University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Final

Thursday, May 5

- 5 Problems with point weightings shown. All 5 problems must be completed.
- Calculators (non-cellphone) allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Final answers here.
- Additional workspace in exam book. Note where to find work in exam book if relevant.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Q5	
Total	Mean: 55.9, Stdev: 20.8

Transform Pairs/Properties and Formulas

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identities:

$e^{j\Theta} = cos(\Theta)+jsin(\Theta)$	$cos(\Theta) = \frac{1}{2}(e^{j\Theta}+e^{-j\Theta})$	$sin(\Theta) = \frac{1}{2j}(e^{j\Theta}-e^{-j\Theta})$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

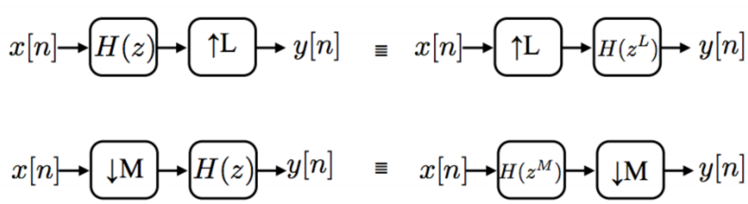
Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M}-\frac{2\pi}{M}i)})$

Generalized Linear Phase Systems:

	Type I	Type II
Symmetry	Even, $h[n] = h[M-n]$	Even, $h[n] = h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]cos(\omega k)\right)$ $a[0] = h[M/2]$ $a[k] = 2h[(M/2)-k]$ for $k = 1, 2, ..., M/2$	$e^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} b[k]cos(\omega(k-\frac{1}{2}))\right)$ $b[k] = 2h[(M+1)/2-k]$ for $k = 1, 2, ..., (M+1)/2$
	Type III	Type IV
Symmetry	Odd, $h[n] = -h[M-n]$	Odd, $h[n] = -h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$je^{-j\omega M/2}\left(\sum_{k=1}^{M/2} c[k]sin(\omega k)\right)$ $c[k] = 2h[(M/2)-k]$ for $k = 1, 2, ..., M/2$	$je^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} d[k]sin(\omega(k-\frac{1}{2}))\right)$ $d[k] = 2h[(M+1)/2-k]$ for $k = 1, 2, ..., (M+1)/2$

Interchange Identities:

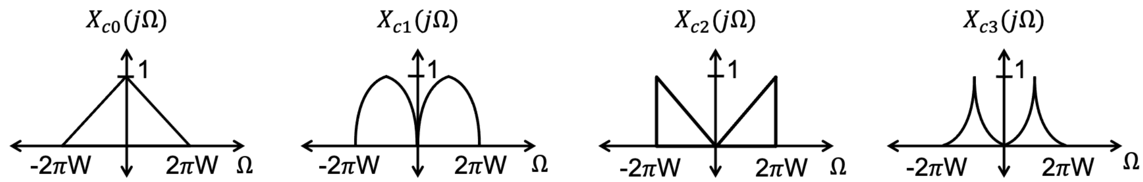


DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

1. (25 pts) Consider four different continuous-time signals with the spectral graphs shown below:

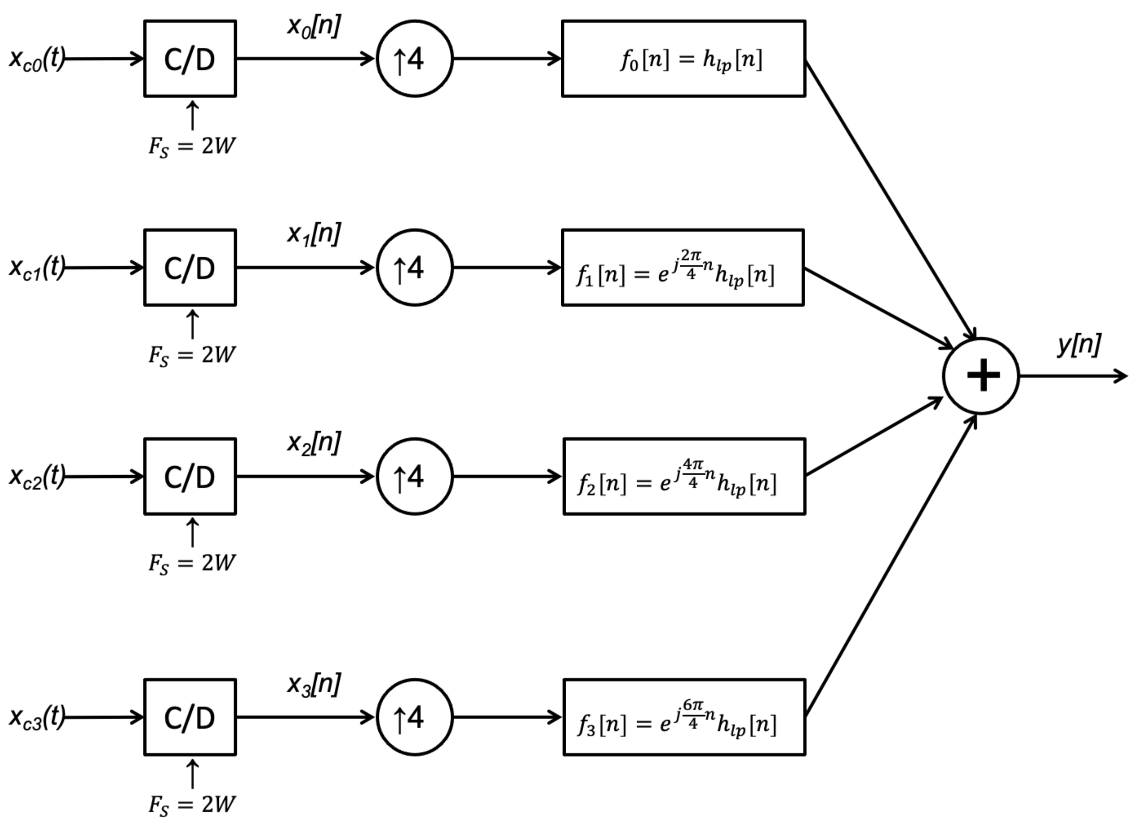


The signals are all sampled at the Nyquist rate and put into the digital subbanding system shown below. The impulse responses for the four filters

$$f_k[n] = e^{j(k\frac{2\pi}{4})n} h_{lp}[n], \text{ for } k = 0, 1, 2, 3$$

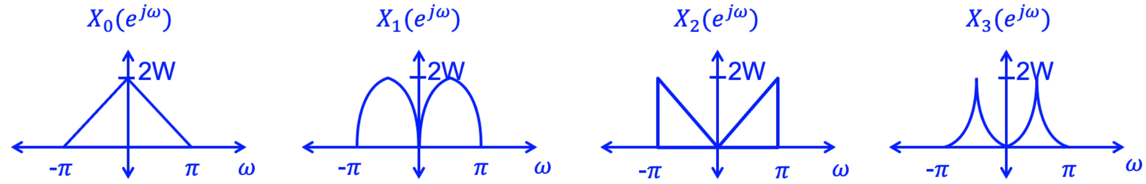
are defined in terms of the ideal low pass filter given as

$$h_{lp}[n] = 4\frac{sin(\frac{\pi}{4}n)}{\pi n}$$

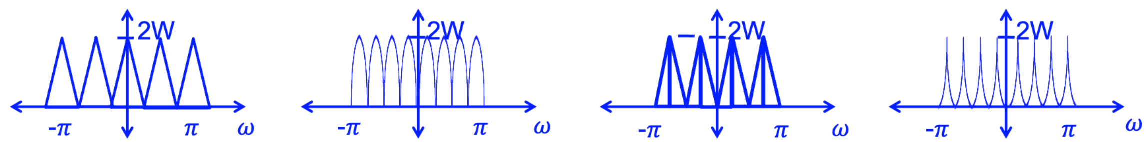


Plot the magnitude of the DTFT $Y(e^{j\omega})$ of the final sum output $y[n]$ over $-\pi < \omega < \pi$. Show your work and any intermediate signals for partial credit.

