ESE 531: Digital Signal Processing

Week 10

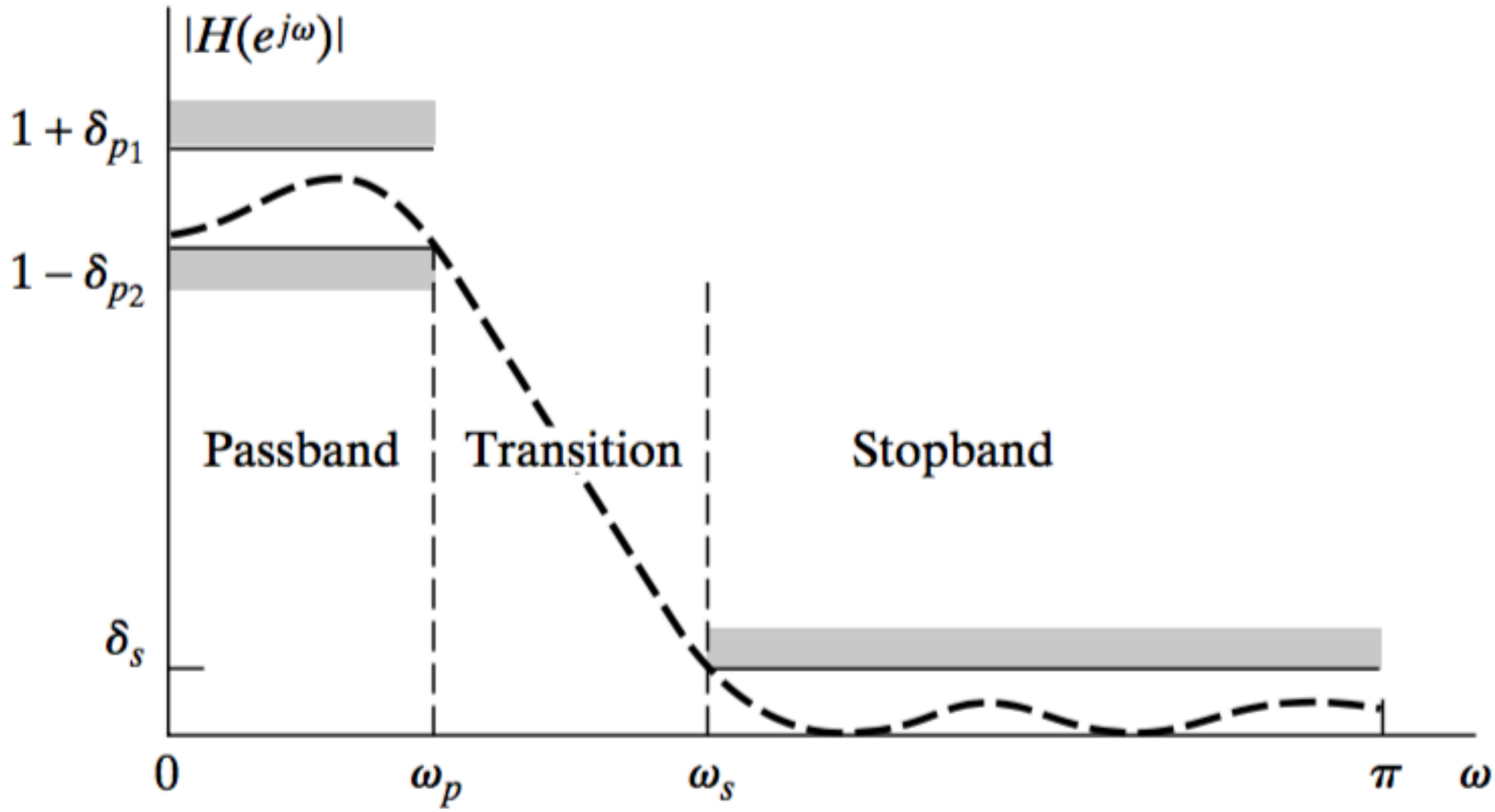
Lecture 17: March 21, 2021

Generalized Linear Phase Systems

Lecture Outline

- General Linear Phase Systems

Filter Specifications



Generalized Linear Phase Systems

Generalized Linear Phase

- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi$$

- Where A(ω) is a real function.

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- Where A(ω) is a real function.

- What is the group delay?

Causal FIR Systems

$$y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M], \quad \text{all } n$$

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_M z^{-M} = b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})$$

$$h[n] = \begin{cases} b_n, & n = 0,1,\ldots,M \\ 0, & otherwise \end{cases}$$

Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length (M+1) and some symmetry
- It can be shown if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise}, \end{cases}$$

- then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

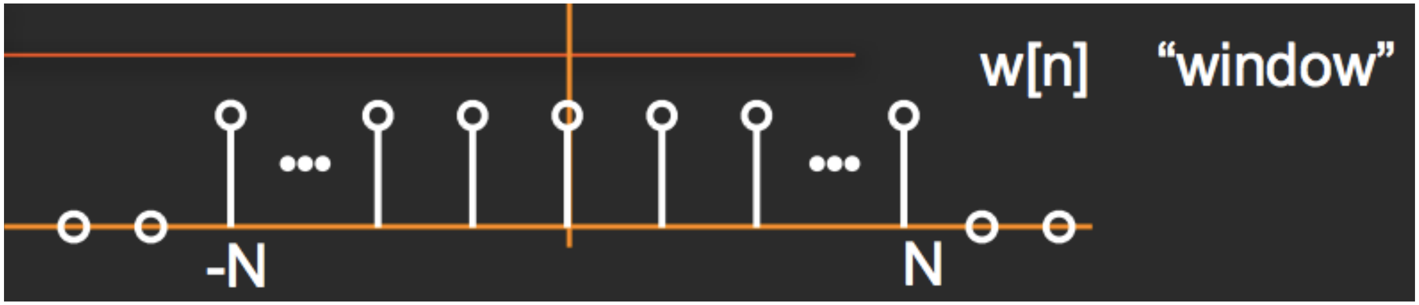
Example: Moving Average

- Moving Average Filter
  - Causal: $M_1=0$, $M_2=M$

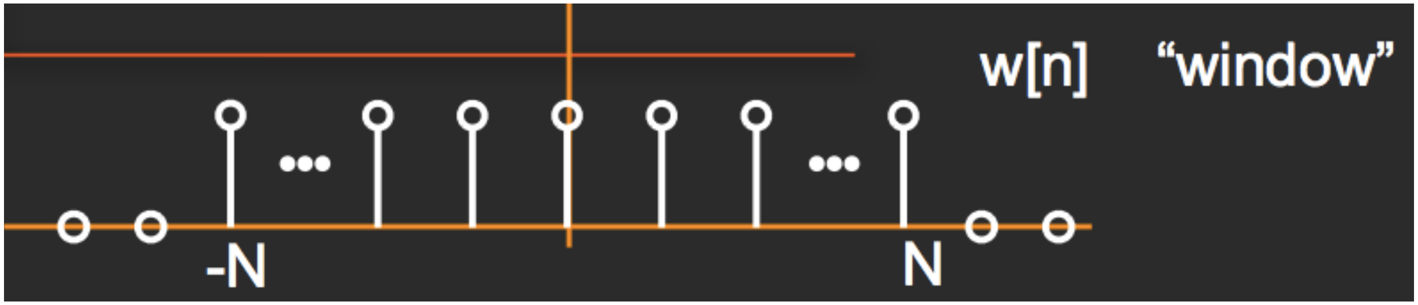
$$y[n] = \frac{x[n-M] + \ldots + x[n]}{M+1}$$

Impulse response

Scaled &Time Shifted window



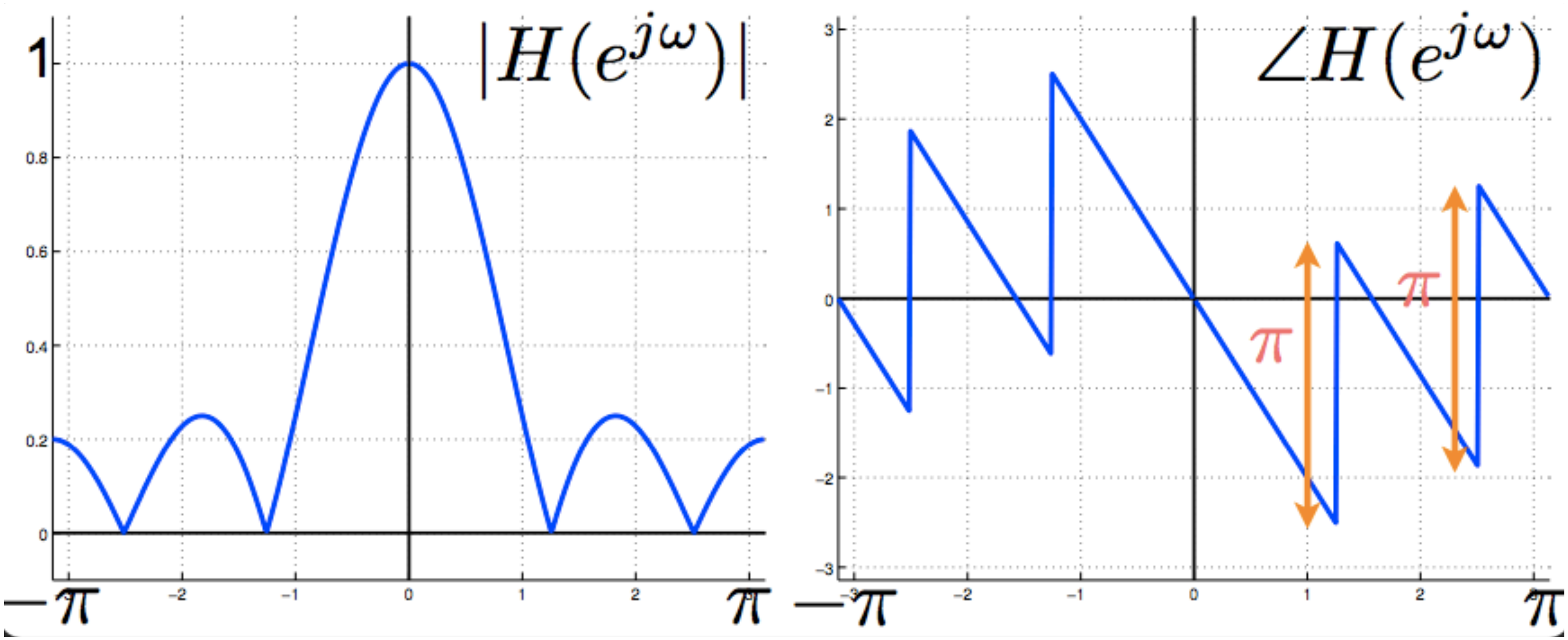
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$\frac{1}{M+1} w[n-M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average



Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length (M+1)
- It can be shown if

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

- Then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

- Sufficient conditions to guarantee GLP, not necessary
  - Causal IIR can also have GLP

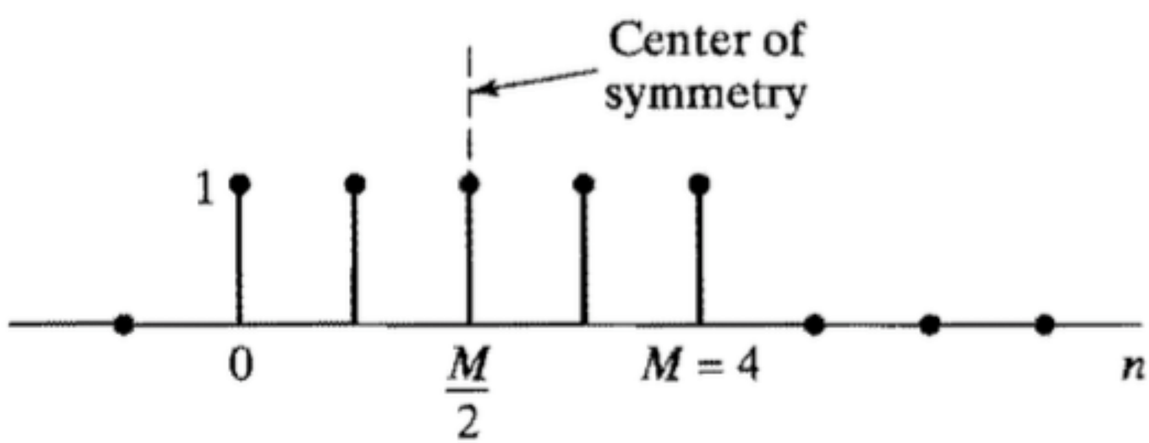
FIR GLP Systems

- Four types of FIR GLP
  - Type I
    - Even Symmetry, M even
  - Type II
    - Even Symmetry, M odd
  - Type III
    - Odd Symmetry, M even
  - Type IV
    - Odd Symmetry, M odd

FIR GLP: Type I

Type I Even Symmetry, M even

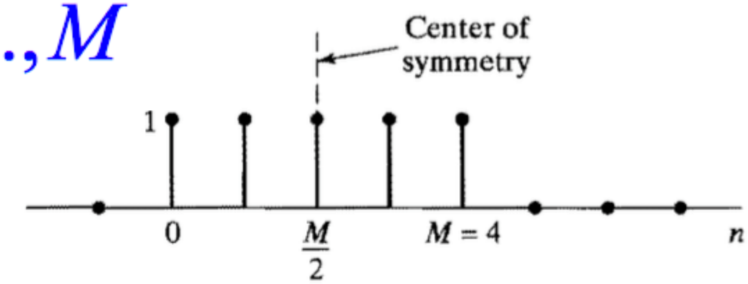
$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

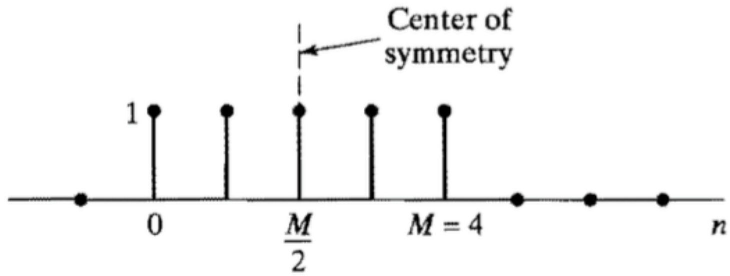


Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

integer delay

FIR GLP: Type I – Example, M=4

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$

Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$

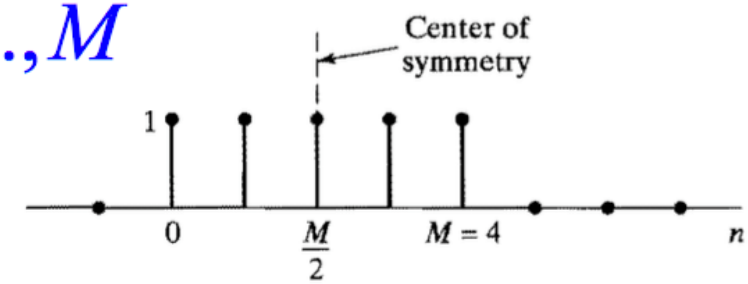
$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega} \right]$$

$$= \underbrace{\left[2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2]\right]}_{A(\omega)\ (even)} \; e^{-j2\omega}$$

FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M-n], \quad n = 0, 1, \ldots, M$$



Then $H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$ ← integer delay

$$H(e^{j\omega}) = e^{-j\omega M/2} \left(\sum_{k=0}^{M/2} a[k] \cos \omega k \right)$$

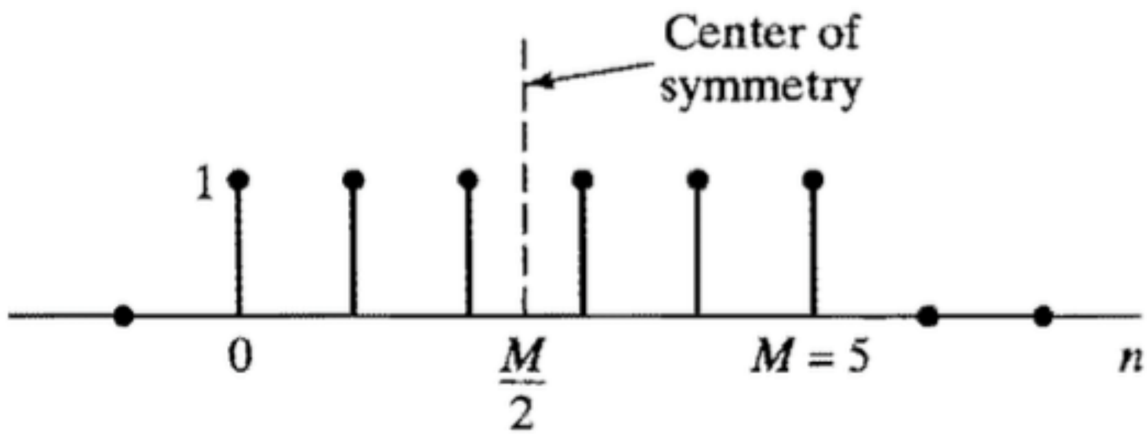
$$a[0] = h[M/2],$$

$$a[k] = 2h[(M/2) - k], \qquad k = 1, 2, \ldots, M/2.$$

FIR GLP: Type II

Type II Even Symmetry, M odd

$$h[n] = h[M-n], \quad n = 0,1,\ldots,M$$



FIR GLP: Type II

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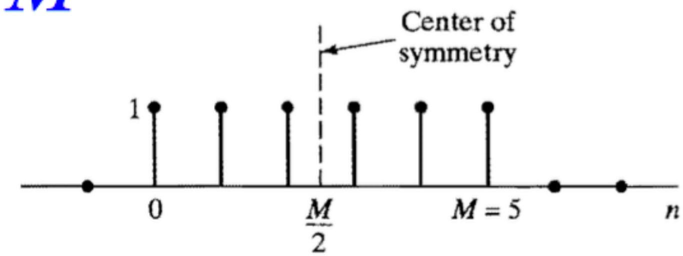

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

integer delay

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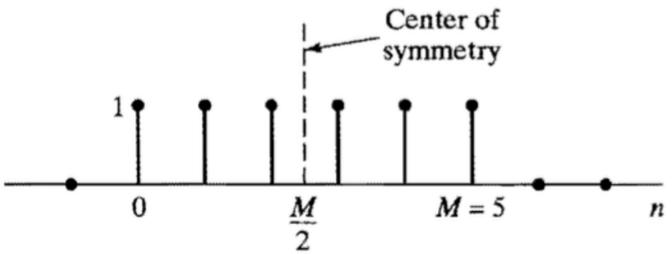


Then $$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

integer delay

FIR GLP: Type II – Example, M=3

Type II Even Symmetry, M odd

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

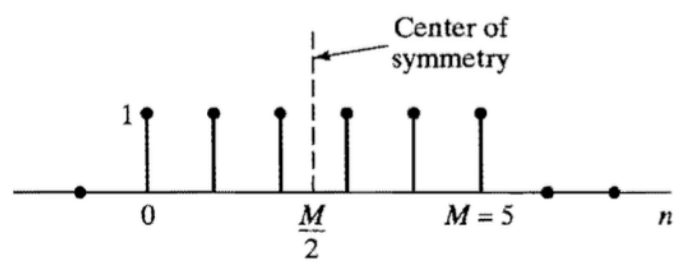
← integer delay

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$

$$H(e^{j\omega}) = \underbrace{\left[2h[0]\cos(3\omega/2) + 2h[1]\cos(\omega/2)\right]}_{A(\omega)} \; e^{-j3\omega/2}$$

