ESE 531: Digital Signal Processing

Week 4

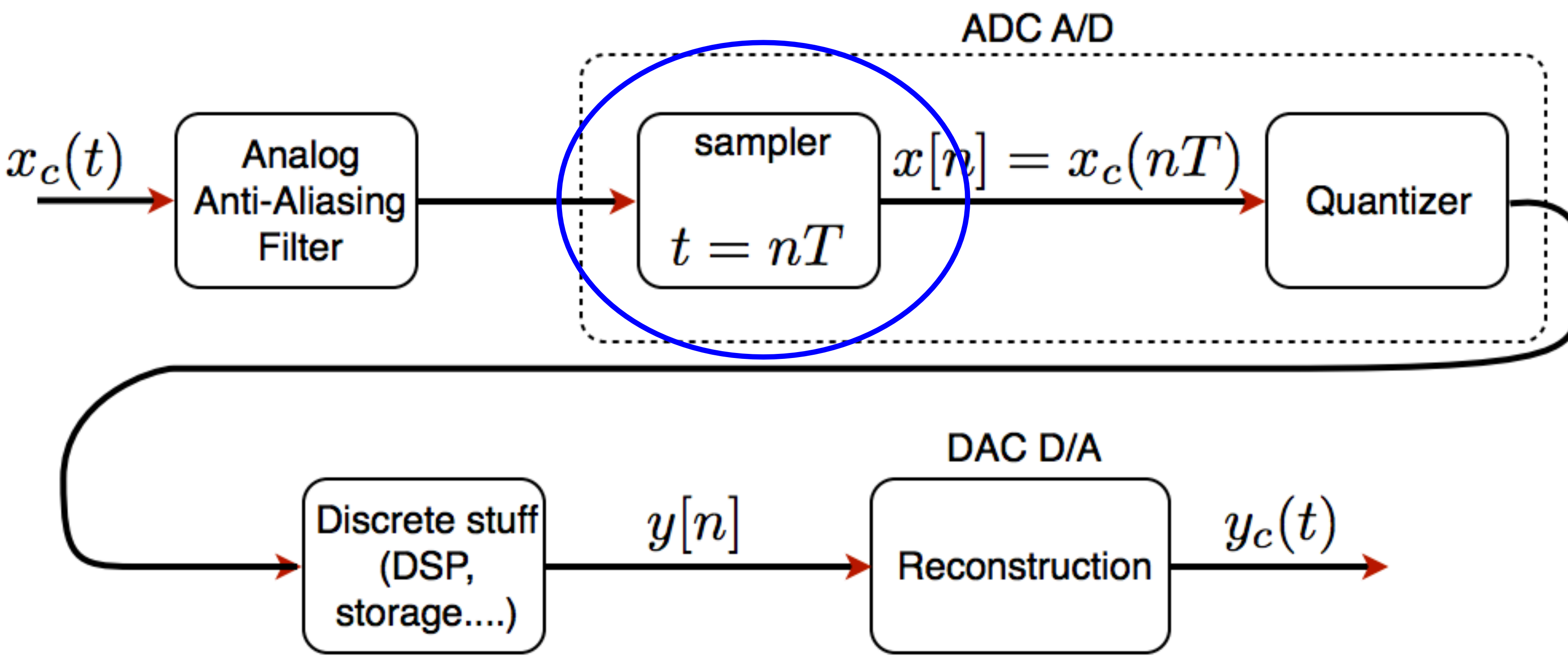
Lecture 8: February 7, 2021

Reconstruction

Lecture Outline

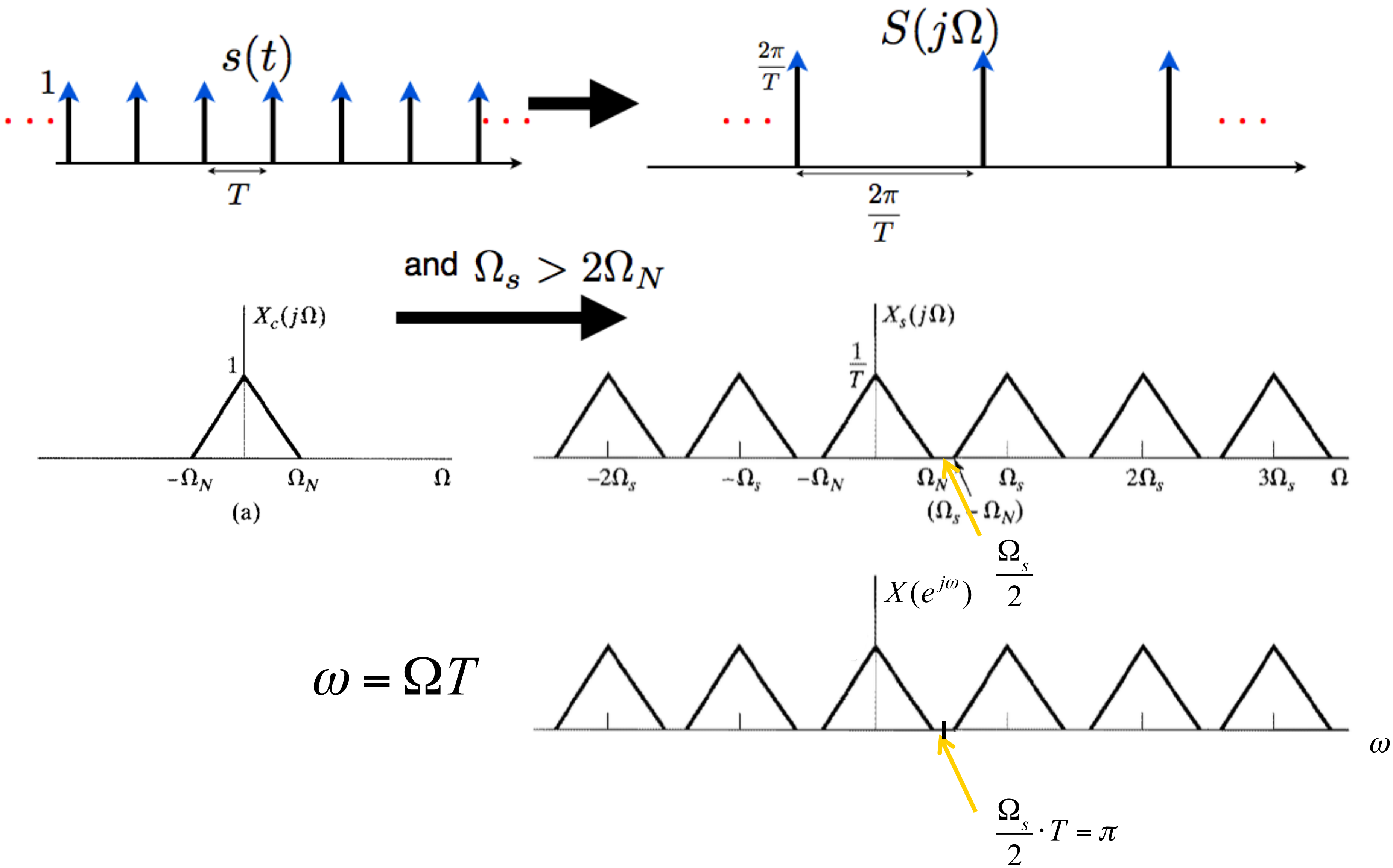
- Reconstruction
- Anti-aliasing Filter

DSP System

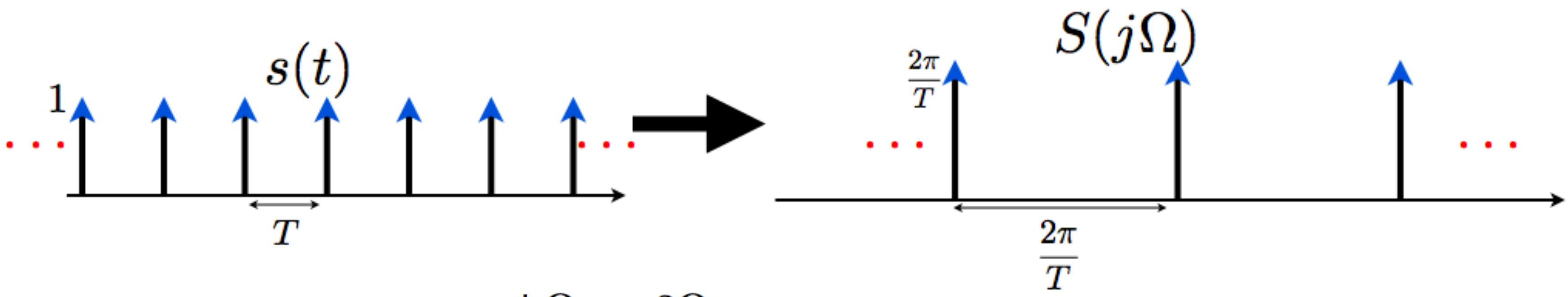


ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

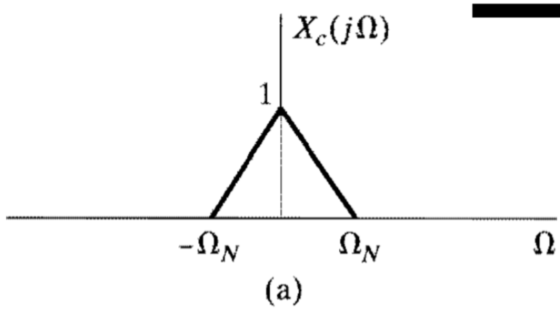