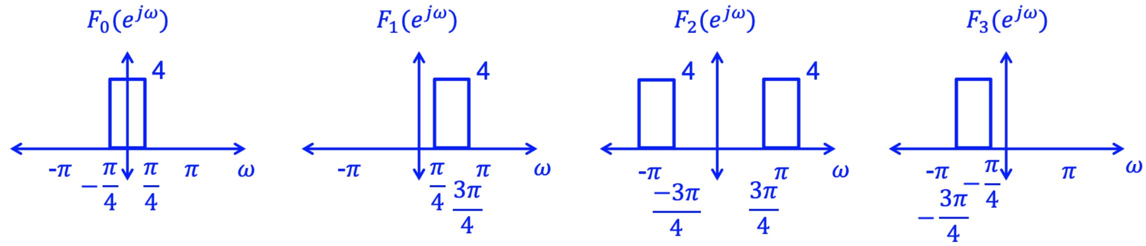
After sampling:



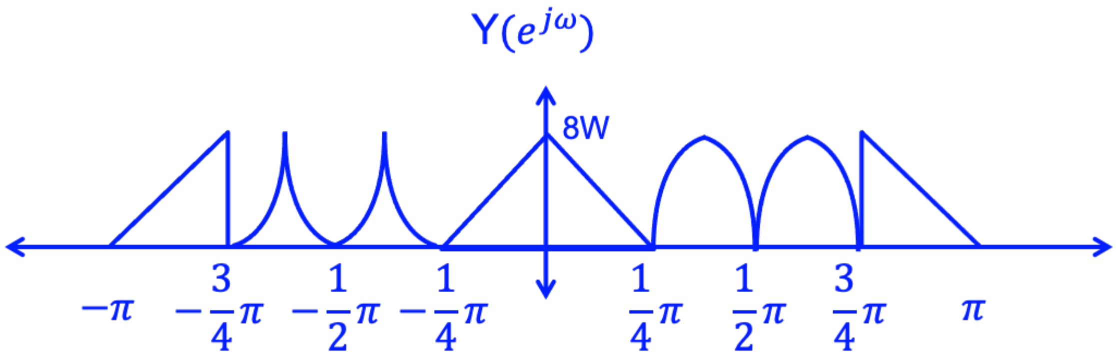
After upsampling by 4:



Filter frequency responses:



After filtering and summing:



2. (15 pts) Consider a causal system $h_1[n]$ with the z-transform

$$H_1(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{5}z\right)}{\left(1 - \frac{1}{6}z\right)}$$

Note the different exponents on the z variables. A new system $H_2(z)$ is designed such that $h_2[n] = z_0^n h_1[n]$. For what values of z_0 is $H_2(z)$ a minimum-phase system?

$$H_1(z) = \frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{5}z\right)}{\left(1 - \frac{1}{6}z\right)} = \frac{6}{5}\frac{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - 5z^{-1}\right)}{\left(1 - 6z^{-1}\right)}$$

$$h_2[n] = z_0^n h_1[n] \leftrightarrow H_2(z) = H_1\left(\frac{z}{z_0}\right) = \frac{6}{5}\frac{\left(1 - \frac{1}{2}z_0z^{-1}\right)\left(1 - \frac{1}{4}z_0z^{-1}\right)\left(1 - 5z_0z^{-1}\right)}{\left(1 - 6z_0z^{-1}\right)}$$

A minimum phase system has all its poles and zeros inside the unit circle.

