ESE 531: Digital Signal Processing

Week 11

Lecture 20: March 24, 2021

Optimal Filter Design

Penn ESE 531 Spring 2021 – Khanna
Adapted from M. Lustig, EECS Berkeley

Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
  - Choose order and window type
  - Check DTFT to see if filter specs are met
- Optimal design
  - Design a filter h[n] with $H(e^{j\omega})$
  - Approximate $H_d(e^{j\omega})$ with some optimality criteria - or satisfies specs.

Mathematical Optimization

(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

- $x = (x_1, \ldots, x_n)$: optimization variables
- $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$: constraint functions

optimal solution $x^\star$ has smallest value of f_0 among all vectors that satisfy the constraints

Solving Optimization Problems

general optimization problem

- very difficult to solve
- methods involve some compromise, *e.g.*, very long computation time, or not always finding the solution

exceptions: certain problem classes can be solved efficiently and reliably

- least-squares problems
- linear programming problems
- convex optimization problems

Least-Squares Optimization

$$\text{minimize} \quad \|Ax - b\|_2^2$$

solving least-squares problems

- analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (*e.g.*, including weights, adding regularization terms)

Linear Programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \geq n$; less with structure
- a mature technology

using linear programming

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs (*e.g.*, problems involving ℓ_1- or ℓ_∞-norms, piecewise-linear functions)

Convex Optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

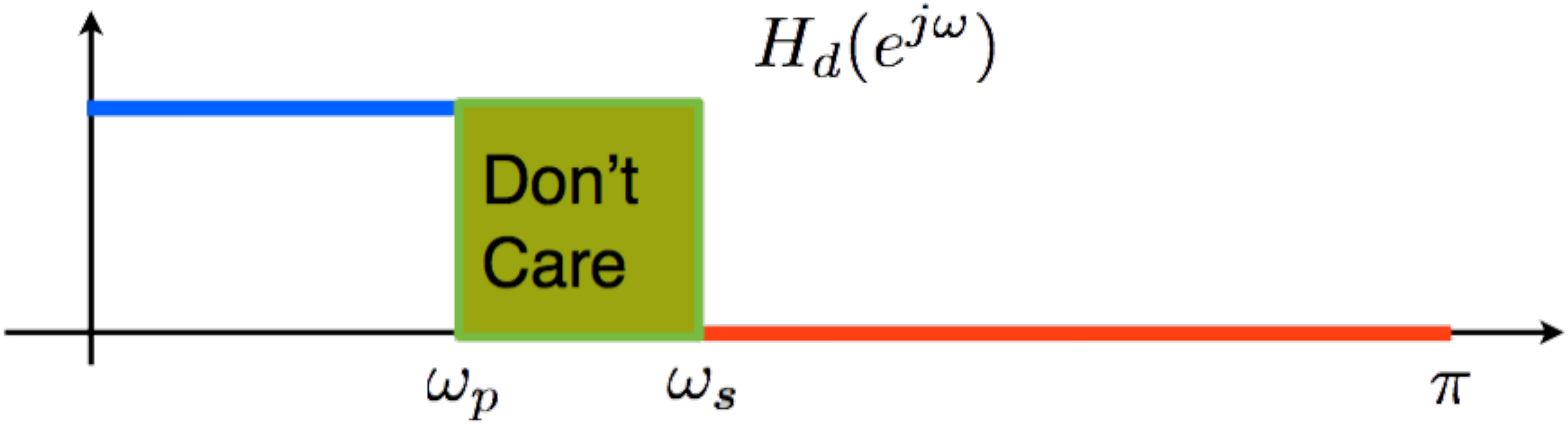
- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

- includes least-squares problems and linear programs as special cases

Optimality – Least Squares



- Least Squares:

$$\text{minimize} \quad \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Design Through Optimization

- Idea: Sample/discretize the frequency response

$$H(e^{j\omega}) \Rightarrow H(e^{j\omega_k})$$

- Sample points are fixed $\omega_k = k\frac{\pi}{P}$

$$-\pi \leq \omega_1 < \cdots < \omega_p \leq \pi$$

- M+1 is the filter order
- P >> M + 1 (rule of thumb P=15M)
- Yields a (good) approximation of the original problem

Example: Least Squares

- Target: Design M+1= 2N+1 filter
- First design non-causal $\tilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$

Example: Least Squares

- Target: Design M+1= 2N+1 filter
- First design non-causal $\tilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$

- Then, shift to make causal

$$h[n] = \tilde{h}[n - M/2]$$

$$H(e^{j\omega}) = e^{-j\frac{M}{2}}\tilde{H}(e^{j\omega})$$

Example: Least Squares

$$\tilde{h} = \left[\tilde{h}[-N], \tilde{h}[-N+1], \cdots, \tilde{h}[N]\right]^T$$

$$b = \left[H_d(e^{j\omega_1}), \cdots, H_d(e^{j\omega_P})\right]^T$$

$$A = \begin{bmatrix} e^{-j\omega_1(-N)} & \dots & e^{-j\omega_1(+N)} \\ \vdots & & \\ e^{-j\omega_P(-N)} & \dots & e^{-j\omega_P(+N)} \end{bmatrix}$$

$$\text{argmin}_{\tilde{h}} \; ||A\tilde{h} - b||^2$$

Least-Squares

$$\text{argmin}_{\tilde{h}} \quad ||A\tilde{h} - b||^2$$

Solution:

$$\tilde{h} = (A^*A)^{-1}A^*b$$

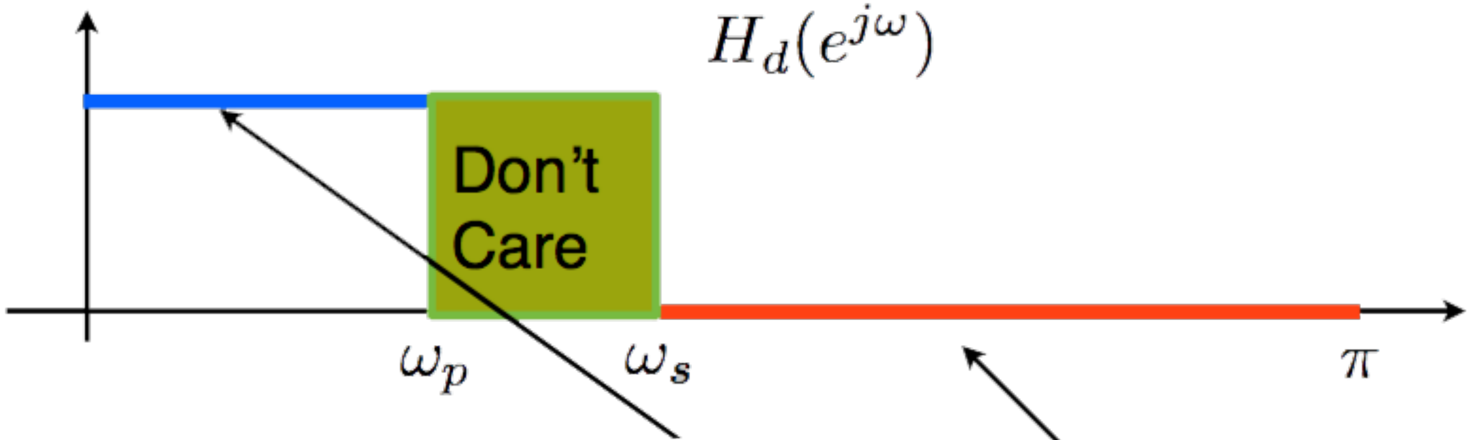
- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}(e^{j\omega})$ is real, $\tilde{h}[n]$ should have symmetry!

Design of Linear-Phase L.P Filter

- Suppose:
  - $\tilde{H}(e^{j\omega})$ is real-symmetric
  - M is even (M+1 length)
- Then:
  - $\tilde{h}[n]$ is real-symmetric around midpoint
- So:

$$
\begin{aligned}
\tilde{H}(e^{j\omega}) &= \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{+j\omega} \\
&\quad + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-2]e^{+j2\omega} \dots \\
&= \tilde{h}[0] + 2\cos(\omega)\tilde{h}[1] + 2\cos(2\omega)\tilde{h}[2] + \cdots
\end{aligned}
$$

Least-Squares Linear Phase Filter



Given M, ω_P, ω_s find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix}$$

$$b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

Capital P

Least-Squares Linear Phase Filter

Given M, ω_P, ω_s find the best LS filter:

$$A = \begin{bmatrix} 1 & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_p) \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ \vdots & & \\ 1 & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix} \qquad b = [1, 1, \cdots, 1, 0, 0, \cdots, 0]^T$$

$$\tilde{h}_+ = [\tilde{h}[0], \cdots, \tilde{h}[\frac{M}{2}]]^T = (A^*A)^{-1}A^*b$$

$$\tilde{h} = \begin{cases} \tilde{h}_+[n] & n \geq 0 \\ \tilde{h}_+[-n] & n < 0 \end{cases}$$

$$h[n] = \tilde{h}[n - M/2]$$

Extension:

- LS has no preference for pass band or stop band
- Use weighting of LS to change ratio

want to solve the discrete version of:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

where W(ω) is δp in the pass band and δs in stop band

Similarly: W(ω) is 1 in the pass band and δp/δs in stop band

Weighted Least-Squares

$$\mathrm{argmin}_{\tilde{h}_+} \quad (A\tilde{h}_+ - b)^* W^2 (A\tilde{h}_+ - b)$$

Solution:

$$\tilde{h}_+ = (A^* W^2 A)^{-1} W^2 A^* b$$

$$W = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \cdots & & & \\ & & & \frac{\delta_p}{\delta_s} & & \\ & & & & \cdots & \\ 0 & & & & & \frac{\delta_p}{\delta_s} \end{bmatrix}$$

Optimality – min-max

- Chebychev Design (min-max)

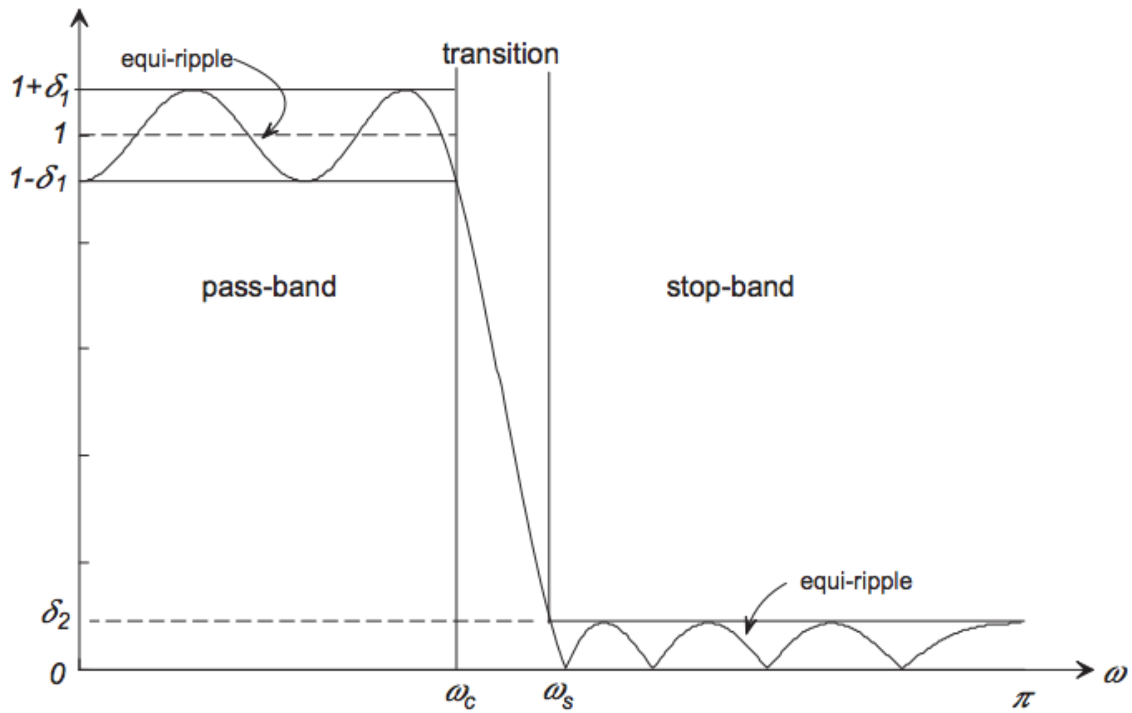
$$\text{minimize}_{\omega \in \text{care}} \quad \max \left| H(e^{j\omega}) - H_d(e^{j\omega}) \right|$$

  - Parks-McClellan algorithm - equiripple
  - Also known as Remez exchange algorithms (signal.remez)
  - Can also use convex optimization

Parks-McClellan

- Allows for multiple pass- and stop-bands.
- Is an equi-ripple design in the pass- and stop-bands, but allows independent weighting of the ripple in each band.
- Allows specification of the band edges.

