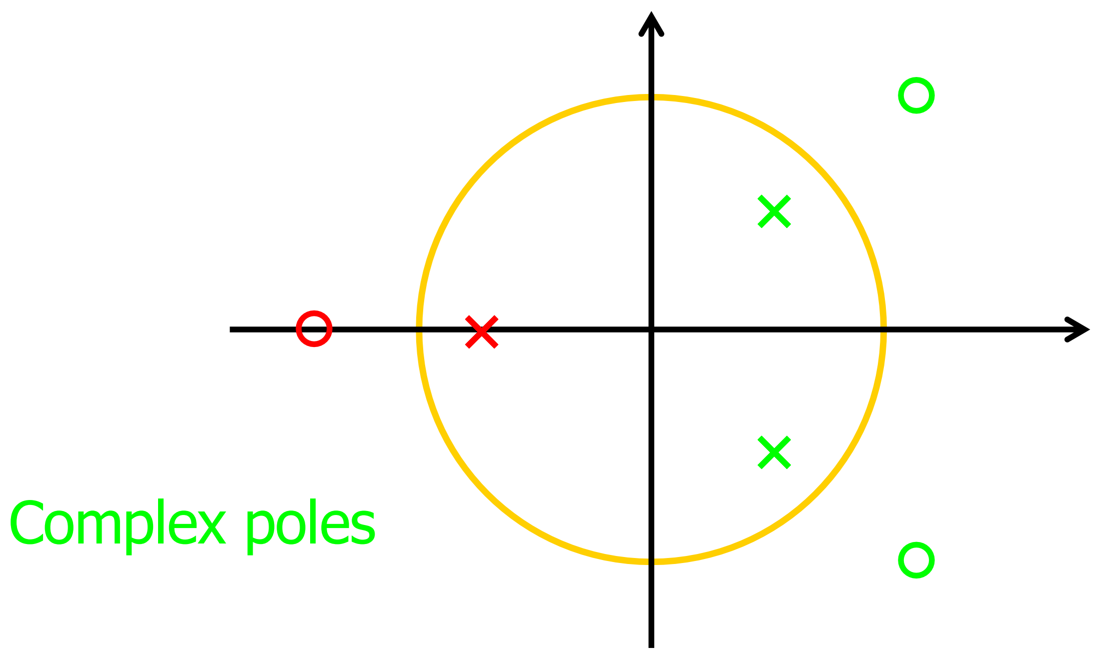


All Pass Filter Phase Response

- First order system

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase

All Pass Filter Phase Response

- First order system
$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$
$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase
$$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$

All Pass Filter Phase Response

- First order system
$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$
$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase
$$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$
$$= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right)$$
$$= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$
$$= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$
$$= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega})$$

Group Delay Math

$$\operatorname{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \operatorname{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \operatorname{grd}[1 - d_k e^{-j\omega}]$$

