ESE 531: Digital Signal Processing

Lecture 12: February 22, 2022

Data Converters, Noise Shaping

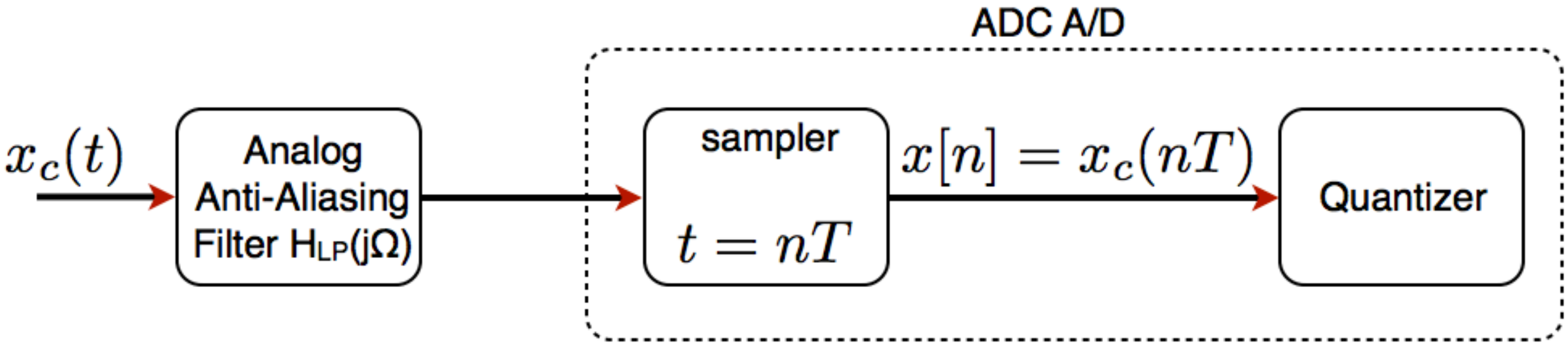
Lecture Outline

- Data Converters
  - Anti-aliasing
  - ADC
    - Quantization
  - Practical DAC
- Noise Shaping

ADC

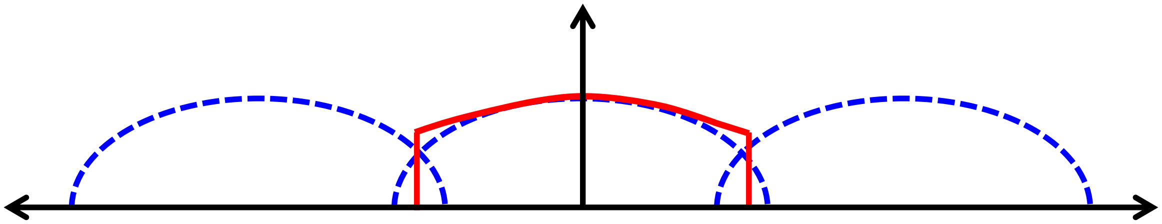
Analog to Digital Converter

Anti-Aliasing Filter with ADC



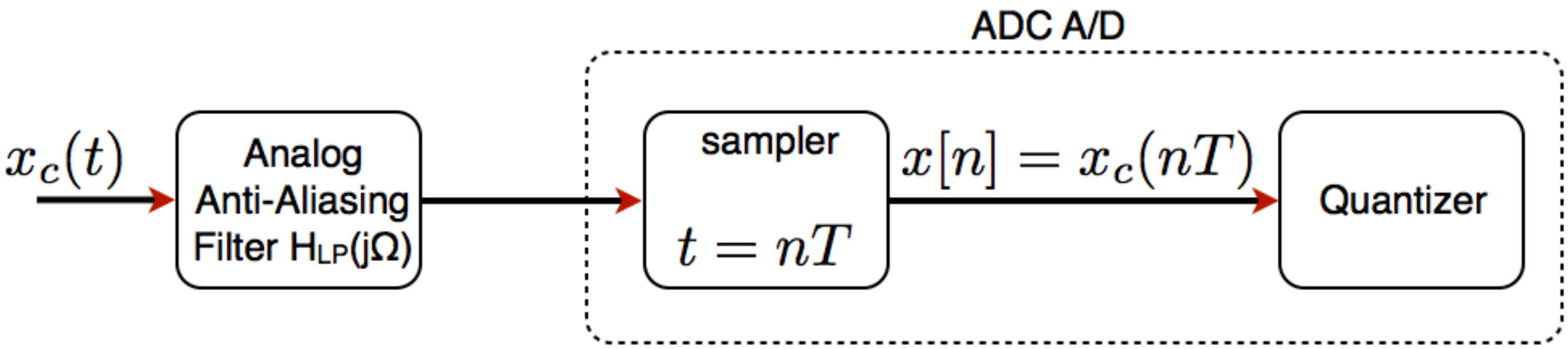
Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ an aliased version of $x_c(t)$



$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if} \quad |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Anti-Aliasing Filter with ADC