Parks-McClellan: LP Filter



- For the low-pass filter shown above the specifications are

$$\begin{array}{rcll} 1-\delta_1 & < & H(e^{j\omega}) < 1+\delta_1 & \text{in the pass-band } 0 < \omega \leq \omega_c \\ -\delta_2 & < & H(e^{j\omega}) < \delta_2 & \text{in the stop-band } \omega_s < \omega \leq \pi. \end{array}$$

Parks-McClellan: LP Filter

- Need to determine M+1 (length of the filter) and the filter coefficients $\{h_n\}$

Parks-McClellan: LP Filter

- Need to determine M+1 (length of the filter) and the filter coefficients $\{h_n\}$

- If we assume M even and even symmetry FIR filter (Type I), then

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n).$$

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}.$$

Parks-McClellan: LP Filter

- Reformulate

$$A_e(e^{j\omega}) = h_e[0] + \sum_{n=1}^{L} 2h_e[n]\cos(\omega n).$$

- To fitting a polynomial

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k,$$

$$A_e(e^{j\omega}) = P(x)|_{x=\cos\omega}, \qquad P(x) = \sum_{k=0}^{L} a_k x^k.$$

Parks-McClellan: LP Filter

- Define approximation error function

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})],$$

$$W(e^{j\omega}) = \begin{cases} \delta_2/\delta_1 & \text{in the pass-band} \\ 1 & \text{in the stop-band} \\ 0 & \text{in the transition band.} \end{cases}$$

Parks-McClellan: LP Filter

- Define approximation error function

$$E(\omega) = W(\omega)[H_d(e^{j\omega}) - A_e(e^{j\omega})],$$

- Apply min-max or Chebyshev criteria

$$\min_{\{h_e[n]:0\le n\le L\}} \left(\max_{\omega \in F} |E(\omega)| \right),$$

Min-Max Filter Design

- Constraints:
  - min-max pass-band ripple

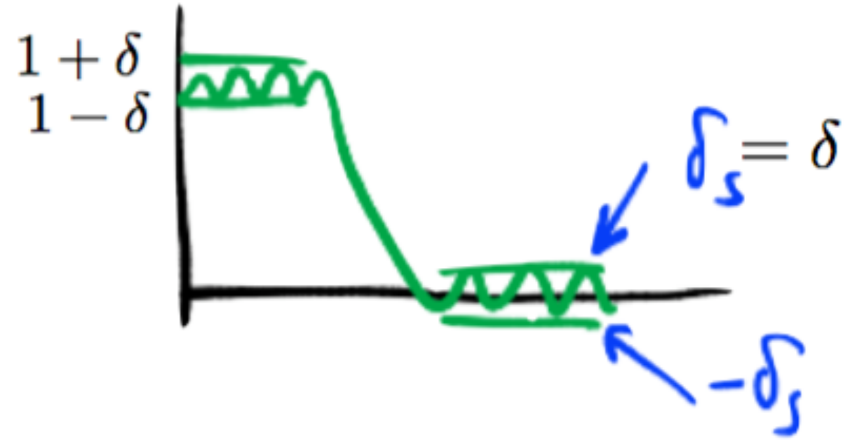
$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p, \qquad 0 \leq w \leq \omega_p$$

  - min-max stop-band ripple

$$|H(e^{j\omega})| \leq \delta_s, \qquad \omega_s \leq w \leq \pi$$

Min-Max Ripple Design

- Given ω_p, ω_s, M, find δ, $\tilde{h}_+$



$$
\begin{aligned}
&\text{minimize} \quad \delta \\
&\text{Subject to :} \\
&\quad 1-\delta \le \tilde{H}(e^{j\omega_k}) \le 1+\delta \qquad 0 \le \omega_k \le \omega_p \\
&\quad -\delta \le \tilde{H}(e^{j\omega_k}) \le \delta \qquad \omega_s \le \omega_k \le \pi \\
&\quad \delta > 0
\end{aligned}
$$

- Formulation is a linear program with solution δ, $\tilde{h}_+$
- A well studied class of problems with good solvers

Min-Max Ripple via LP

$$\text{minimize} \quad \delta$$

$$\text{subject to :}$$

$$1 - \delta \preceq A_p \tilde{h}_+ \preceq 1 + \delta$$

$$-\delta \preceq A_s \tilde{h}_+ \preceq \delta$$

$$\delta > 0$$

$$A_p = \begin{bmatrix} 1 & 2\cos(\omega_1) & \cdots & 2\cos(\frac{M}{2}\omega_1) \\ & \vdots & & \\ 1 & 2\cos(\omega_p) & \cdots & 2\cos(\frac{M}{2}\omega_p) \end{bmatrix}$$

$$A_s = \begin{bmatrix} 1 & 2\cos(\omega_s) & \cdots & 2\cos(\frac{M}{2}\omega_s) \\ & \vdots & & \\ 1 & 2\cos(\omega_P) & \cdots & 2\cos(\frac{M}{2}\omega_P) \end{bmatrix}$$

capital P

Parks-McClellan

- The method is based on reformulating the problem as one in polynomial approximation, using Chebyshev polynomials

$$A_e(e^{j\omega}) = \sum_{k=0}^{L} a_k(\cos\omega)^k,$$

$$A_e(e^{j\omega}) = P(x)|_{x=\cos\omega}, \qquad P(x) = \sum_{k=0}^{L} a_k x^k.$$

Parks-McClellan – Alternation Theorem

- The algorithm uses Chebyshev's alternation theorem to recognize the optimal solution.

