# ESE 531: Digital Signal Processing

Week 9

Lecture 16: March 14th, 2021

Min Phase Decomposition



Penn

# Lecture Outline

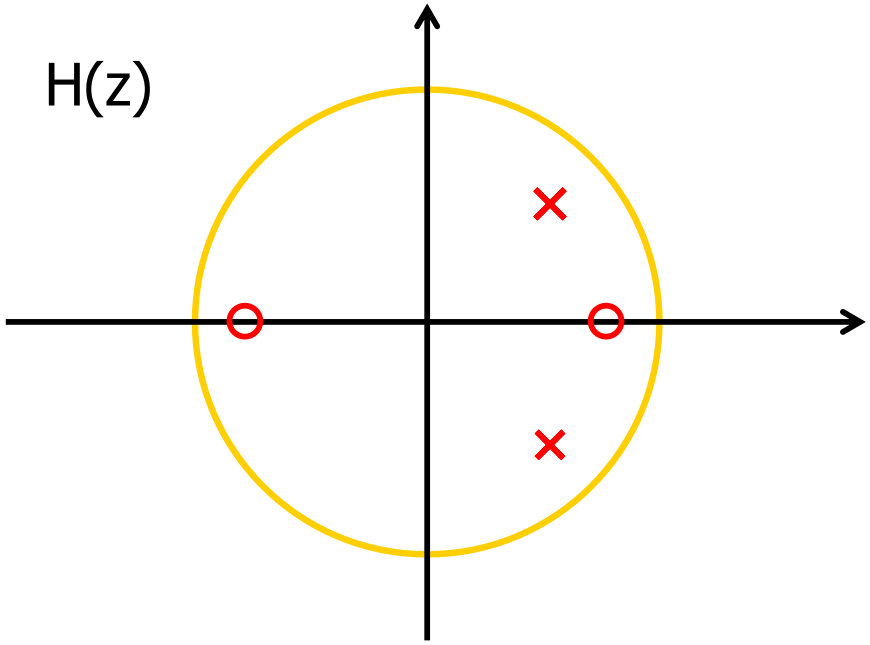
- Minimum Phase Systems

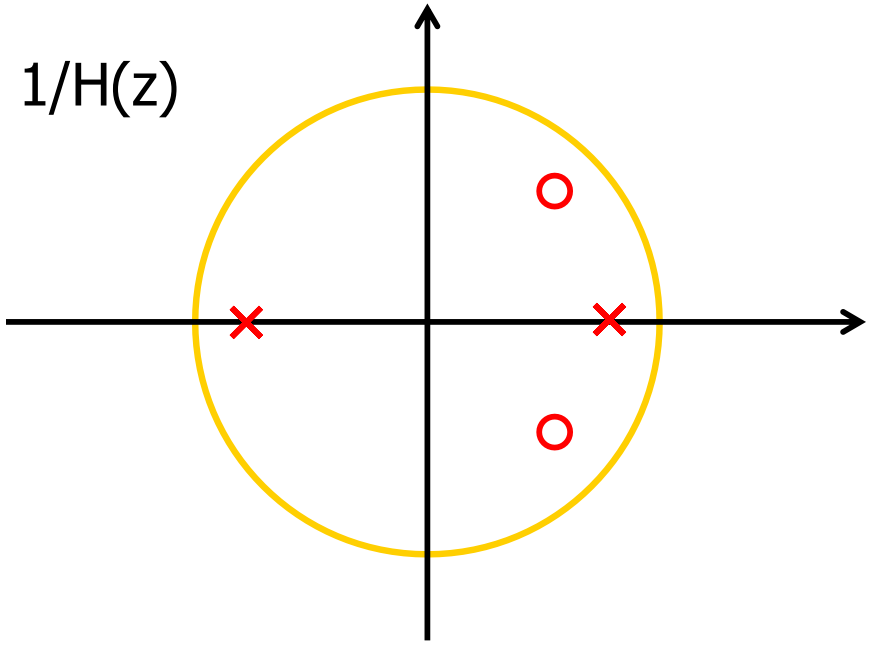
# Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)

# Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



# All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

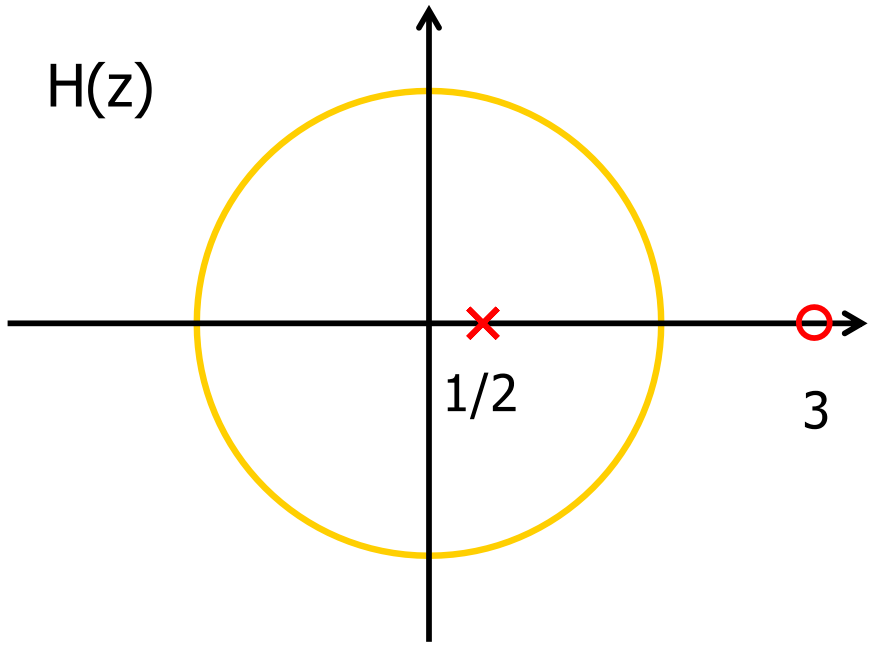
$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
  - (1) First construct $H_{ap}$ with all zeros outside unit circle
  - (2) Compute