FIR GLP: Type II

Type II Even Symmetry, M odd



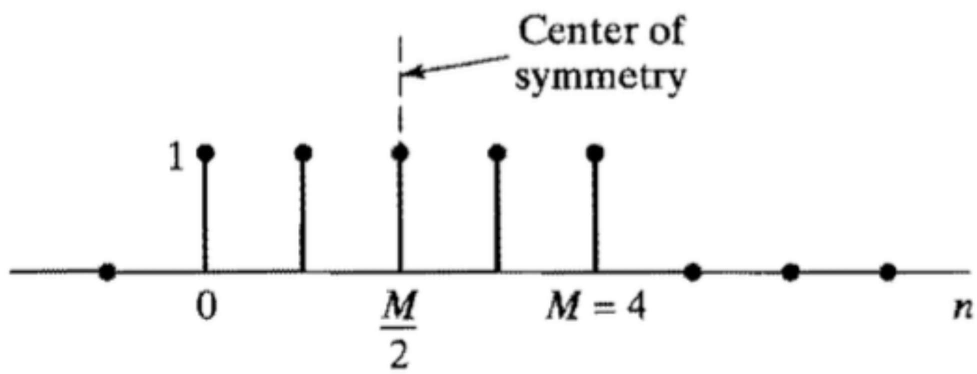
$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Even}} \; e^{-j\omega M/2}$$

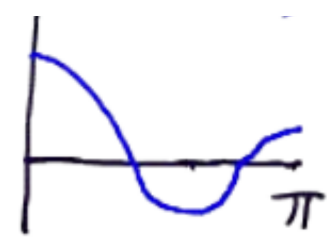
integer delay

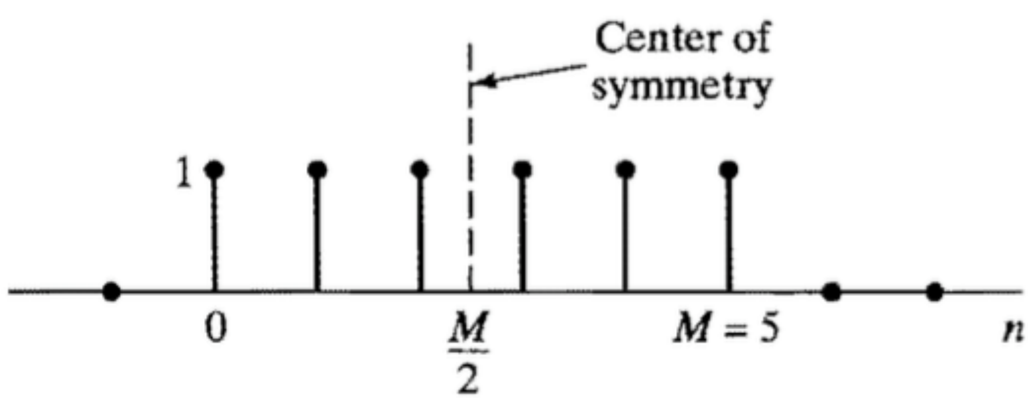
$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{(M+1)/2} b[k] \cos\left[\omega\left(k - \tfrac{1}{2}\right)\right] \right\}$$

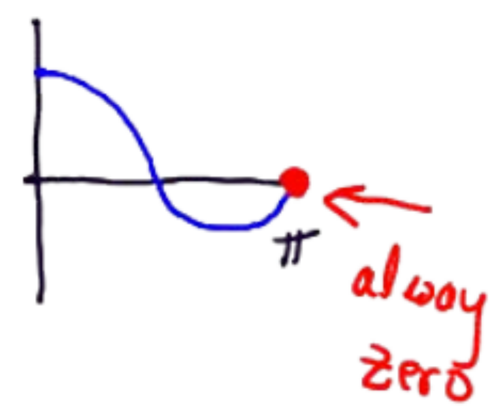
$$b[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2.$$

FIR GLP: Type I and II



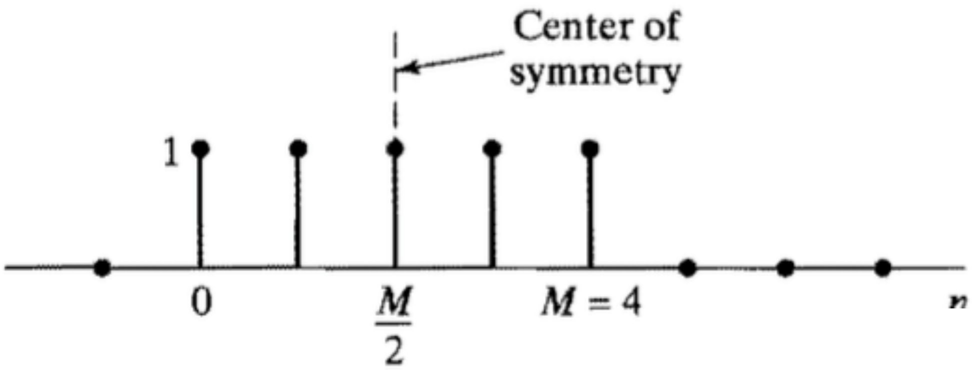


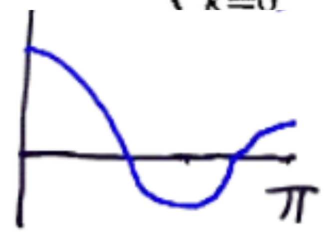




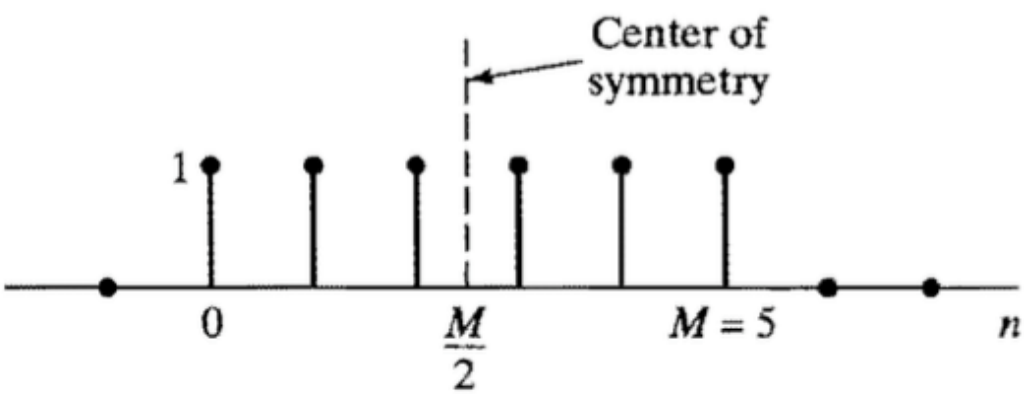
FIR GLP: Type I and II

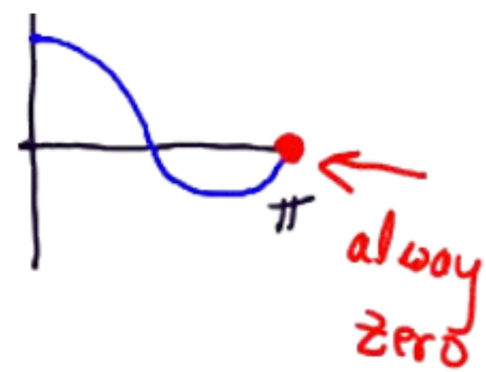
$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$





$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k-\frac{1}{2}\right)\right]\right\}$$

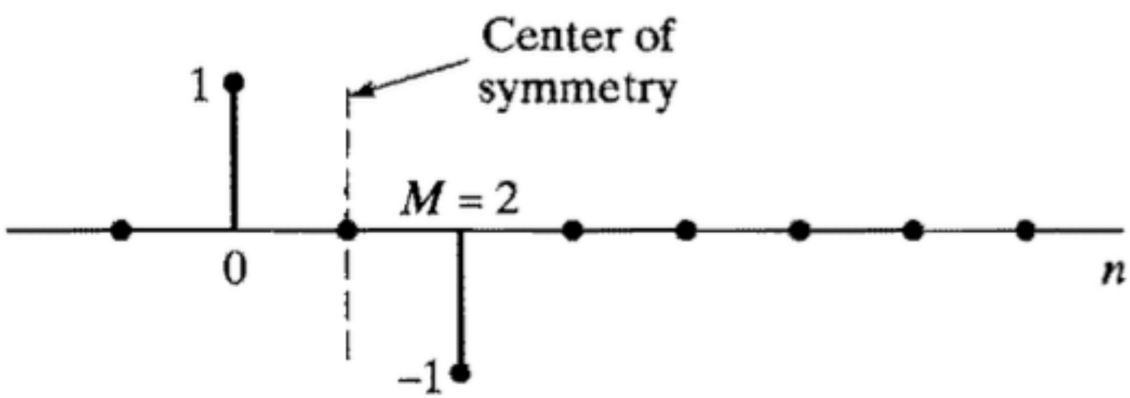




FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note\ \ h[M/2] = 0)$$



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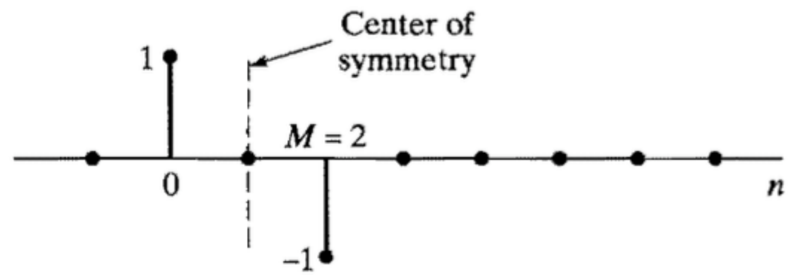