Define the error $E(x)$ as above, namely

$$E(\mathrm{e}^{\mathrm{j}\,\omega}) = W(\mathrm{e}^{\mathrm{j}\,\omega})\left(H_d(\mathrm{e}^{\mathrm{j}\,\omega}) - H(\mathrm{e}^{\mathrm{j}\,\omega})\right)$$

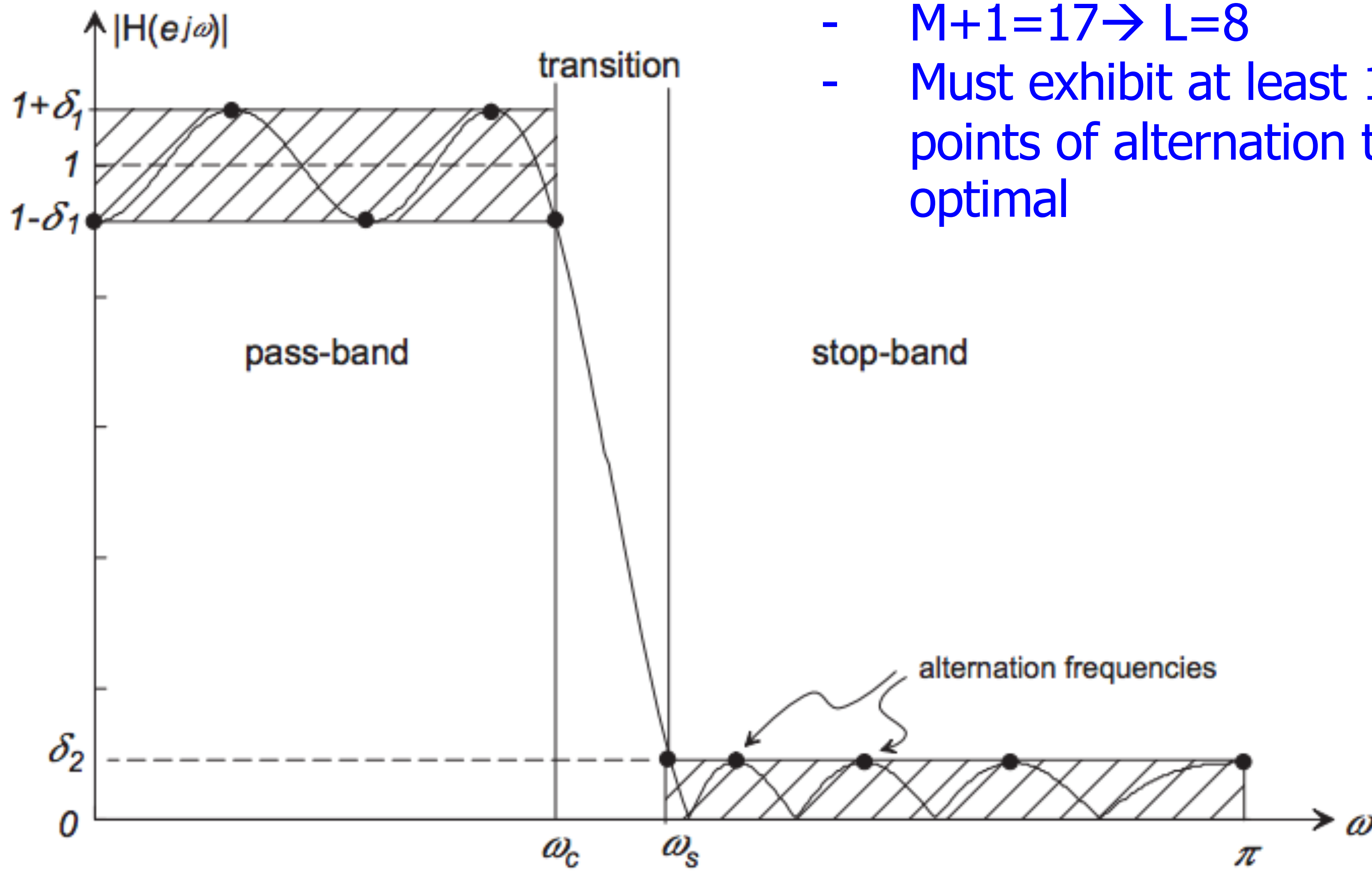
and the maximum error as

$$\|E(\mathrm{e}^{\mathrm{j}\,\omega})\|_\infty = \mathrm{argmax}_{x\in\Omega}\left|E(\mathrm{e}^{\mathrm{j}\,\omega})\right|$$

A necessary and sufficient condition that $H(\mathrm{e}^{\mathrm{j}\,\omega})$ is the unique Lth-order polynomial minimizing $\|E(\mathrm{e}^{\mathrm{j}\,\omega})\|_\infty$ is that $E(\mathrm{e}^{\mathrm{j}\,\omega})$ exhibit at least $L+2$ extremal frequencies, or "alternations", that is there must exist at least $L+2$ values of ω, $\omega_k \in \Omega$, $k = [0, 1, \ldots, L+1]$, such that $\omega_0 < \omega_1 < \ldots < \omega_{L+1}$, and such that

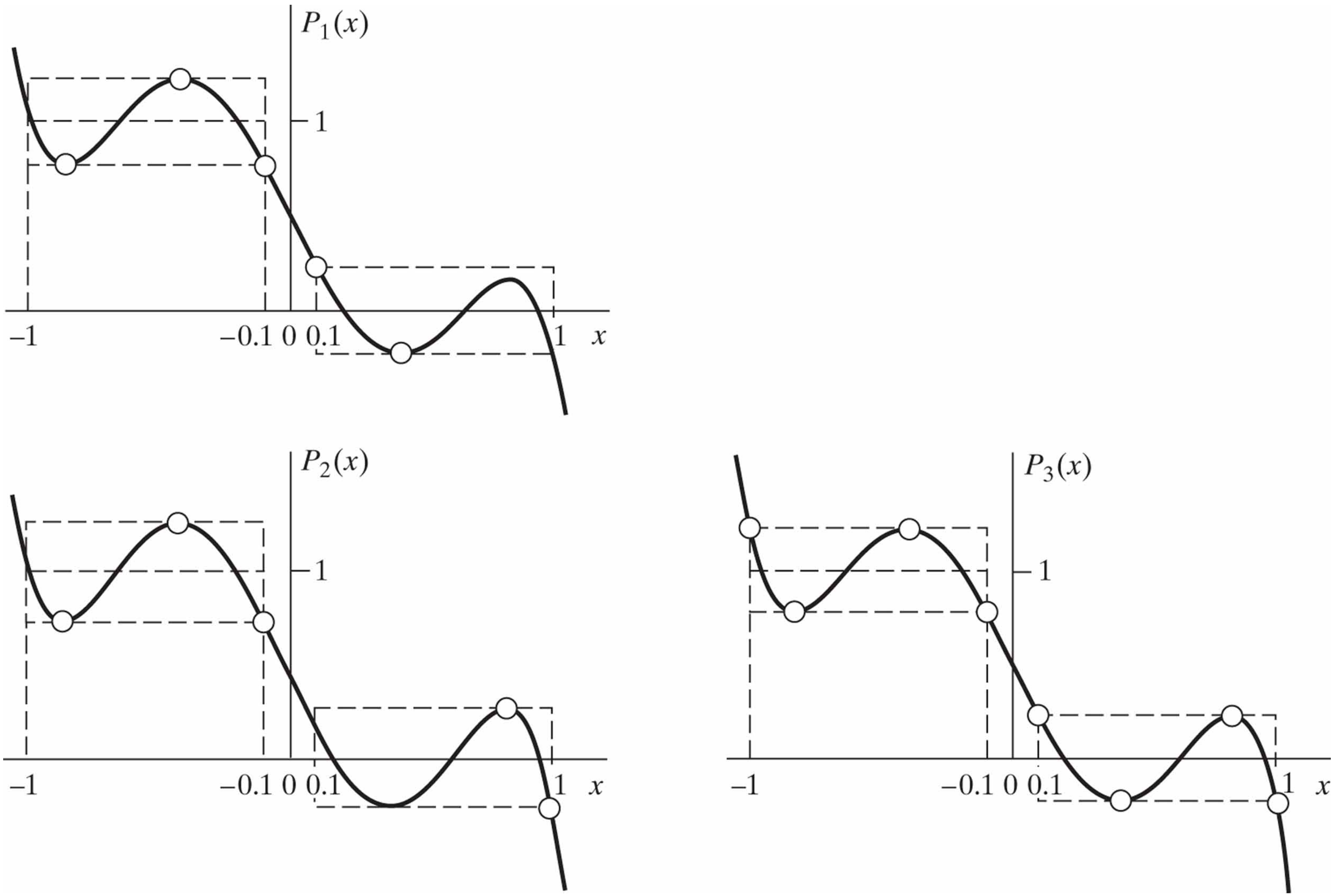
$$E(\mathrm{e}^{\mathrm{j}\,\omega_k}) = -E(\mathrm{e}^{\mathrm{j}\,\omega_{k+1}}) = \pm\left(\|E(\mathrm{e}^{\mathrm{j}\,\omega})\|_\infty\right).$$