ADC A/D

$x_c(t)$

Analog Anti-Aliasing Filter $H_{LP}(j\Omega)$

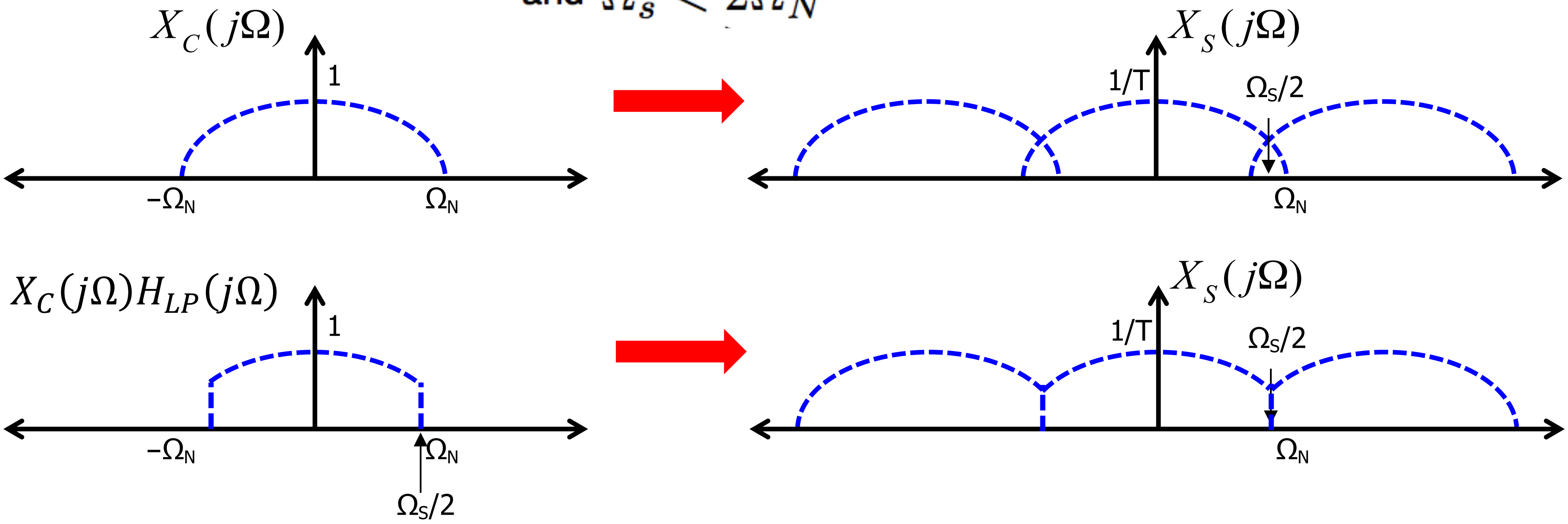
sampler

$t = nT$

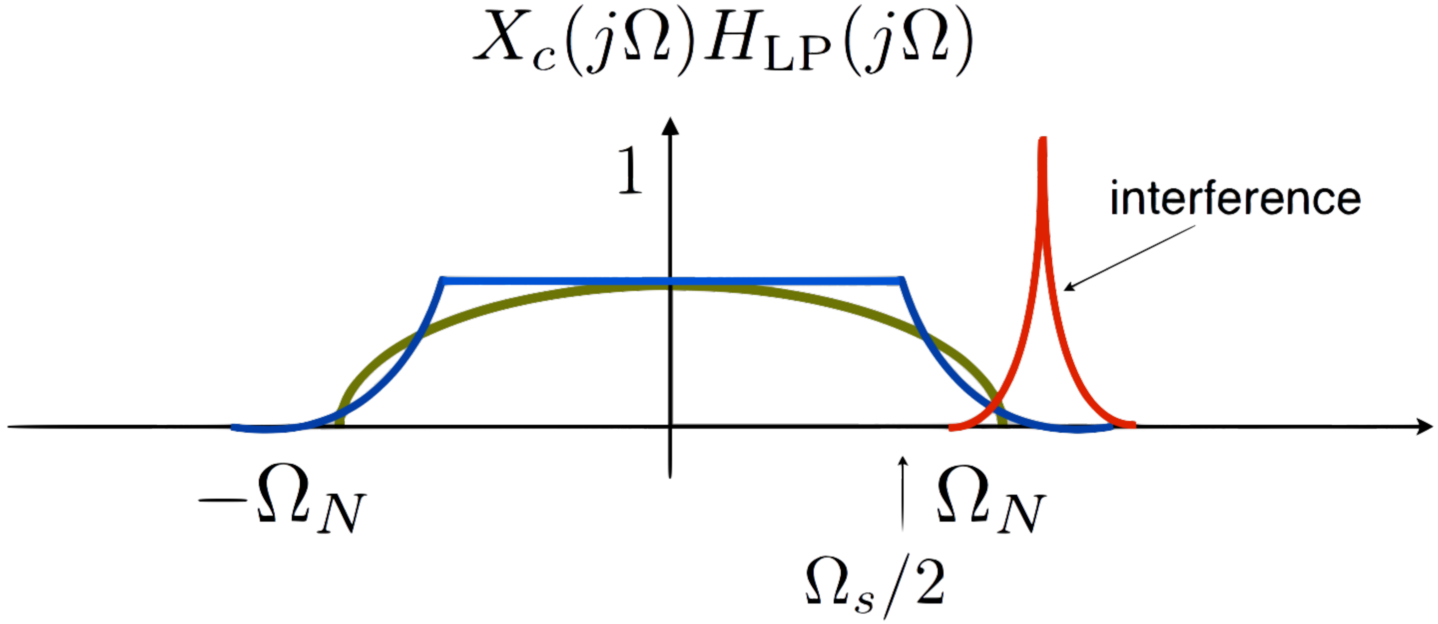
$x[n] = x_c(nT)$

Quantizer

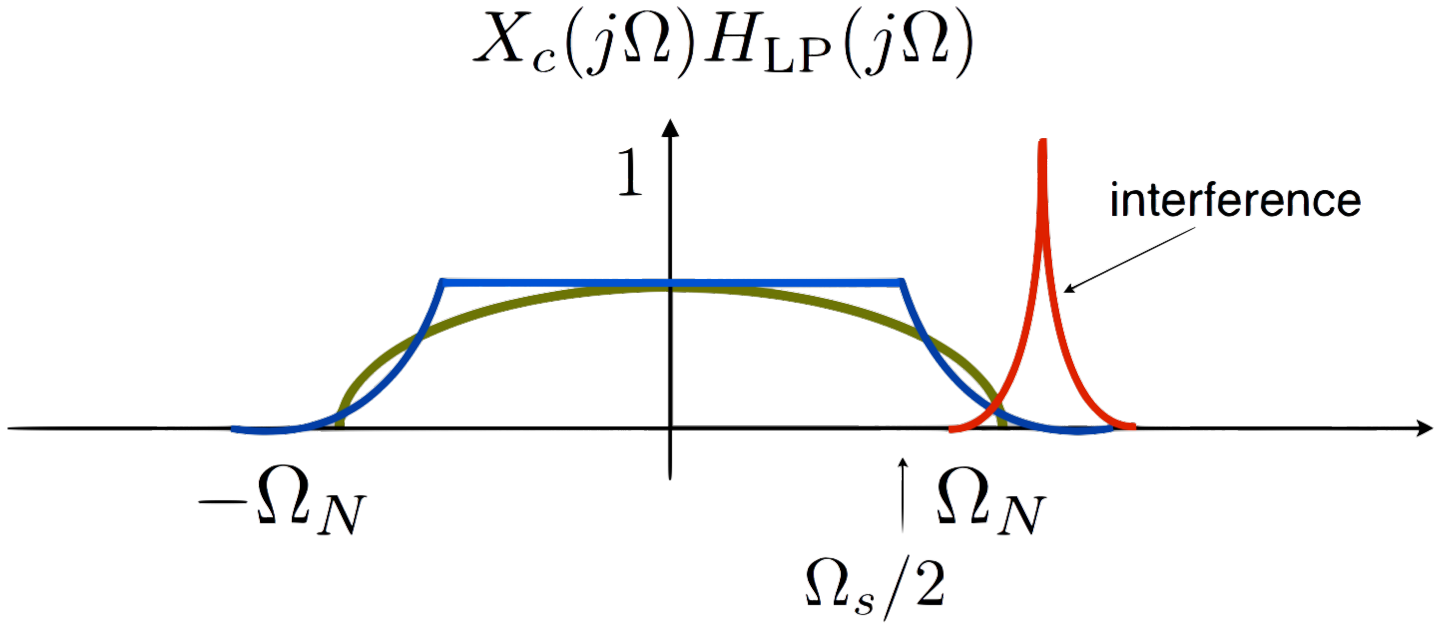
and $\Omega_s < 2\Omega_N$

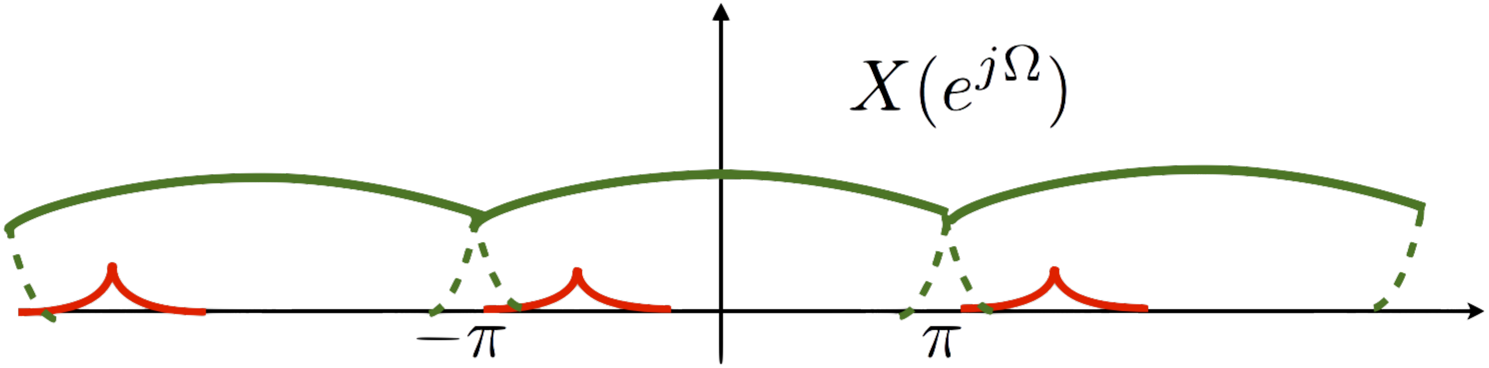


Non-Ideal Anti-Aliasing Filter

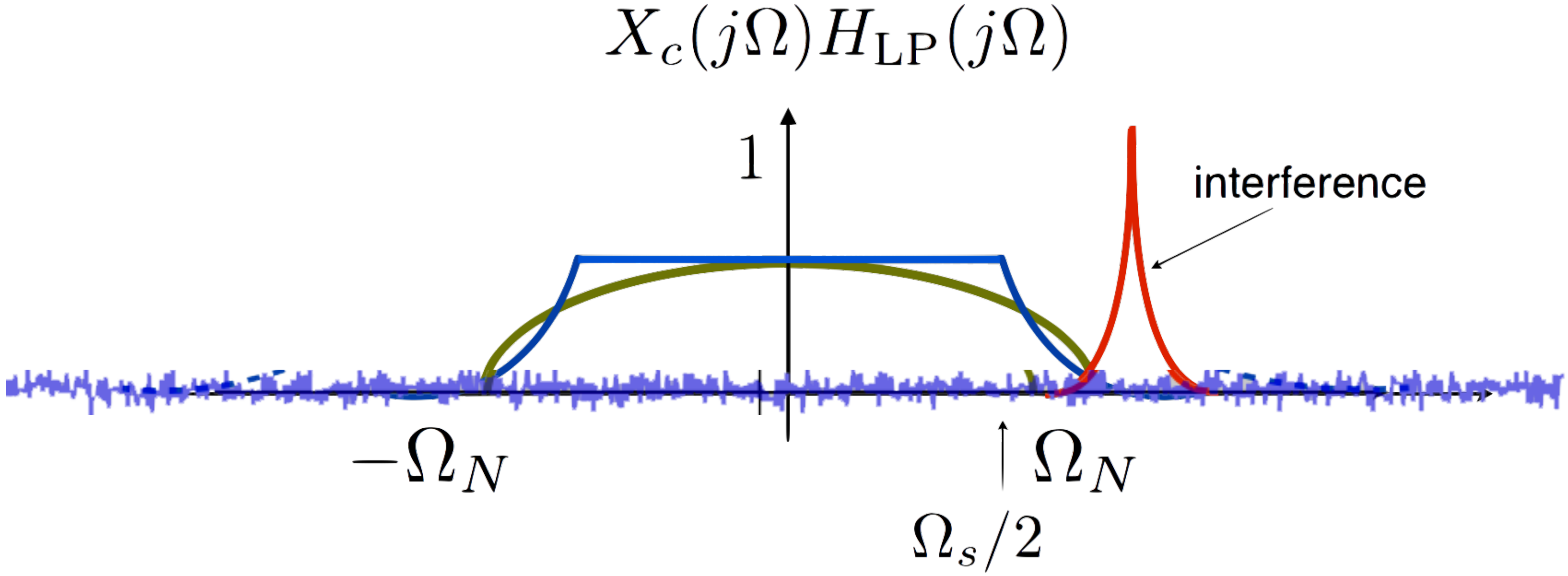


Non-Ideal Anti-Aliasing Filter



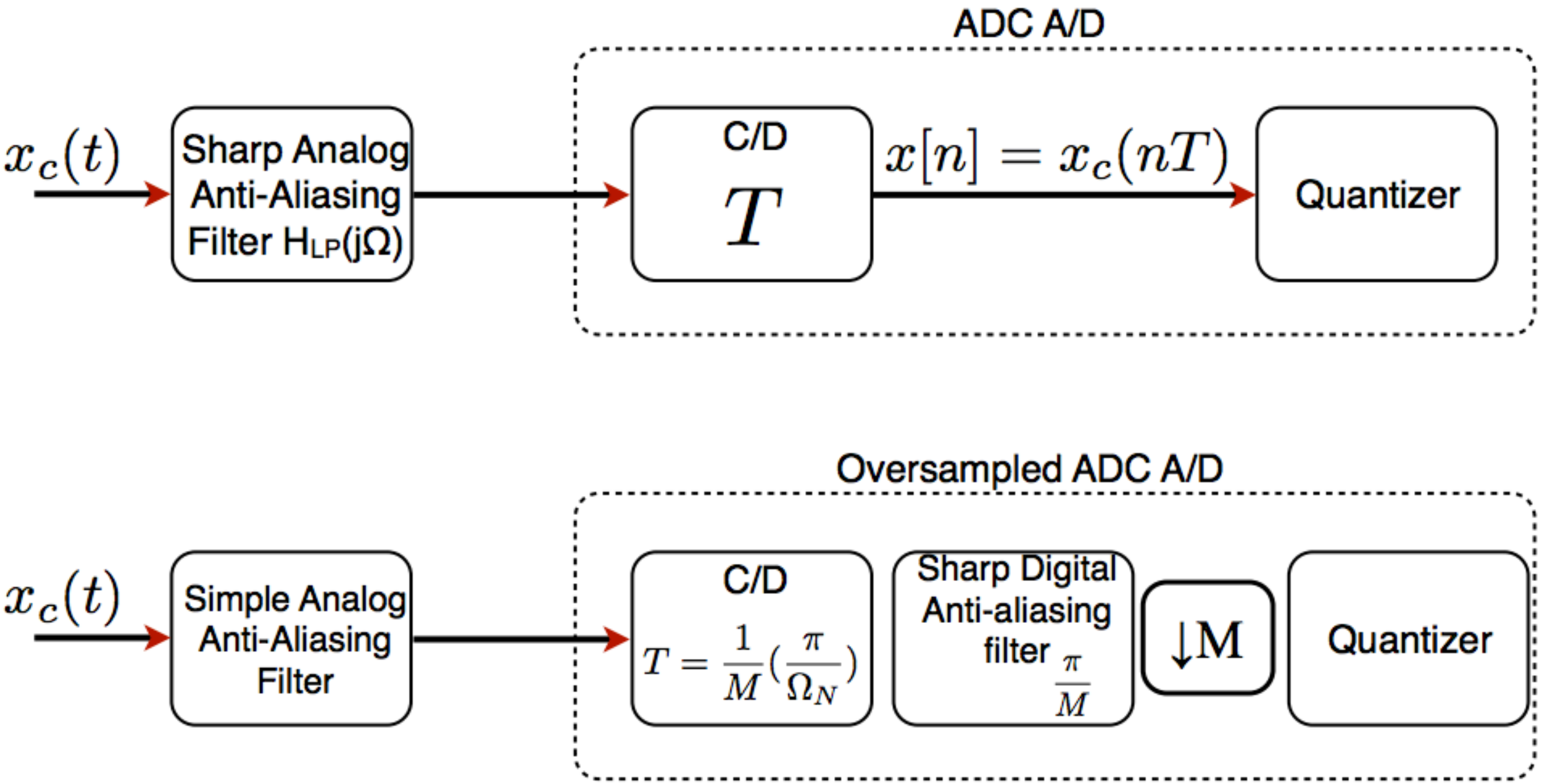


Non-Ideal Anti-Aliasing Filter

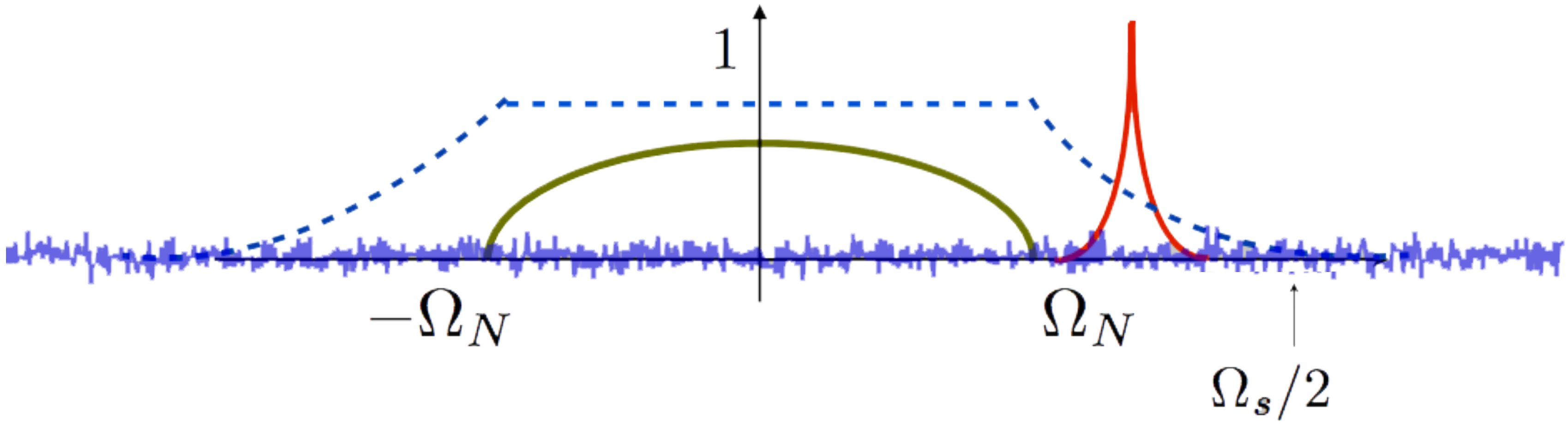


- Problem: Hard to implement sharp analog filter
- Consequence: Crop part of the signal and suffer from noise and interference

