# ESE 531: Digital Signal Processing

Week 3:

Lecture 5: January 31, 2021

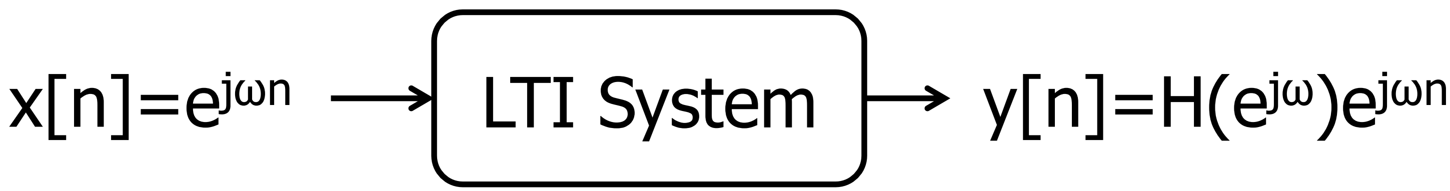
z-Transform

# Lecture Outline

- z-Transform
  - Regions of convergence (ROC) & properties
  - z-Transform properties

# LTI System Frequency Response

- (DT)Fourier Transform of impulse response




$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

# z-Transform

# z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats

# Reminder: DTFT Definition

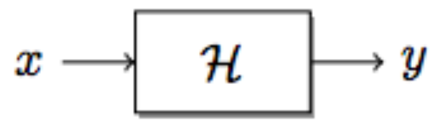
- The core "basis functions" (I.e eigenfunctions) of the DTFT are the complex sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad -\infty \le n < \infty$$

# Reminder: Frequency Response of LTI System

- We can use the DTFT to characterize an LTI system




$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

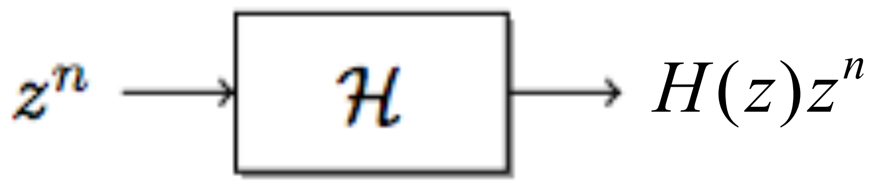
- and relate the DTFTs of the input and output

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n]\, e^{-j\omega n}, \quad H(\omega) = \sum_{m=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

$$Y(\omega) = X(\omega) H(\omega)$$

# Complex Exponentials as Eigenfunctions

- Fact: A more general set of eigenfunctions of an LTI system are the complex exponentials $z^n$, $z \in C$



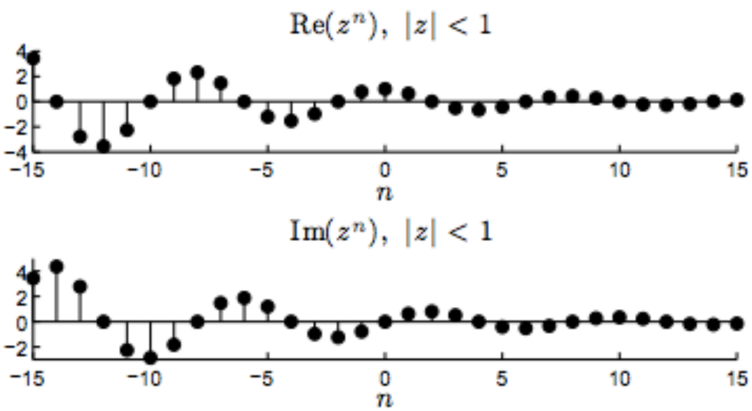
# Reminder: Complex Exponentials

$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

$|z|^n$ is a **real exponential** envelope ($a^n$ with $a = |z|$)

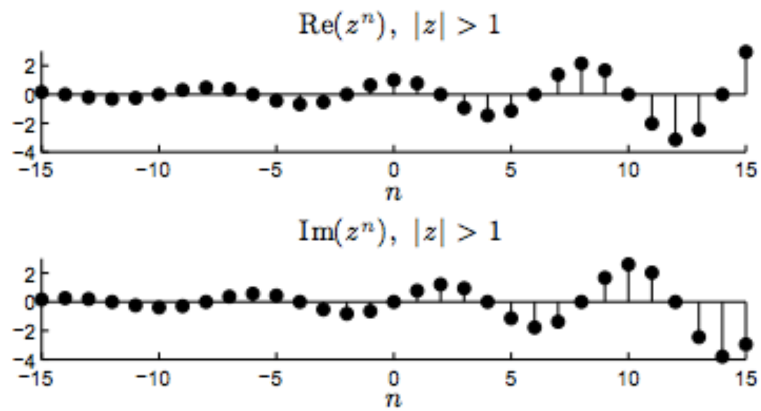
$e^{j\omega n}$ is a **complex sinusoid**

$|z| < 1$



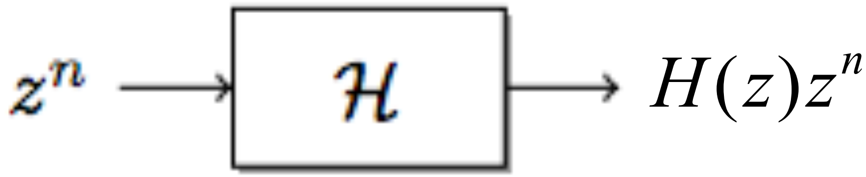

Bounded

$|z| > 1$



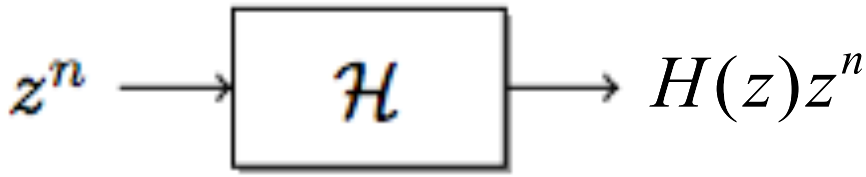

Unbounded

# Proof: Complex Exponentials as Eigenfunctions




- Prove by computing the convolution with input $x[n] = z^n$

# Proof: Complex Exponentials as Eigenfunctions




- Prove by computing the convolution with input x[n] = $z^n$

$$z^n * h[n] = \sum_{m=-\infty}^{\infty} z^{n-m}\, h[m] = \sum_{m=-\infty}^{\infty} z^n\, z^{-m}\, h[m]$$

$$= \left( \sum_{m=-\infty}^{\infty} h[m]\, z^{-m} \right) z^n$$

$$= H(z)z^n \checkmark$$

# z-Transform

- Define the **forward z-transform** of x[n] as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