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III

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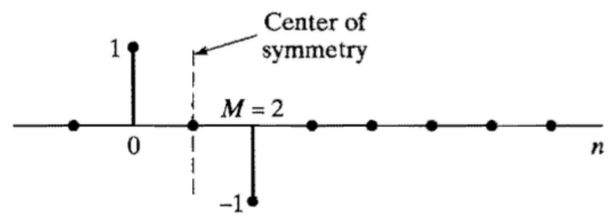

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}} \; e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III – Example, M=4

Type III Odd Symmetry, M even

$$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note\ \ h[M/2] = 0)$$

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$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$

$$= e^{-j2\omega}\left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega}\right]$$

$$= \underbrace{\left[2h[0]\sin(2\omega) + 2h[1]\sin(\omega)\right]}_{A(\omega)\ (odd)}\ j\,e^{-j2\omega}$$

FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (note\ \ h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

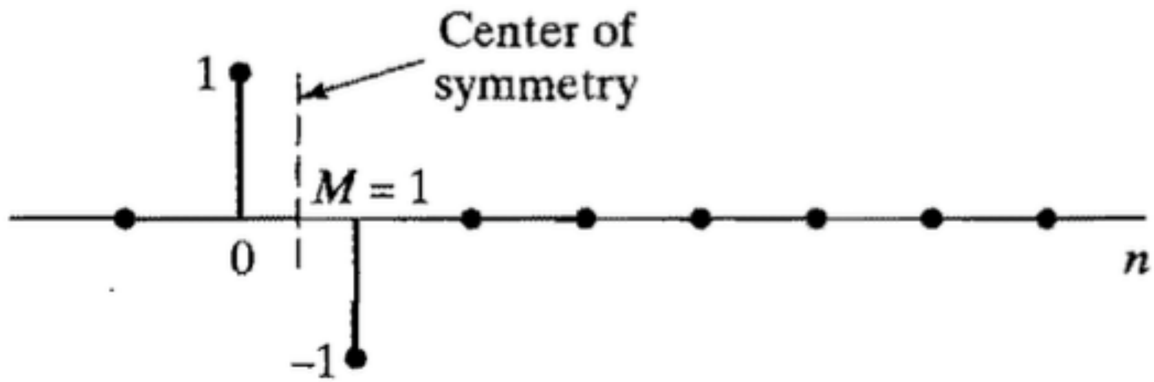
$$H(e^{j\omega}) = j e^{-j\omega M/2} \left[\sum_{k=1}^{M/2} c[k] \sin \omega k \right]$$

$$c[k] = 2h[(M/2) - k], \qquad k = 1, 2, \ldots, M/2.$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$



FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

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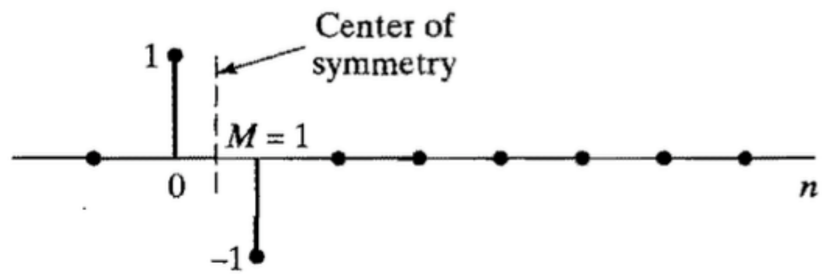
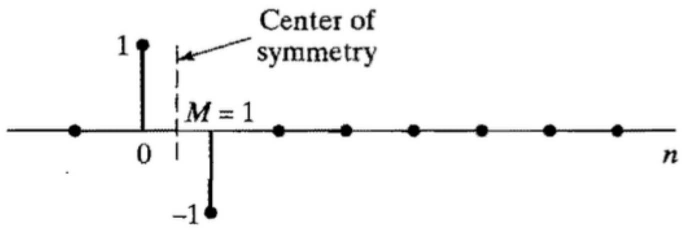


$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}} \, e^{-j\omega M/2 + j\pi/2}$$

← fractional delay

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd



$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Odd}} \; e^{-j\omega M/2 + j\pi/2}$$

fractional delay

FIR GLP: Type IV – Example, M=3

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$$

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real, Odd}} \; e^{-j\omega M/2 + j\pi/2}$$

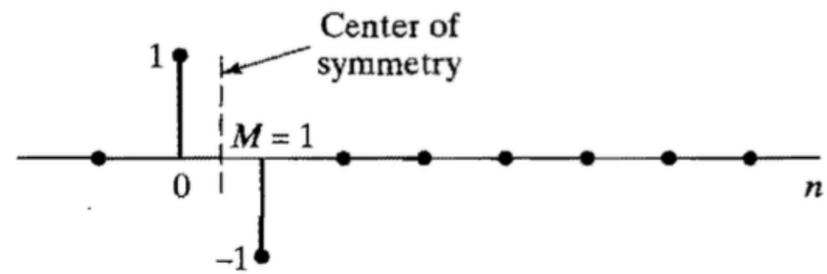
← fractional delay

$$H(e^{j\omega}) = \underbrace{\left[2h[0]\sin(3\omega/2) + 2h[1]\sin(\omega/2)\right]}_{A(\omega)} \; je^{-j3\omega/2}$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M-n], \quad n = 0,1,\ldots,M \qquad (M/2 \text{ not an integer})$$



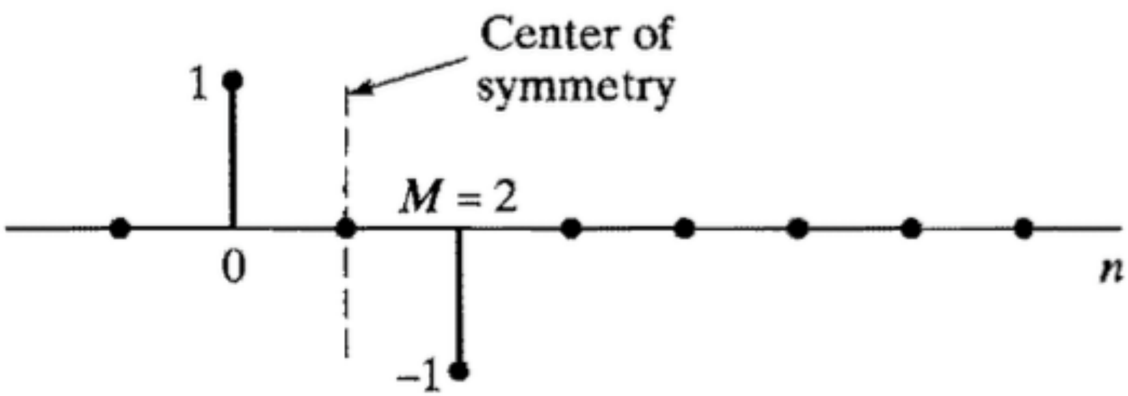
$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \underbrace{A(w)}_{\text{Real,Odd}}\ e^{-j\omega M/2 + j\pi/2}$$

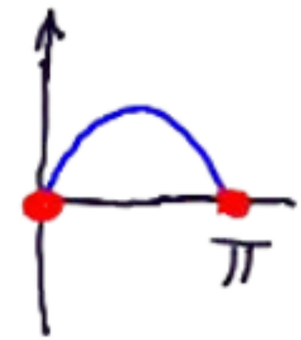
fractional delay

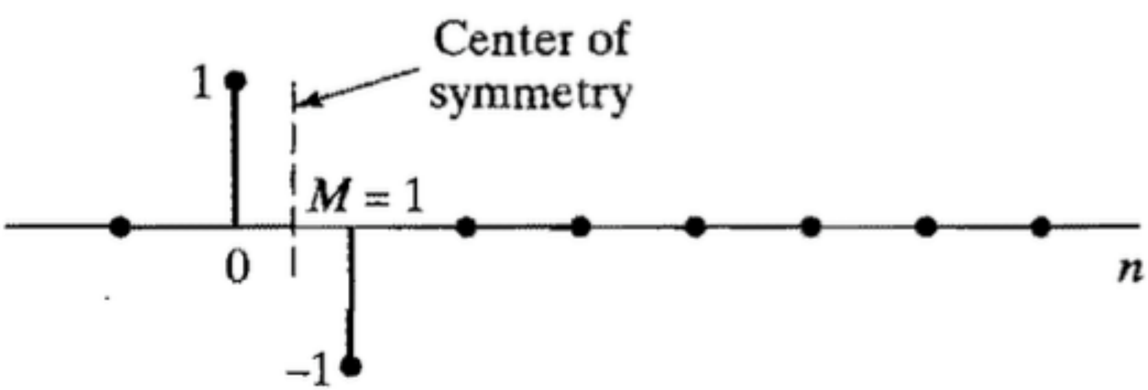
$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right]$$

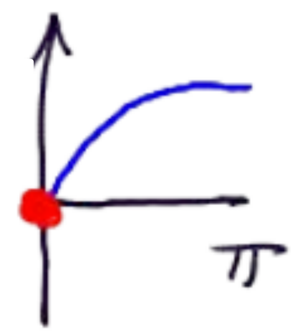
$$d[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2.$$