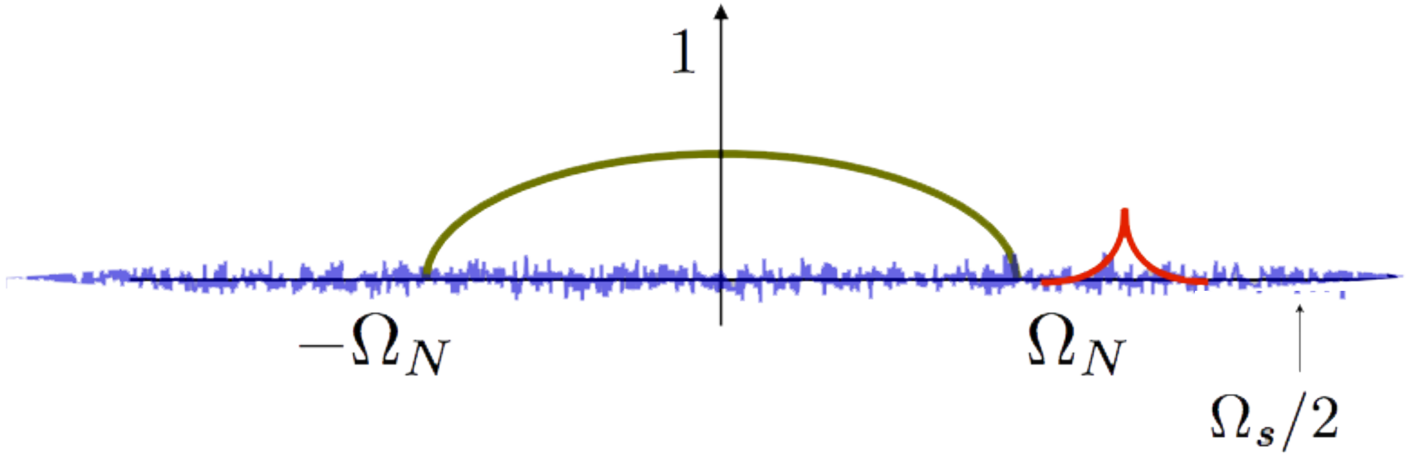
Oversampled ADC



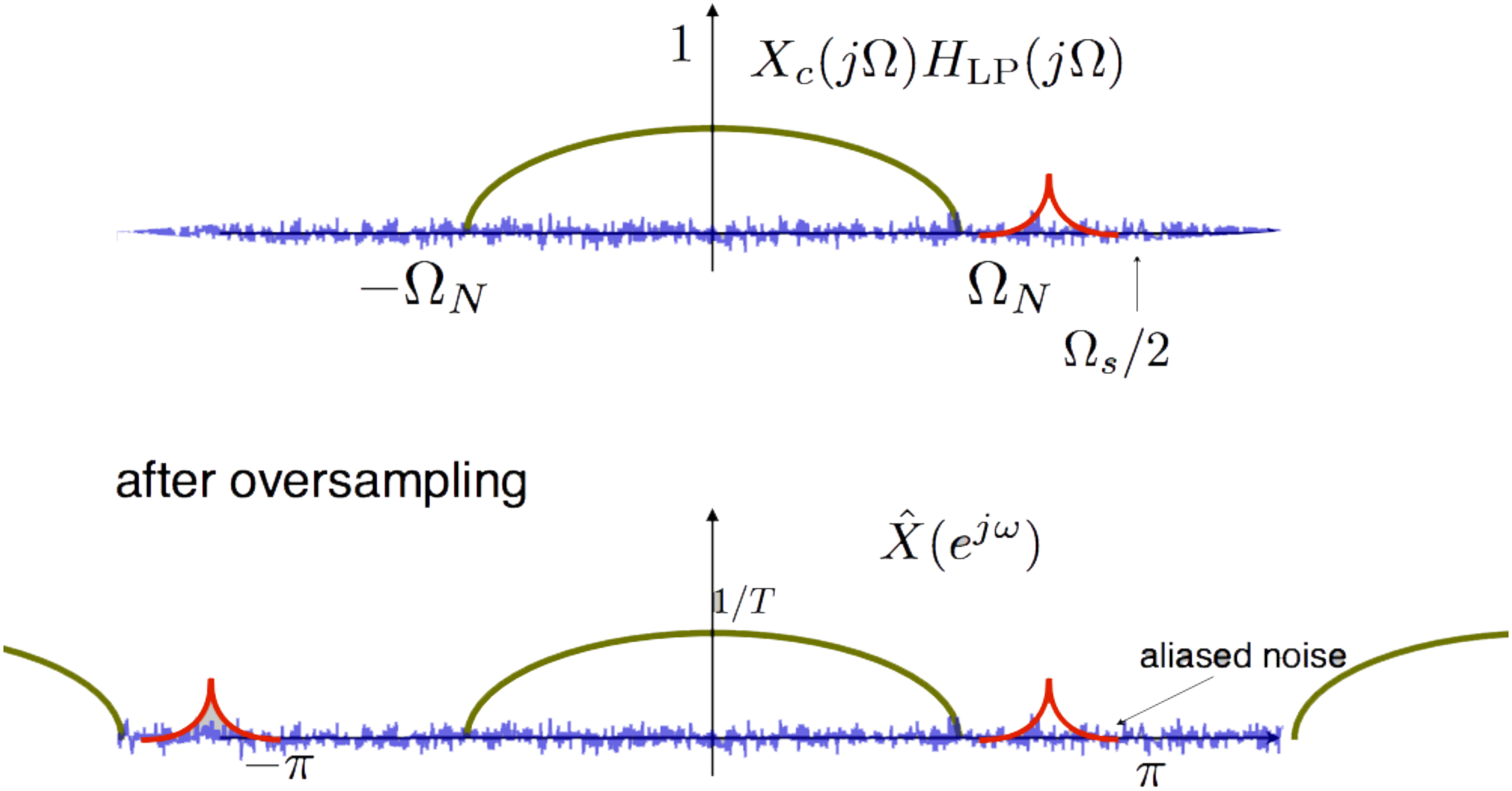
Oversampled ADC – Simple filter



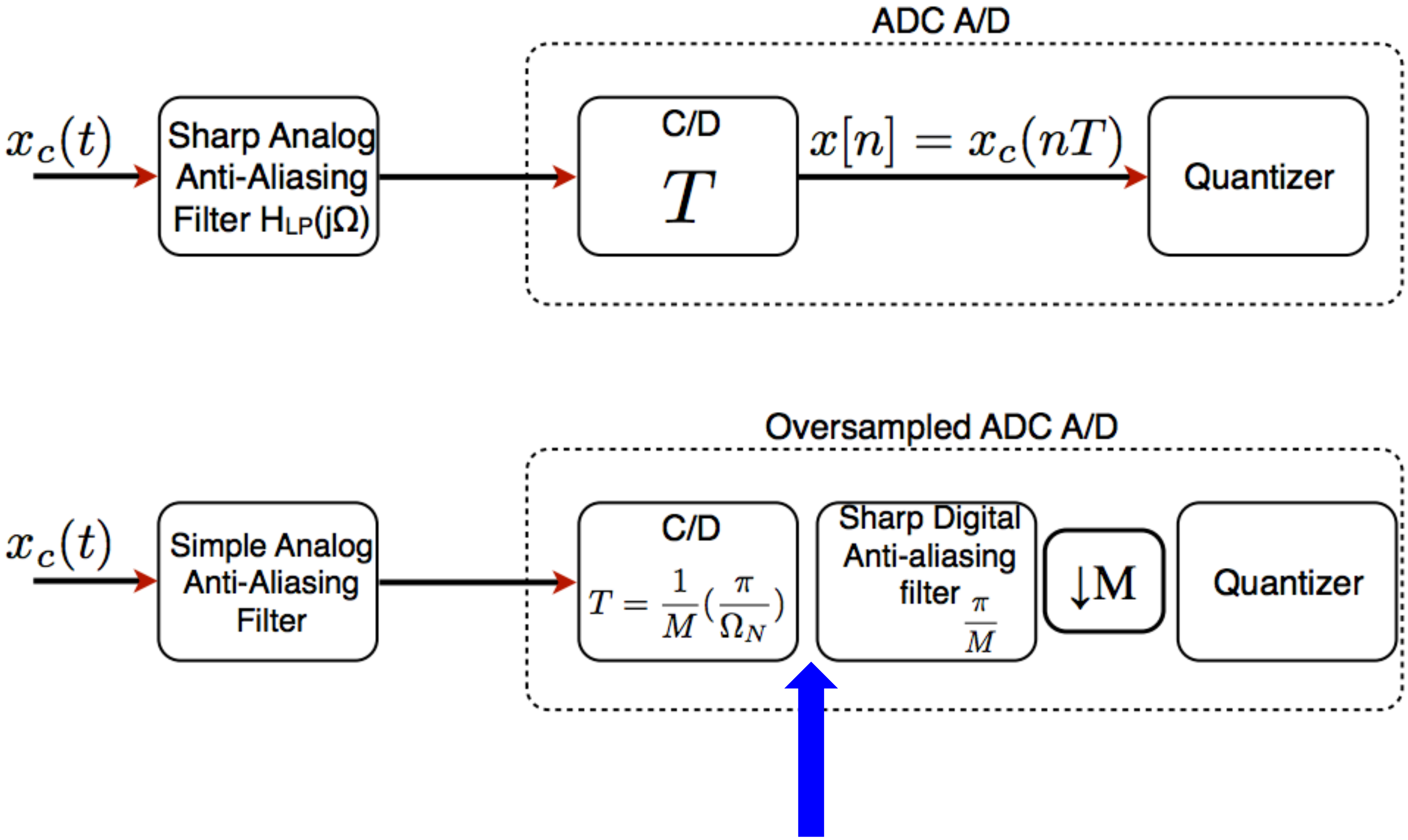
$X_c(j\Omega)H_{\mathrm{LP}}(j\Omega)$



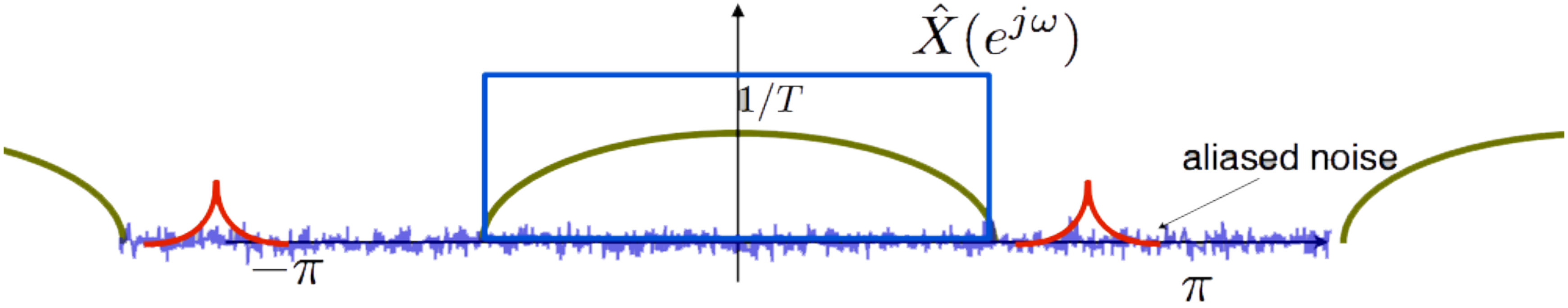