$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

# Min-Phase Decomposition Example

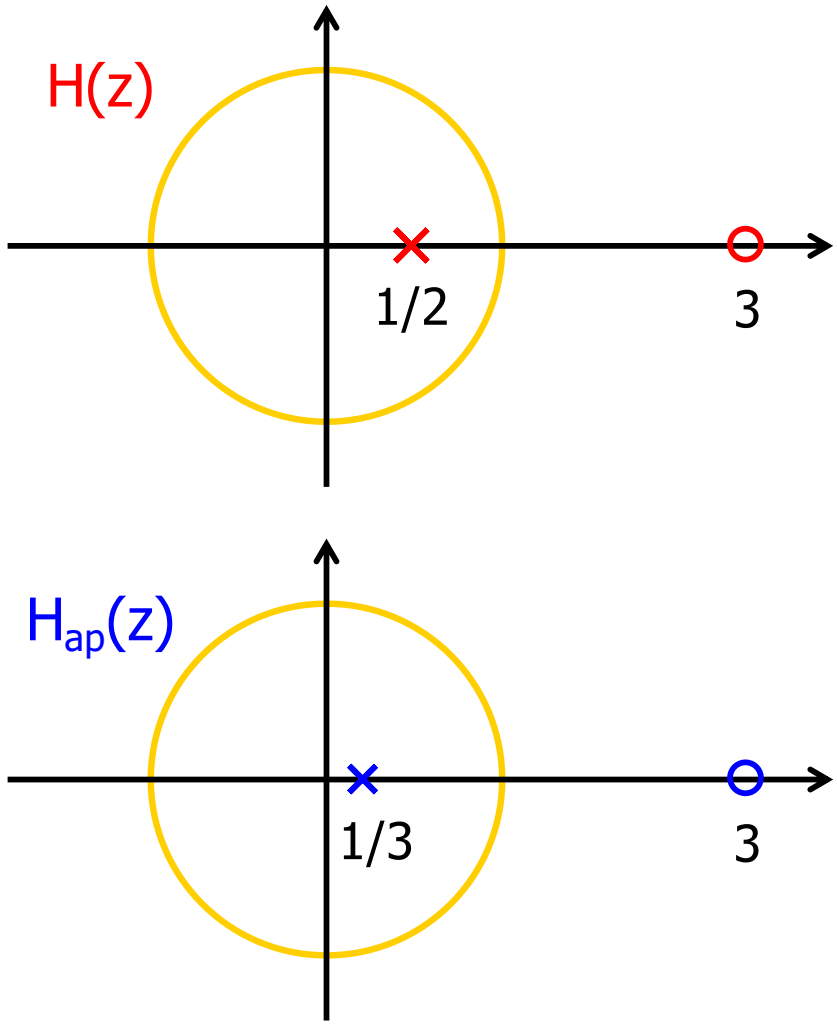
$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



# Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

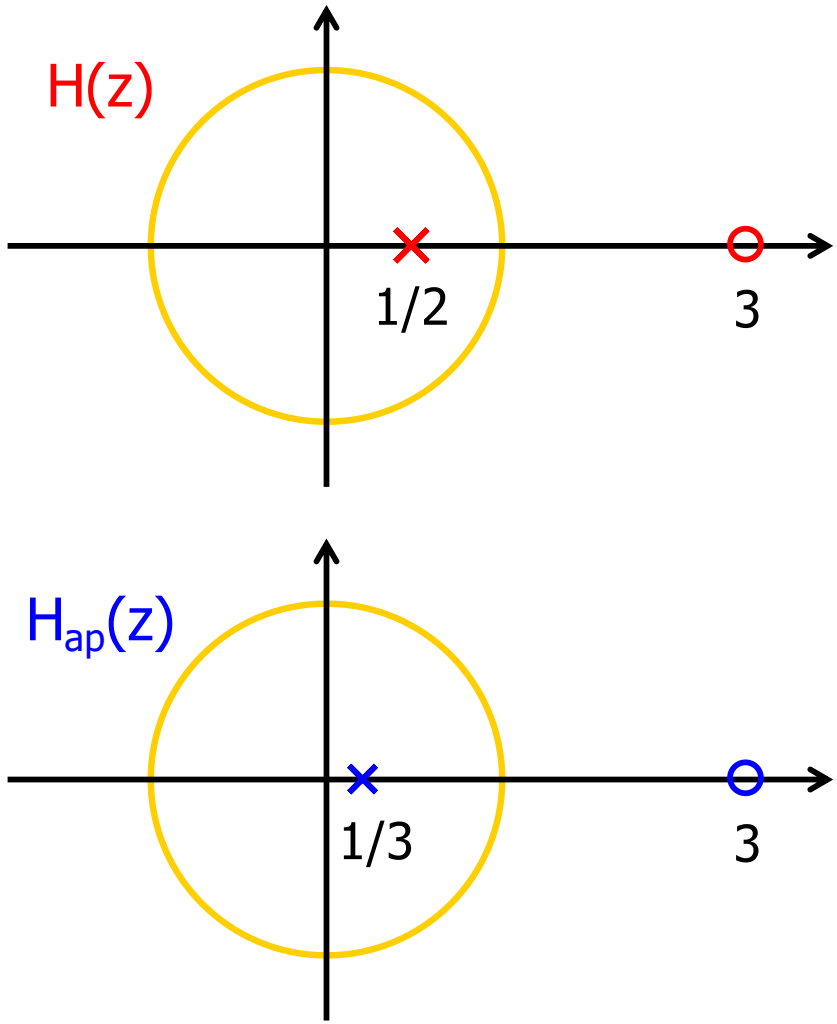
- Set $H_{ap}(z) = \dfrac{z^{-1} - a^*}{1 - az^{-1}}$



# Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$



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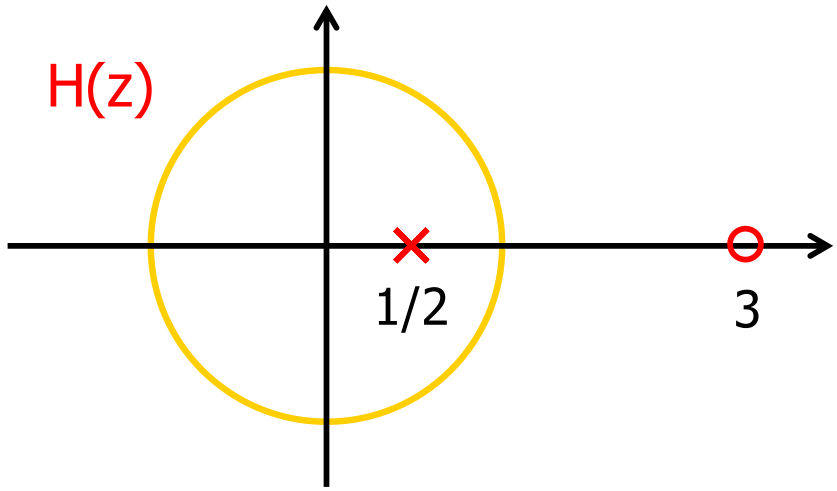
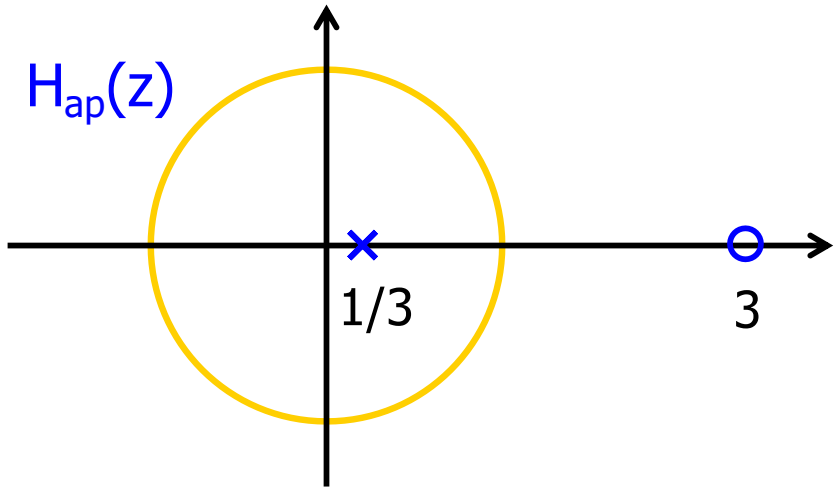
- Set $H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$