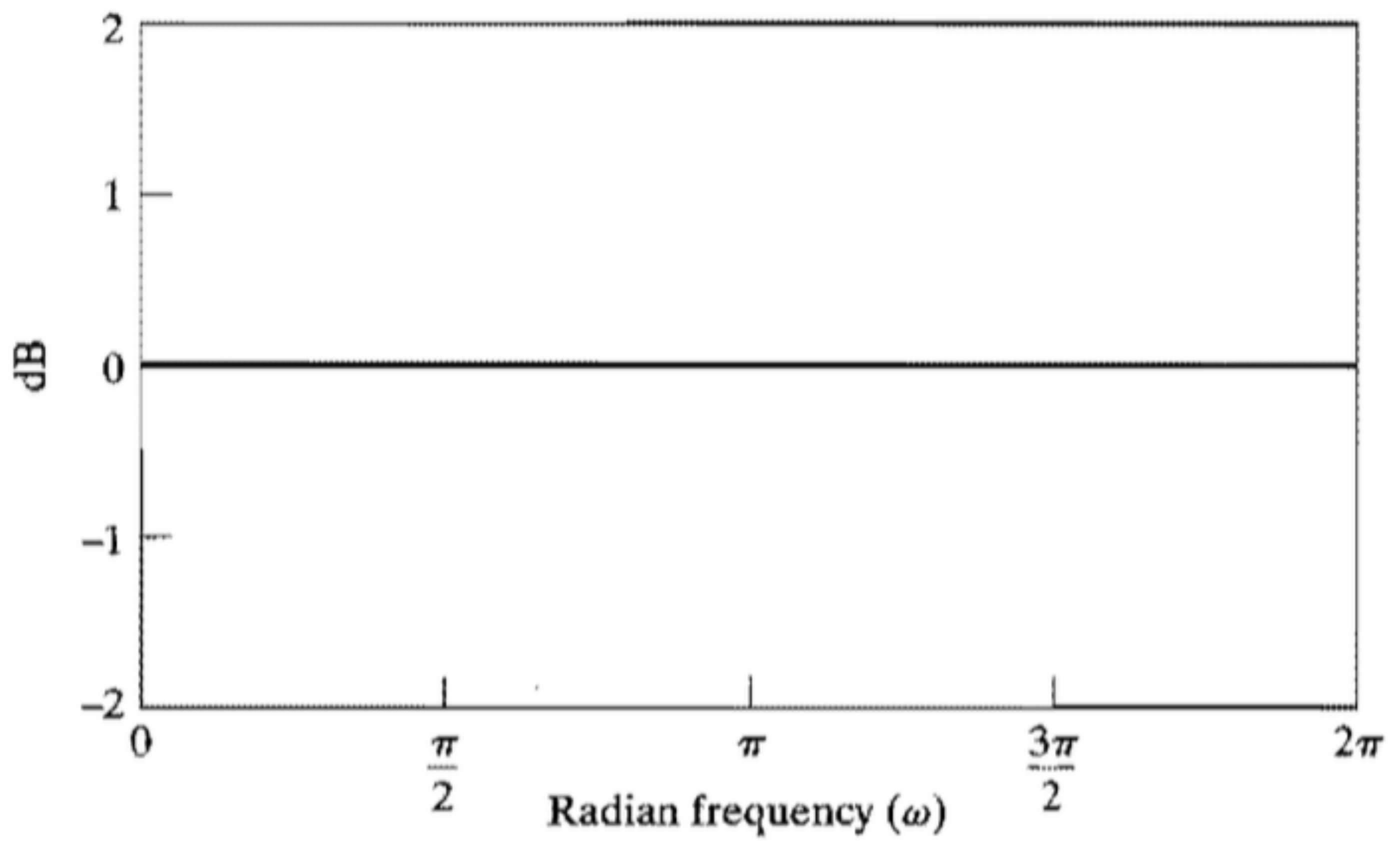
- Look at each factor:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\operatorname{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