Frequency Domain Analysis

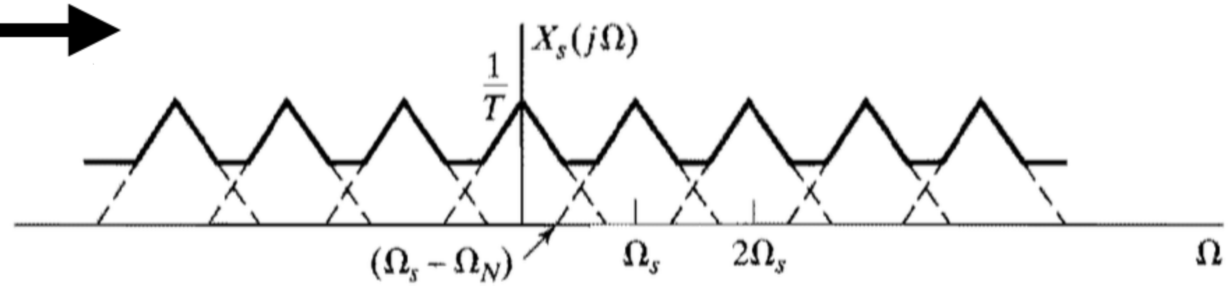


Frequency Domain Analysis w/Aliasing

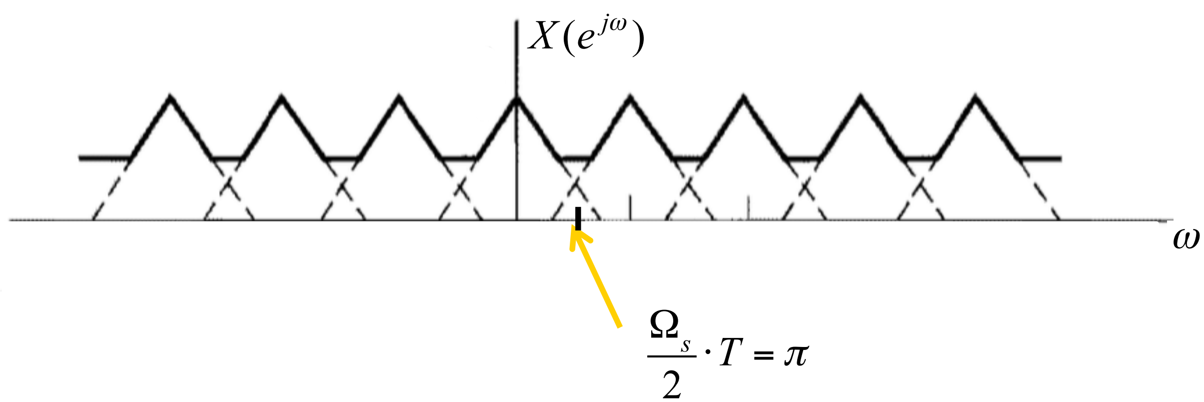


and $\Omega_s < 2\Omega_N$

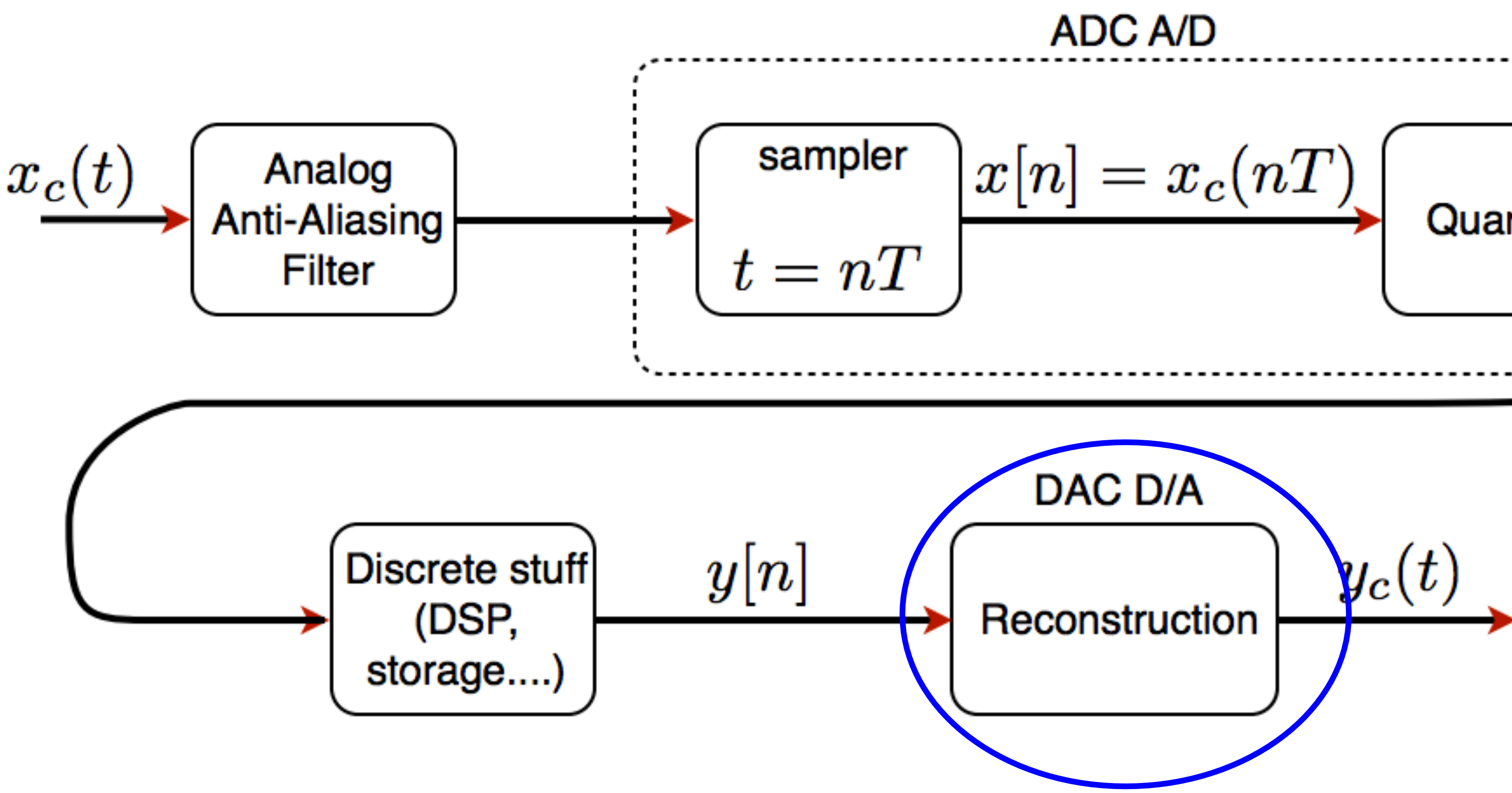




$\omega = \Omega T$



DSP System



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(DSP,
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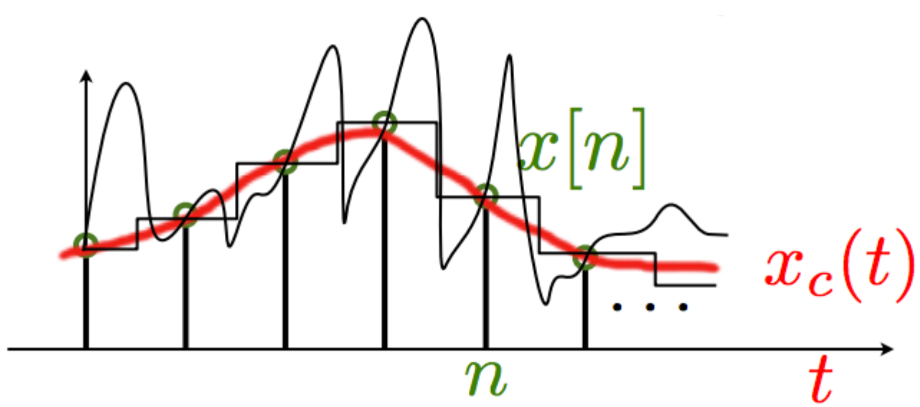