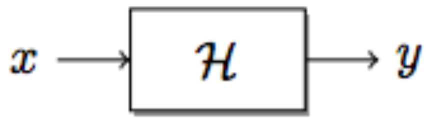
- The core "basis functions" of the z-transform are the complex exponentials $z^n$ with arbitrary $z \in C$; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT $X(e^{j\omega})$ and the z-transform X(z); they are, in fact, intimately related

$$X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

# Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system




$$y[n] \;=\; x[n] * h[n] \;=\; \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}, \qquad H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

$$Y(z) \;=\; X(z)\, H(z)$$

# Z-transform

What are we missing?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

# Z-Transform

$$z^n = \left(|z|\, e^{j\omega n}\right)^n = |z|^n\, e^{j\omega n}$$

$|z|^n$ is a **real exponential** envelope $(a^n$ with $a = |z|)$

$e^{j\omega n}$ is a **complex sinusoid**

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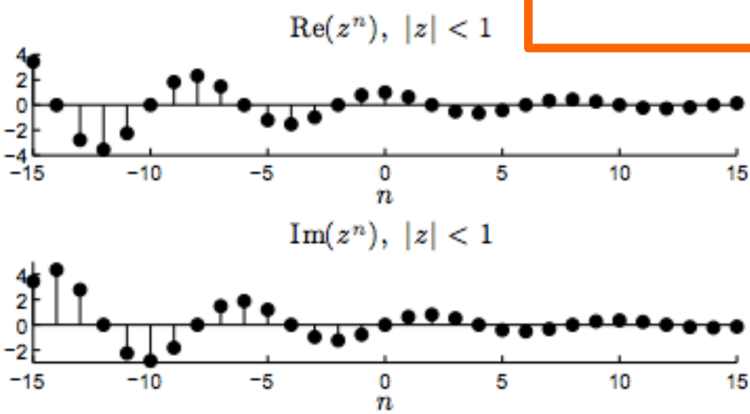
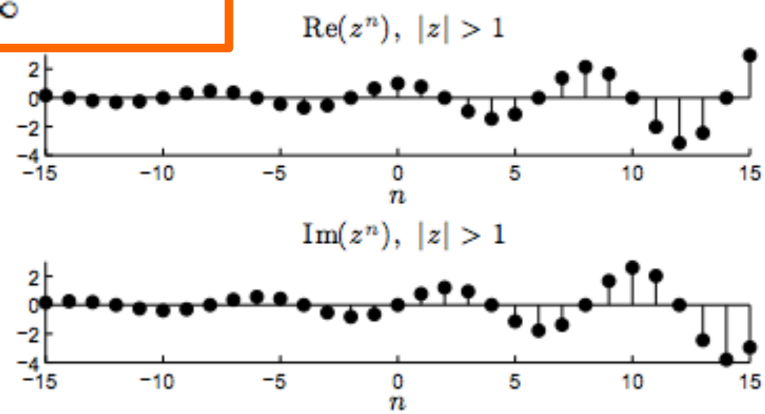



Bounded

$|z| > 1$




Unbounded

# Region of Convergence (ROC)

# Region of Convergence (ROC)

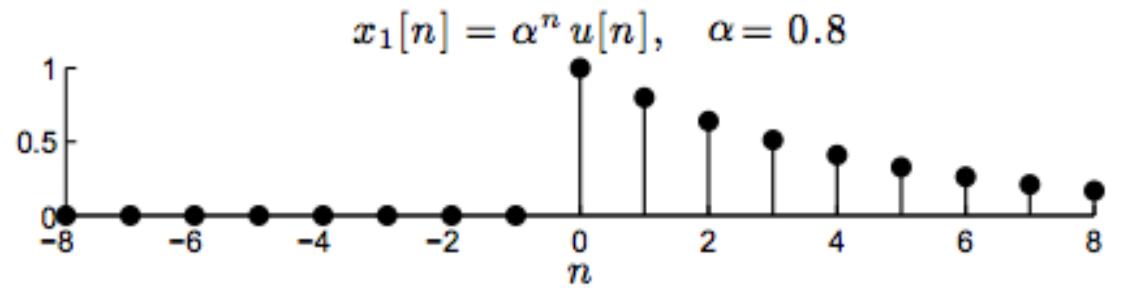
DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its $z$-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\,z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\,z^{-n}| \;<\; \infty$$

# ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence
- Example for $\alpha = 0.8$



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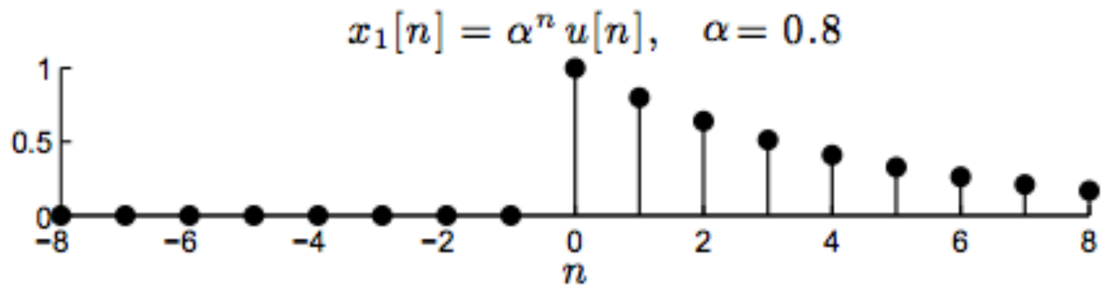