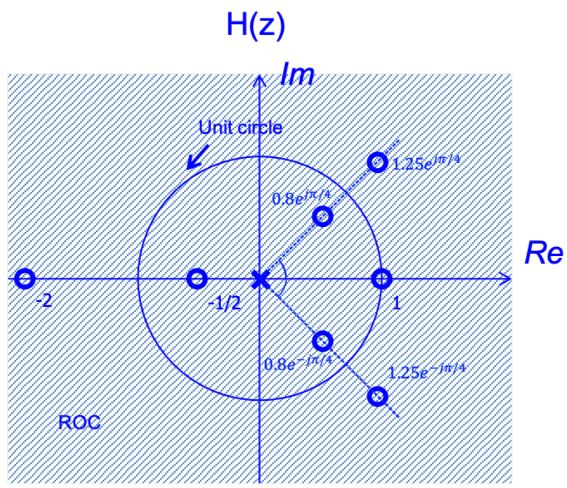
$$|\frac{z_0}{2}| < 1 \rightarrow |z_0| < 2$$
$$|\frac{z_0}{4}| < 1 \rightarrow |z_0| < 4$$
$$|5z_0| < 1 \rightarrow |z_0| < \frac{1}{5}$$
$$|6z_0| < 1 \rightarrow |z_0| < \frac{1}{6}$$

Therefore, $|z_0| < \frac{1}{6}$ for $H_2(z)$ to be minimum phase.

3. (20 pts) A generalized linear-phase FIR system has an impulse response with real values and $h[n] = 0$ for $n < 0$ and for $n \geq 8$, and $h[n] = -h[7-n]$. The system function of this system has a zero at $z = 0.8e^{\frac{j\pi}{4}}$ and another zero at $z = -2$. Find $H(z)$ and draw its pole-zero diagram with the region of convergence specified.

We know that $h[n]$ is length 8 ($M = 7$) and therefore has 7 zeros. Since $M = 7$ is odd and this has odd symmetry, it is a Type IV GLP filter with real coefficients. It therefore has the property that its zeros come in conjugate and reciprocal pairs. The zero at $z = -2$ implies another zero at $z = -\frac{1}{2}$, and the zero at $z = 0.8e^{\frac{j\pi}{4}}$ implies zeros at $z = 0.8e^{-\frac{j\pi}{4}}$, $z = 1.25e^{\frac{j\pi}{4}}$, and $z = 1.25e^{-\frac{j\pi}{4}}$. Because it is a Type IV filter it also has a zero at $z = 1$. Putting all this together gives

$$\begin{aligned} H(z) = &(1+2z^{-1})(1+0.5z^{-1})(1-0.8e^{j\frac{\pi}{4}}z^{-1})(1-0.8e^{-j\frac{\pi}{4}}z^{-1}) \\ &(1-1.25e^{j\frac{\pi}{4}}z^{-1})(1-1.25e^{-j\frac{\pi}{4}}z^{-1})(1-z^{-1}) \end{aligned}$$



4. (20 pts) Read each part of this problem carefully to note the differences among the two parts.

   (a) Consider the signal

$$x[n] = \begin{cases} 1 + cos(\frac{\pi}{4}n) - 0.5cos(\frac{3\pi}{4}n), & \text{if } 0 \le n \le 7 \\ 0, & \text{else} \end{cases}$$

   which can be represented by the IDFT equation as

$$x[n] = \begin{cases} \frac{1}{8}\sum\limits_{k=0}^{7} X_8[k]e^{j\frac{2\pi}{8}kn}, & \text{if } 0 \le n \le 7 \\ 0, & \text{else} \end{cases}$$

   where $X_8[k]$ is the 8-point DFT of x[n]. Plot $X_8[k]$ for $0 \le k \le 7$.

   (b) Determine $V_{16}[k]$, the 16-point DFT of the 16-point sequence $v[n]$, where

$$v[n] = \begin{cases} 1 + cos(\frac{\pi}{4}n) - 0.5cos(\frac{3\pi}{4}n), & \text{if } 0 \le n \le 15 \\ 0, & \text{else} \end{cases}$$

   Plot $V_{16}[k]$ for $0 \le k \le 15$.