Oversampled ADC – M=2



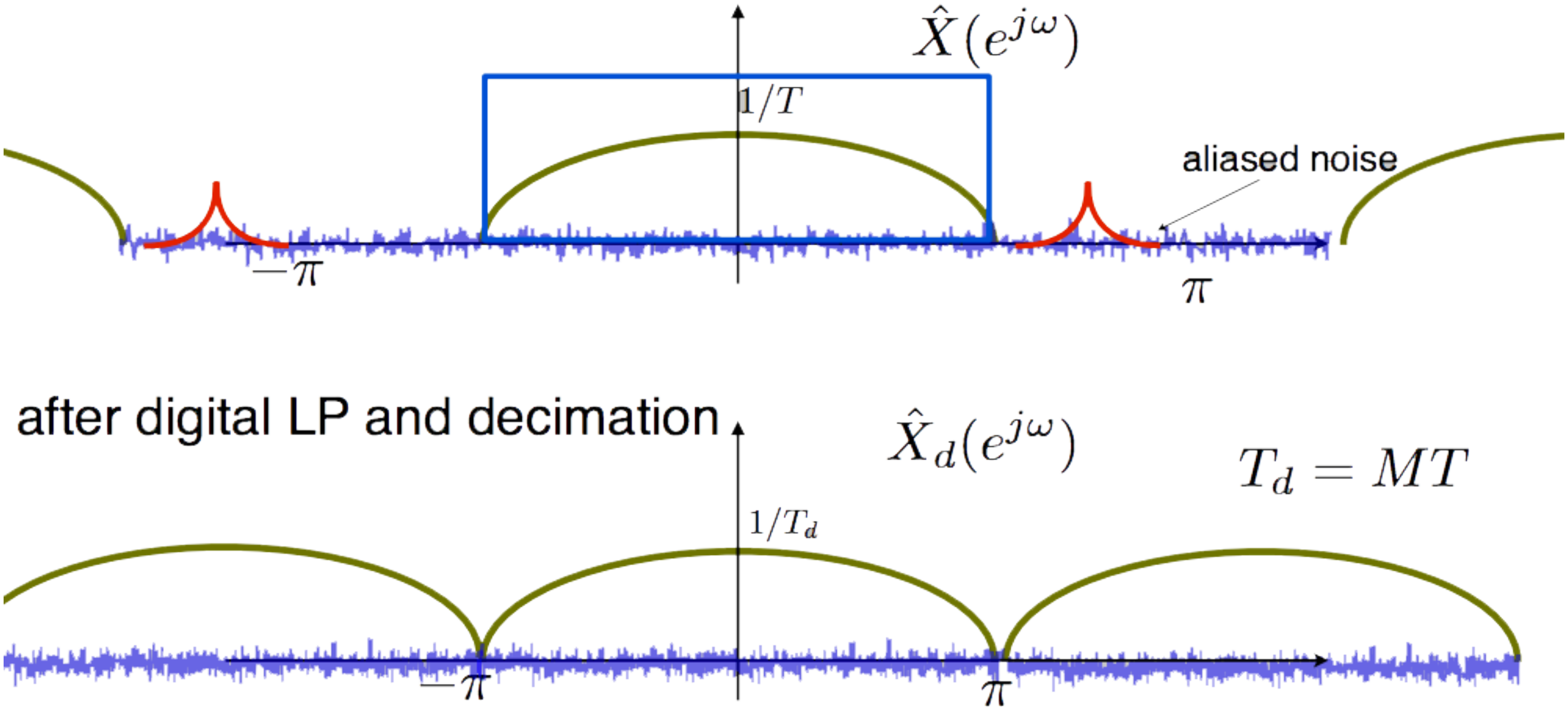
Oversampled ADC



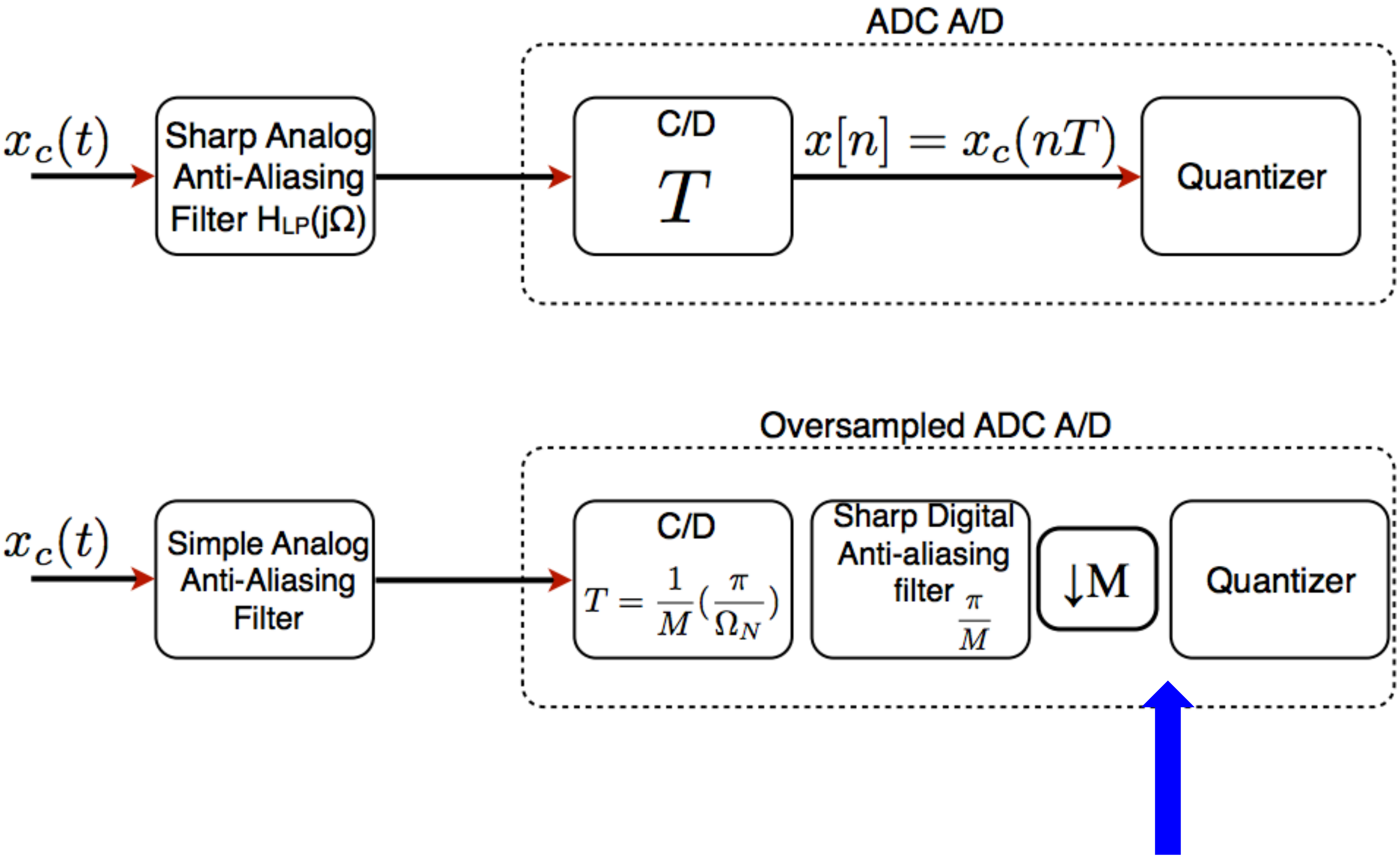
Oversampled ADC – Sharp digital filter/Downsample



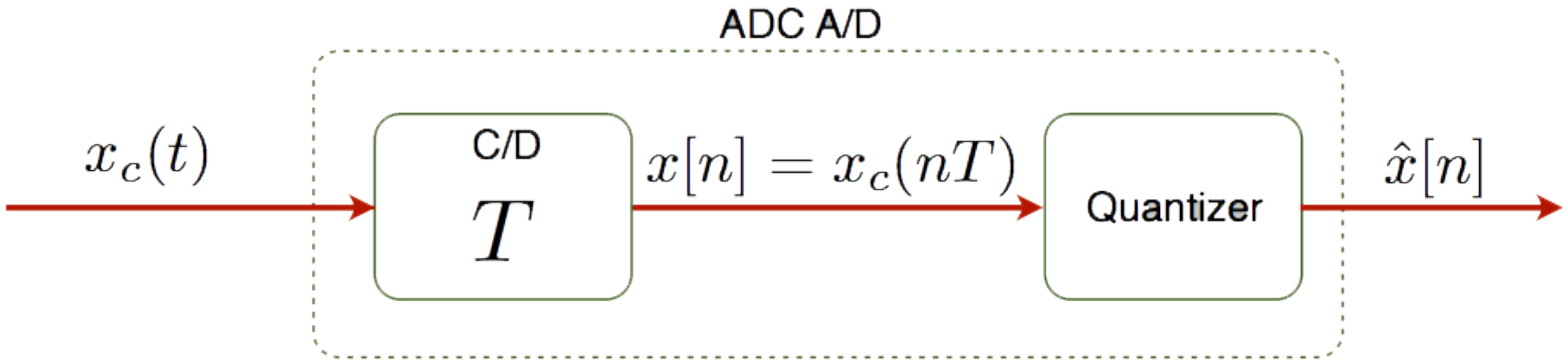
Oversampled ADC – Sharp digital filter/Downsample