- The **forward** $z$**-transform** of $x_1[n]$

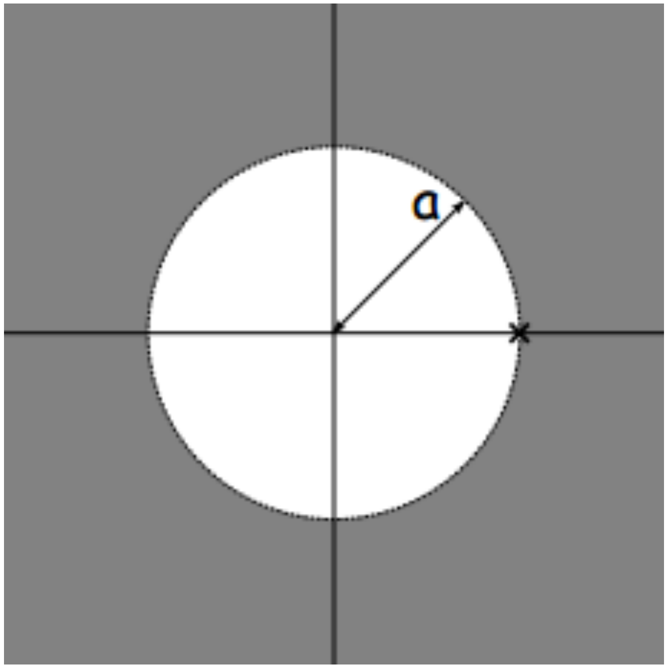
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} = \sum_{n=0}^{\infty} \alpha^n\, z^{-n} = \sum_{n=0}^{\infty} (\alpha\, z^{-1})^n = \frac{1}{1-\alpha\, z^{-1}} = \frac{z}{z-\alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha\, z^{-1}| < 1$ or $|z| > |\alpha|$

# ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)

$$ROC = \{z : \ |z| > |a|\}$$

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- **Important:** We can apply the geometric sum formula only when $|\alpha\, z^{-1}| < 1$ or $|z| > |\alpha|$

# ROC Example 1

- What is the DTFT of $x_1[n]=a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \qquad -\infty \le n < \infty$$

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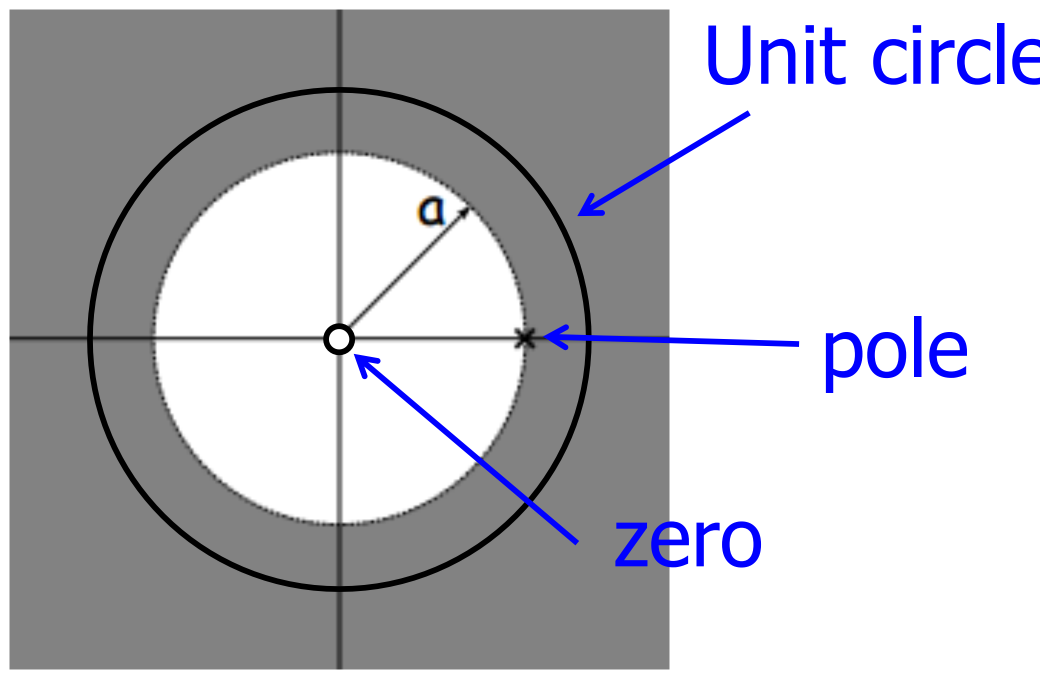
$$X_1(z) = \frac{z}{z-a}$$

$$ROC = \{z: \ |z| > |a|\}$$



# ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

# ROC Example 2

- What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

- Hint: $x_1[n] = a^n u[n] \stackrel{Z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$

# ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Left-sided sequence

# ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The z-transform without ROC does not uniquely define a sequence!

# ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

# ROC Example 4

- What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

- Hint:

$$x_1[n] = a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \quad |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC = \{z: \quad |z| < |a|\}$$

# ROC Example 5

- What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z : \quad |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \qquad ROC = \{z : \quad |z| < |a|\}$$