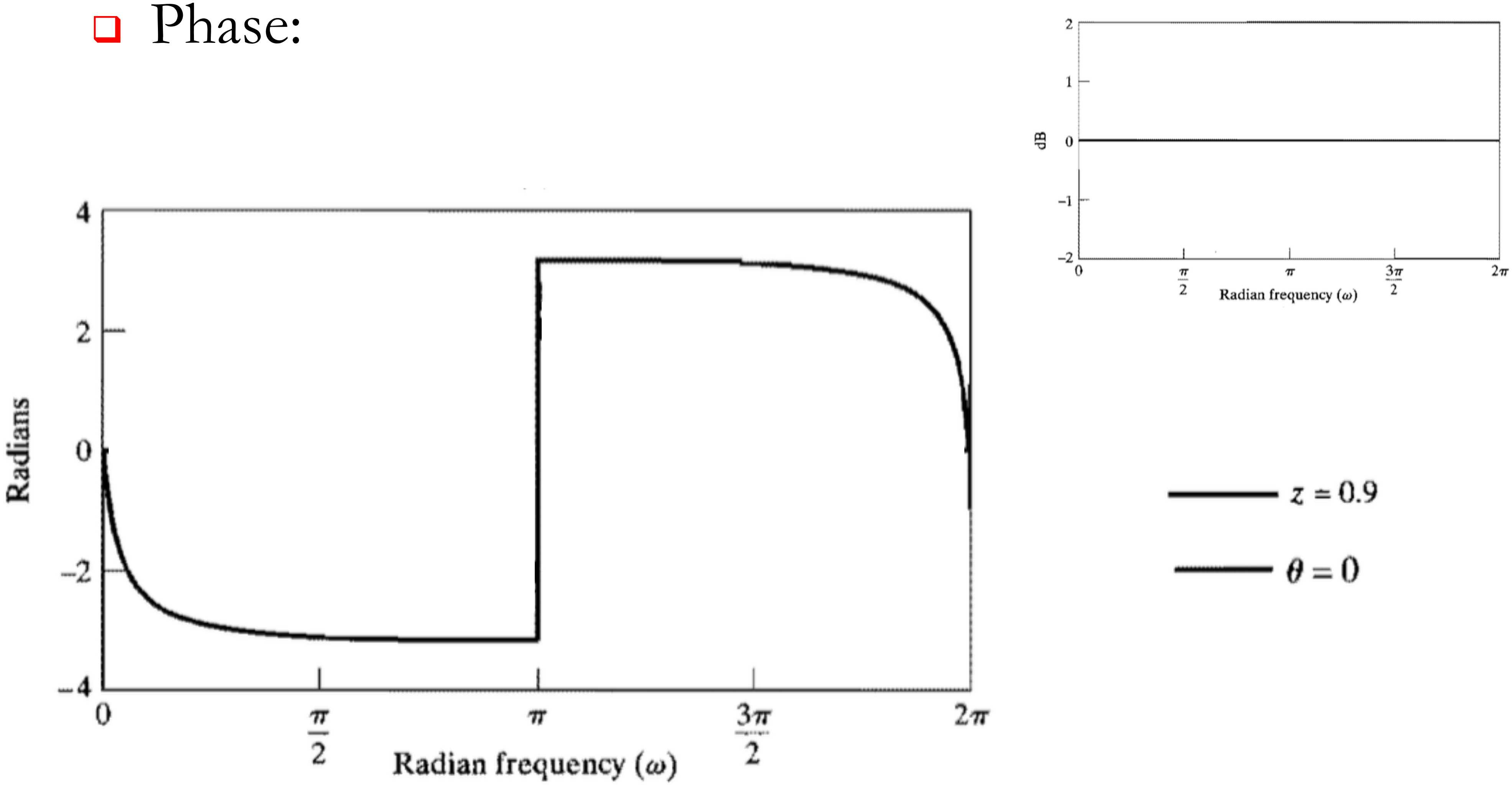
First Order Example

- Magnitude:



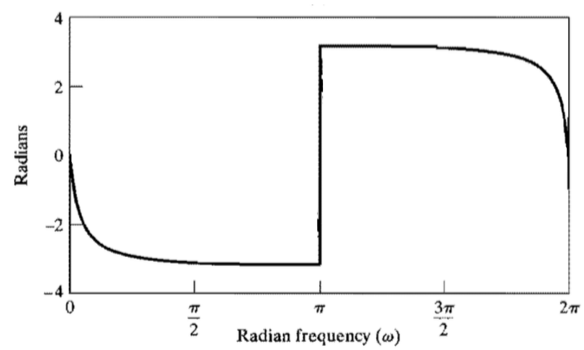
First Order Example

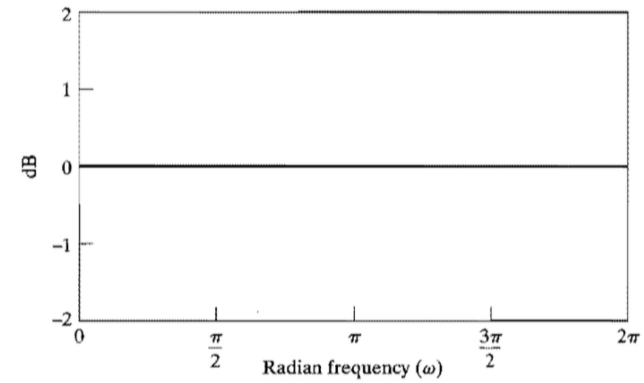
- Phase:

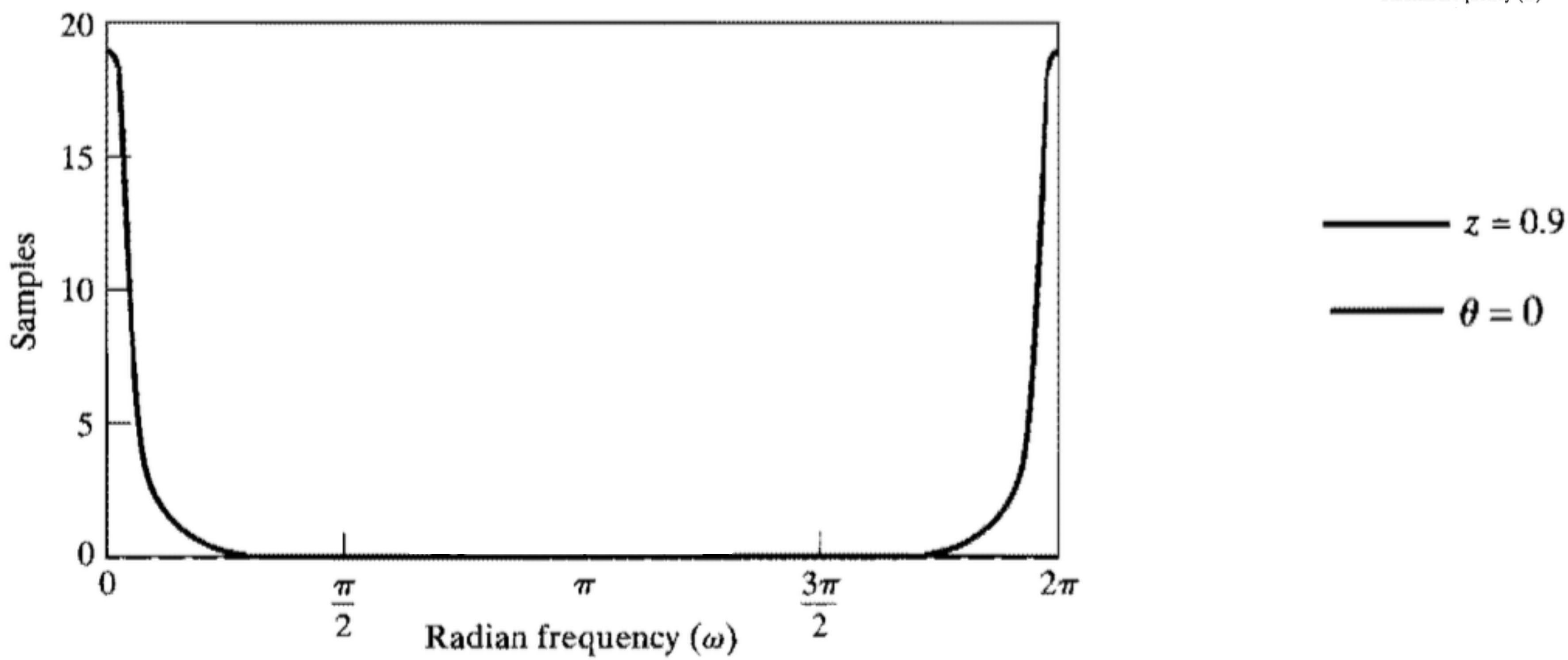


First Order Example

- Group Delay:







All Pass Filter Phase Response

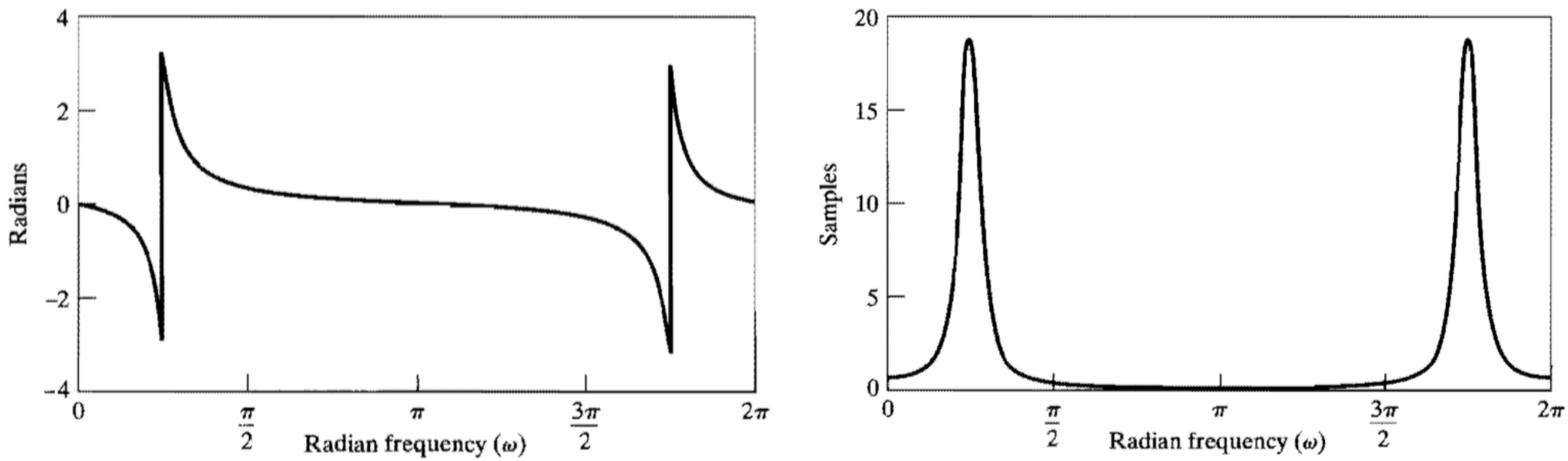
- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