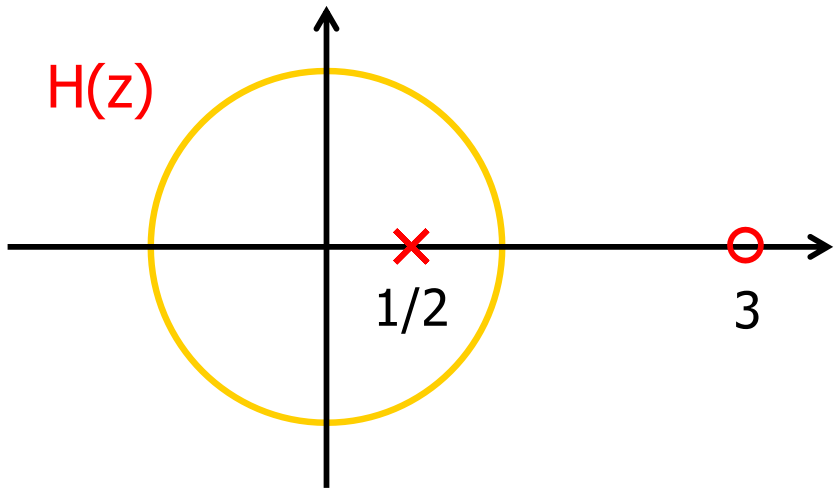
$$H_{\min}(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}} \left( \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}} \right)^{-1}$$

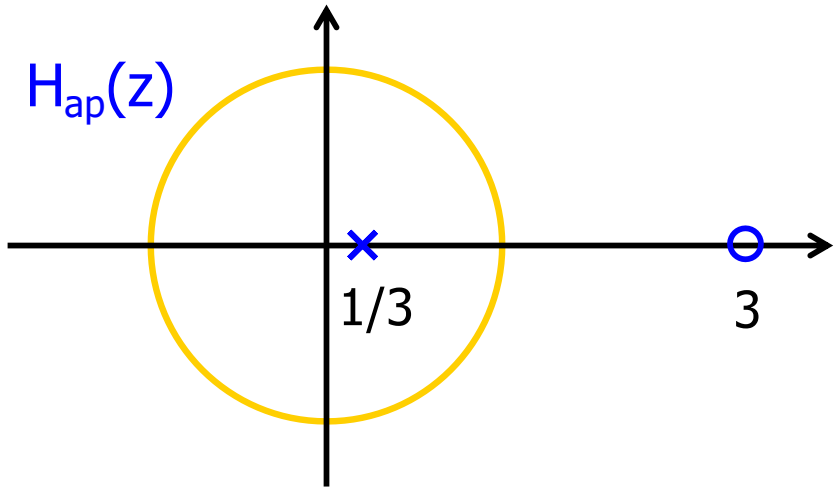


# Min-Phase Decomposition Example

$$H_{\min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}\left(\frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}\right)^{-1}$$



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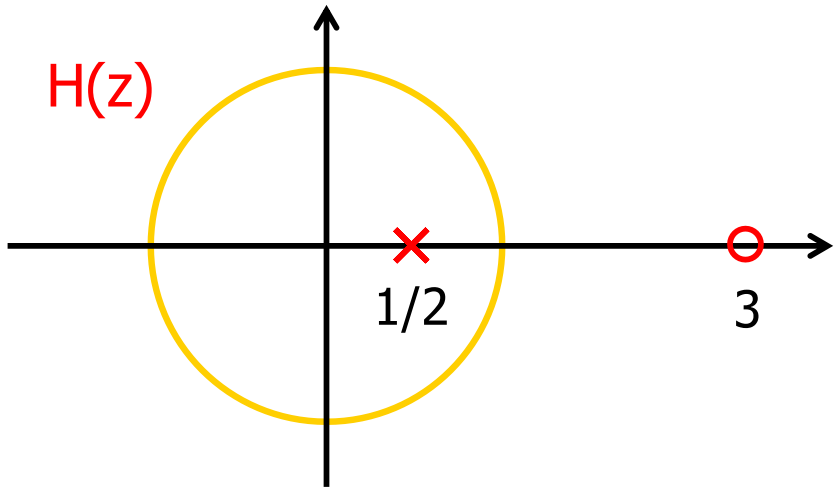


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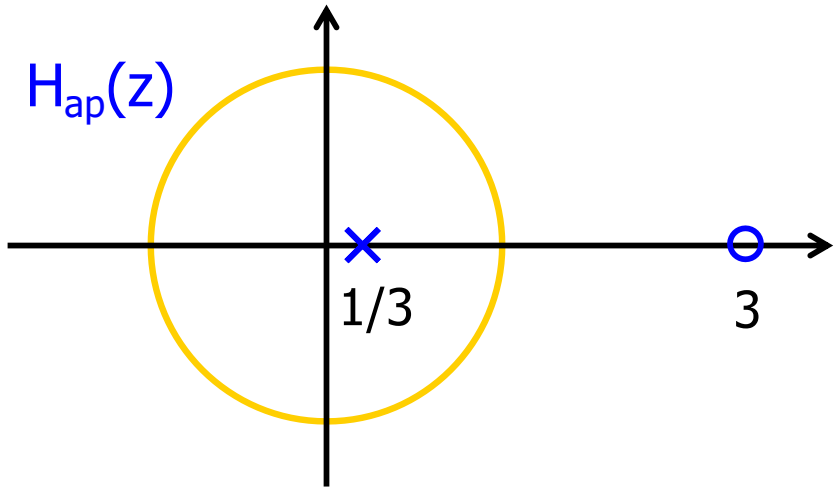


$$H_{\min}(z) = -3\frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

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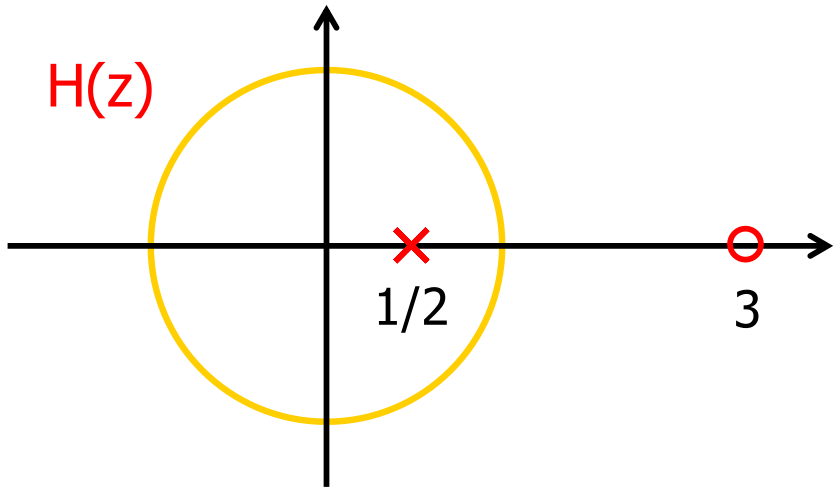