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n+M-1]$$

finite length sequence

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$

Zero cancels pole

$$= \prod_{k=1}^{M-1} \left(1 - a e^{j2\pi k / M} z^{-1}\right)$$

# ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

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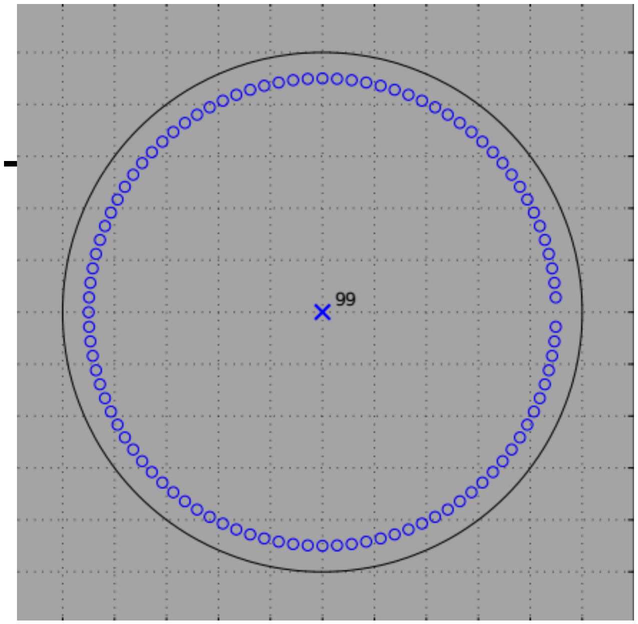



$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - a z^{-1}}$$

$$= \prod_{k=1}^{M-1} \left(1 - a e^{j2\pi k/M} z^{-1}\right)$$

Zero cancels pole

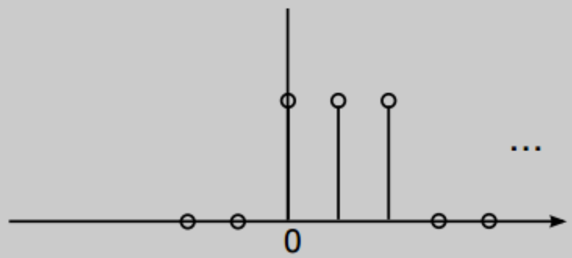
M=100

# Properties of ROC

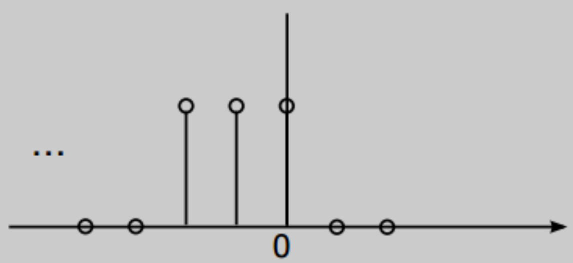
- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1,2
- For left-sided: inwards from inner most pole to zero
  - Example 3
- For two-sided, ROC is a ring - or do not exist
  - Examples 4,5

# Properties of ROC

- For finite duration sequences, ROC is the entire z-plane, except possibly z=0, z=∞ (Example 6)




$X(z) = 1 + z^{-1} + z^{-2}$ ROC excludes $z = 0$




$X(z) = 1 + z^{1} + z^{2}$ ROC excludes $z = \infty$

# Formal Properties of the ROC

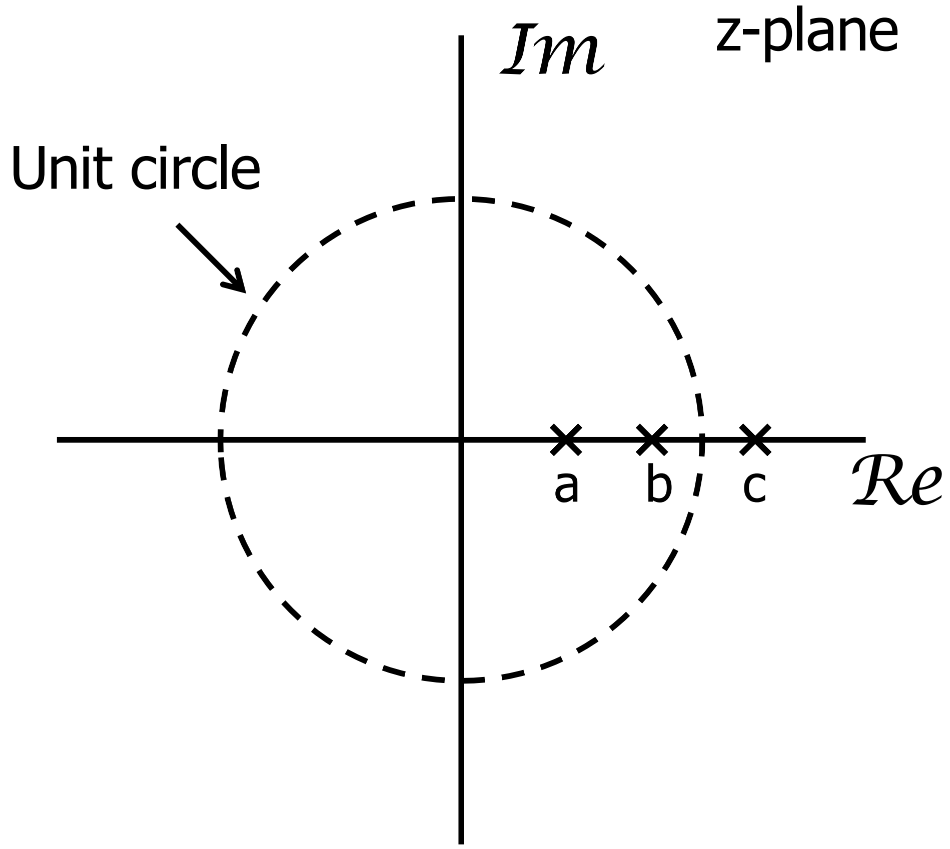
- PROPERTY 1:
  - The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.
- PROPERTY 2:
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
- PROPERTY 3:
  - The ROC cannot contain any poles.
- PROPERTY 4:
  - If x[n] is *a finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

# Formal Properties of the ROC

- PROPERTY 5:
  - If x[n] is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in X(z) to (and possibly including) $z = \infty$.
- PROPERTY 6:
  - If x[n] is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in X(z) to (and possibly including) z=0.
- PROPERTY 7:
  - A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.
- PROPERTY 8:
  - The ROC must be a connected region.