Oversampled ADC



Sampling and Quantization



ADC A/D

$x_c(t)$

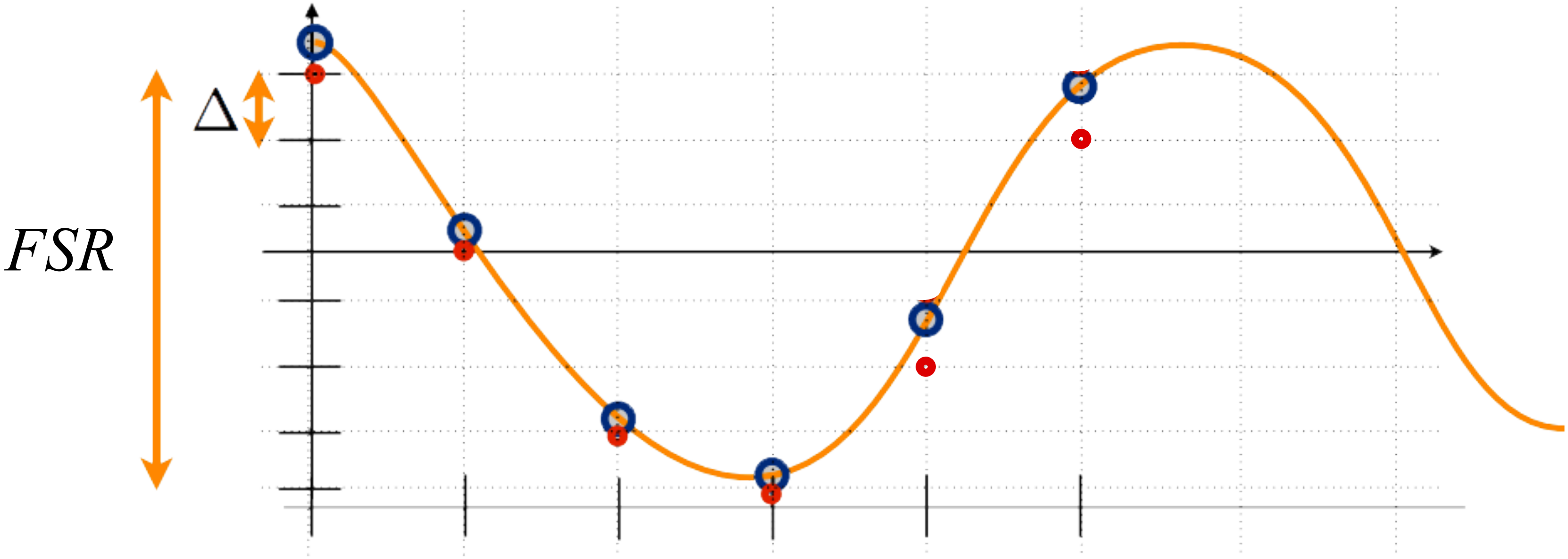
C/D

T

$x[n] = x_c(nT)$

Quantizer

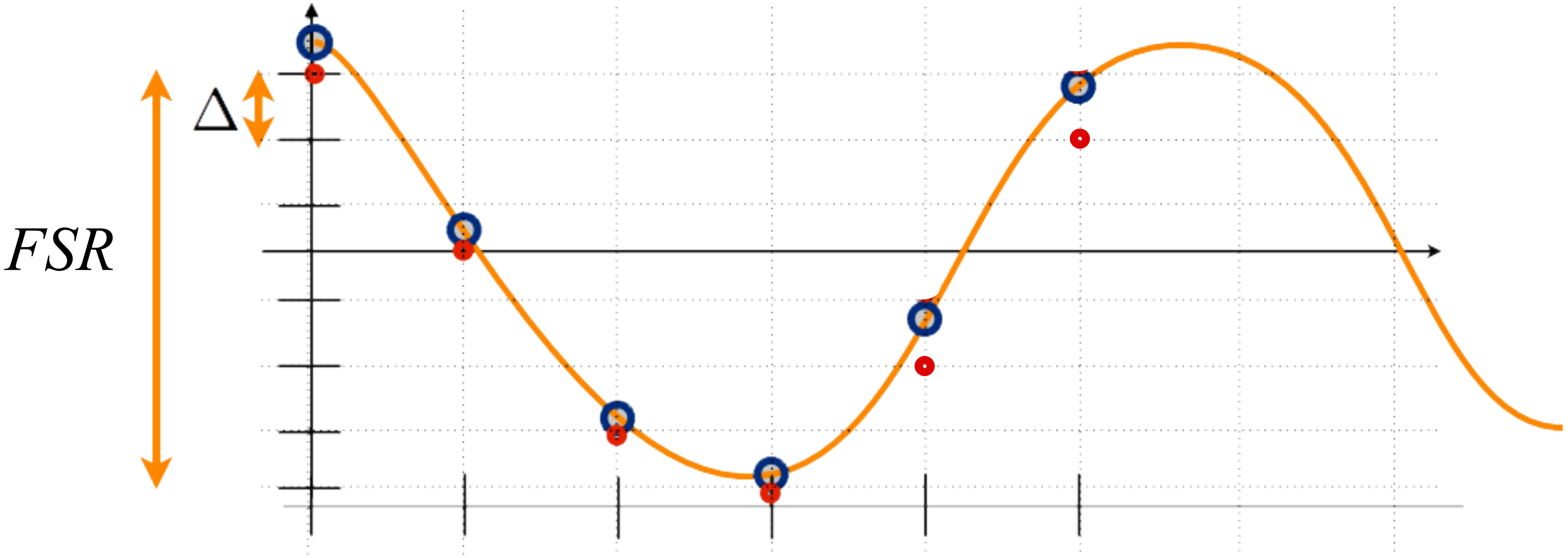
$\hat{x}[n]$



Sampling and Quantization

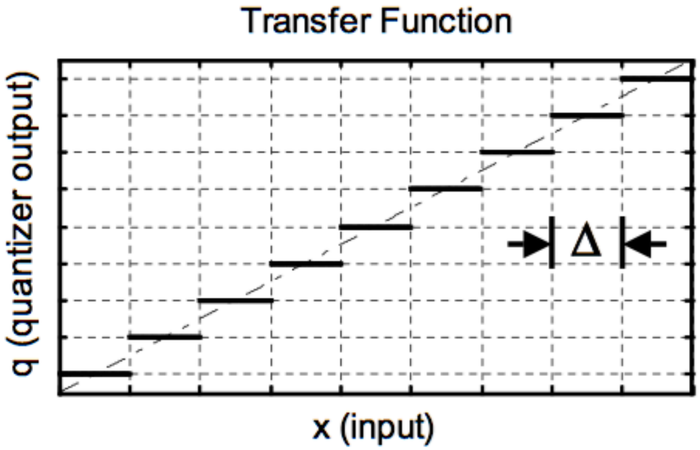
- For an input signal with V_{pp}=FSR with B bits

$$\Delta = \frac{FSR}{2^B}$$



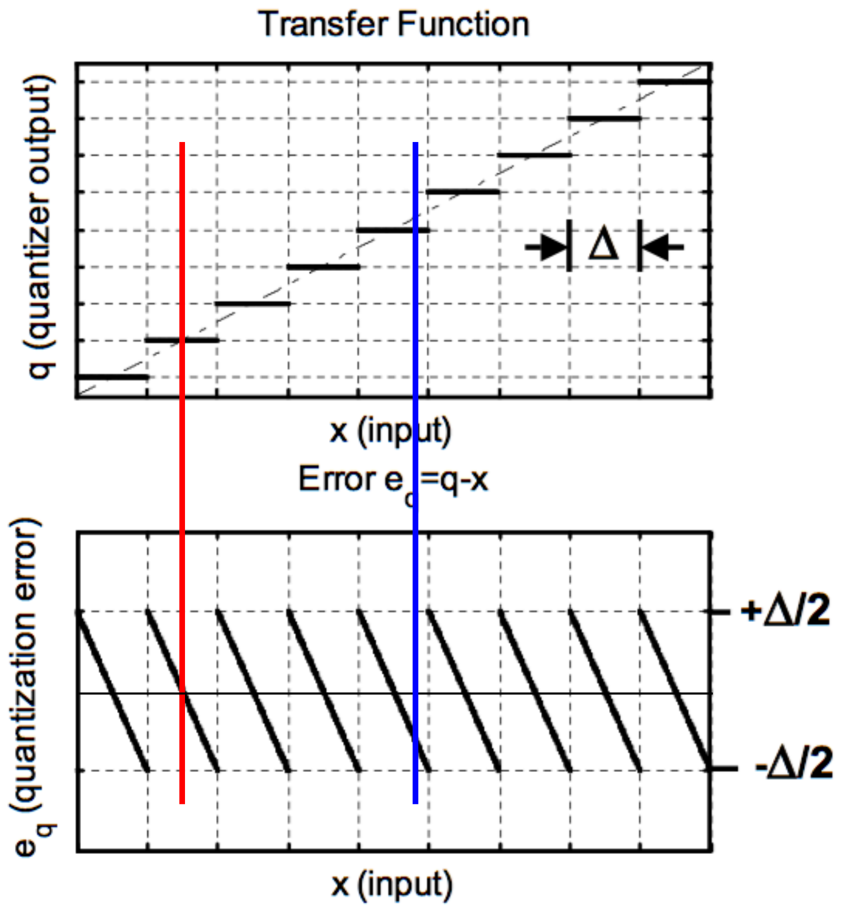
Ideal Quantizer

- Quantization step Δ



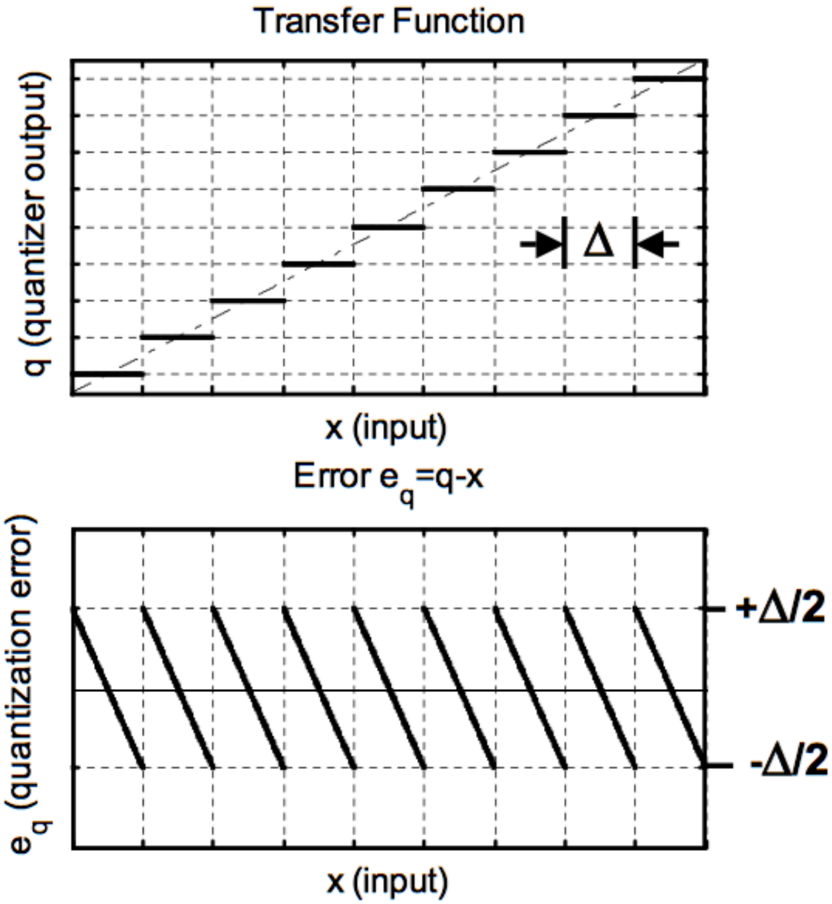
Ideal Quantizer

- Quantization step Δ
- Quantization error has sawtooth shape
  - Bounded by –Δ/2, +Δ/2



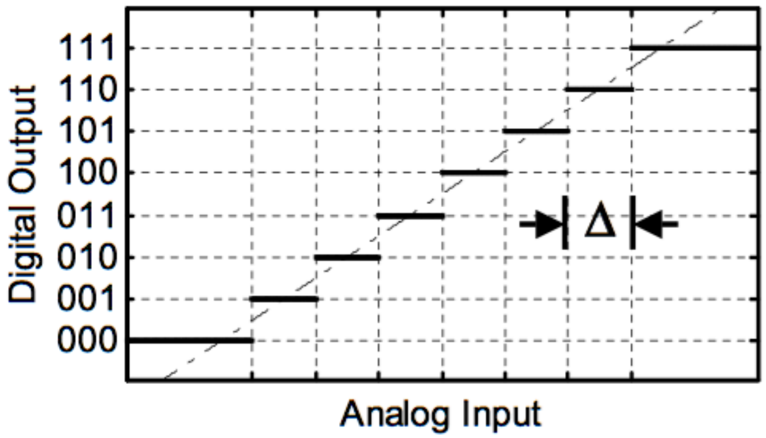
Ideal Quantizer

- Quantization step Δ
- Quantization error has sawtooth shape
  - Bounded by –Δ/2, +Δ/2
- Ideally infinite input range and infinite number of quantization levels



Ideal B-bit Quantizer

- Practical quantizers have a limited input range and a finite set of output codes
- E.g. a 3-bit quantizer can map onto $2^3=8$ distinct output codes



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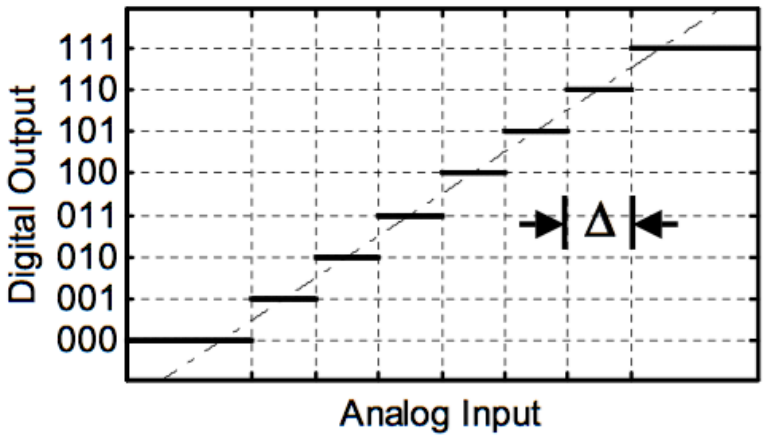
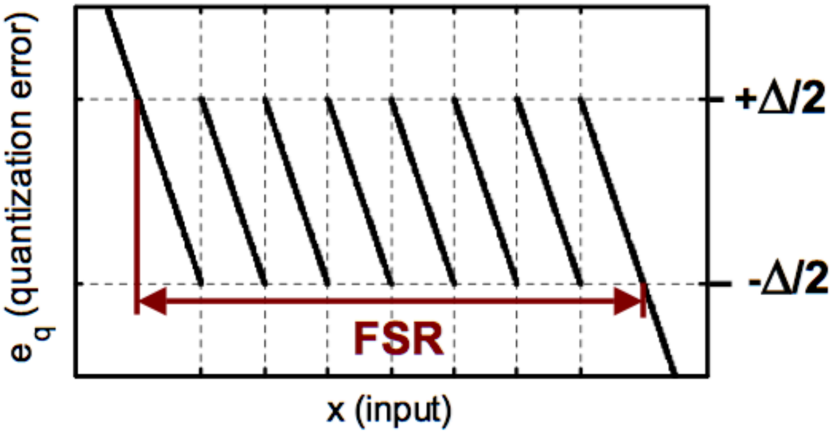