FIR GLP: Type III and IV



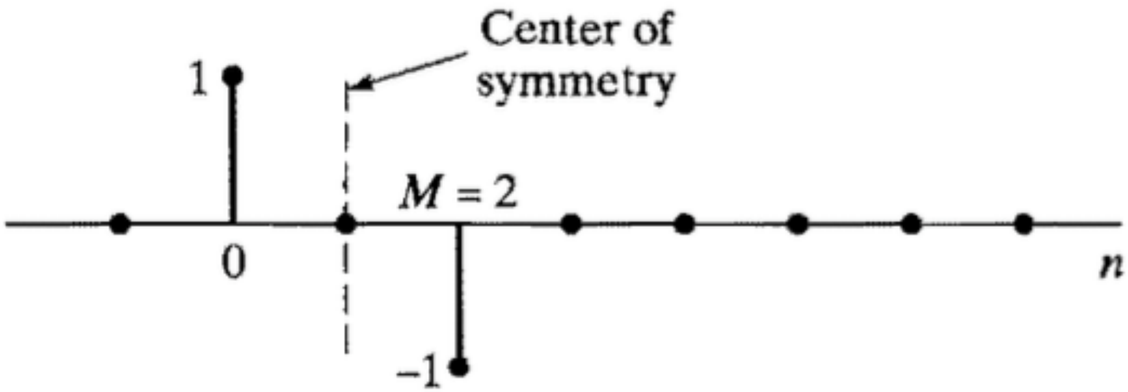


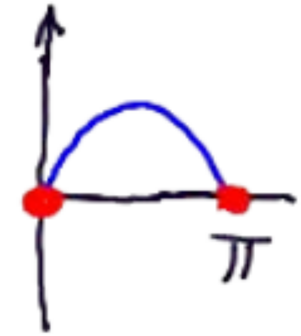


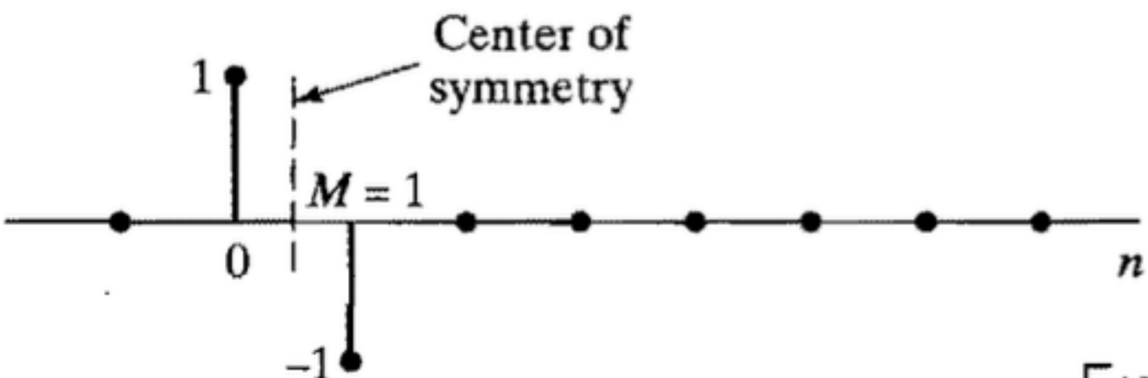


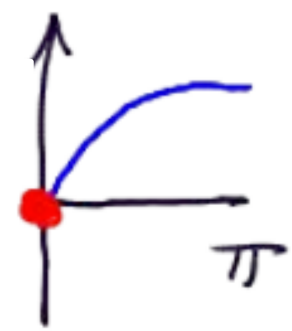
FIR GLP: Type III

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right]$$









$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right]$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

$$H(z) = \sum_{n=0}^{M} h[M-n] z^{-n} = \sum_{k=M}^{0} h[k] z^{k} z^{-M}$$

$$= z^{-M} H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

- If h[n] is real,

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

If z_0 is a zero then z_0^{-1} is also a zero.

- If h[n] is real,

If z_0 is a zero then z_0^{*} is also a zero.

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

z_0 $(z_0)^*$ $(z^{-1}_0)^*$ z^{-1}_0

Zeros of GLP System – Type I and II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

- If zero is on unit circle (r=1)

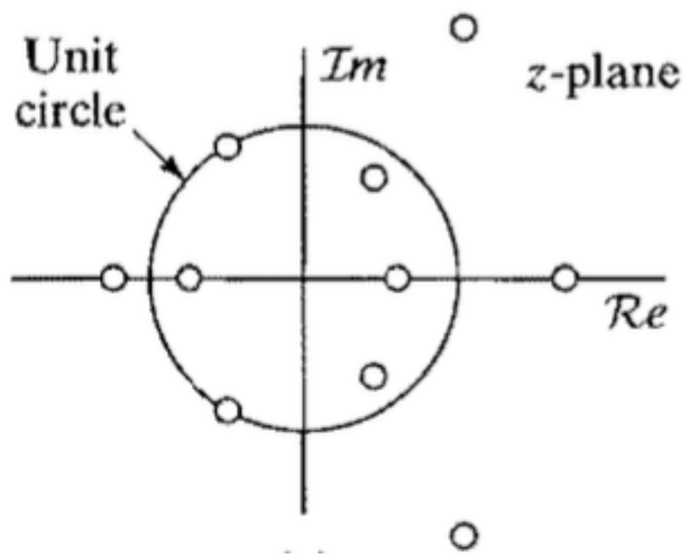
$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

- If zero is real and not on unit circle (θ=0)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

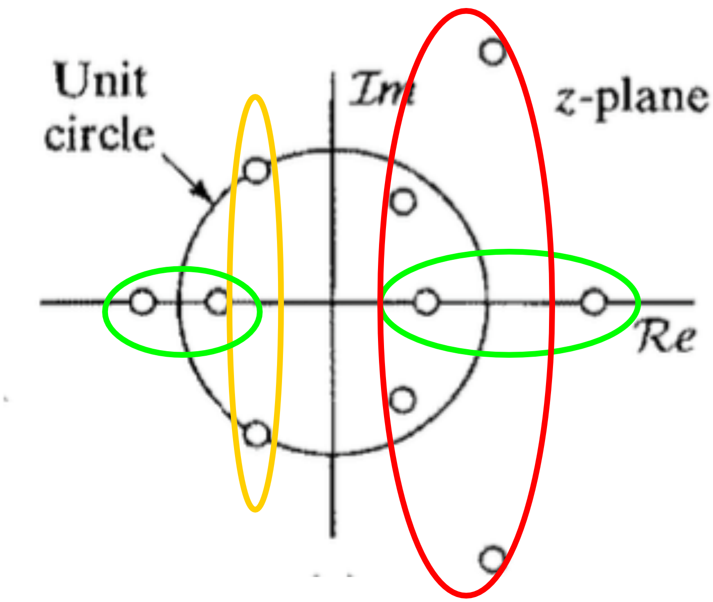
Zeros of GLP System – Type I

Type I



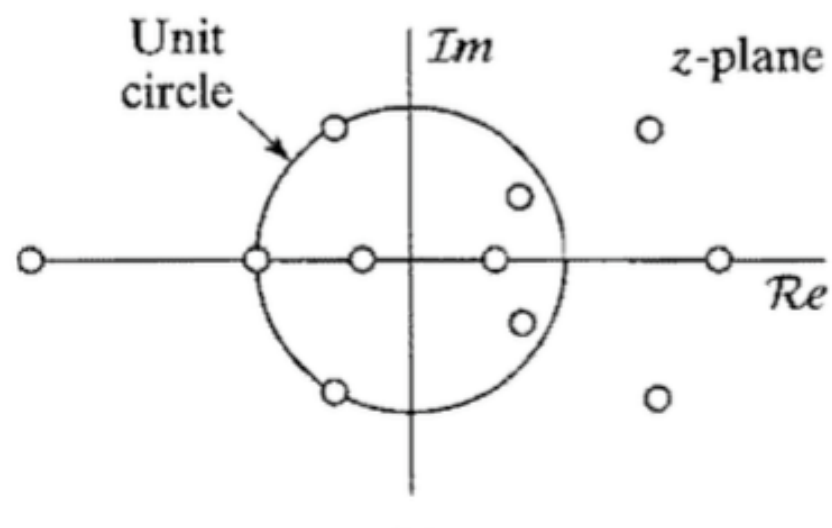
Zeros of GLP System – Type I

Type I



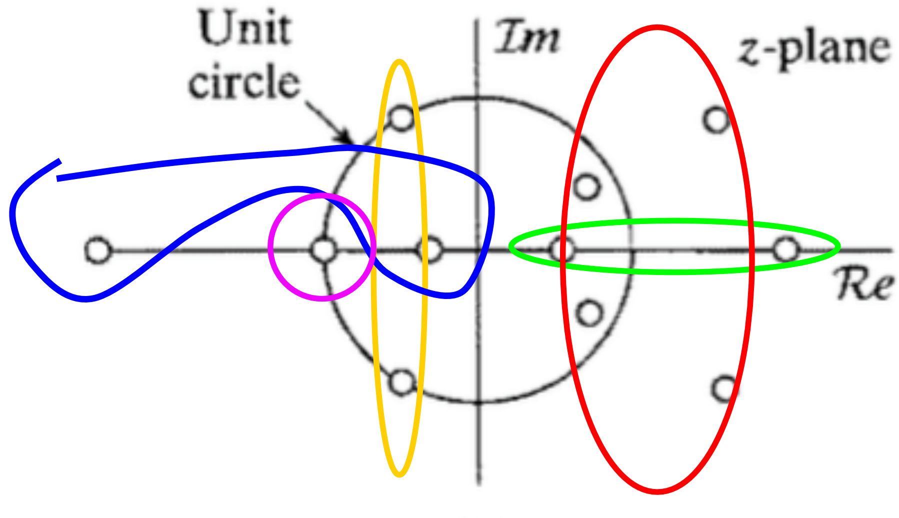
Zeros of GLP System – Type II

Type II



Zeros of GLP System – Type II

Type II



Zeros of GLP System – Type II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$
$$= z^{-M}H(z^{-1}).$$

Consider $z = -1$:

Zeros of GLP System – Type II

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M}$$

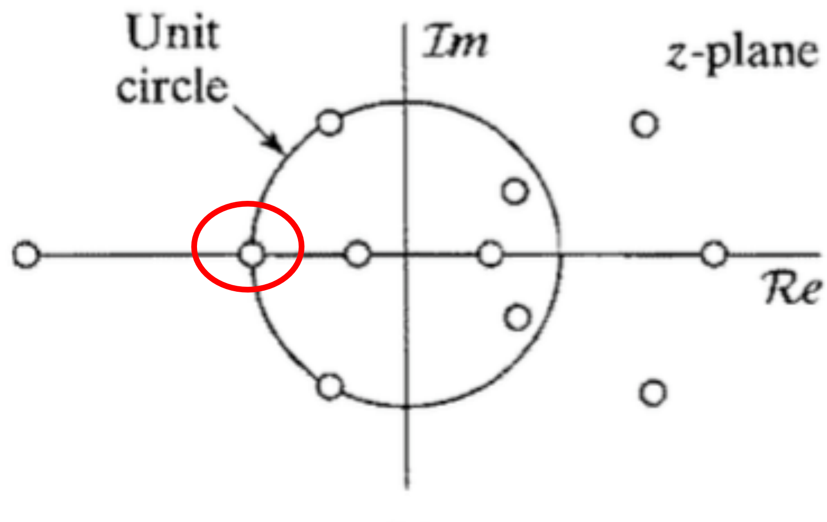
$$= z^{-M}H(z^{-1}).$$

Consider $z = -1$: $H(-1) = (-1)^{-M}H(-1)$

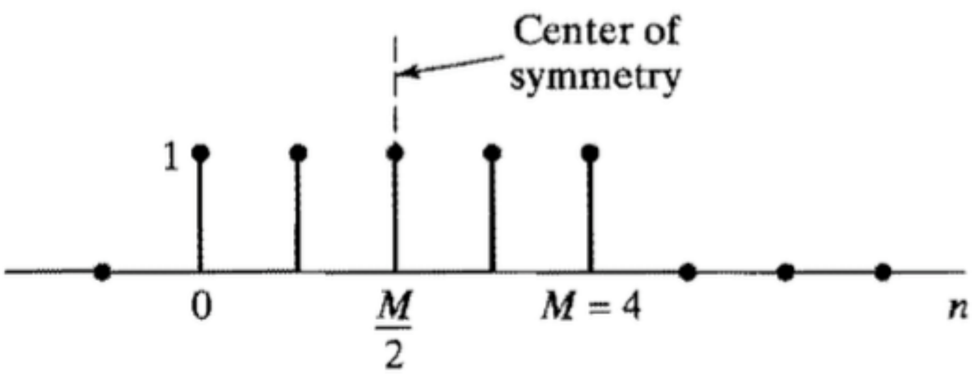
$\Rightarrow$ for M odd, $z = -1$ must be a zero (Type II)

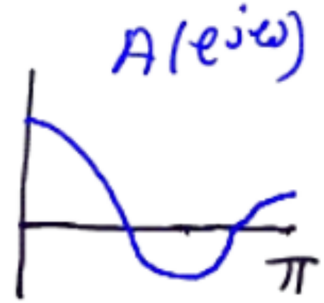
Zeros of GLP System – Type II

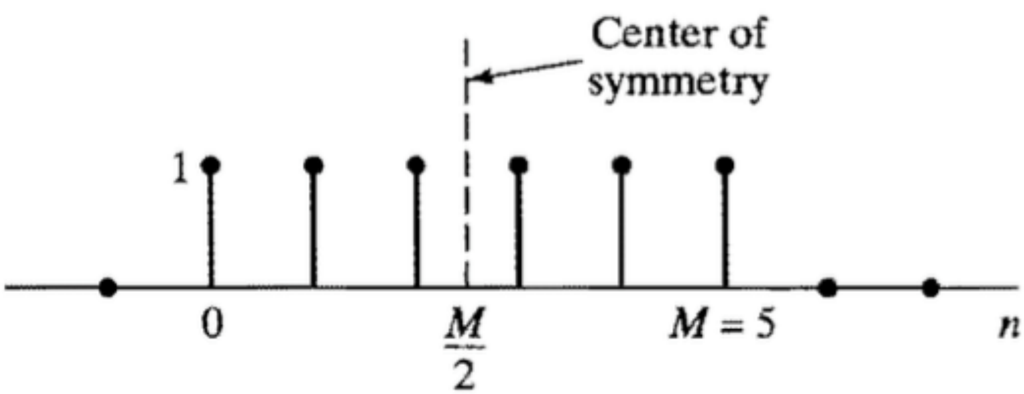
Type II

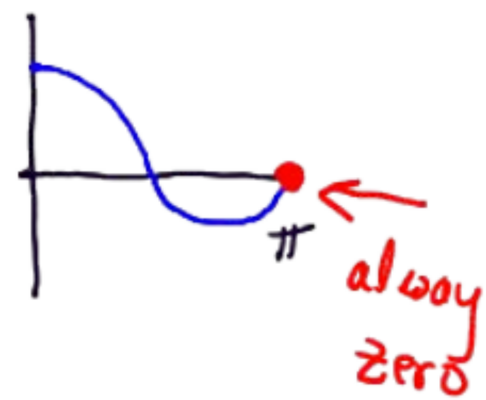


FIR GLP: Type I and II









Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

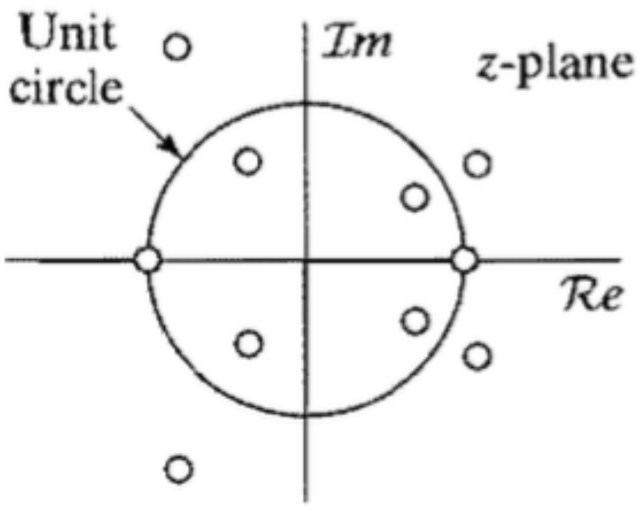
- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

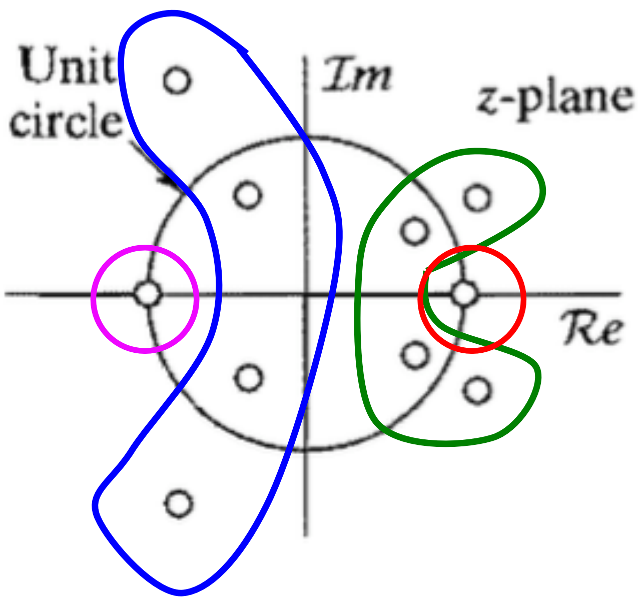
- If zero is real and not on unit circle (θ=0)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