$$\angle\left[\frac{(e^{-j\omega} - re^{-j\theta})(e^{-j\omega} - re^{j\theta})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

$$-2\arctan\left[\frac{r\sin(\omega + \theta)}{1 - r\cos(\omega + \theta)}\right].$$

Second Order Example

- Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)



All-pass Example

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- Example:

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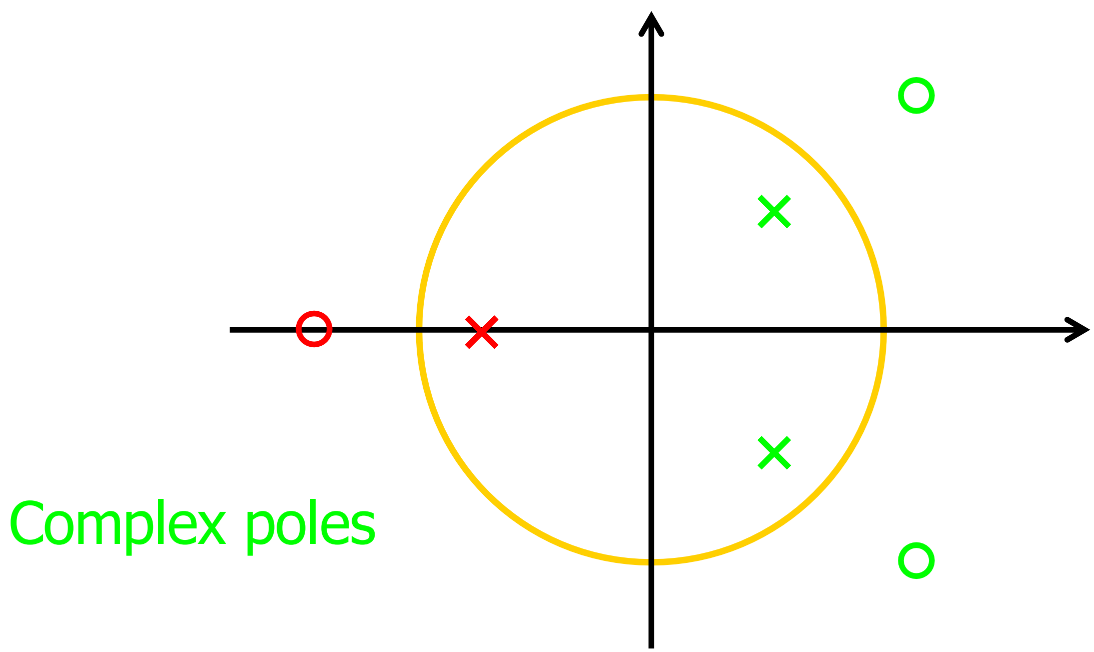
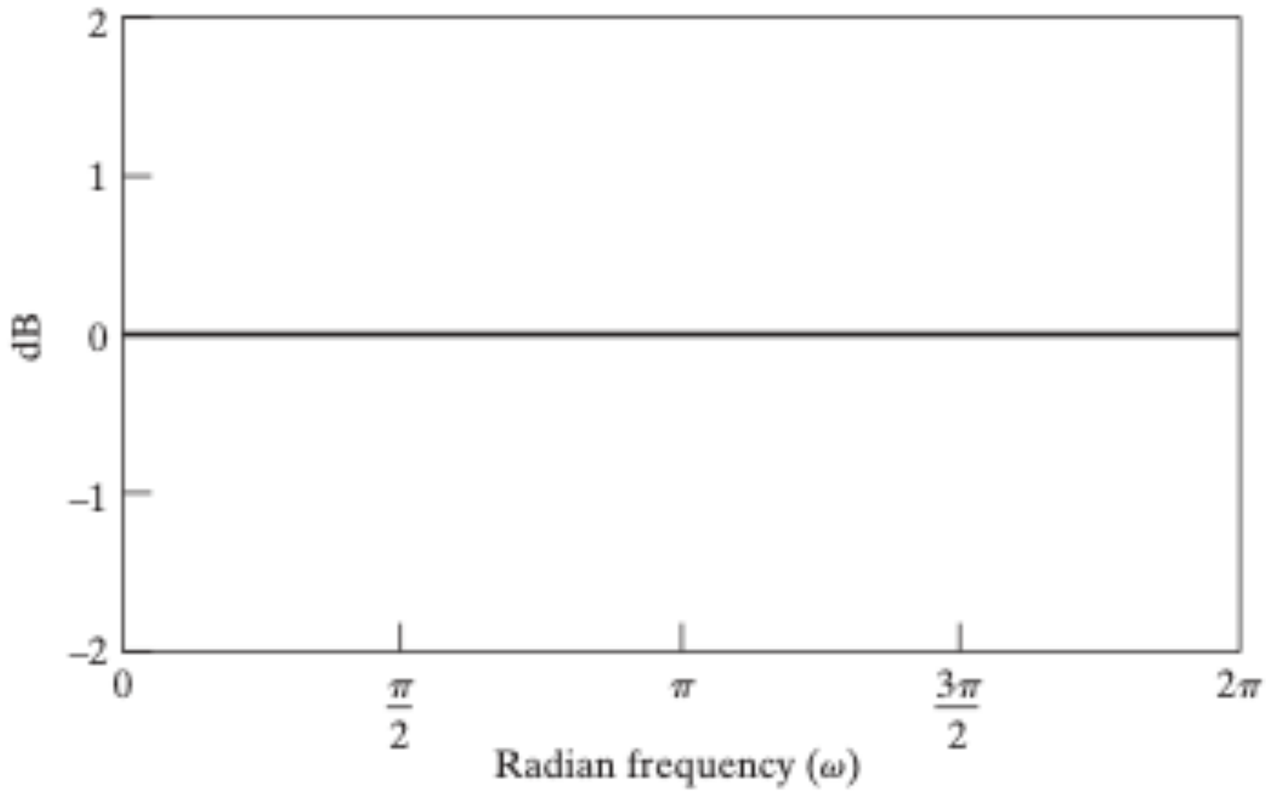
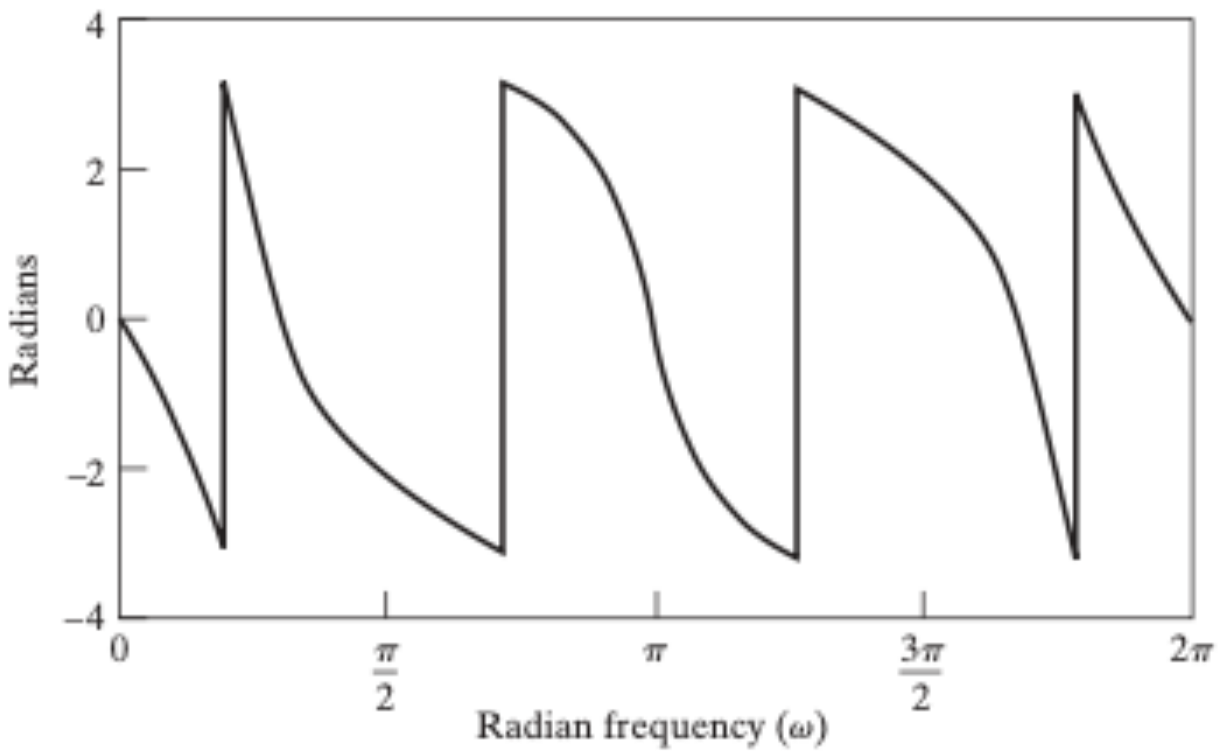