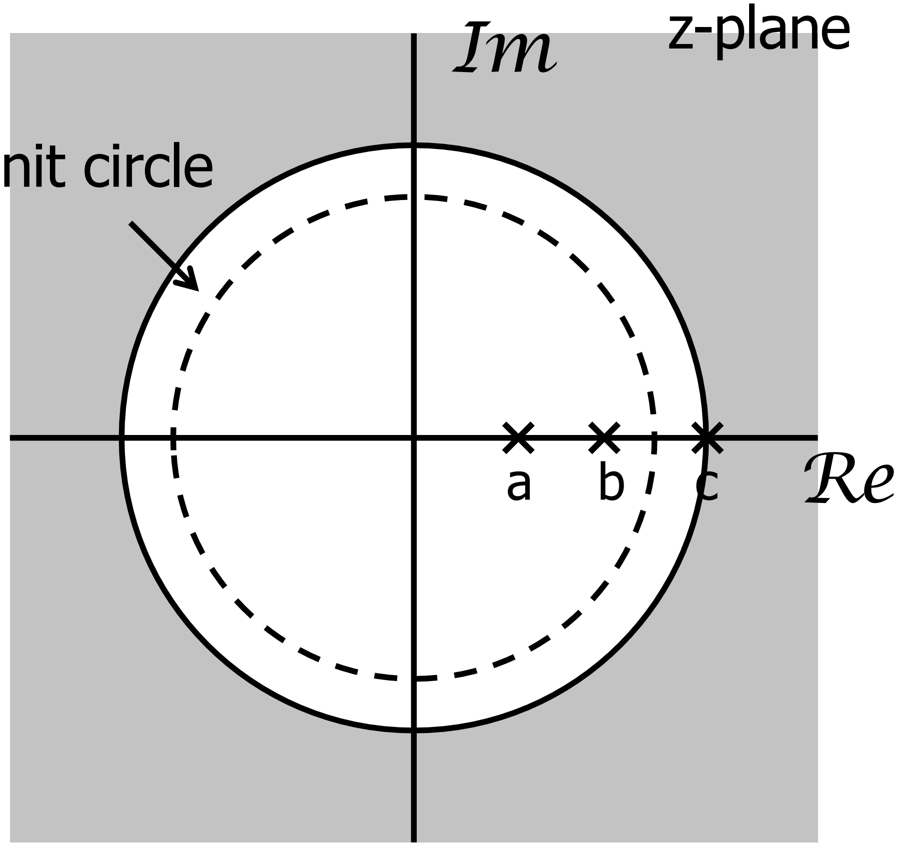
# Example: ROC from Pole-Zero Plot

- How many possible ROCs?