Parks-McClellan – Alternation Theorem

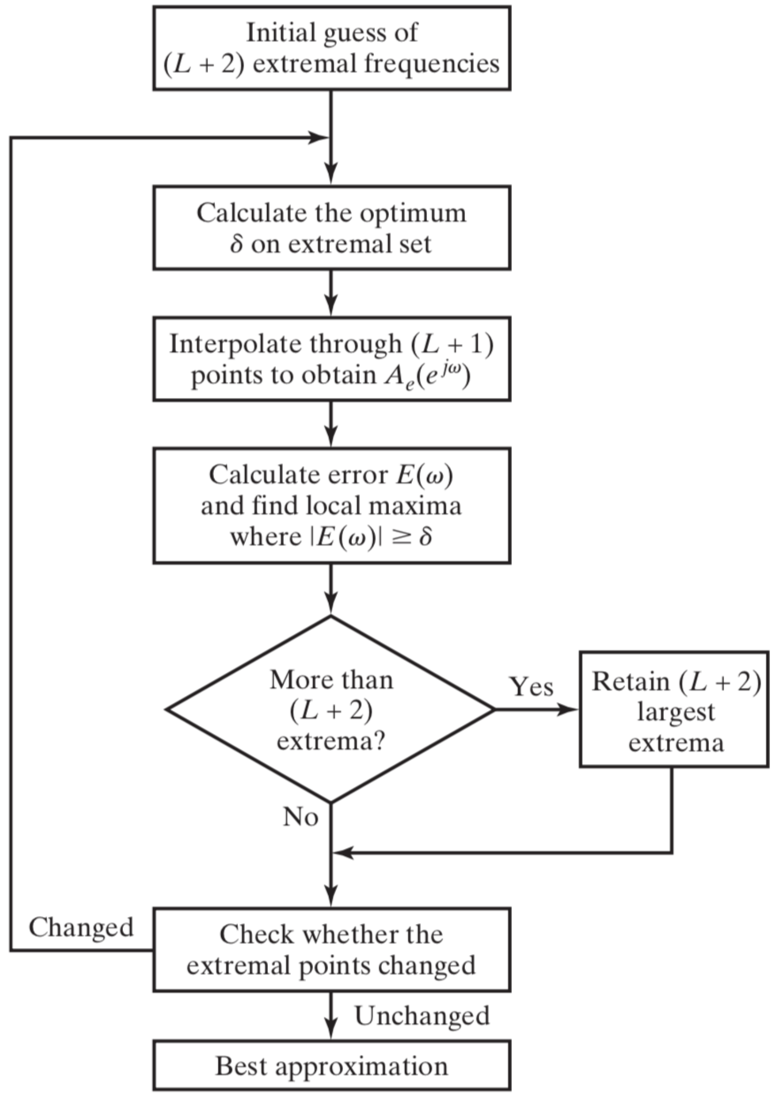


- M+1=17→ L=8
- Must exhibit at least 10 points of alternation to be optimal

Alternation Theorem Example – 5^{th} order



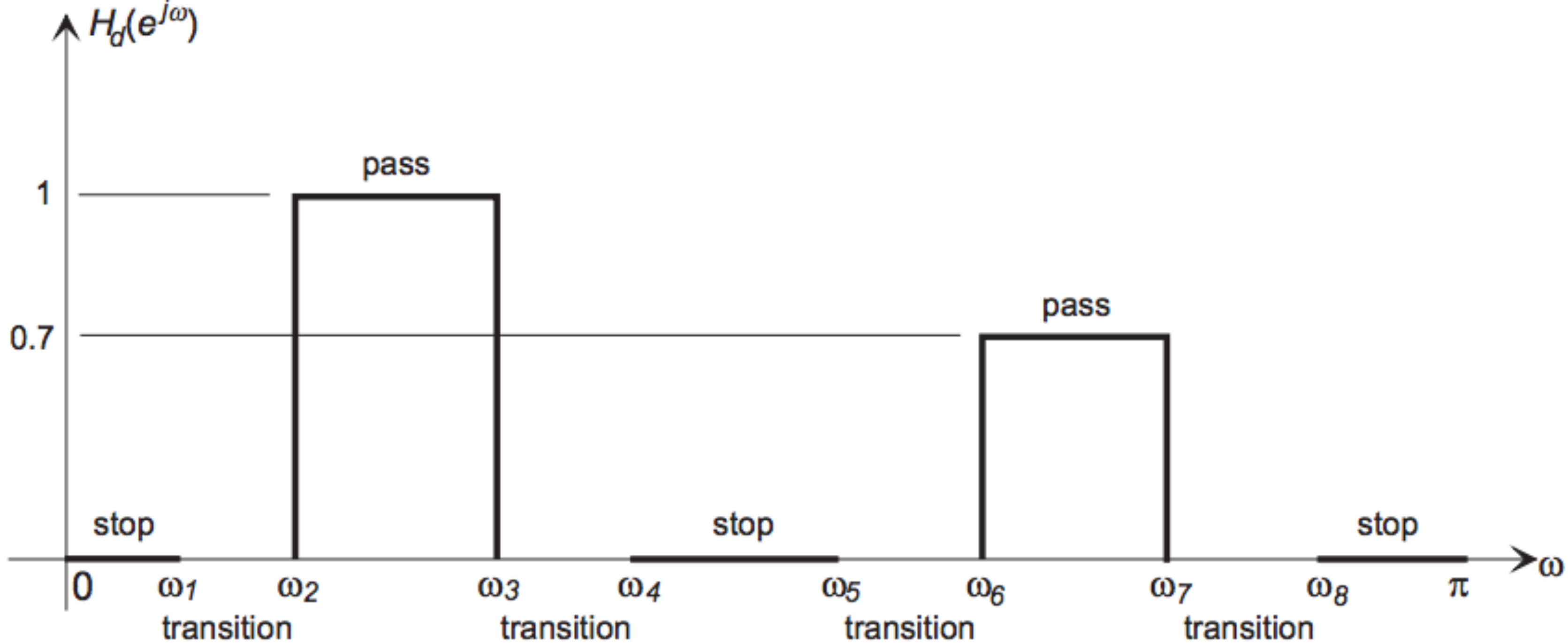
Parks–McClellan algorithm



MATLAB Parks-McClellan Function

- `b = firpm(M,F,A,W)`
  - `b` is the array of filter coefficients (impulse response)
  - `M` is the filter order (`M+1` is the length of the filter),
  - `F` is a vector of band edge frequencies in ascending order
  - `A` is a set of filter gains at the band edges