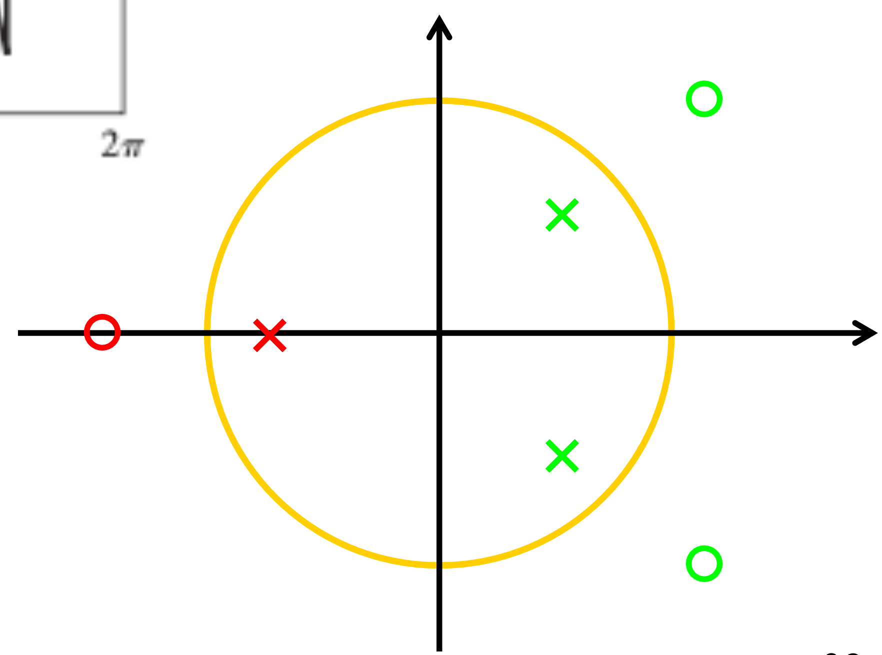
All-pass Example: Magnitude



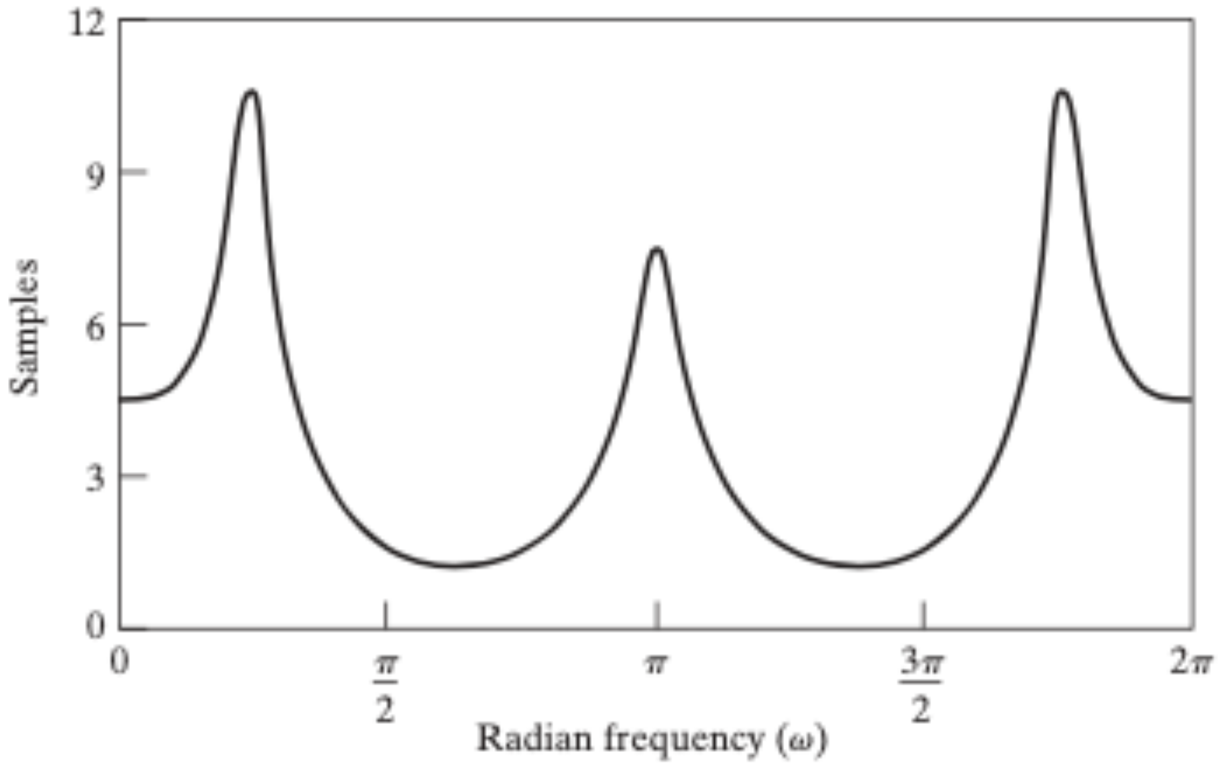
All-pass Example: Phase



(b)



All-pass Example: Group Delay



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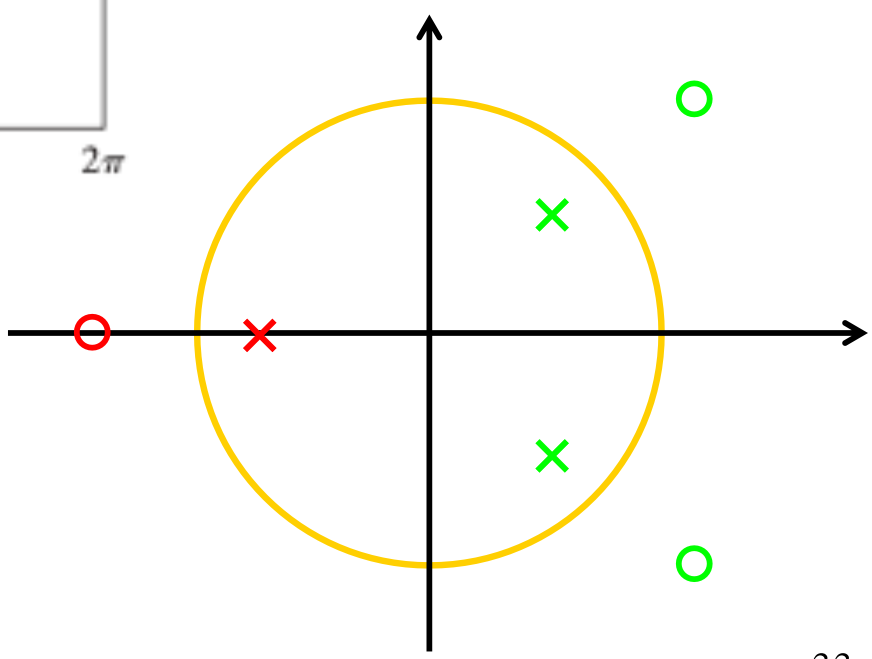

All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

Big Ideas

- Review: Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- Review: LTI Stability and Causality
  - If all poles inside unit circle
- All Pass Systems
  - Used for delay compensation

Admin

- HW 5
  - Due Monday 3/15
- HW 6
  - Due Sunday 3/22