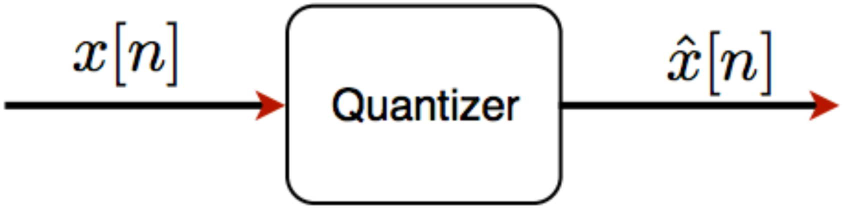
- Quantization error grows out of bounds beyond code boundaries
- We define the full scale range (FSR) as the maximum input range that satisfies $|e_q| \leq \Delta/2$
    - Implies that FSR = $2^B \cdot \Delta$



Effect of Quantization Error on Signal

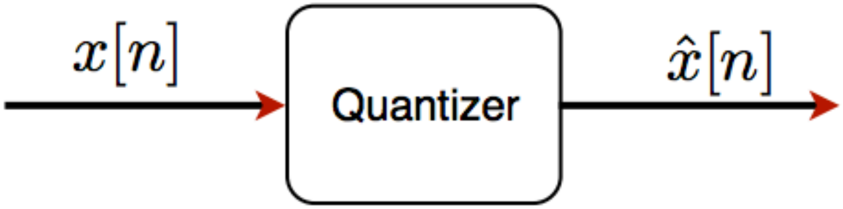
- Quantization error is a deterministic function of the signal
  - Consequently, the effect of quantization strongly depends on the signal itself
- Unless, we consider fairly trivial signals, a deterministic analysis is usually impractical
  - More common to look at errors from a statistical perspective
  - "Quantization noise”

Quantization Error

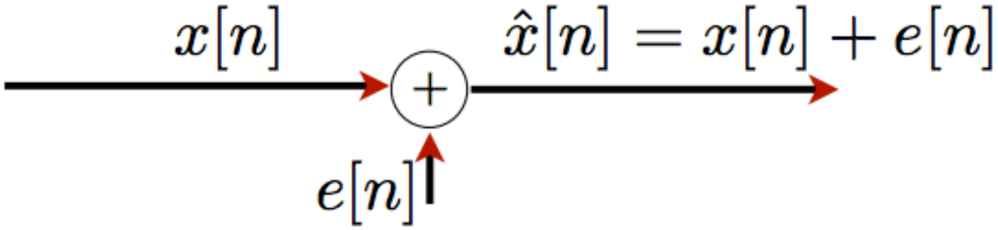


- Model quantization error as noise:

Quantization Error



- Model quantization error as noise:



- In that case:

$$-\Delta/2 \leq e[n] < \Delta/2$$

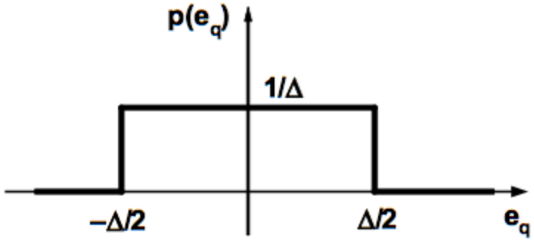
Noise Model for Quantization Error

- Assumptions:
  - Model e[n] as a sample sequence of a stationary random process
  - e[n] is not correlated with x[n]
  - e[n] not correlated with e[m] where m≠n
  - e[n] ~ U[-Δ/2, Δ/2] (uniform pdf)

- Result:
- Variance is: $\sigma_e^2 = \frac{\Delta^2}{12}$
- Assumptions work well for signals that change rapidly, are not clipped, and for small Δ

Quantization Error Statistics

- Crude assumption: $e_q(x)$ has uniform probability density
- This approximation holds reasonably well in practice when
  - Signal spans large number of quantization steps
  - Signal is "sufficiently active”
  - Quantizer does not overload

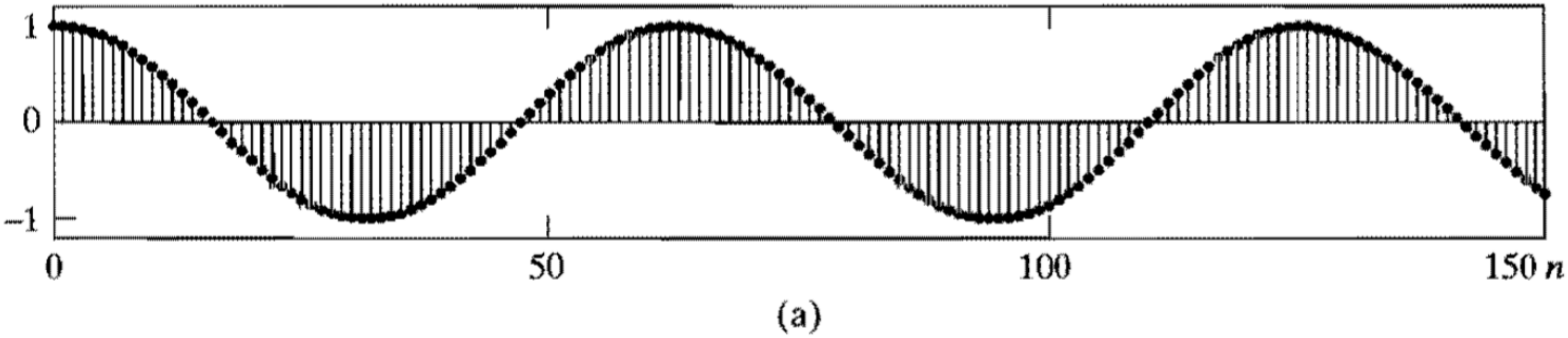


Mean $$\bar{e} = \int_{-\Delta/2}^{+\Delta/2} \frac{e}{\Delta} de = 0$$

Variance $$\overline{e^2} = \int_{-\Delta/2}^{+\Delta/2} \frac{e^2}{\Delta} de = \frac{\Delta^2}{12}$$

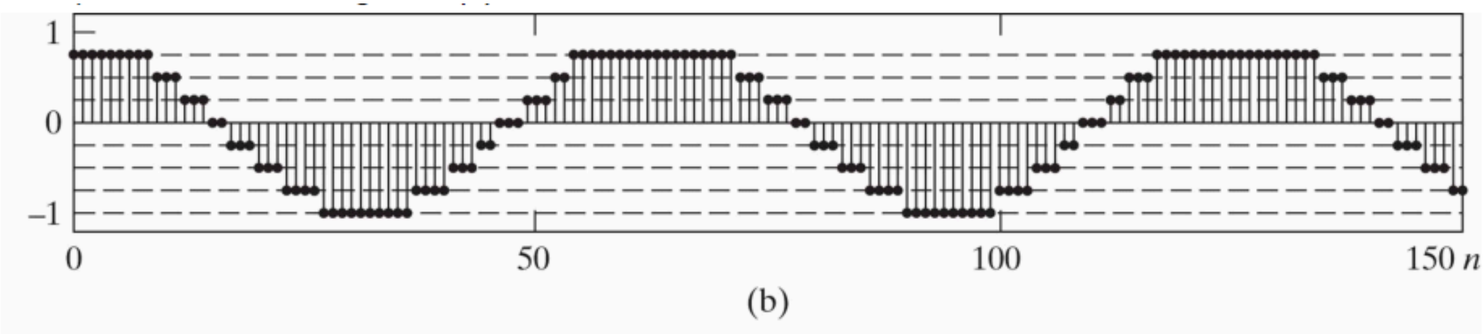
Quantization Noise

- **Figure 4.57** Example of quantization noise. (a) Unquantized samples of the signal x[n] = 0.99cos(n/10).



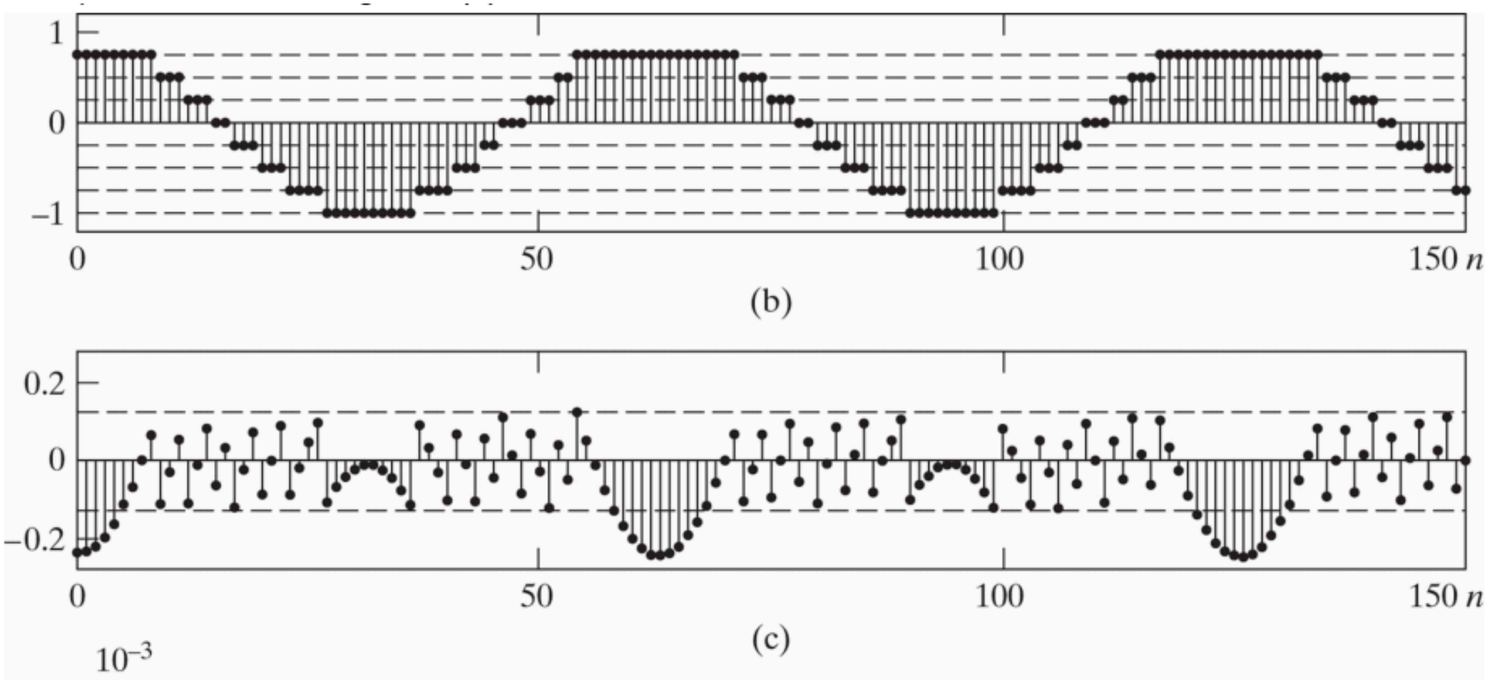
Quantization Noise

- **Figure 4.57**(continued) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer.