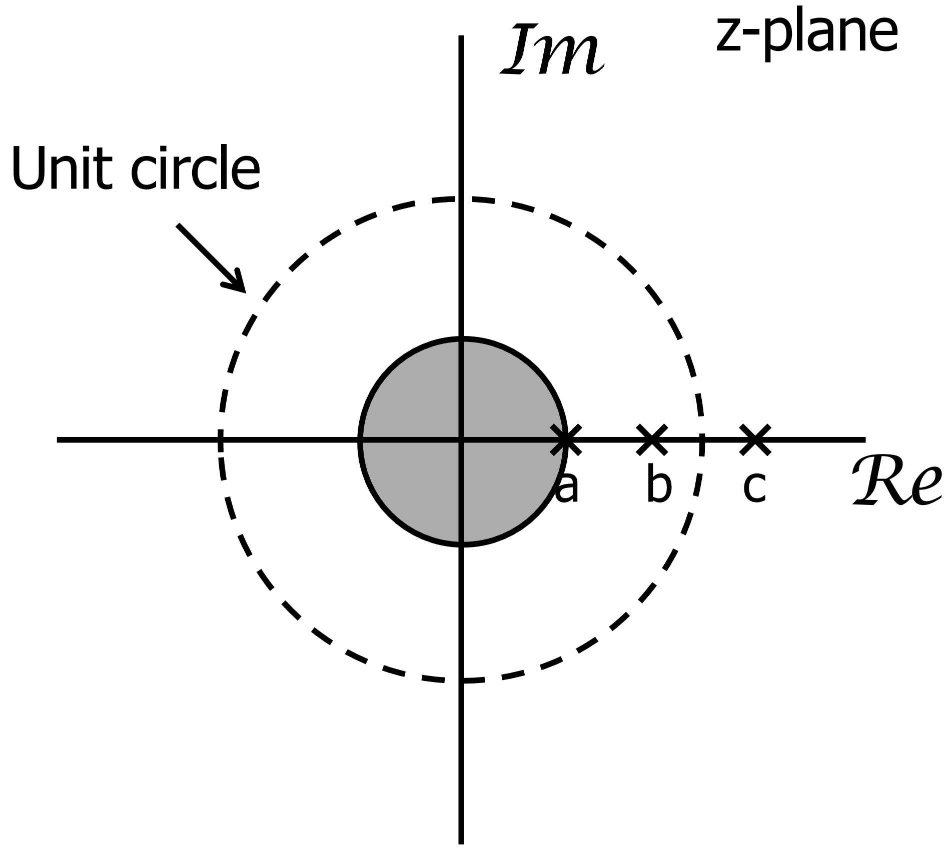
# Example: ROC from Pole-Zero Plot

## ROC 1: right-sided



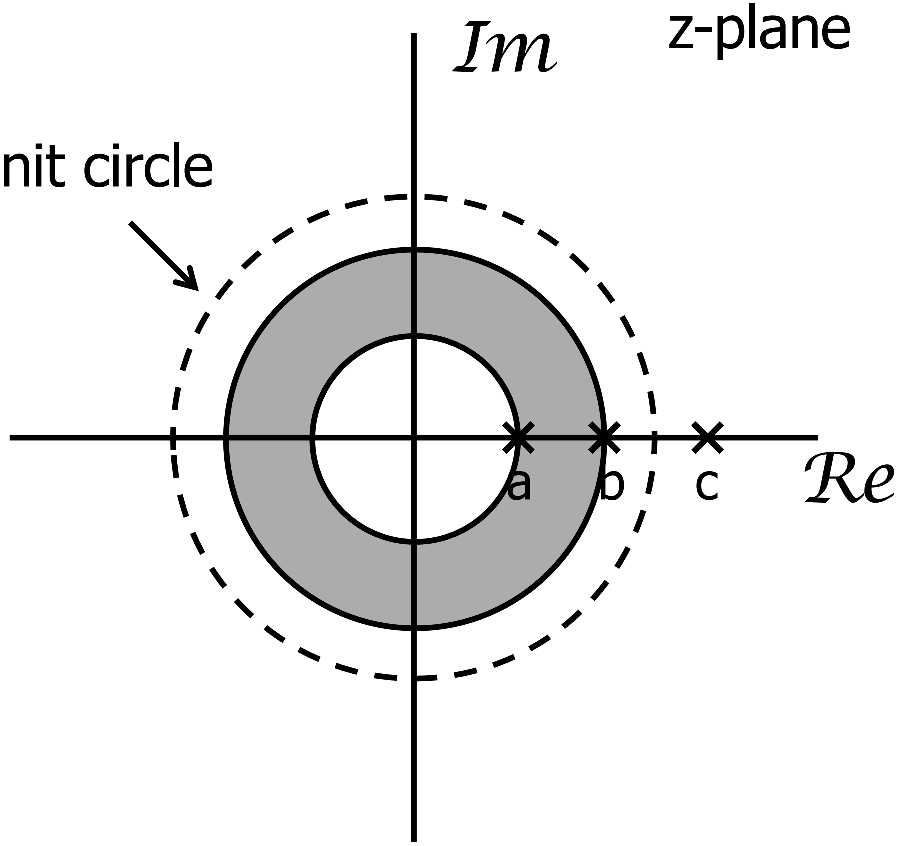
# Example: ROC from Pole-Zero Plot

## ROC 2: left-sided



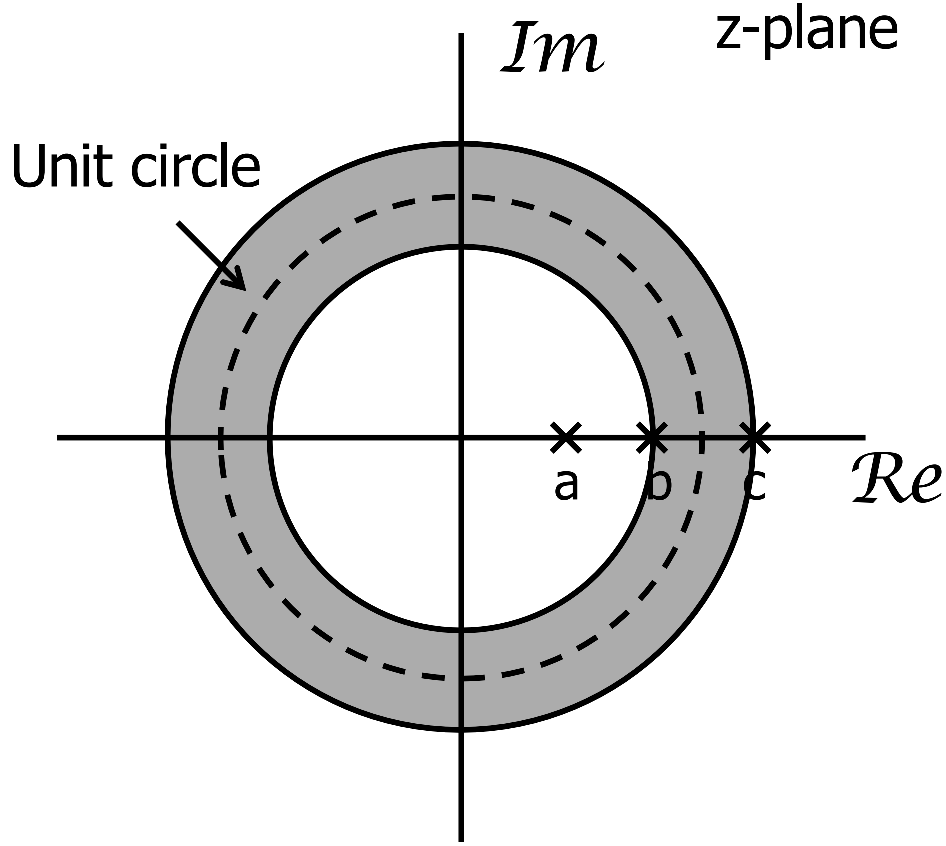