  - `W` is an optional set of relative weights to be applied to each of the bands

MATLAB Parks-McClellan Function



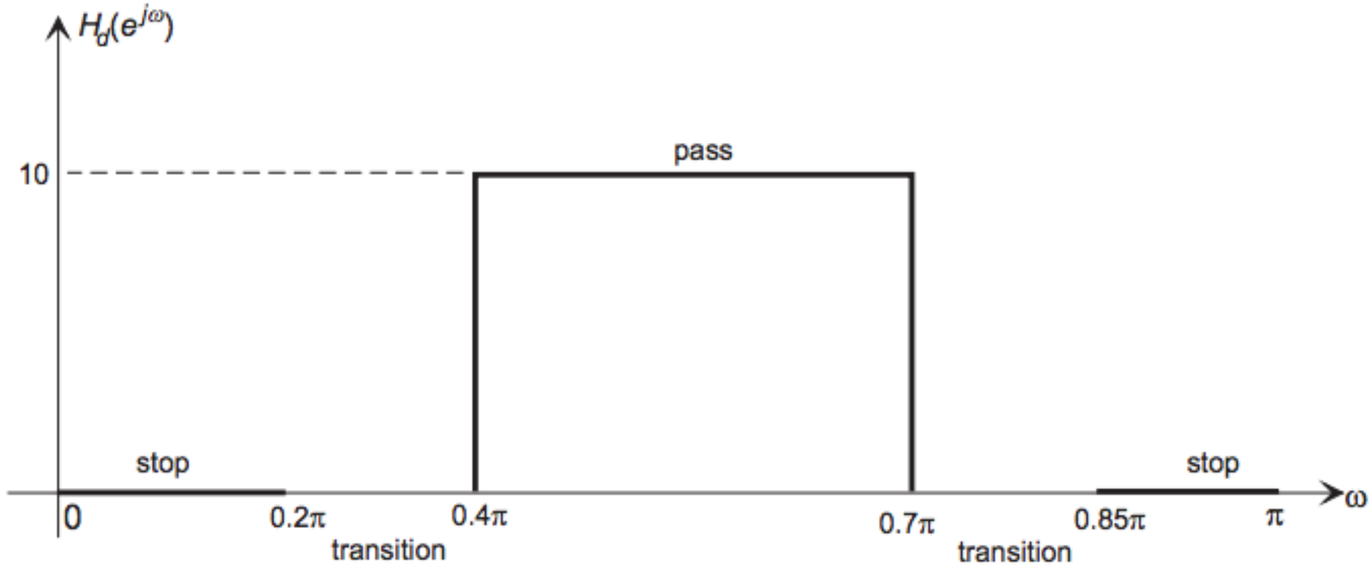
$$F = [0 \;\; \omega_1 \;\; \omega_2 \;\; \omega_3 \;\; \omega_4 \;\; \omega_5 \;\; \omega_6 \;\; \omega_7 \;\; \omega_8 \;\; 1]$$

$$A = [0 \;\; 0 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0.7 \;\; 0.7 \;\; 0 \;\; 0]$$

$$W = [10 \;\; 1 \;\; 10 \;\; 1 \;\; 10]$$

MATLAB Example

- Design a 33 length PM band-pass filter and weight the stop-band ripple 10x more than the pass-band ripple