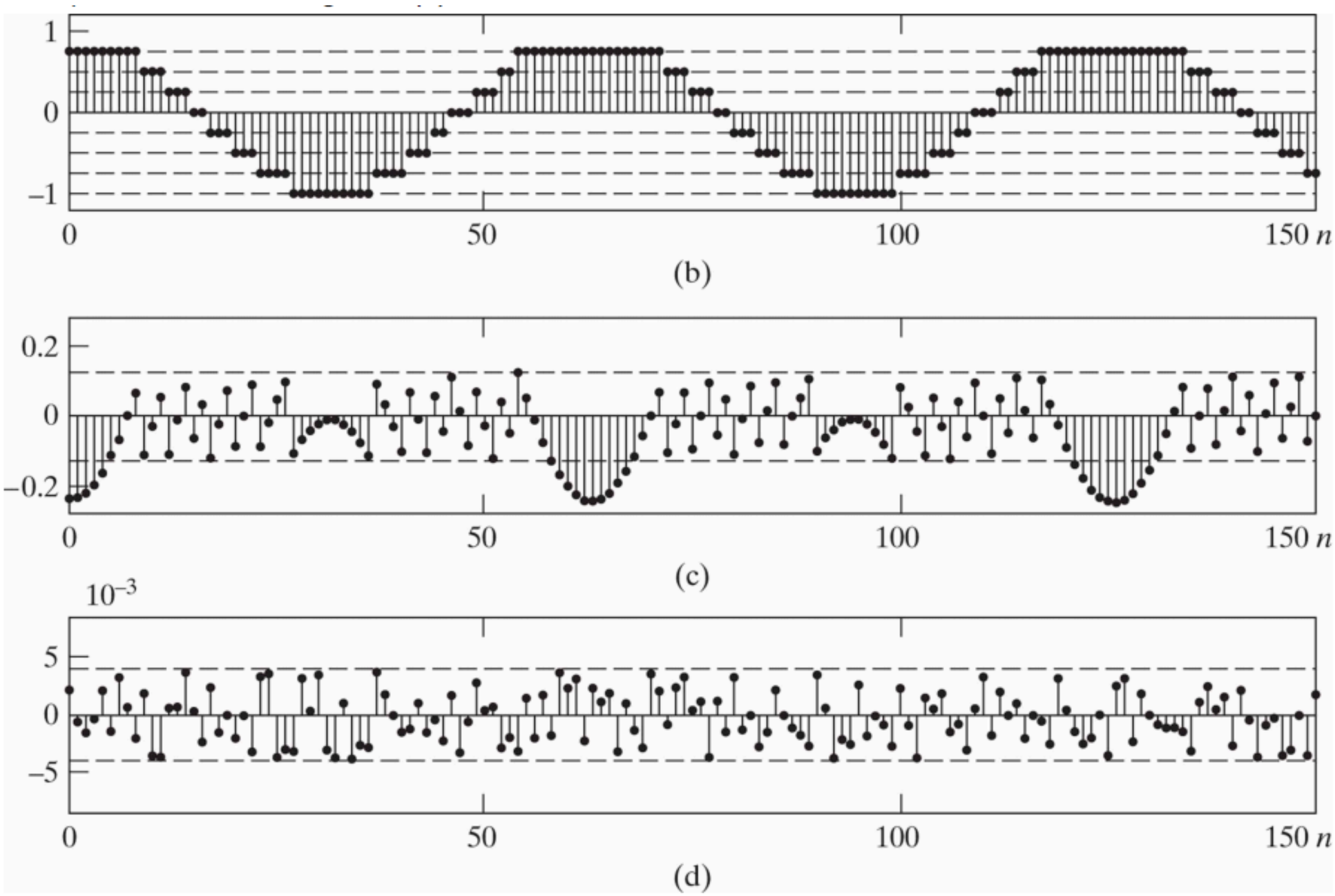
Quantization Noise

- **Figure 4.57**(continued) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a).



Quantization Noise

- **Figure 4.57**(continued) (b) Quantized samples of the cosine waveform in part (a) with a 3-bit quantizer. (c) Quantization error sequence for 3-bit quantization of the signal in (a). (d) Quantization error sequence for 8-bit quantization of the signal in (a).



Signal-to-Quantization-Noise Ratio

- For uniform B bits quantizer

$$SNR_Q = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right)$$

Signal-to-Quantization-Noise Ratio

- For uniform B bits quantizer

$$\begin{aligned} SNR_Q &= 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_e^2}\right) \\ &= 10\log_{10}\left(\frac{12\cdot 2^{2B}\sigma_x^2}{FSR^2}\right) \end{aligned}$$

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right)$$

Quantizer range

rms of amp

Signal-to-Quantization-Noise Ratio

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right)$$

Quantizer range

rms of amp

- Improvement of 6dB with every bit
- The range of the quantization must be adapted to the rms amplitude of the signal
  - Tradeoff between clipping and noise!
  - Often use pre-amp
  - Sometimes use analog auto gain controller (AGC)

Signal-to-Quantization-Noise Ratio

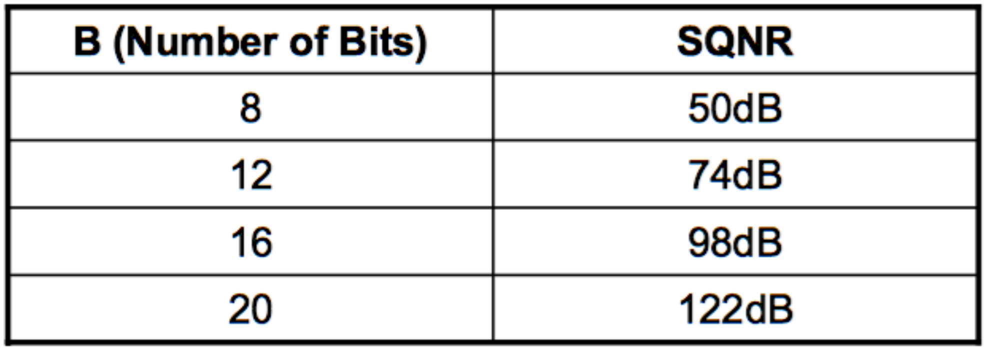
- Assuming full-scale sinusoidal input, we have

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right)^{\text{Quantizer range}}_{\text{rms of amp}}$$

Signal-to-Quantization-Noise Ratio

- Assuming full-scale sinusoidal input, we have

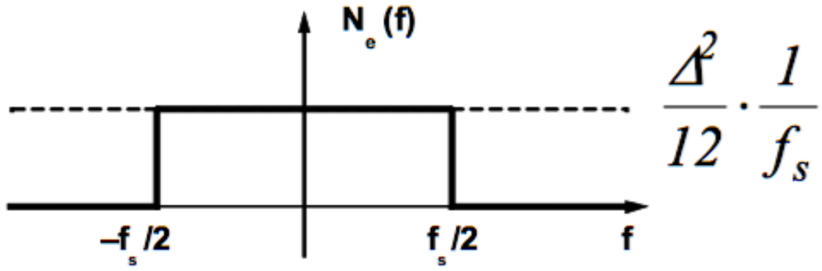
$$\mathrm{SNR_Q} = 6.02\mathrm{B} + 1.76\ \mathrm{dB}$$



B (Number of Bits)	**SQNR**
8	50dB
12	74dB
16	98dB
20	122dB

Quantization Noise Spectrum

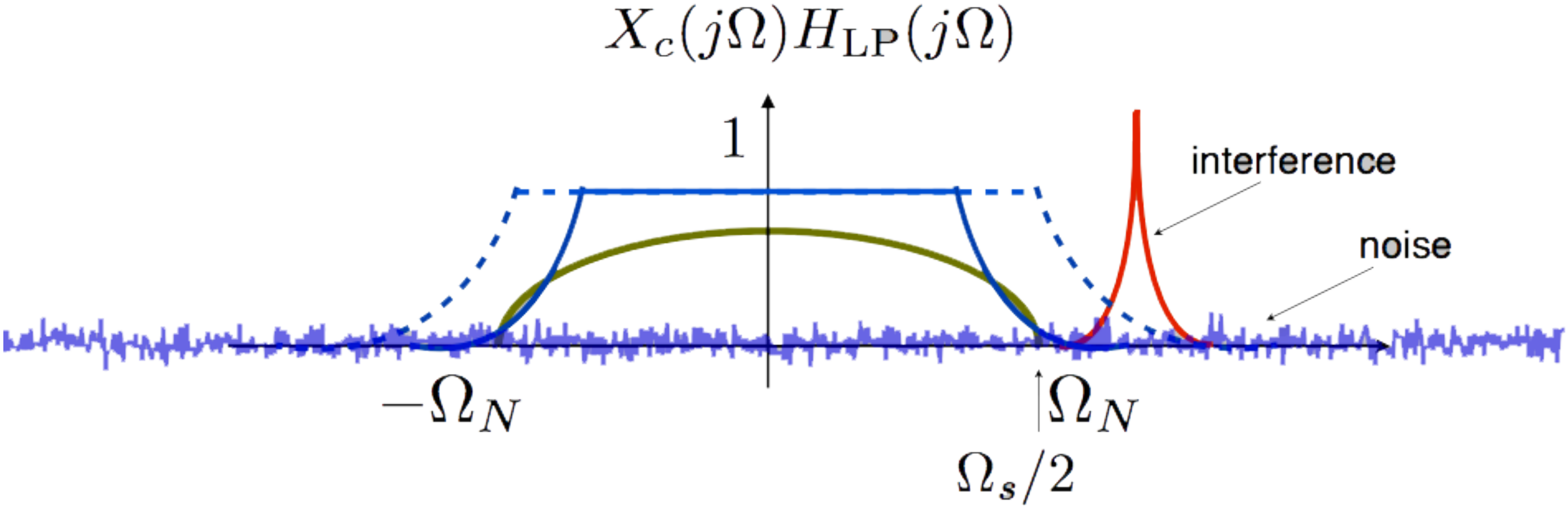
- If the quantization error is "sufficiently random", it also follows that the noise power is uniformly distributed in frequency