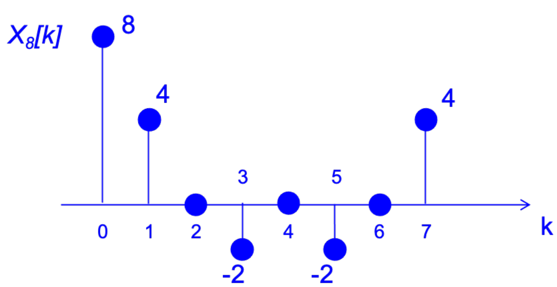
(a) We can rewrite $x[n]$ as

$$\begin{aligned} x[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\ &= 1 + \frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{j\frac{2\pi}{8}n\cdot 7} - \frac{1}{4}e^{j\frac{2\pi}{8}n\cdot 3} - \frac{1}{4}e^{j\frac{2\pi}{8}n\cdot 5} \\ &= \frac{1}{8}\left(8 + 4e^{j\frac{2\pi}{8}n} + 4e^{j\frac{2\pi}{8}n\cdot 7} - 2e^{j\frac{2\pi}{8}n\cdot 3} - 2e^{j\frac{2\pi}{8}n\cdot 5}\right) \end{aligned}$$

Setting this equal to the IDFT representation of $x[n]$ we get

$$\frac{1}{8}\left(8 + 4e^{j\frac{2\pi}{8}n} + 4e^{j\frac{2\pi}{8}n\cdot 7} - 2e^{j\frac{2\pi}{8}n\cdot 3} - 2e^{j\frac{2\pi}{8}n\cdot 5}\right) = \frac{1}{8}\sum_{k=0}^{7} X_8[k]e^{j\frac{2\pi}{8}kn}$$

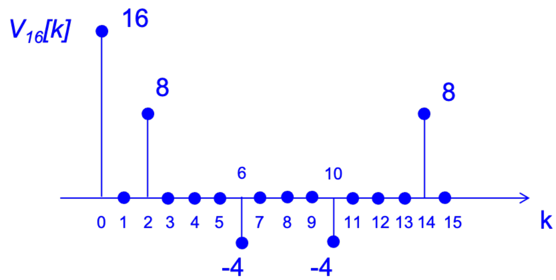
We thus get the following plot for $X_8[k]$



(b) Similarly since $x[n]$ is the same sum of cosines,

$$\begin{aligned} x[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\ &= \frac{1}{16}\left(16 + 8e^{j\frac{2\pi}{16}n\cdot 2} + 8e^{j\frac{2\pi}{16}n\cdot 14} - 4e^{j\frac{2\pi}{16}n\cdot 6} - 4e^{j\frac{2\pi}{16}n\cdot 10}\right) \end{aligned}$$

We thus get the following plot for $V_{16}[k]$



5. (20 pts) You have two discrete-time signals, $x[n]$ and $v[n]$, where $x[n] = 0$ for $n < 0$ and $n \geq 500$ and $v[n] = 0$ for $n < 0$ and $n \geq 450$. Describe what FFT/IFFT operations (including the FFT/IFFT length) you would use in order to efficiently compute the linear convolution $x[n] * v[n]$ and estimate the number of (complex) multiplications your method would need.

In general, the convolution of $x[n]$ and $v[n]$ will have $500 + 450 = 950$ nonzero components. The smallest power of 2 that is larger than 950 is $L = 1024 = 2^10$. Let us pad $x[n]$ and $v[n]$ with zeros so that

$$x[n] = 0, \; n = 500, 501, ..., 1024$$
$$v[n] = 0, \; n = 450, 451, ..., 1024$$

Then to compute $x[n] * v[n]$, we would first use the FFT algorithm to compute the L-point DFTs X_k and V_k of $x[n]$ and $v[n]$, and then use the FFT algorithm to compute the inverse L-point DFT of the product of X_k and V_k. We will need:

- There are $(1/2) \cdot L \cdot log_2(L) = (1/2) \cdot 1024 \cdot 10 = 5120$ multiplications to compute the L-point DFT of $x[n]$.
- There are $(1/2) \cdot L \cdot log_2(L) = (1/2) \cdot 1024 \cdot 10 = 5120$ multiplications to compute the L-point DFT of $v[n]$.
- L = 1024 multiplications to compute the product of X_k and V_k.
- There are $(1/2) \cdot L \cdot log_2(L) = (1/2) \cdot 1024 \cdot 10 = 5120$ multiplications to compute the L-point inverse DFT of the product of X_k and V_k.

Thus, the total number of multiplications is on the order of $5120 + 5120 + 1024 + 5120 = 16384$.