# Example: ROC from Pole-Zero Plot

## ROC 3: two-sided



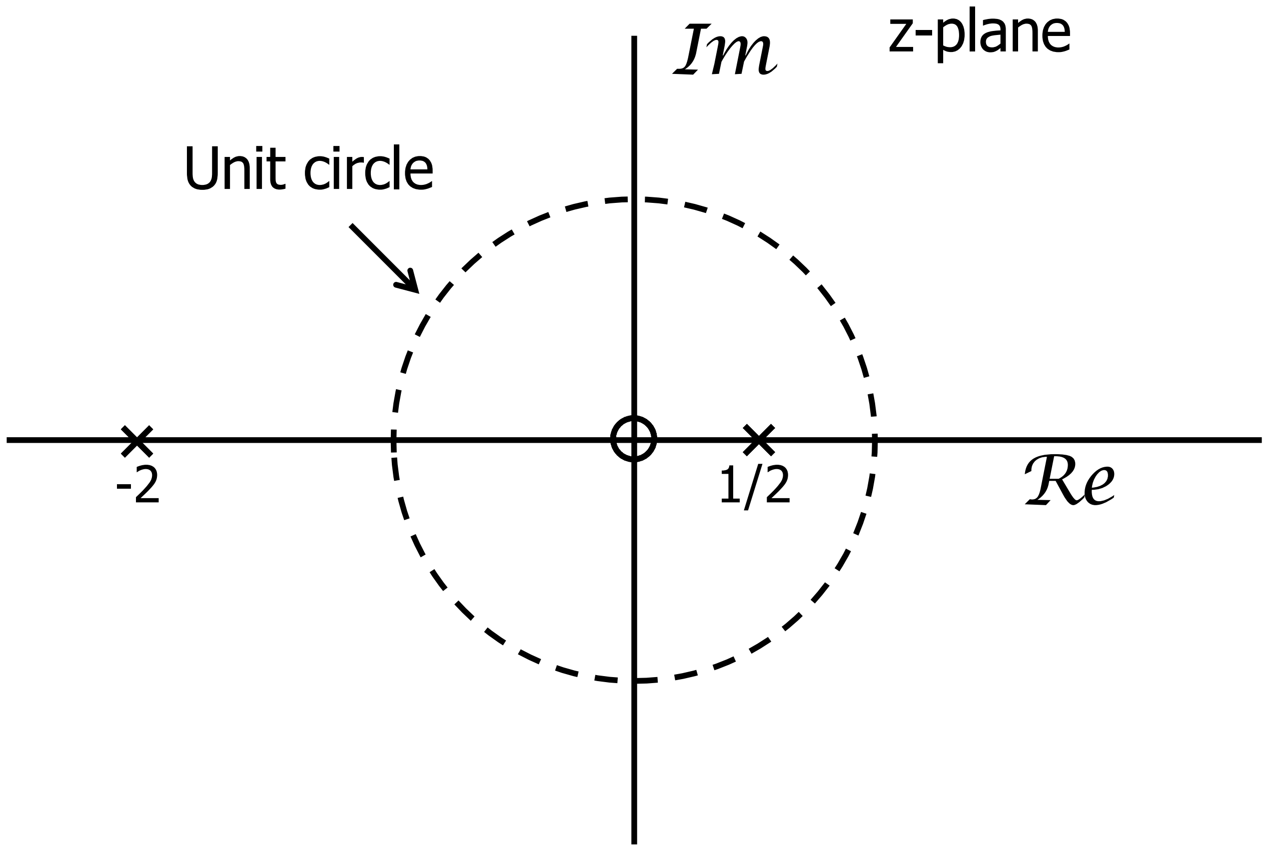
# Example: ROC from Pole-Zero Plot

## ROC 4: two-sided



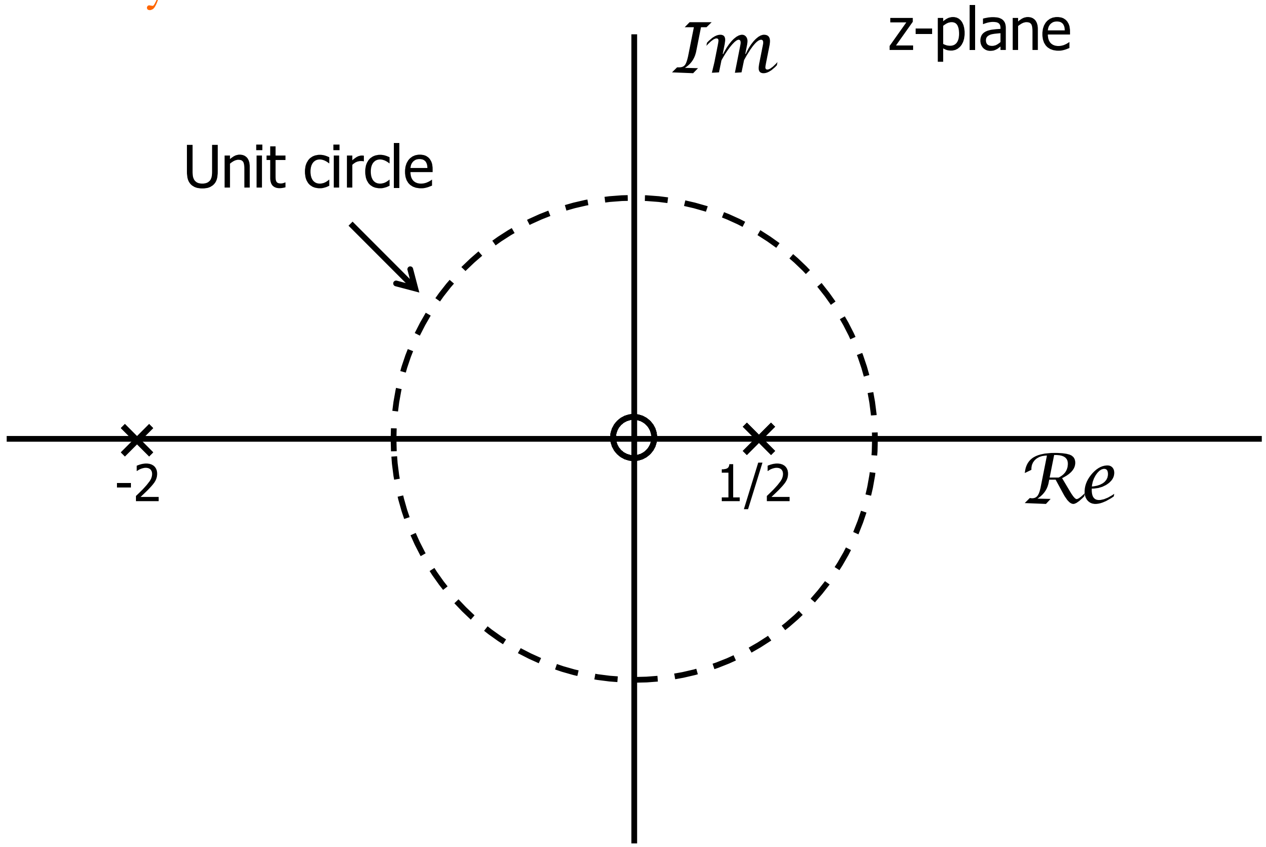
# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?



# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is stable?