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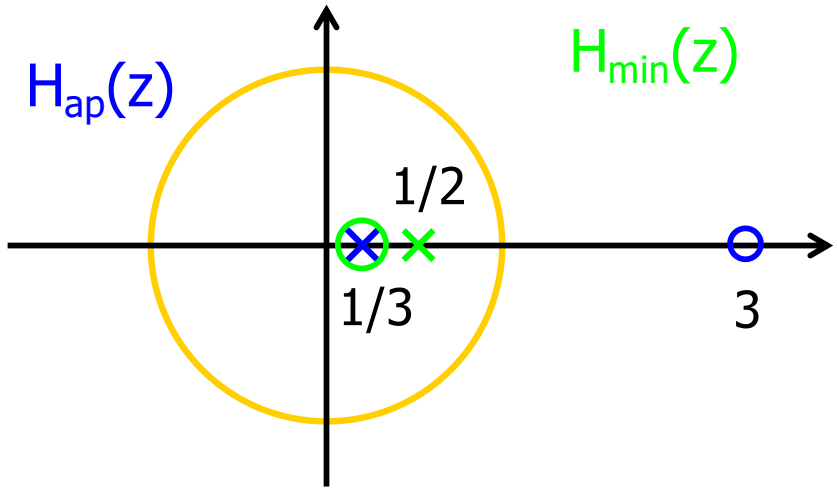
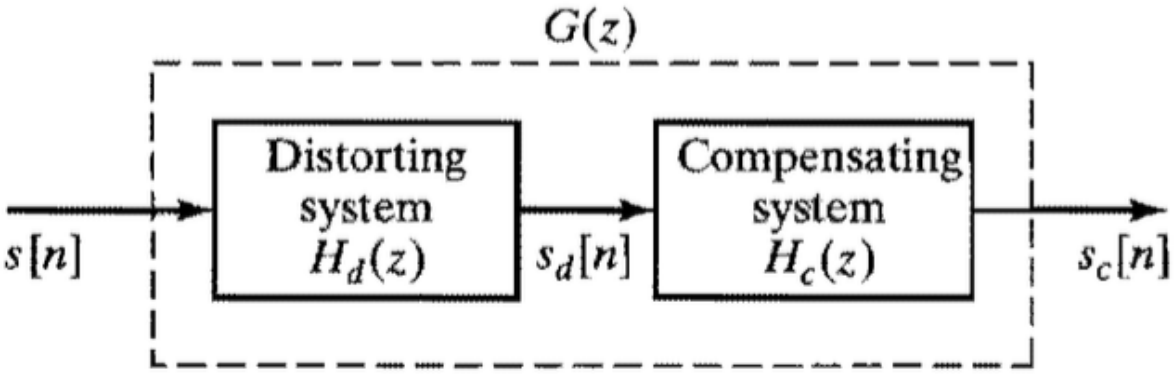


# Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



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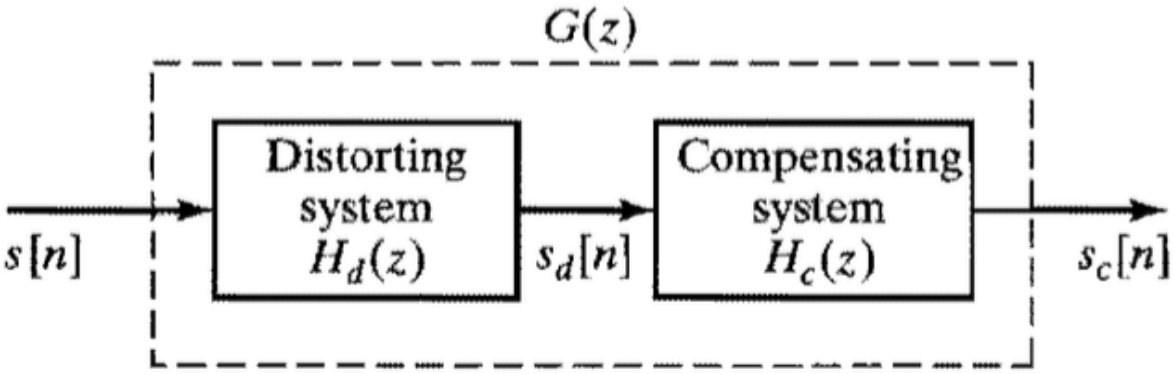


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  - $H_c(z)=1/H_d(z)$ ← also stable and causal

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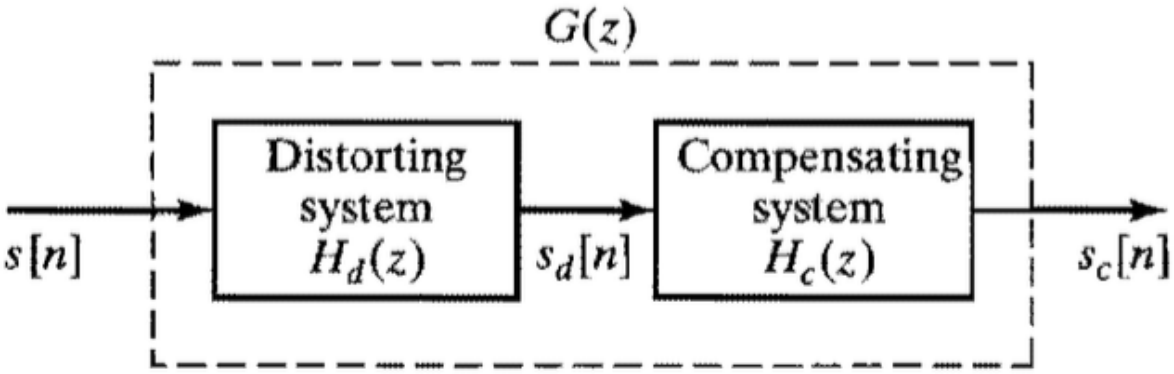
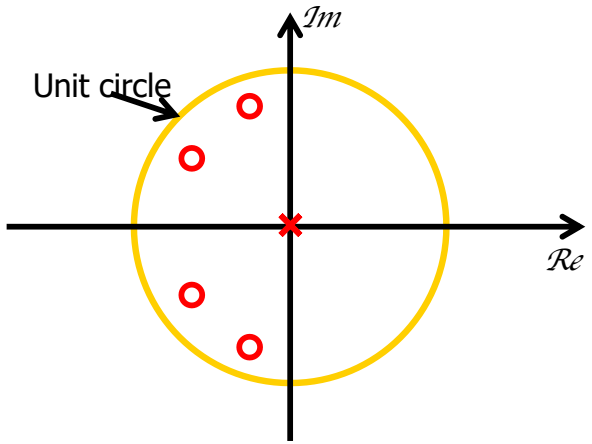