$y[n]$
Reconstruction
$y_c(t)$

Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

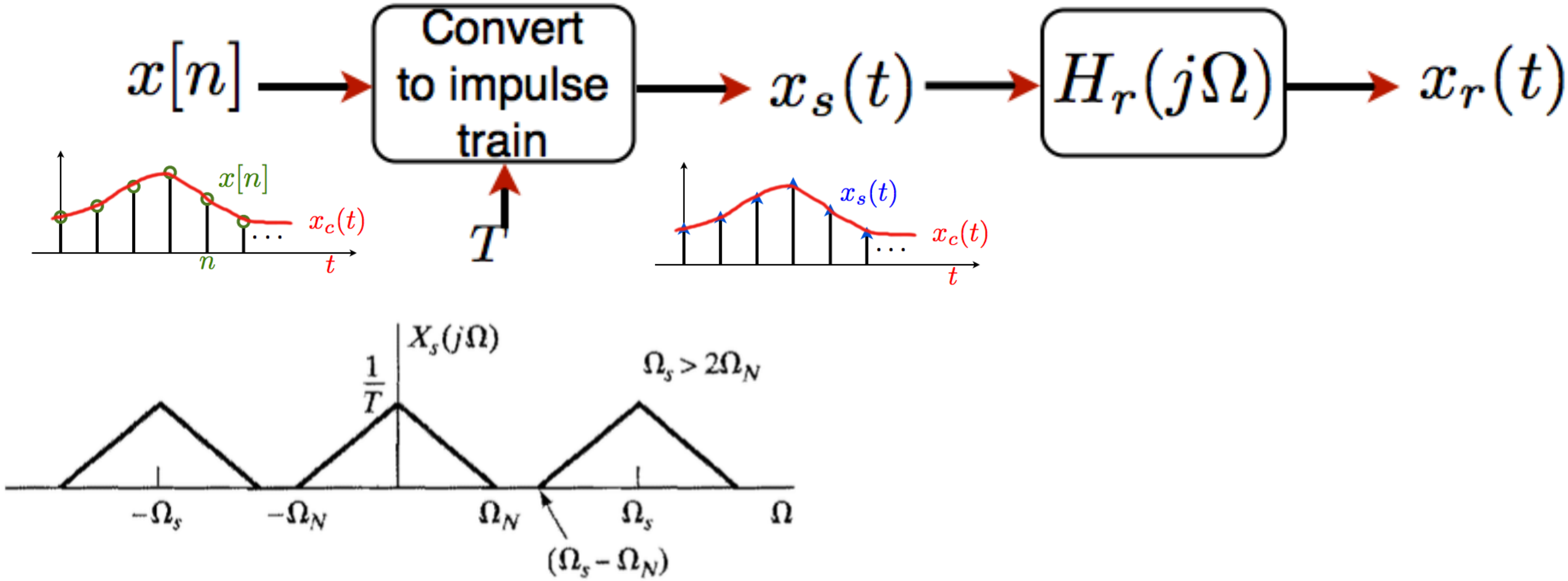
$$X_c(j\Omega) = 0 \ \ \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness

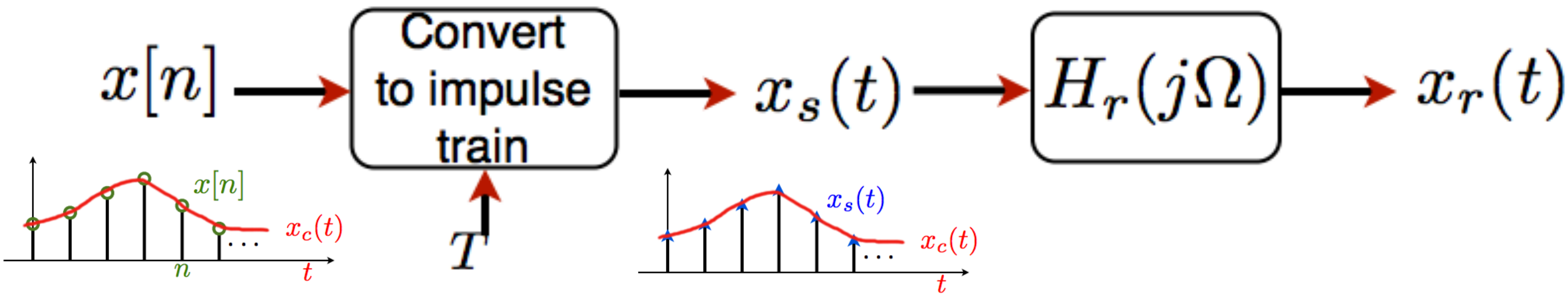


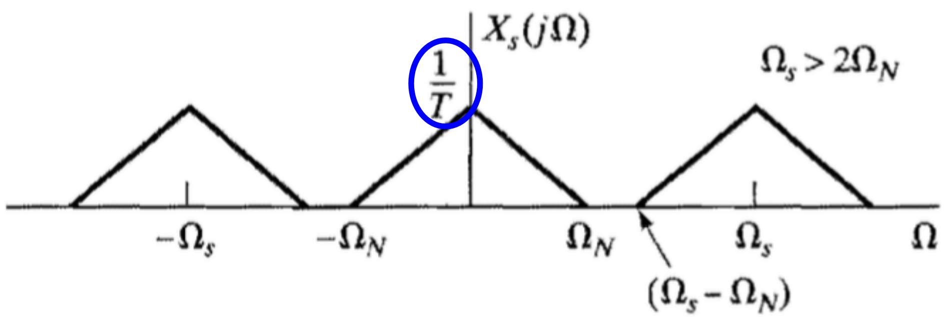
Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