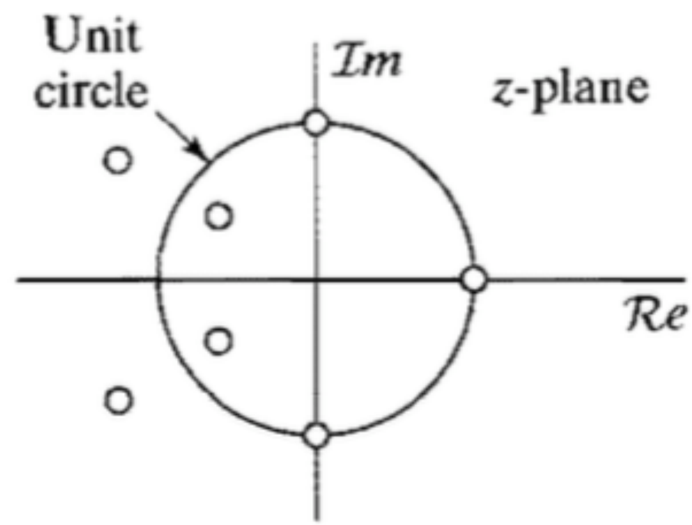
Zeros of GLP System – Type III



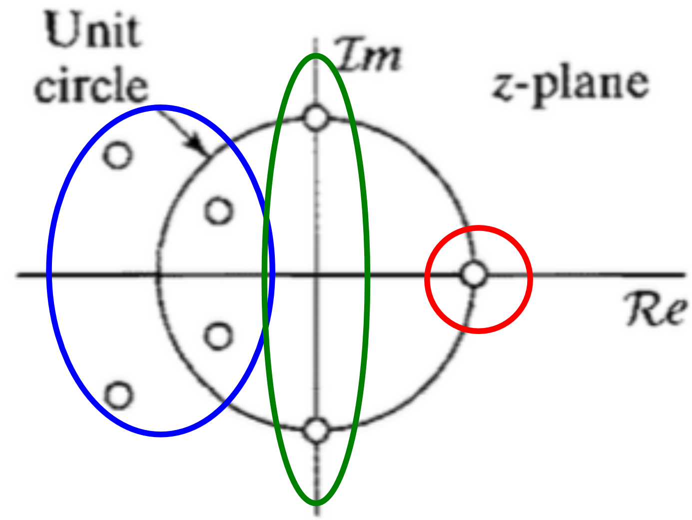
Zeros of GLP System – Type III



Zeros of GLP System – Type IV



Zeros of GLP System – Type IV



Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

$H(1) = -H(1) \Rightarrow z = 1$ must be a zero **(Type III and IV)**

Zeros of GLP System – Type III and IV

- FIR GLP System Function

$$H(z) = -z^{-M} H(z^{-1}).$$

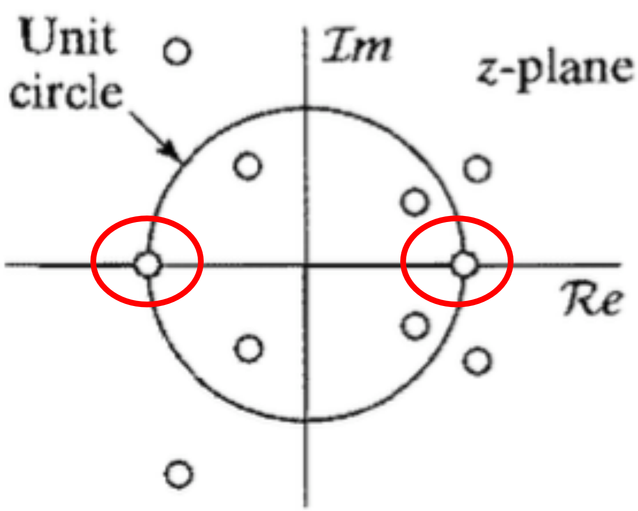
$H(1) = -H(1) \;\Rightarrow\; z = 1$ must be a zero **(Type III and IV)**

$H(-1) = -(-1)^{-M} H(-1)$

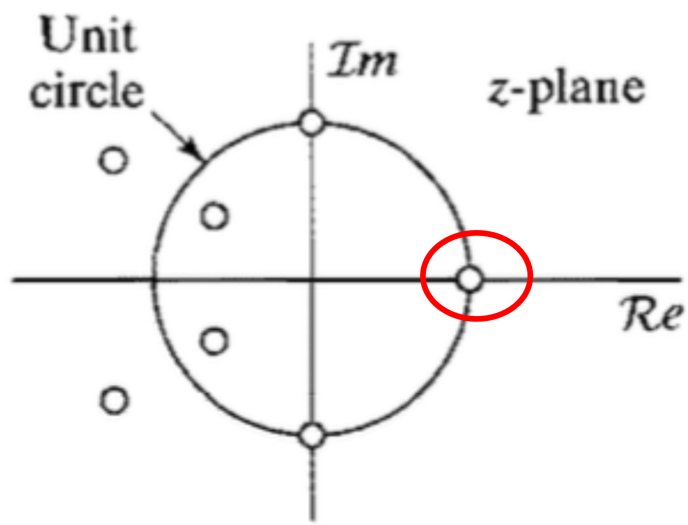
$\Rightarrow$ for M even, $z = -1$ must be a zero (Type III)

Zeros of GLP System – Type III and IV

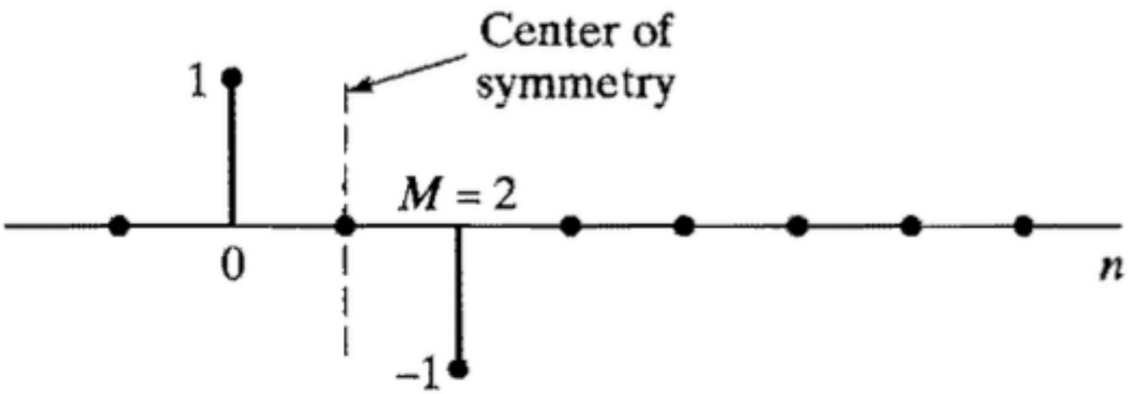
Type III

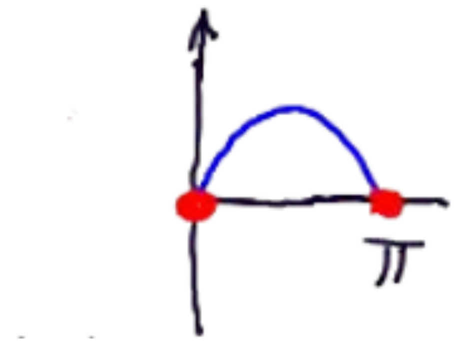


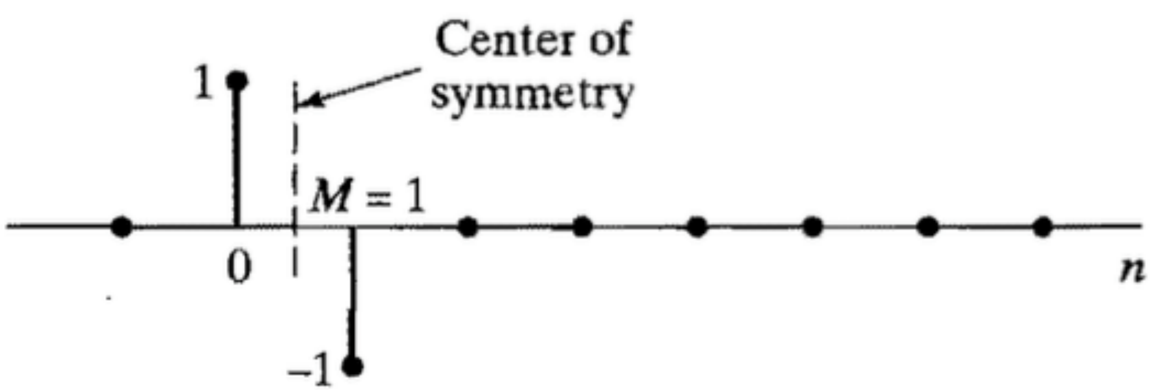
Type IV

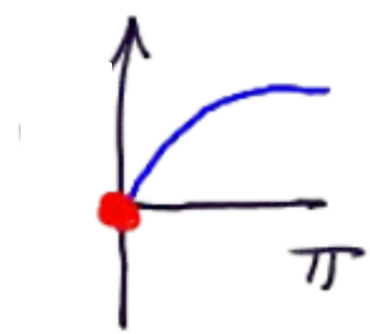


FIR GLP: Type III and IV









GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$$H(z) = H_{\min}(z) H_{uc}(z) H_{\max}(z)$$

- A min phase system, system containing only zeros on unit circle, and max phase system

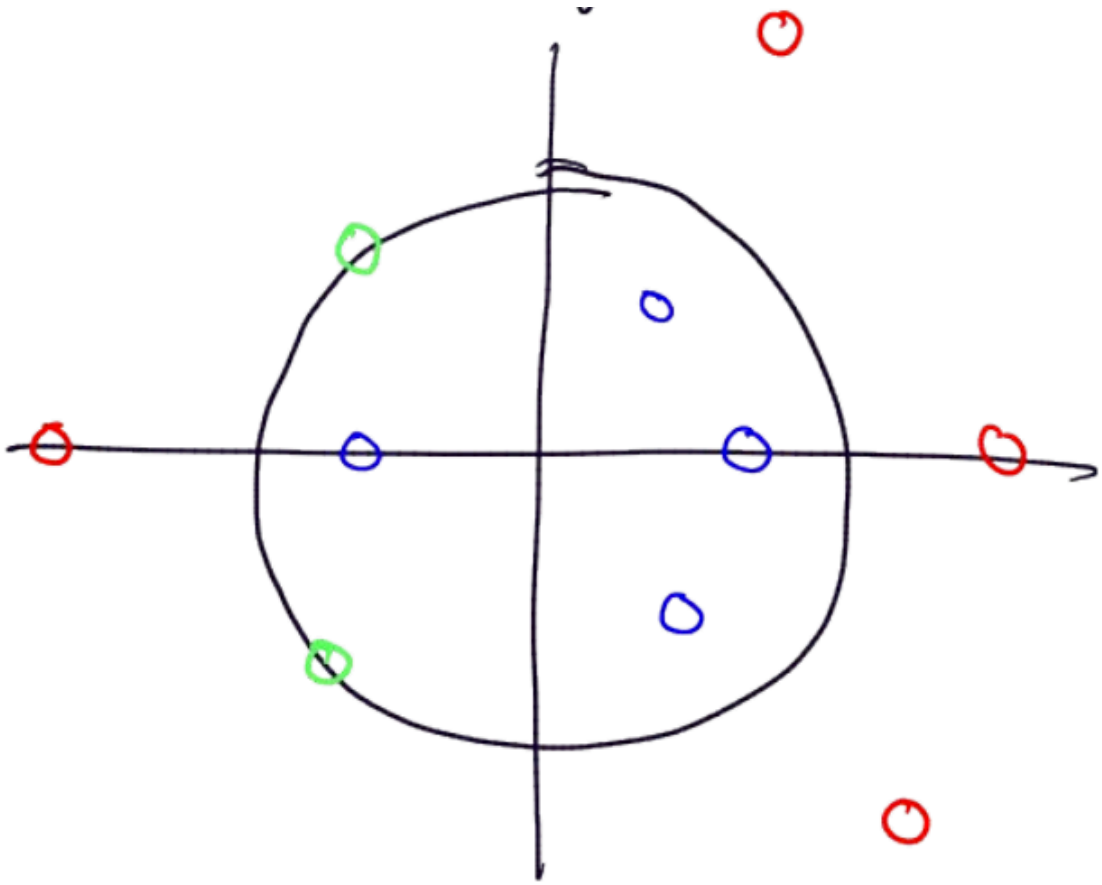
GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