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z)$ →$H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

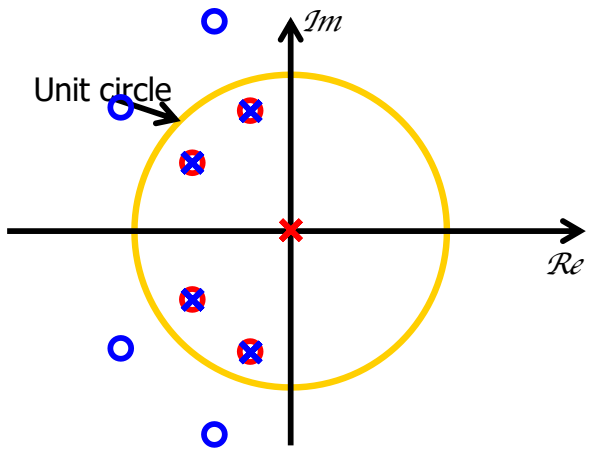
# Minimum Phase to Max Phase

Min phase



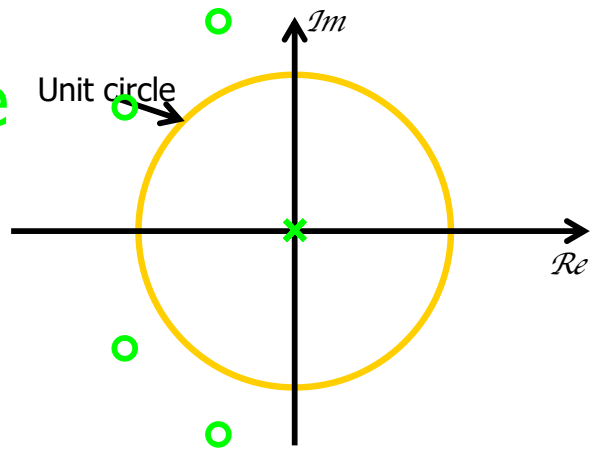
# Minimum Phase to Max Phase

Min phase
x All Pass



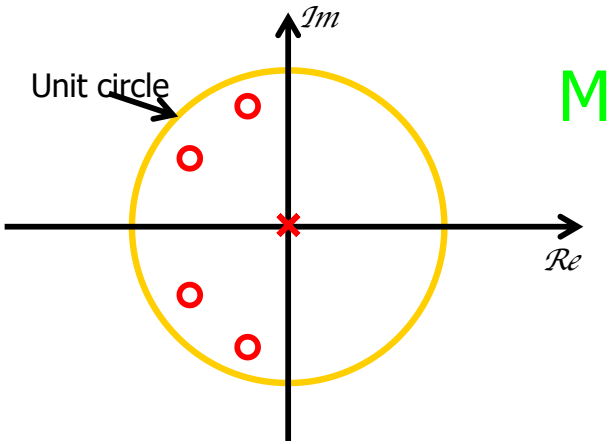
# Minimum Phase to Max Phase

Max phase

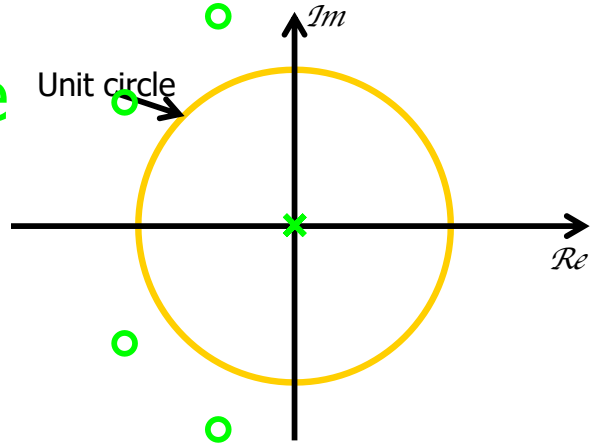


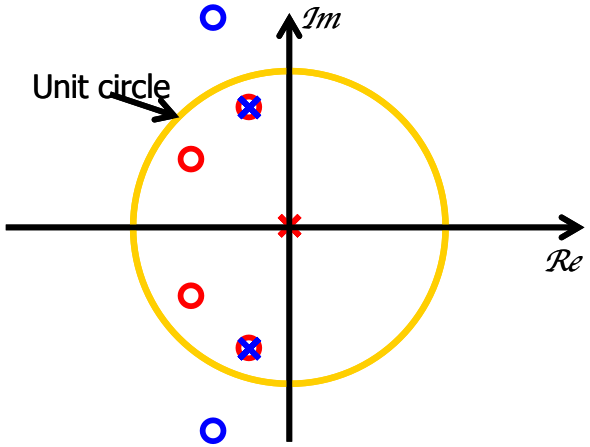
# Minimum Phase

Min phase




Max phase



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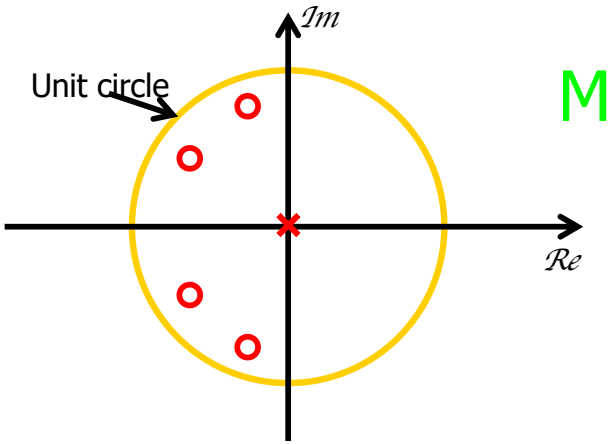


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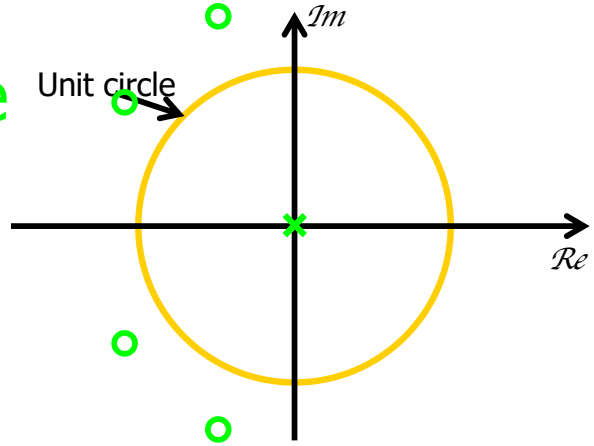