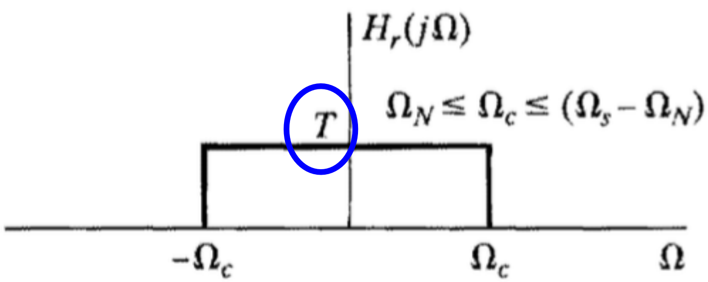
Reconstruction in Frequency Domain



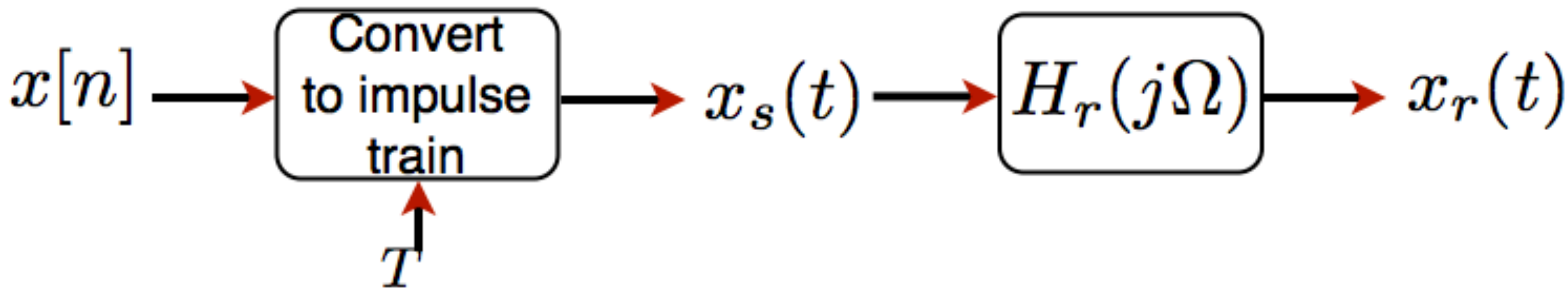
Reconstruction in Frequency Domain

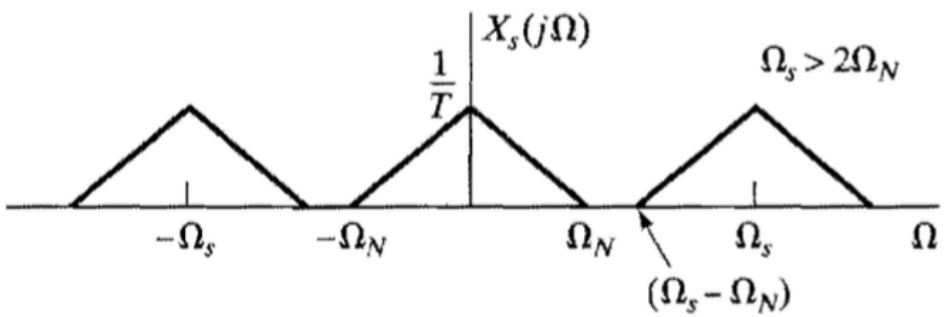


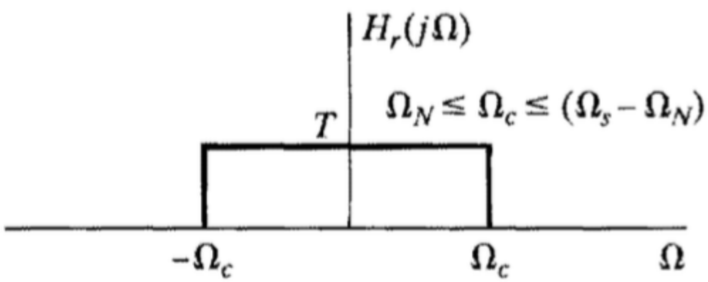


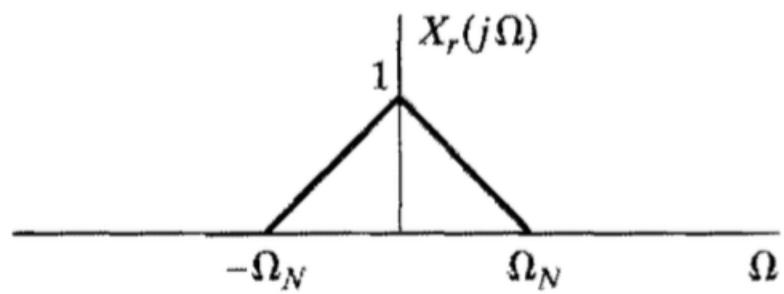


Reconstruction in Frequency Domain









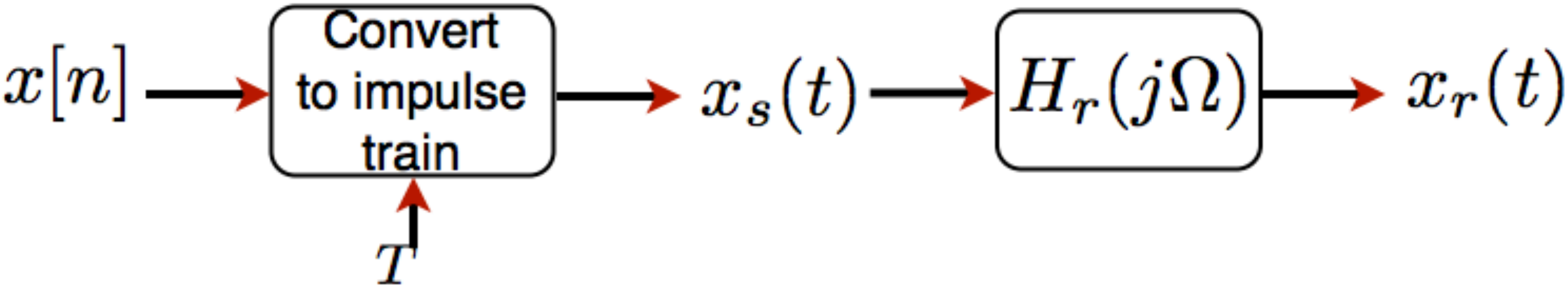
Example: Cosine Input

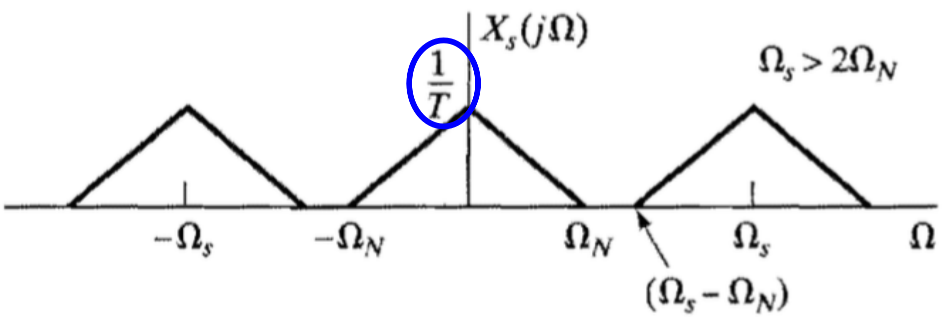
- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period T=1/6000 s (f_s=6kHz)