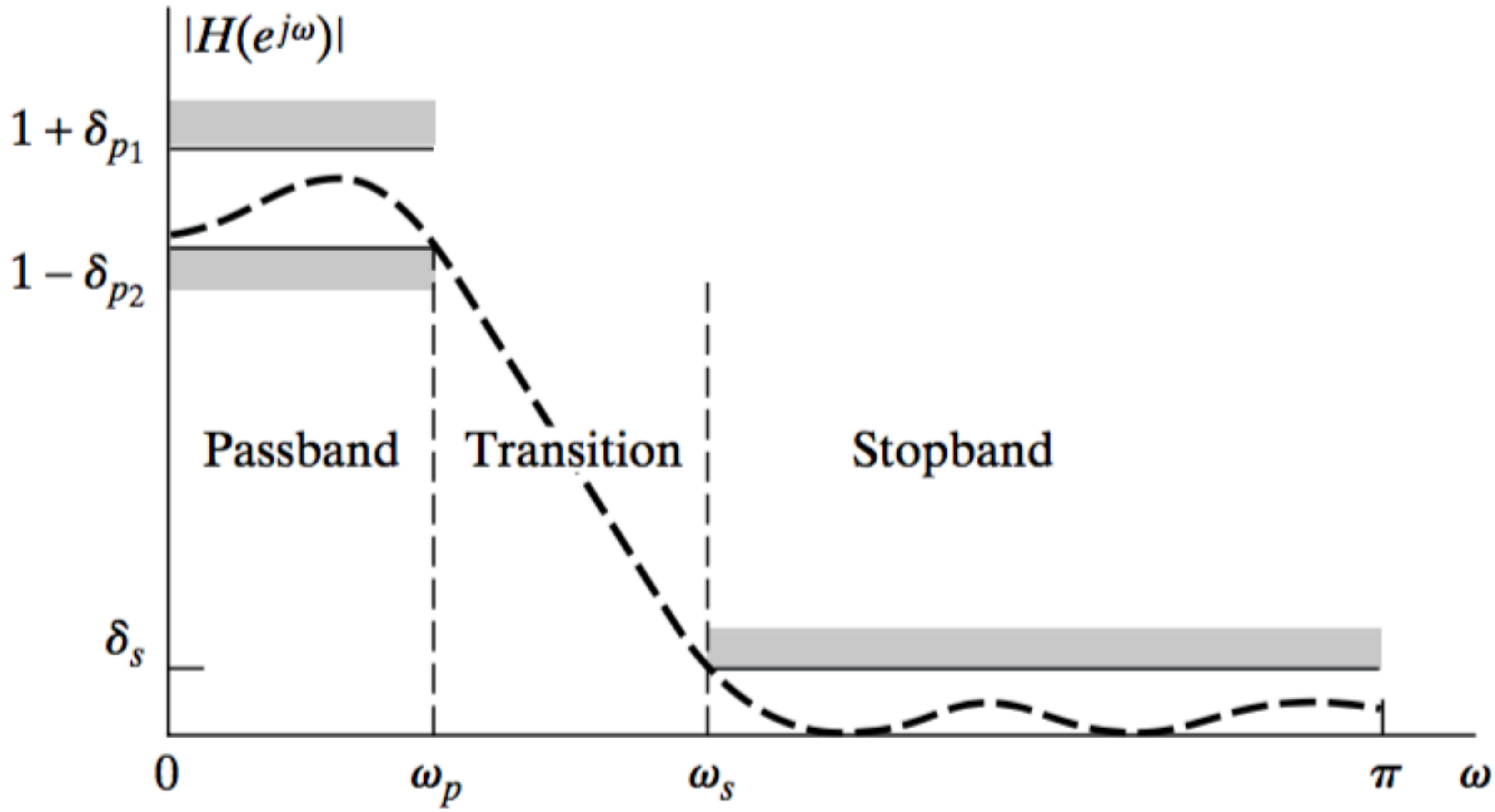
$H(z) =$

- A min phase sy

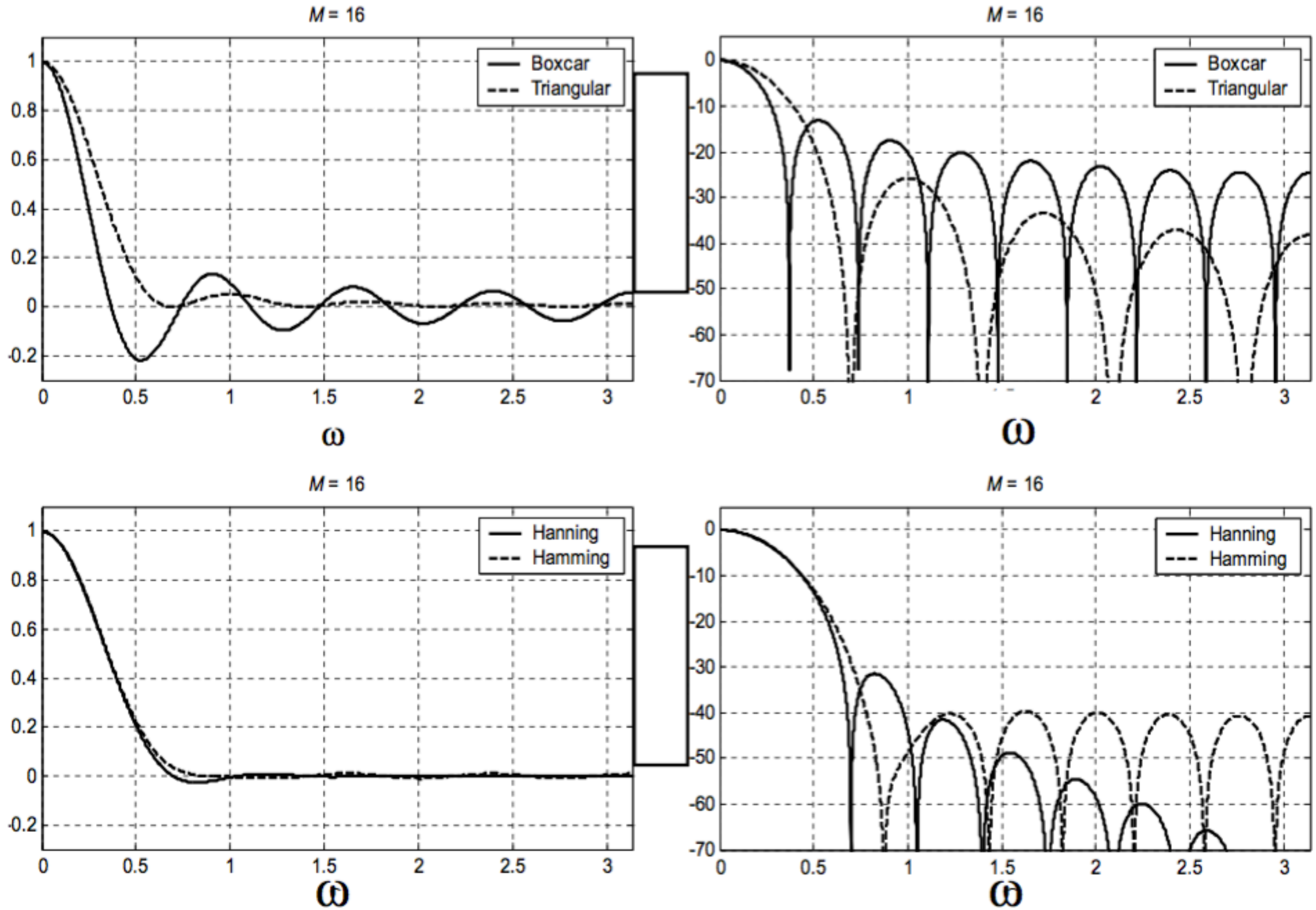
on unit circle,



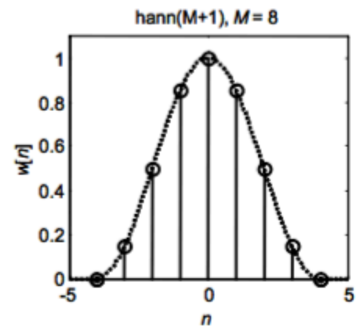
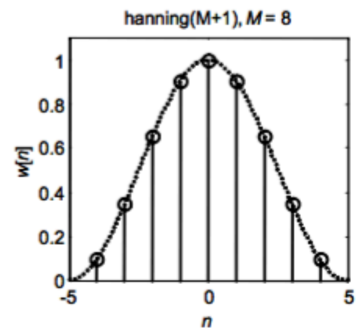
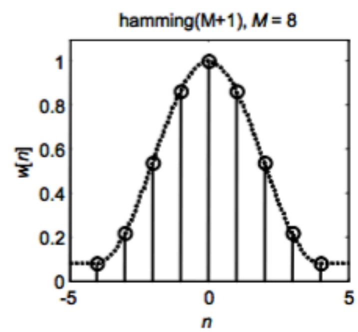
Filter Specifications



Tradeoff – Ripple vs. Transition Width



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Big Ideas

- Generalized Linear Phase Systems
  - Useful for design of causal FIR filters

Admin

- HW 6
  - Due Monday 3/22
- Project1 out now
  - Due Monday 4/5