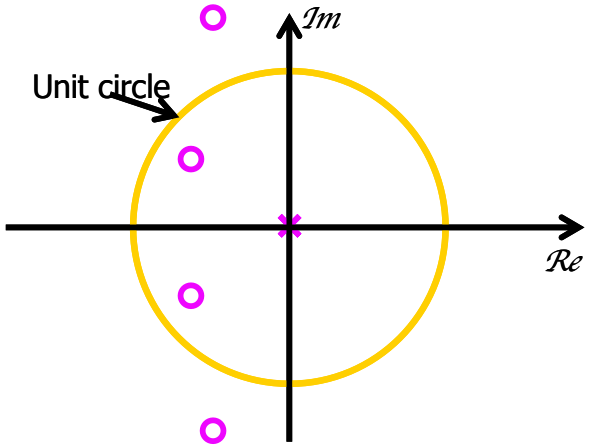


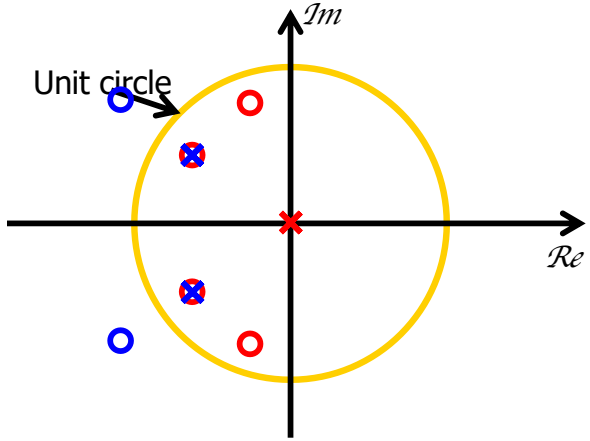

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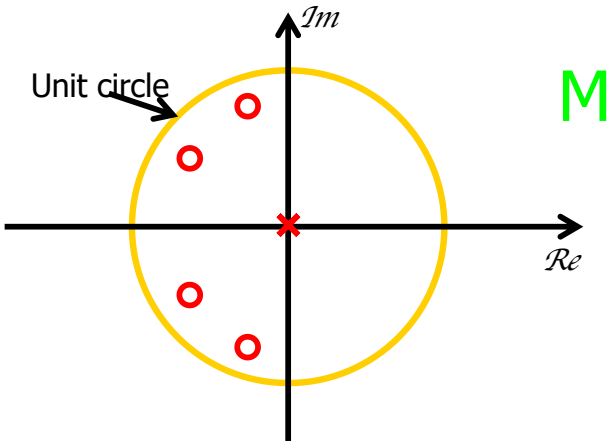


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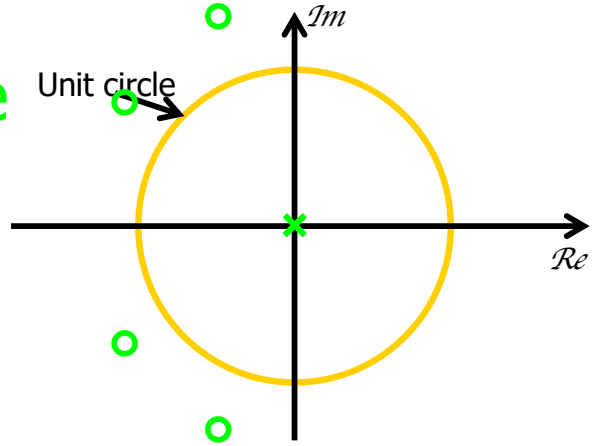


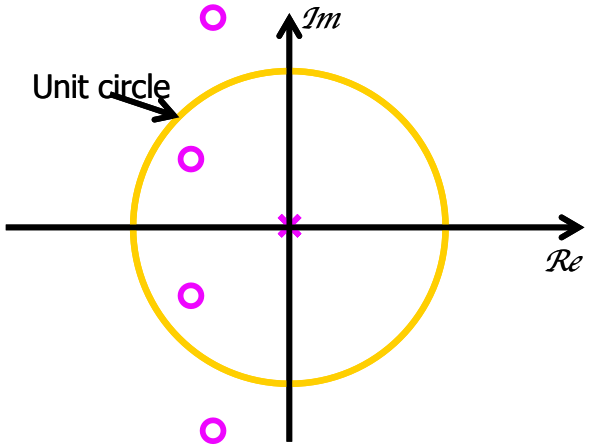


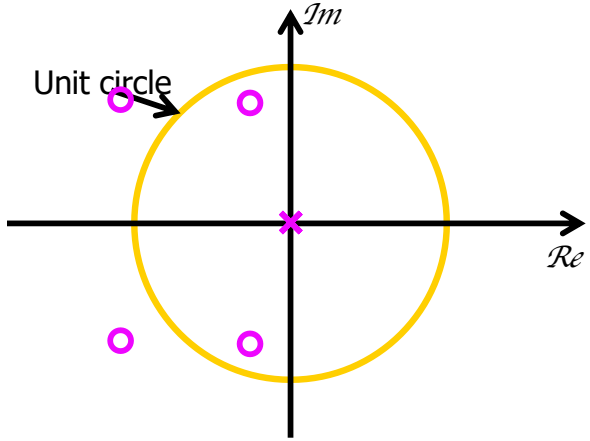

# Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

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  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

- $\arg[H_{max}(e^{j\omega})] = \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

  $= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$

  $= \arg[H_{min}(e^{j\omega})] + \quad \leq 0$

# Minimum Group Delay Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

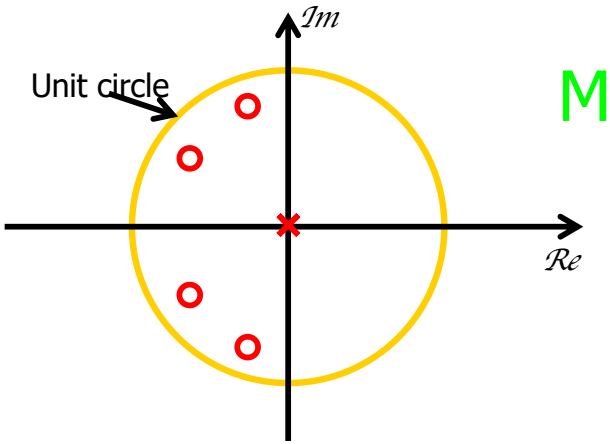
- $\text{grd}[H_{max}(e^{j\omega})] = \text{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \text{grd}[H_{min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})]$$