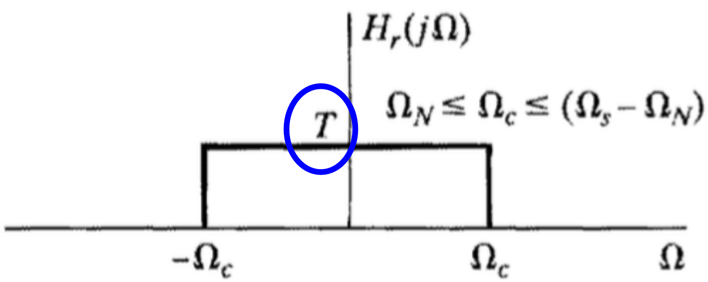
$$\boxed{\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T} \end{aligned}}$$

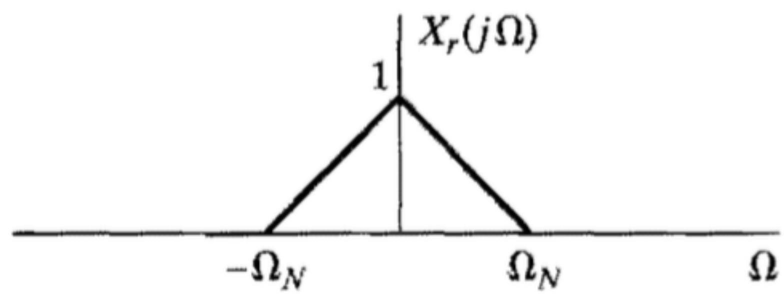
- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period T=1/6000 (f_s=6kHz)

Reconstruction in Frequency Domain









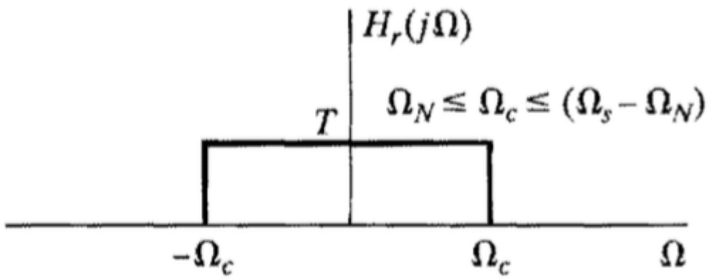
$2\Omega_N = \Omega_s$

$\Omega_N = \Omega_C = \Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

Reconstruction in Time Domain

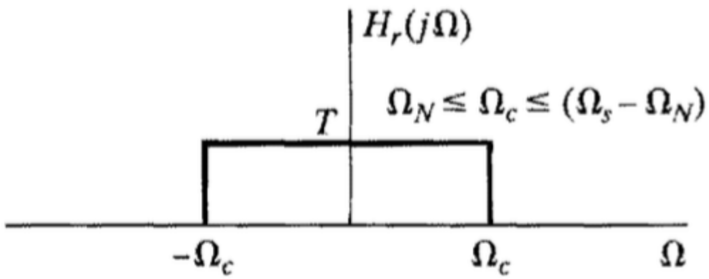
$\Omega_N = \Omega_C = \Omega_s/2$



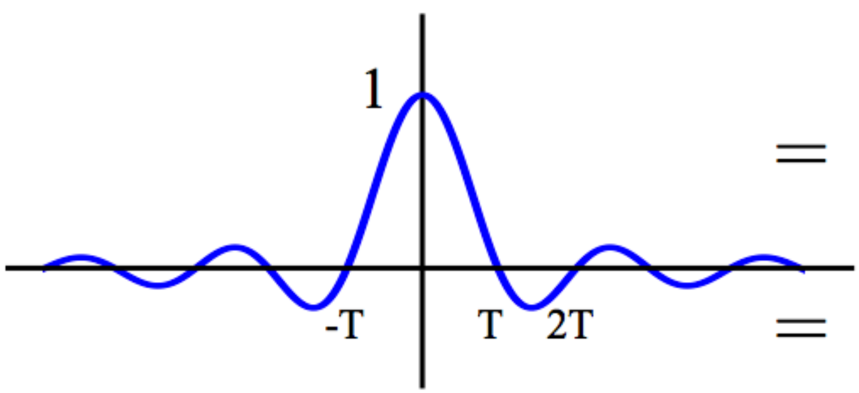
$$h_r(t) = \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t} d\Omega$$

Reconstruction in Time Domain

$\Omega_N = \Omega_C = \Omega_s/2$

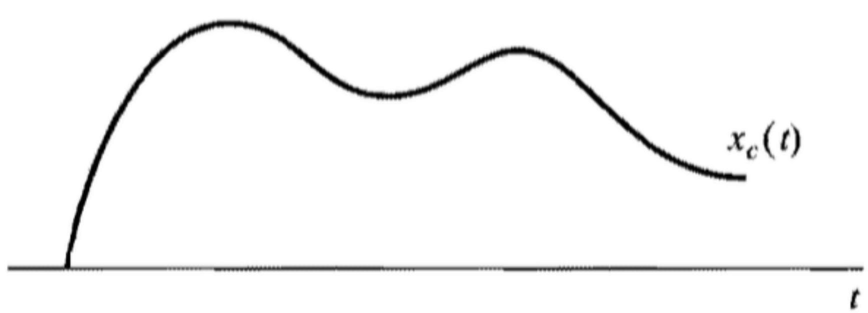


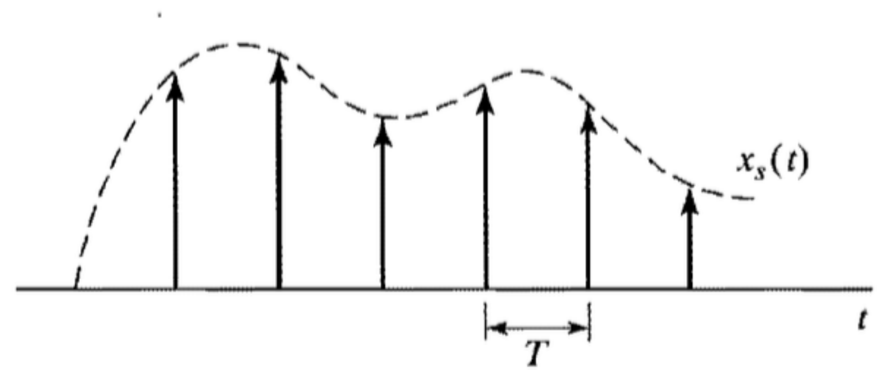
$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t}\, d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\,\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t}\sin\left(\frac{\pi}{T}t\right) \\
&= \mathrm{sinc}\left(\frac{t}{T}\right)
\end{aligned}
$$

