ESE 531: Digital Signal Processing

Week 4

Lecture 6: February 7, 2021

Inverse z-Transform

Lecture Outline

- Inverse z-transform
  - Inspection
  - Partial fraction
  - Power series expansion
- z-transform of difference equations

z-Transform

z-Transform

- Define the **forward z-transform** of x[n] as

$$X(z) \;=\; \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The core "basis functions" of the z-transform are the complex exponentials z^n with arbitrary $z \in C$; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT X(ω) and the z-transform X(z); they are, in fact, intimately related

$$X_{\mathrm{DTFT}}(\omega) \;=\; X_z(z)\big|_{z=e^{j\omega}} \;=\; X_z(e^{j\omega})$$

Inverse z-Transform

Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

- There is a similar formula for the inverse z-transform using a contour integral

$$x[n] = \oint_C X(z)\, z^n\, \frac{dz}{j2\pi z}$$

- Contour integrals are fun but beyond the scope of this course!

Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion

Z-Transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\dfrac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Z-Transform Properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Partial Fraction Expansion

- Let

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- M zeros and N poles at nonzero locations

Partial Fraction Expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- Factored numerator/denominator

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Partial Fraction Expansion

- If M<N and the poles are 1st order

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- where

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \ \frac{1}{2} < |z| \right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z: \ \frac{1}{2} < |z|\right\}$$

$$X(z) = \frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \frac{1}{2} < |z| \right\}$$

$$\Rightarrow \quad X(z) = \frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2} z^{-1}\right)}$$

$$A_1 = (1 - \frac{1}{4} z^{-1}) X(z)\Big|_{z=1/4} = \frac{(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}\Bigg|_{z=1/4} = -1$$

$$A_2 = (1 - \frac{1}{2} z^{-1}) X(z)\Big|_{z=1/2} = \frac{(1 - \frac{1}{2} z^{-1})}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}\Bigg|_{z=1/2} = 2$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z:\ \frac{1}{2} < |z|\right\}$$

Example: 2^{nd}-Order z-Transform

- 2^{nd}-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2} z^{-1}\right)}, \qquad ROC = \left\{ z : \ \frac{1}{2} < |z| \right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

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$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z: \ \frac{1}{2} < |z|\right\}$$

5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \ \frac{1}{2} < |z| \right\}$$

5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

Partial Fraction Expansion

- If M≥N and the poles are 1st order

$$X(z)=\sum_{r=0}^{M-N} B_r z^{-r}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- Where B_k is found by long division

$$A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC=\{z:\ 1<|z|\}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC = \left\{z:\ 1<|z|\right\}$$

$$= \frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}, \qquad ROC = \left\{z:\ 1<\left|z\right|\right\}$$

$$\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \,\overline{\big)\, z^{-2} + 2z^{-1} + 1}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2} z^{-1}} + \frac{A_2}{1 - z^{-1}}, \qquad ROC = \left\{ z:\ 1 < |z| \right\}$$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 2 \\ \frac{1}{2} z^{-2} - \frac{3}{2} z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \quad\quad 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - z^{-1})}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}, \qquad ROC = \{z:\ 1 < |z|\}$$

$$\frac{-1 + 5z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}, \qquad ROC = \{z : 1 < |z|\}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2} z^{-1}} + \frac{8}{1 - z^{-1}}, \qquad ROC = \left\{ z : \; 1 < |z| \right\}$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power Series Expansion

- Expansion of the z-transform definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

4. $\delta[n-m]$ $\qquad z^{-m}$

Reminder: Difference Equations

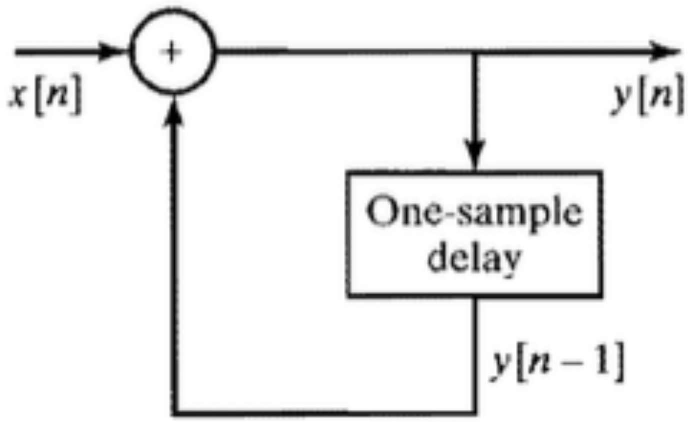
- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$\boxed{y[n] - y[n-1] = x[n]}$$



$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right) x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N} \left(\frac{a_k}{a_0}\right) z^{-k} Y(z) = \sum_{m=0}^{M} \left(\frac{b_m}{a_0}\right) z^{-m} X(z)$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N}\left(\frac{a_k}{a_0}\right) z^{-k} Y(z) = \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right) z^{-m} X(z)$$

$$\Rightarrow Y(z) = \frac{\sum_{m=0}^{M}\left(b_m\right) z^{-m}}{\sum_{k=0}^{N}\left(a_k\right) z^{-k}} X(z)$$

$$\boxed{H(z) = \frac{\sum_{m=0}^{M}\left(b_k\right) z^{-k}}{\sum_{k=0}^{N}\left(a_k\right) z^{-k}}}$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{\overset{b_0}{1}}{\underset{a_0}{1} - \underset{a_1}{a}z^{-1}}$$

$$\boxed{\begin{gathered} \sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m] \\[2em] H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}} \end{gathered}}$$

Example: 1st-Order System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

$$H(z) = \frac{\overset{b_0}{1}}{\underset{a_0}{1} - \underset{a_1}{a} z^{-1}}$$

$$h[n] = a^n u[n]$$

Why right sided?

Big Ideas

- Inverse z-transform
  - Avoid it!
  - Inspection, properties, partial fractions, power series
- Difference equations easy to transform

Admin

- HW 2 due Monday 2/15 at midnight
  - Double check that your submission by viewing it!

- Recitation videos in Canvas and notes on course webpage

- Make sure you have access to Matlab
  - Let us know ASAP if you need help!