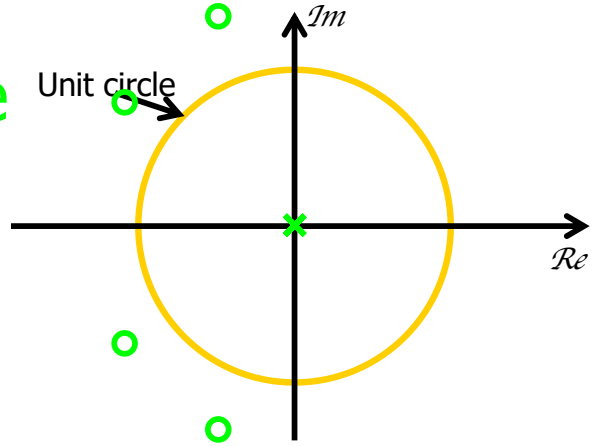
$$= \text{grd}[H_{min}(e^{j\omega})] + \quad \geq 0$$

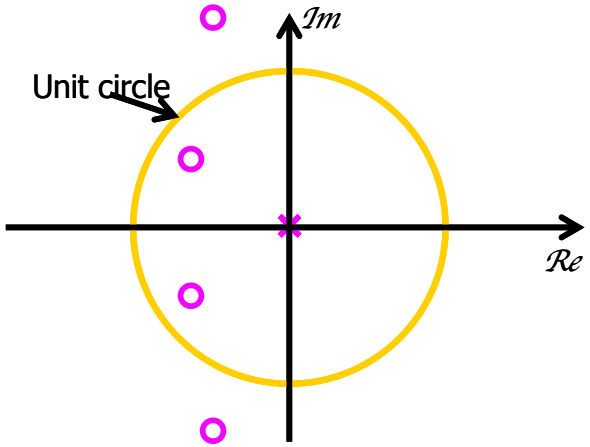
# Minimum Phase

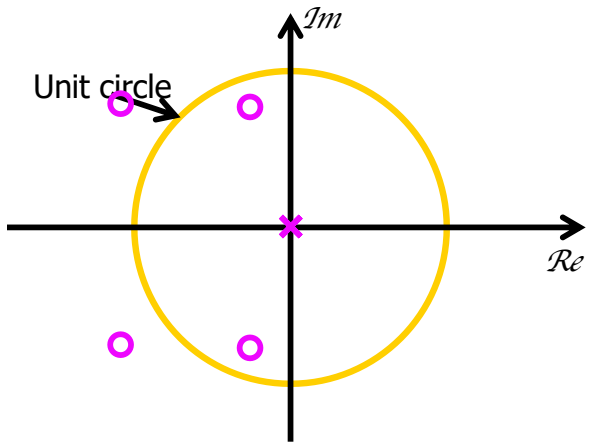
Min phase




Max phase

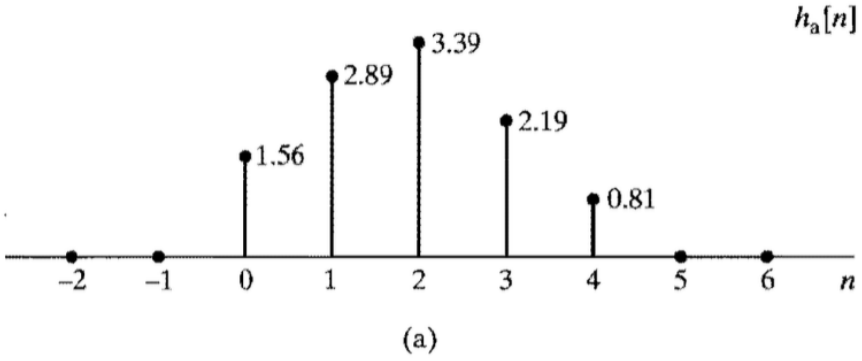


# Minimum Energy-Delay Property

Min phase

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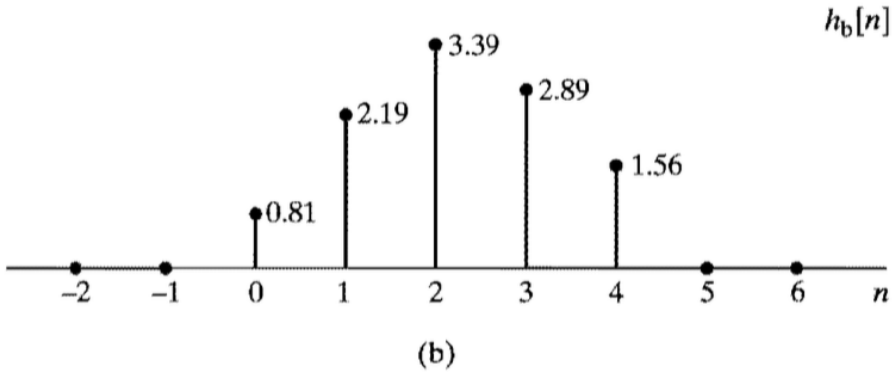
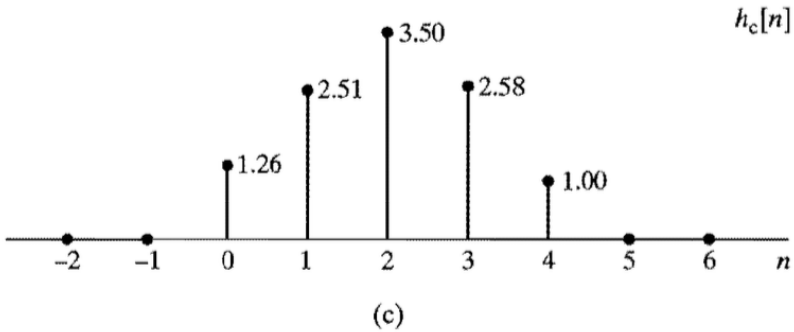
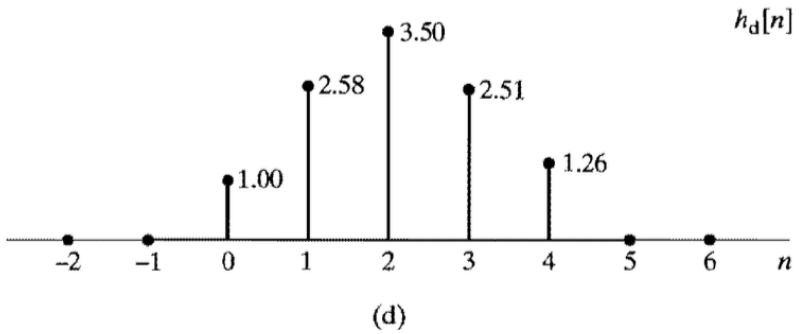