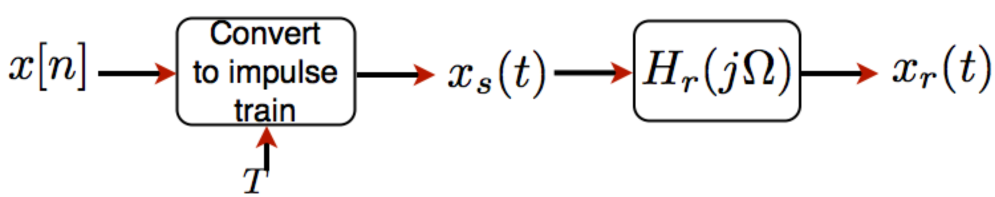


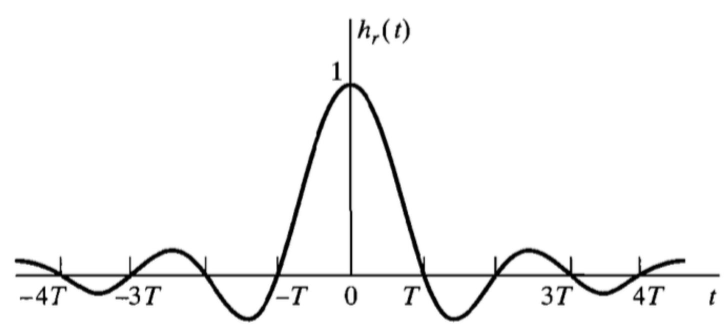
Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$
$$= \sum_n x[n] h_r(t - nT)$$









Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t)$$
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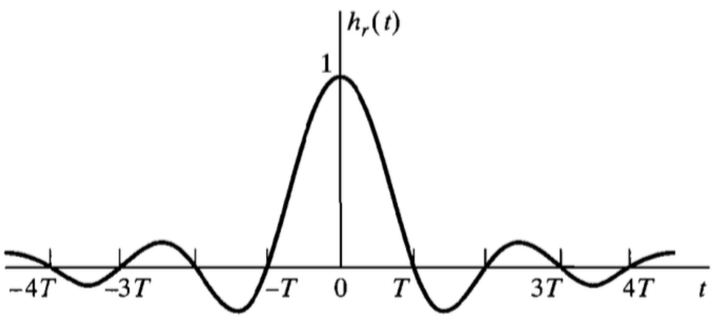

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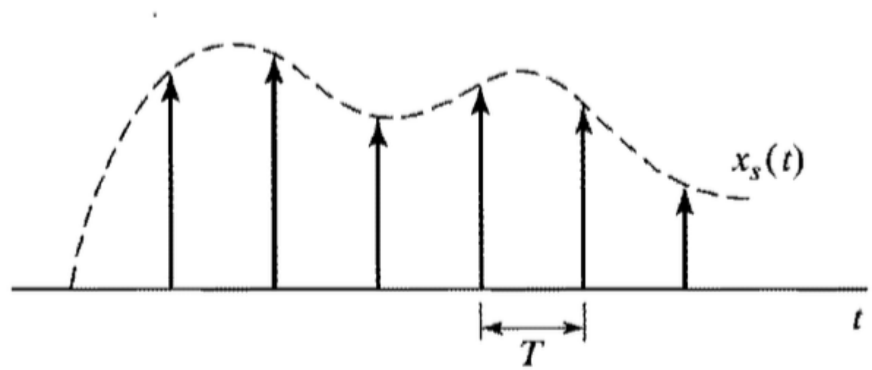

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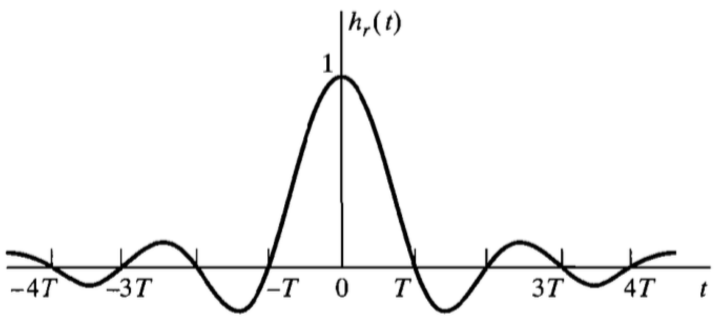

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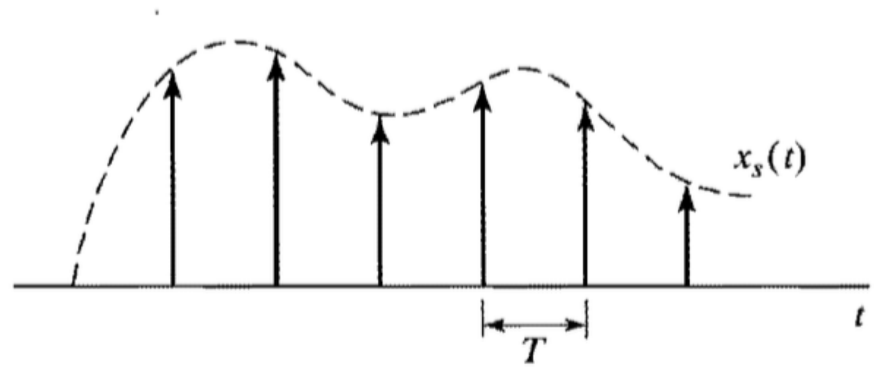

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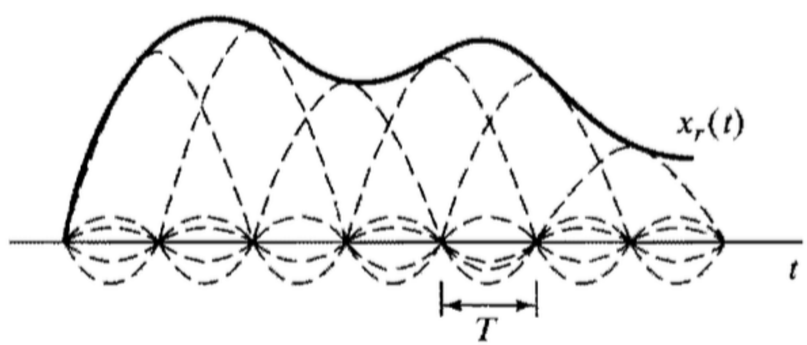
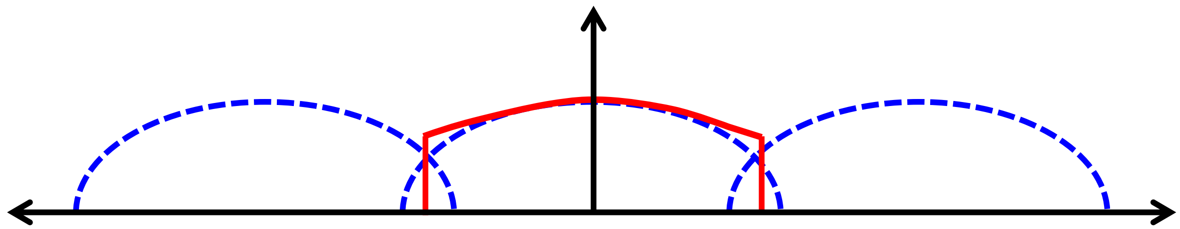