# BIBO Stability Revisited

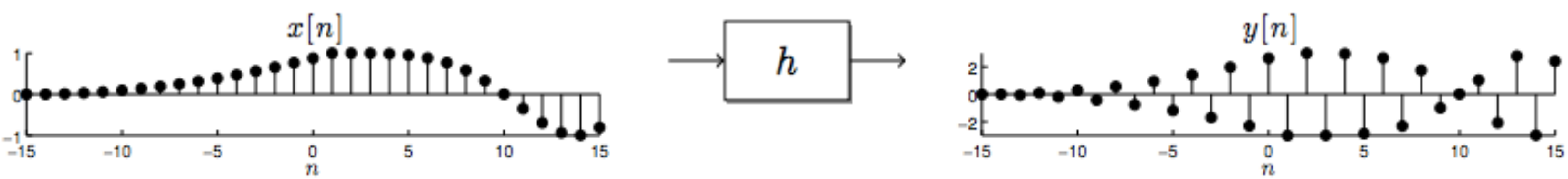
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output** (
input $x$ always produces a bounded output $y$

bounded $x$ → $h$ → bounded $y$

- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$

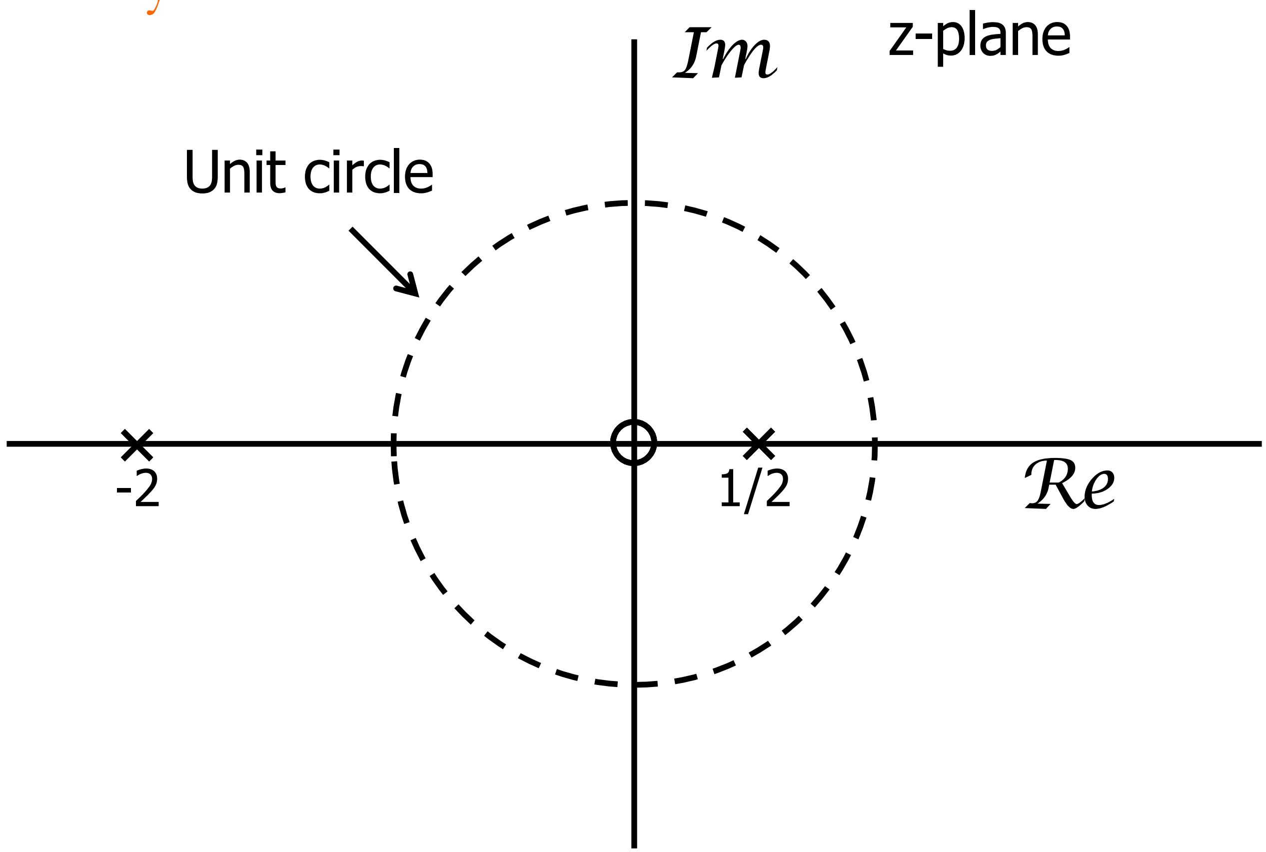



- **Fact:** An LTI system with impulse response $h$ is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

# Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is causal?



# z-transform Pairs

**TABLE 3.1** SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| > \|a\|$ |
| 6. $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| < \|a\|$ |
| 7. $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| > \|a\|$ |
| 8. $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| < \|a\|$ |
| 9. $\cos(\omega_0 n)u[n]$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| 10. $\sin(\omega_0 n)u[n]$ | $\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| 11. $r^n \cos(\omega_0 n)u[n]$ | $\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |
| 12. $r^n \sin(\omega_0 n)u[n]$ | $\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\frac{1-a^N z^{-N}}{1-az^{-1}}$ | $\|z\| > 0$ |

# Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \leftrightarrow X(z)$$

$$x[n - n_d] \leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \leftrightarrow X(z)$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$

# Properties of z-Transform

- Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

- Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

$ROC_Y$ at least $ROC_X \wedge ROC_H$

# Big Ideas

- z-Transform
  - Uses complex exponential eigenfunctions to represent discrete time sequence
    - DTFT is z-Transform where $z=e^{j\omega}$, $|z|=1$
  - Draw pole-zero plots
  - Must specify region of convergence (ROC)

- z-Transform properties
  - Similar to DTFT

# Admin

- HW 1 out now
  - Due 2/8 at midnight
  - Submit in Canvas

- Updated Course Schedule
- Updated Office hours

- See course calendar for full details!!