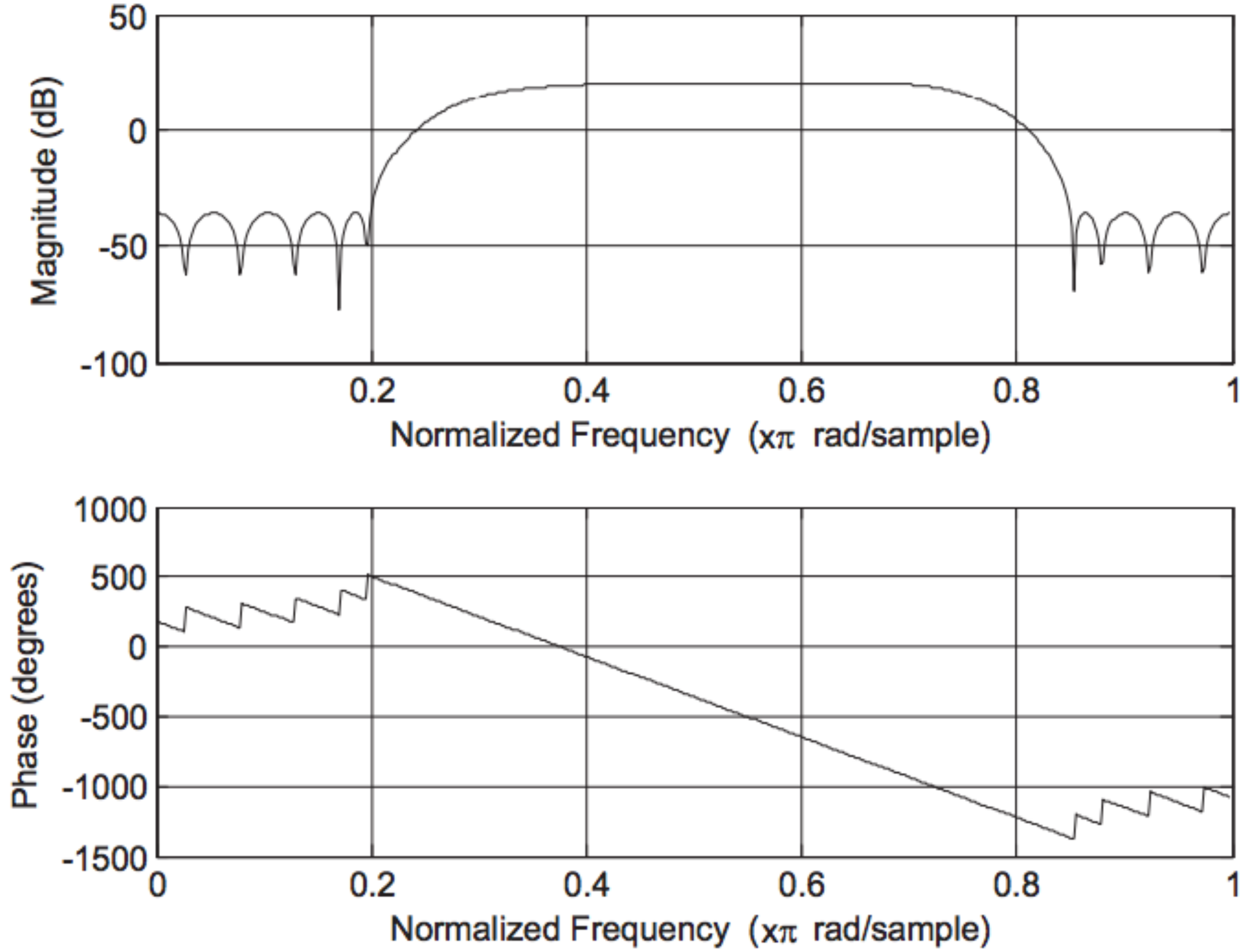



```
h=firpm(32,[0 0.2 0.4 0.7 0.85 1],[0 0 10 10 0 0],[10 1 10])
freqz(h,1)
```

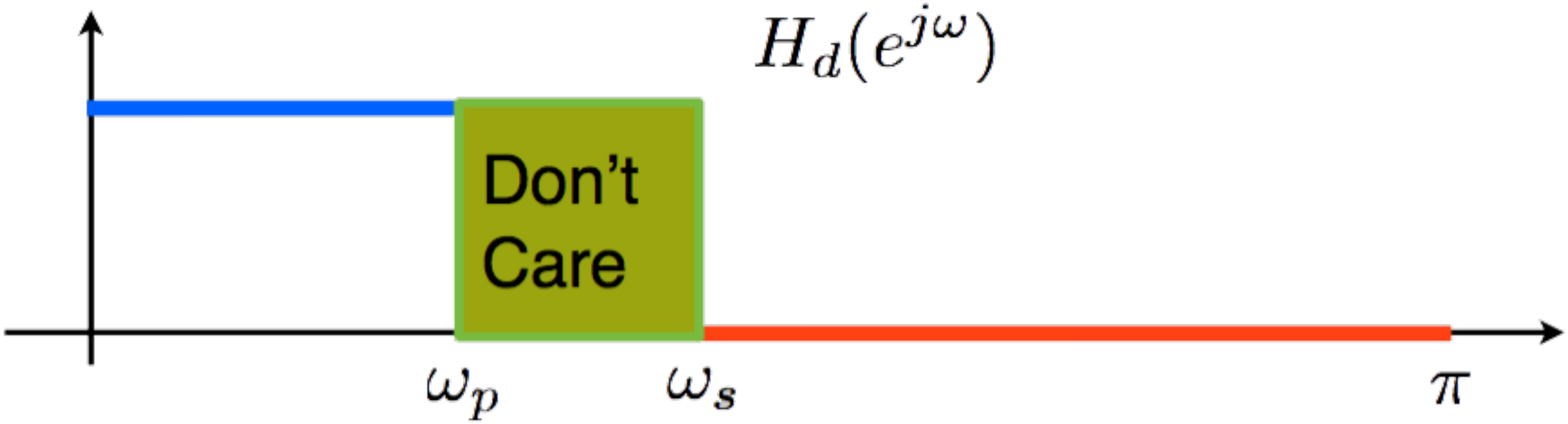
MATLAB Example

- Des… …op-
  ban…

h=firp… …1 10])
freqz(…



Optimality – Least Squares



- Least Squares:

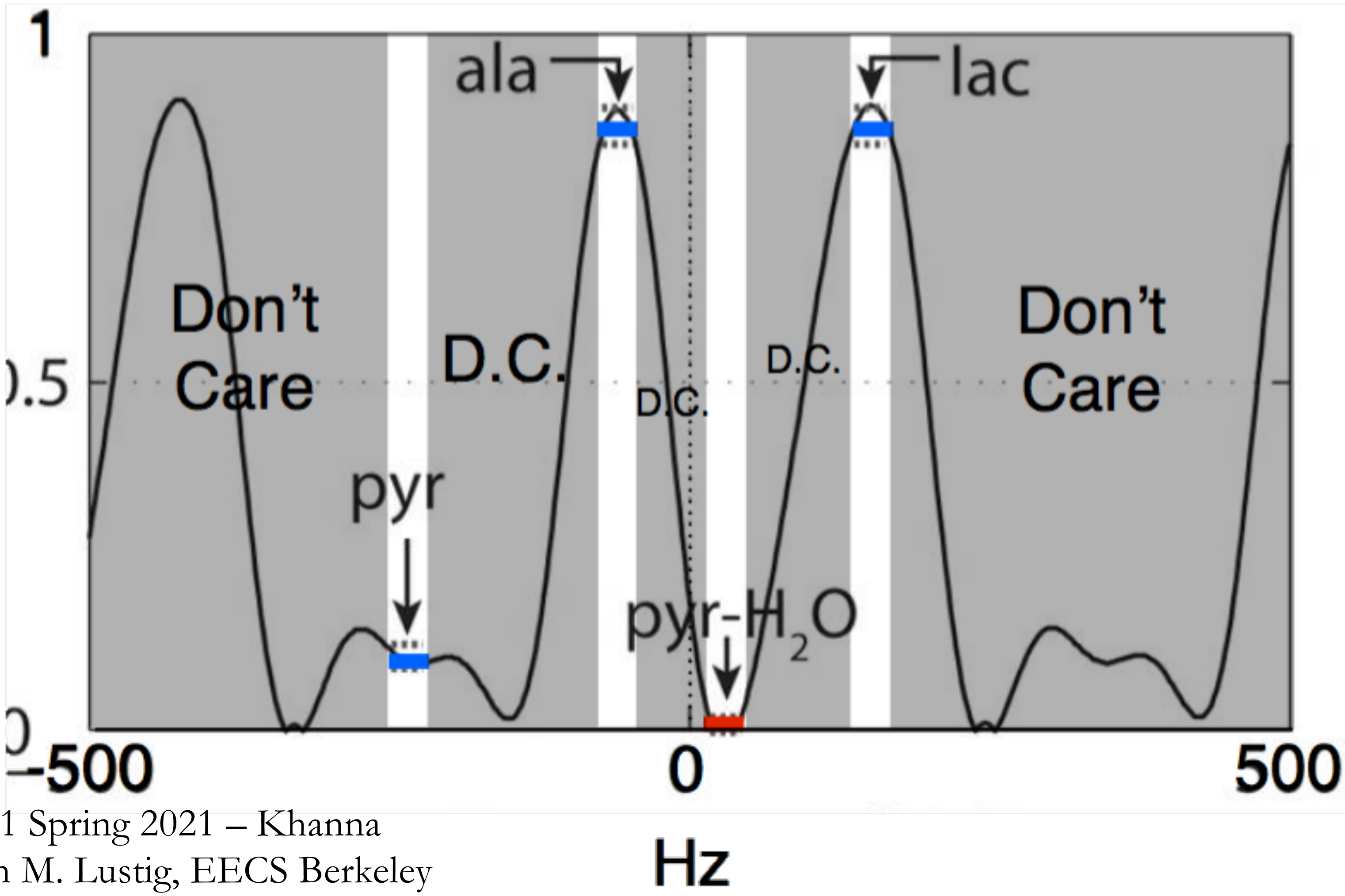
$$\text{minimize} \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Parks-McClellan

$$\min_{\{h_e[n]:0\leq n\leq L\}} \left(\max_{\omega\in F} |E(\omega)| \right),$$

Example of Complex Filter

- Larson et. al, "Multiband Excitation Pulses for Hyperpolarized 13C Dynamic Chemical Shift Imaging" JMR 2008;194(1):121-127
- Need to design length 11 filter with following frequency response:

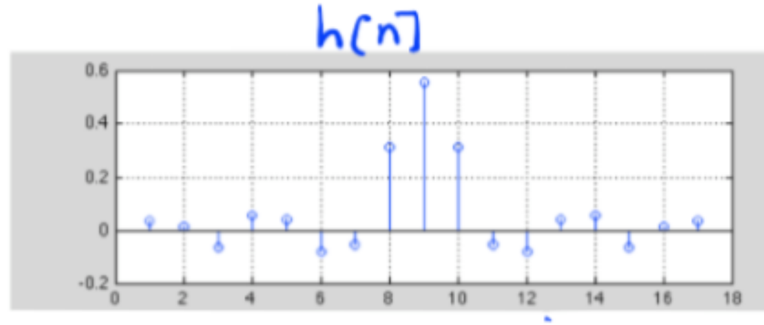


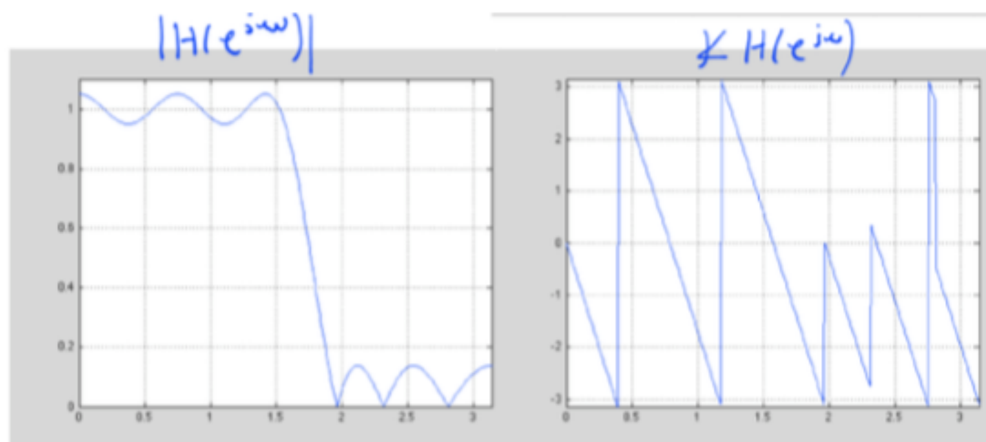
Convex Optimization

- Many tools and Solvers
- Tools:
  - CVX (Matlab) http://cvxr.com/cvx/
  - CVXOPT, CVXMOD (Python)
- Engines:
  - Sedumi (Free)
  - MOSEK (commercial)

Using CVX (in Matlab)

```
M = 16;
wp = 0.5*pi;
ws = 0.6*pi;
MM = 15*M;
w = linspace(0,pi,MM);

idxp = find(w <=wp);
idxs = find(w >=ws);

Ap = [ones(length(idxp),1) 2*cos(kron(w(idxp)',
[1:M/2]))];
As = [ones(length(idxs),1) 2*cos(kron(w(idxs)',
[1:M/2]))];

% optimization
cvx_begin
   variable hh(M/2+1,1);
   variable d(1,1);

   minimize(d)
   subject to
      Ap*hh <=1+d;
      Ap*hh >=1-d;
      As*hh <  d;
      As*hh > -d;
      ds>0;
cvx_end
h = [hh(end:-1:1) ; hh(2:end)];
```





Admin

- Project1 out now
  - Due Monday 4/5