The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

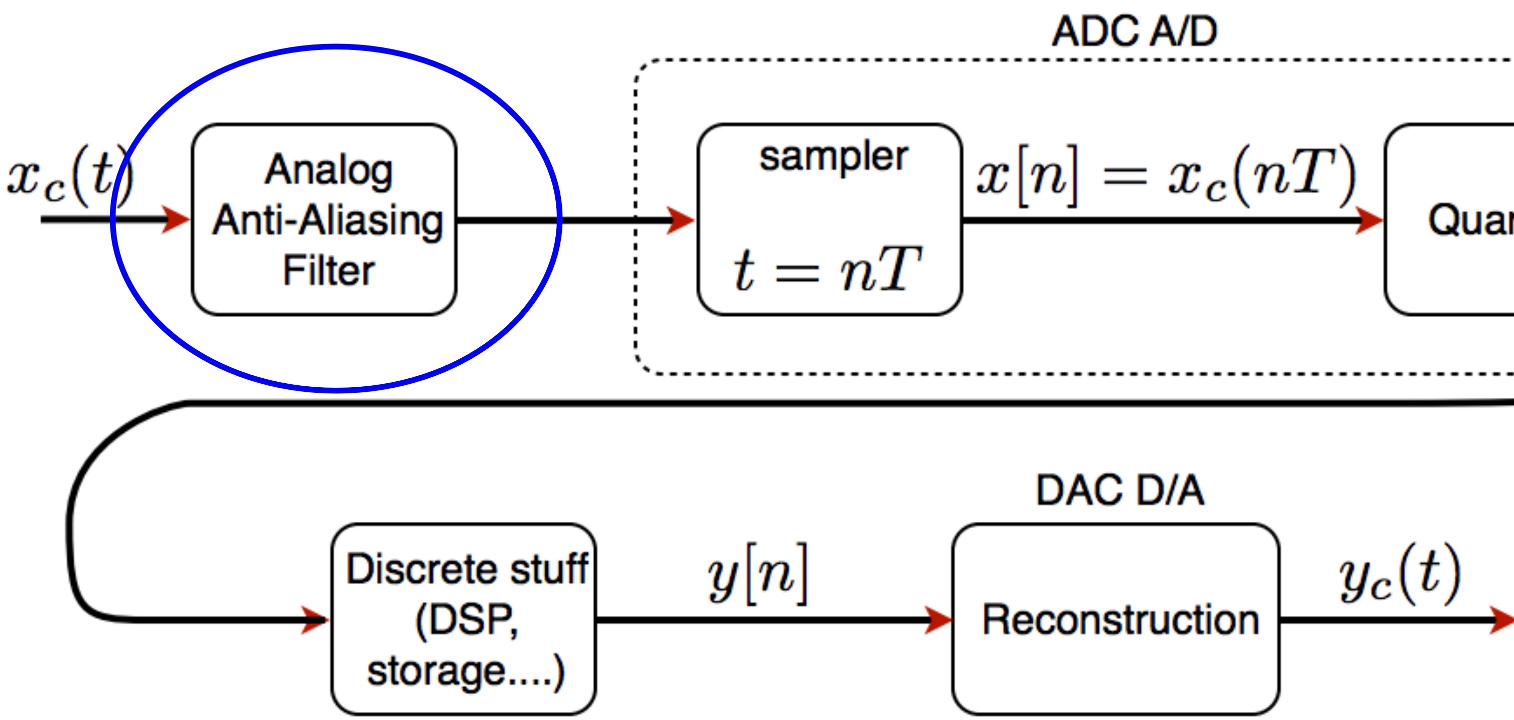
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ is an aliased version of $x_c(t)$



$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if} \quad |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

DSP System



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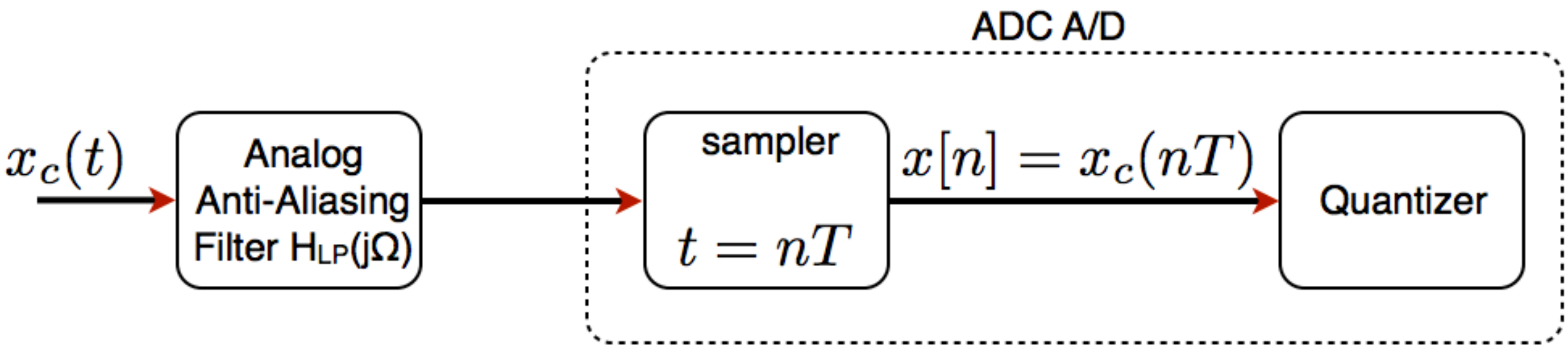
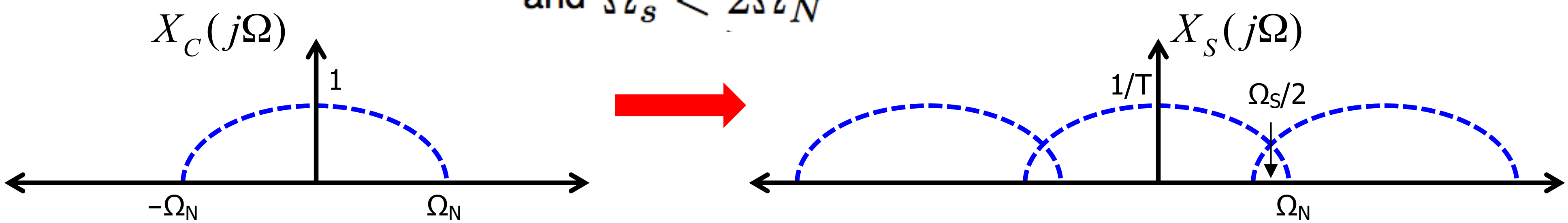
Anti-Aliasing Filter



and $\Omega_s < 2\Omega_N$



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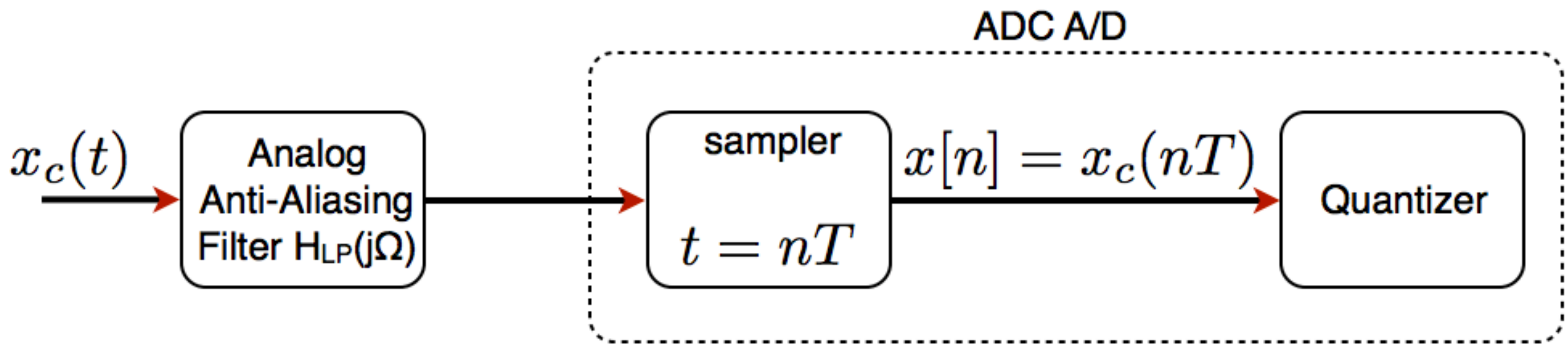