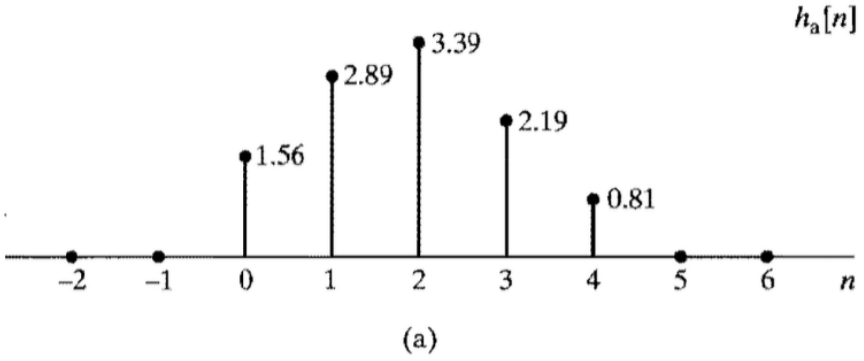




Total energy: $E = \sum_{n=0}^{\infty} |h[n]|^2$

# Minimum Energy-Delay Property

Min phase

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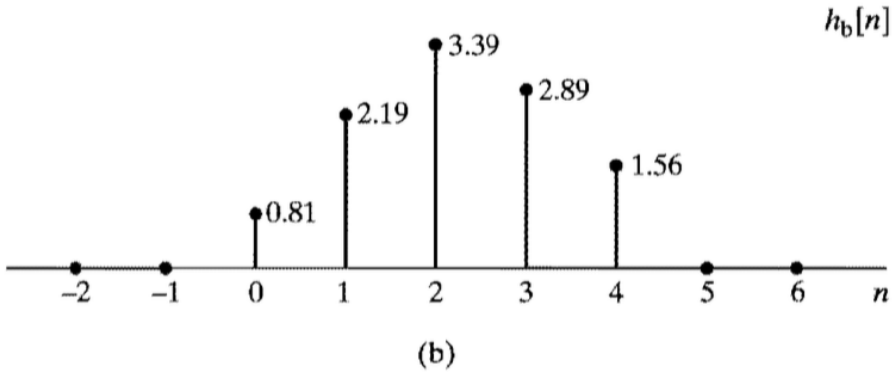
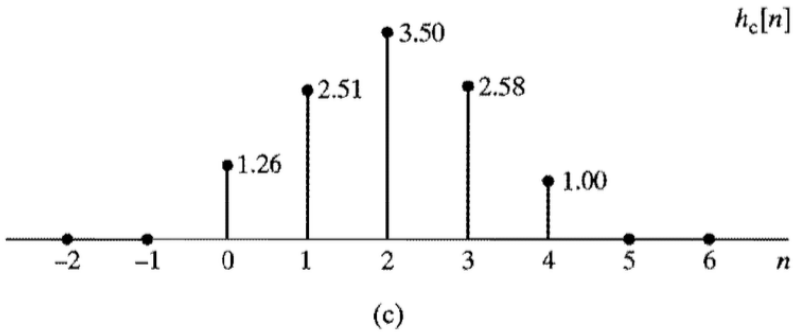
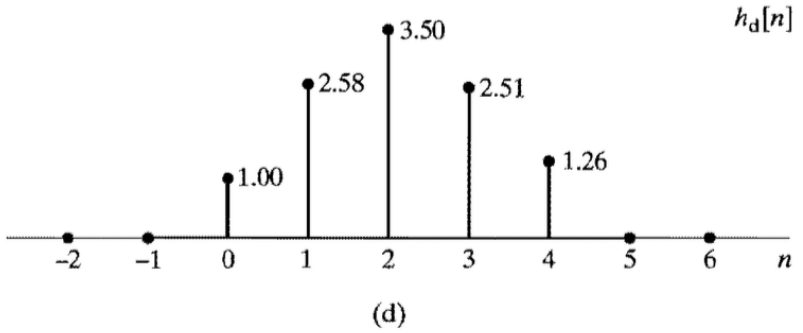
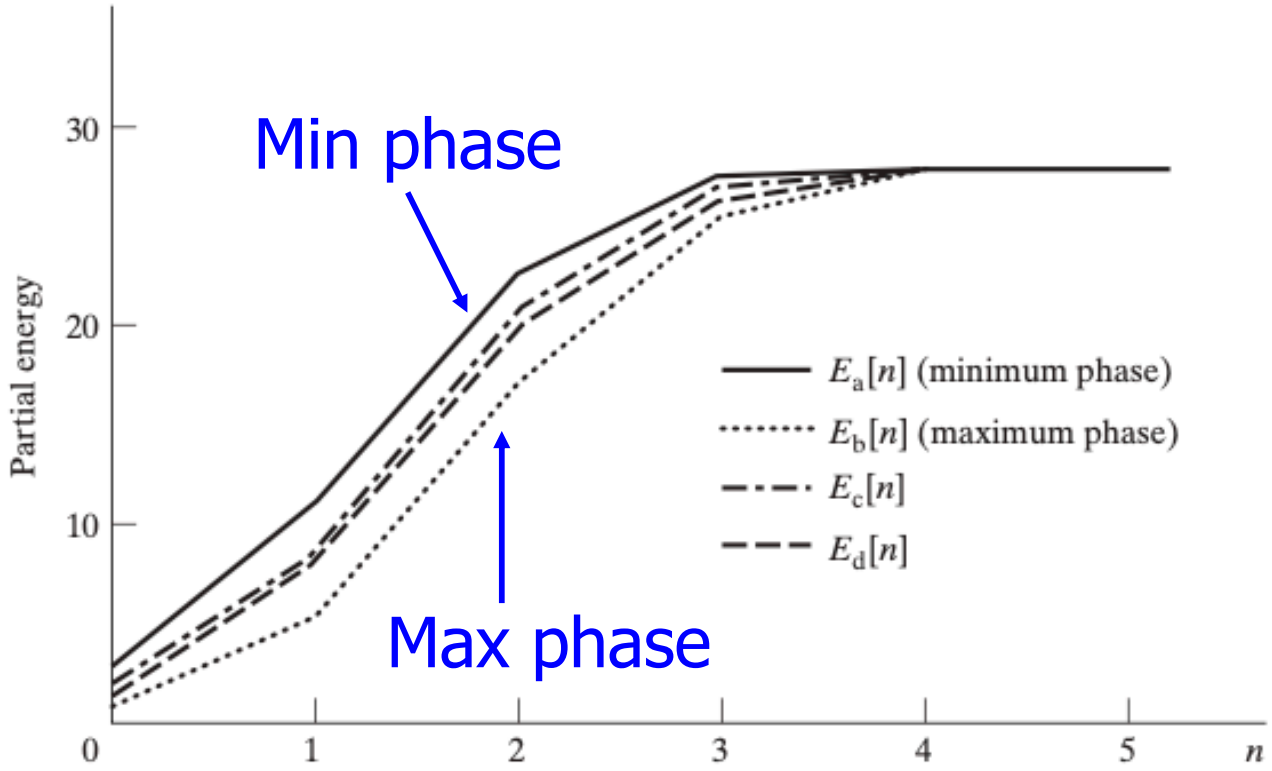



Partial energy: $E[n] = \sum_{m=0}^{n} |h[m]|^2$

# Minimum Energy-Delay Property

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# Big Ideas

- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property

# Admin

- HW 5
  - Due Monday 3/15
- HW 6
  - Due Sunday 3/22