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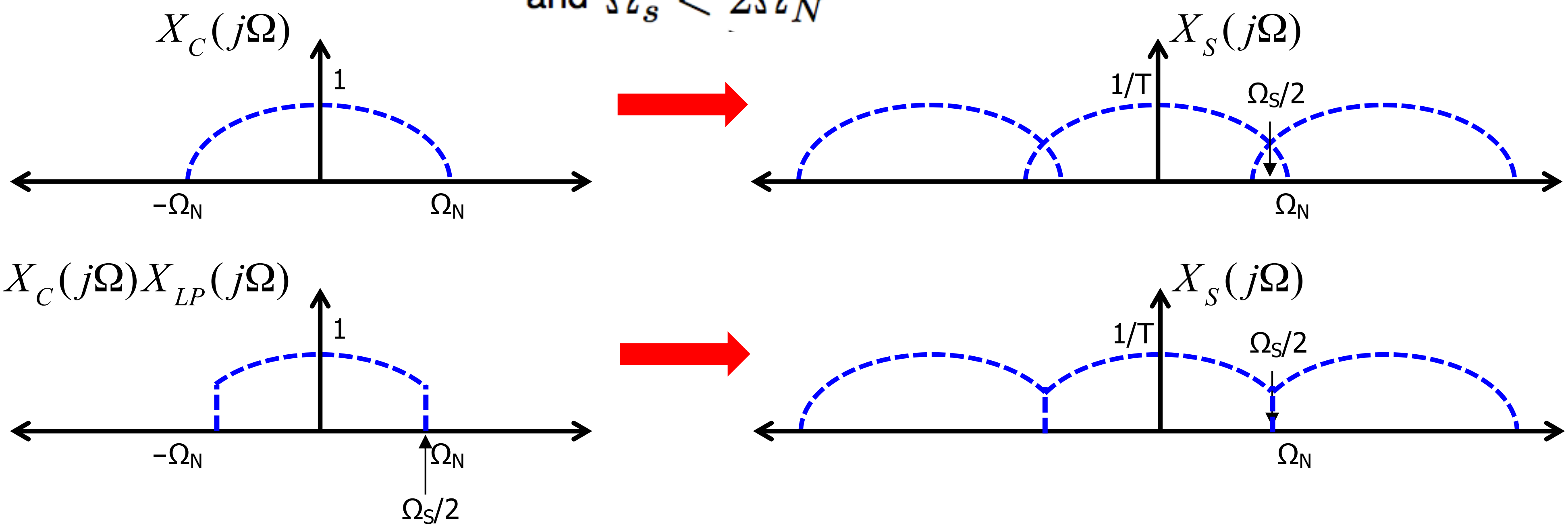
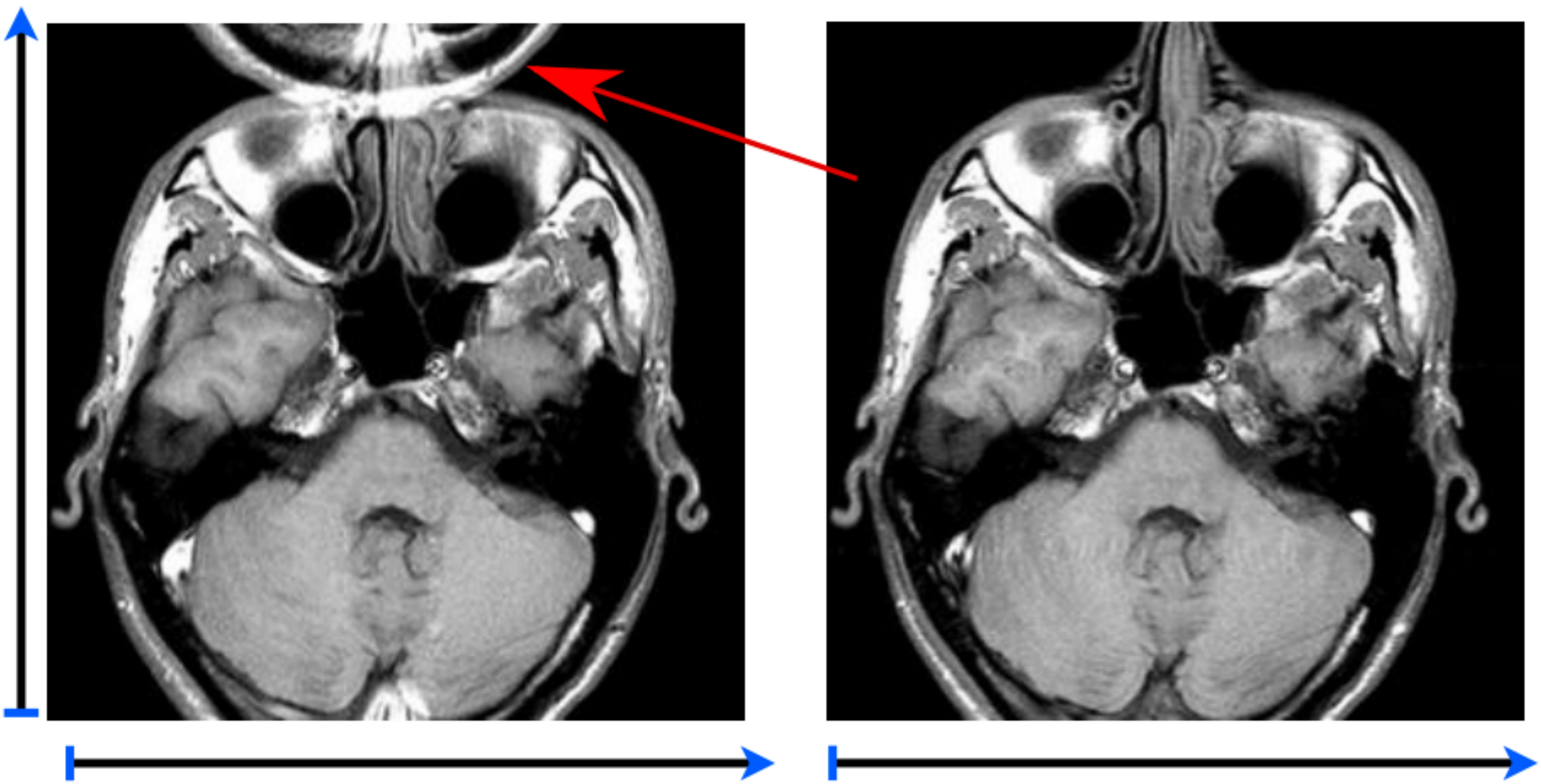


MRI aliasing example



MRI anti-aliasing example

MRI anti-aliasing example

Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited

Admin

- HW 2 due Monday 2/15 at midnight
  - Double check that your submission by viewing it!

- Recitation videos in Canvas and notes on course webpage

- Make sure you have access to Matlab
  - Let us know ASAP if you need help!