$$\frac{\Delta^2}{12}\cdot\frac{1}{f_s}$$

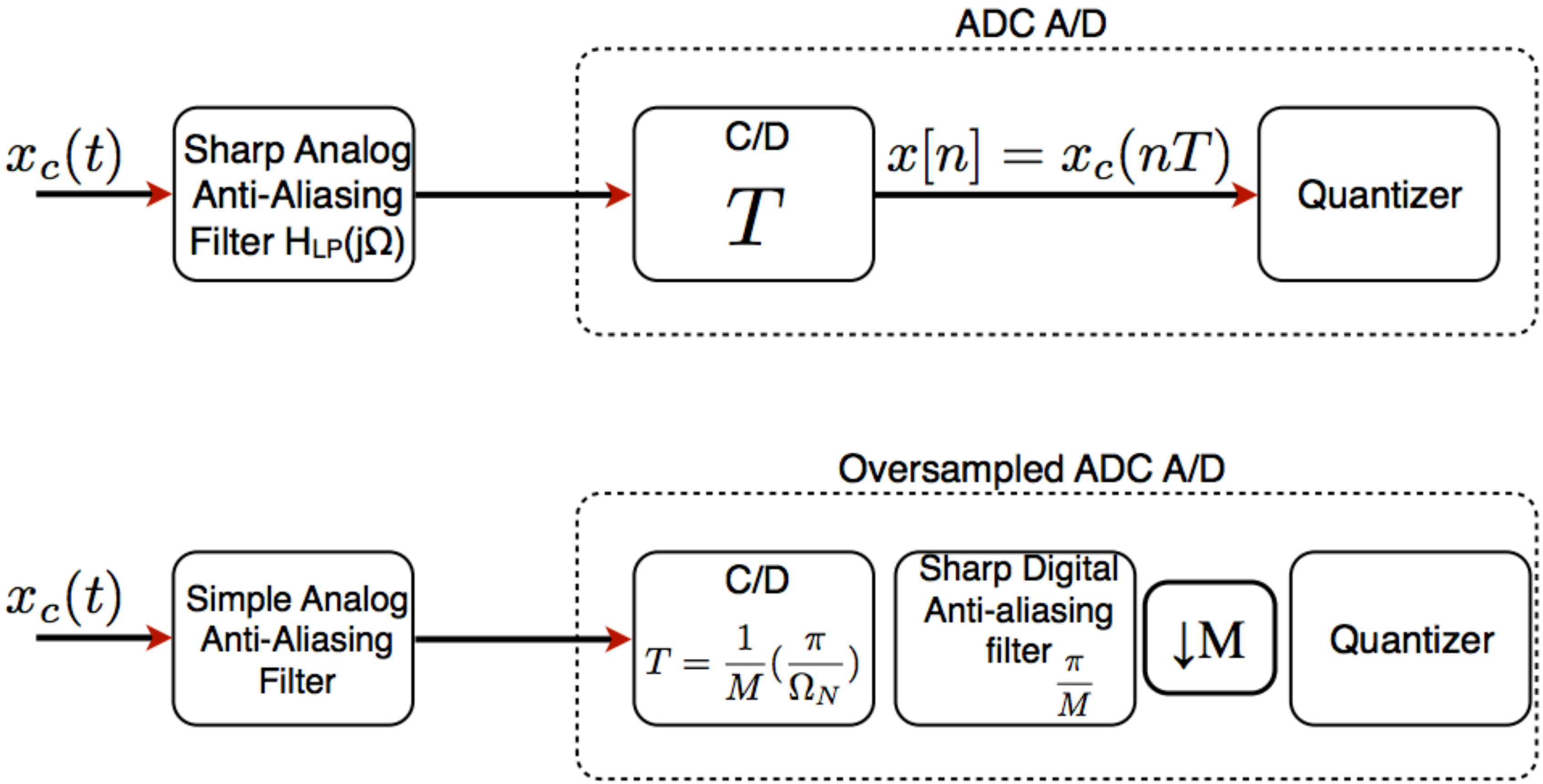
- References
  - W. R. Bennett, "Spectra of quantized signals," Bell Syst. Tech. J., pp. 446-72, July 1988.
  - B. Widrow, "A study of rough amplitude quantization by means of Nyquist sampling theory," IRE Trans. Circuit Theory, vol. CT-3, pp. 266-76, 1956.

Non-Ideal Anti-Aliasing Filter



- Problem: Hard to implement sharp analog filter
- Consequence: Crop part of the signal and suffer from noise and interference

Oversampled ADC



ADC A/D

$x_c(t)$

Sharp Analog Anti-Aliasing Filter $H_{LP}(j\Omega)$

C/D

T

$x[n] = x_c(nT)$

Quantizer

Oversampled ADC A/D

$x_c(t)$

Simple Analog Anti-Aliasing Filter

C/D

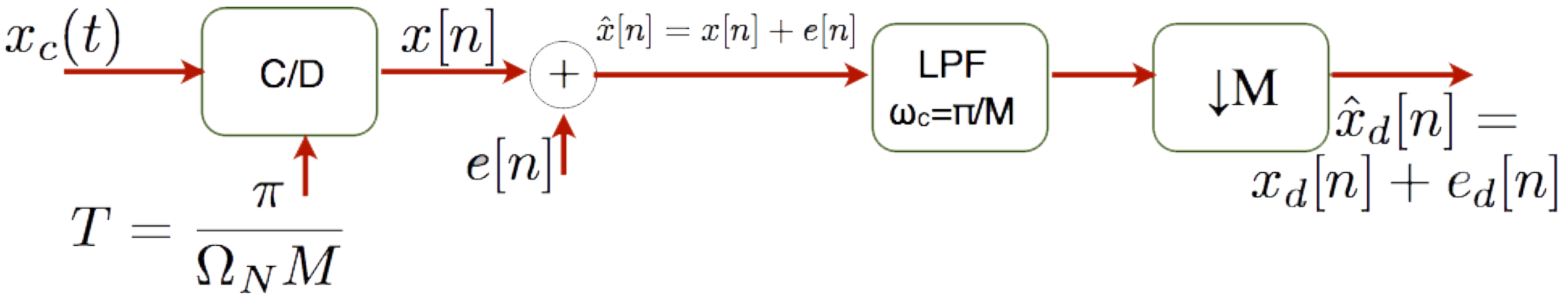
$T = \frac{1}{M}(\frac{\pi}{\Omega_N})$

Sharp Digital Anti-aliasing filter $\frac{\pi}{M}$

$\downarrow$M

Quantizer

Quantization Noise with Oversampling



$x_c(t)$

C/D

$x[n]$

$\hat{x}[n] = x[n] + e[n]$

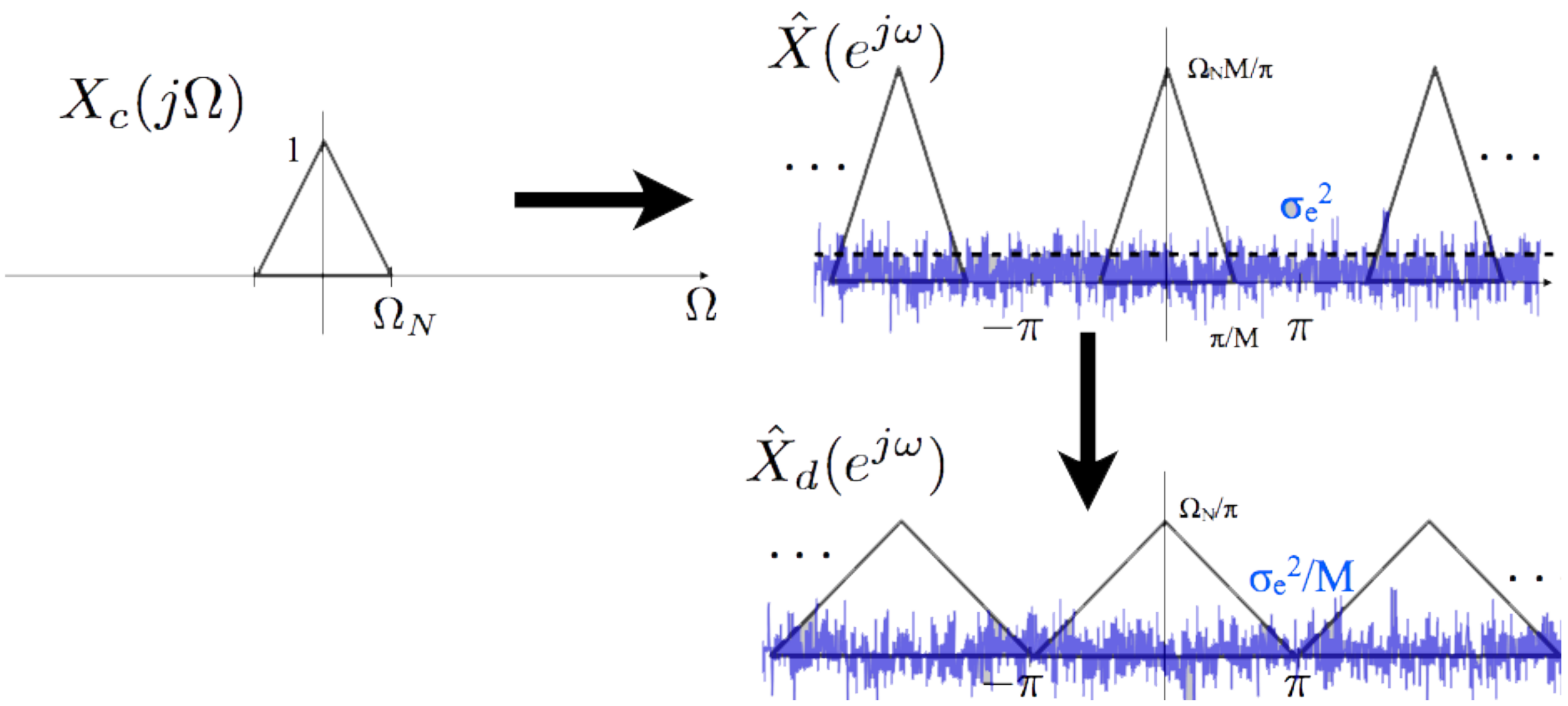
LPF
$\omega_c = \pi/M$

$\downarrow M$

$\hat{x}_d[n] = x_d[n] + e_d[n]$

$e[n]$

$T = \dfrac{\pi}{\Omega_N M}$



Quantization Noise with Oversampling

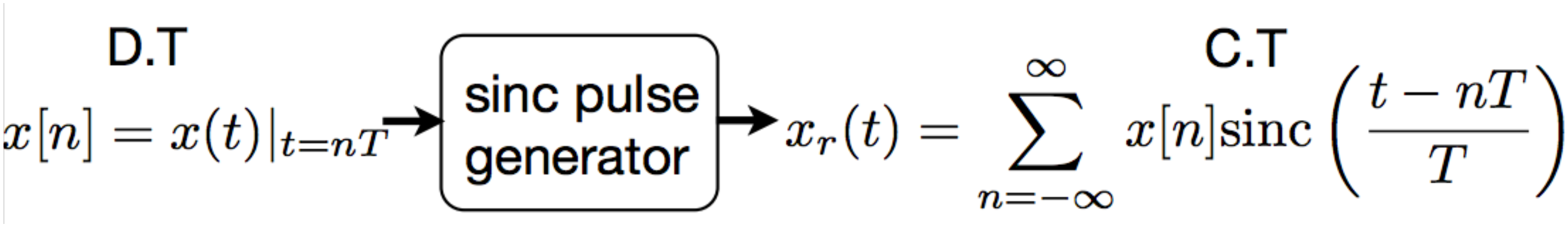
- Energy of $x_d[n]$ equals energy of x[n]
  - No filtering of signal!
- Noise variance is reduced by factor of M

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{FSR}{\sigma_x}\right) + \boxed{10\log_{10} M}$$

- For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
  - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Practical DAC

Practical DAC



D.T

$$x[n] = x(t)|_{t=nT}$$

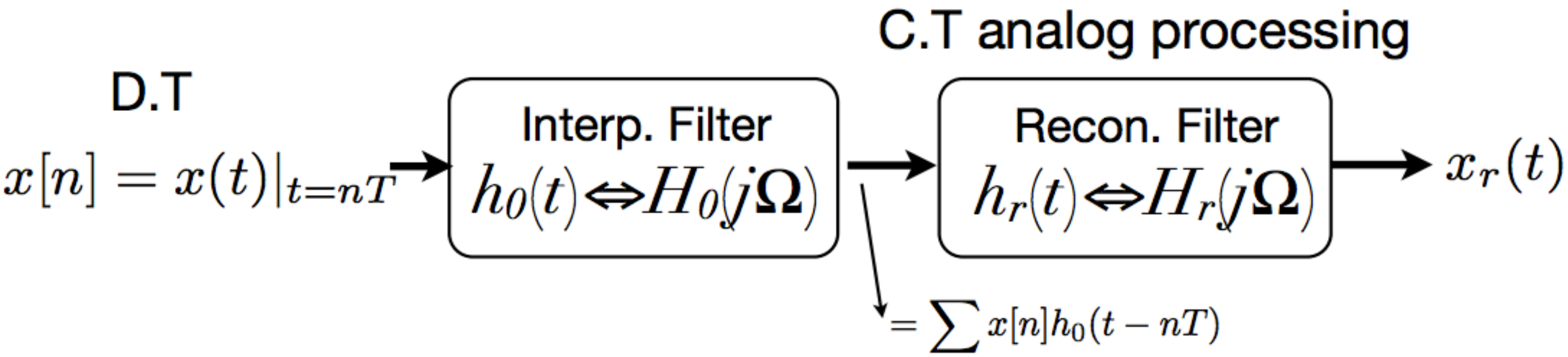
sinc pulse generator

C.T

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\text{sinc}\left(\frac{t-nT}{T}\right)$$

- Scaled train of sinc pulses
- Difficult to generate sinc → Too long!

Practical DAC



D.T

$x[n] = x(t)|_{t=nT}$

C.T analog processing

Interp. Filter

$h_0(t) \Leftrightarrow H_0(j\Omega)$

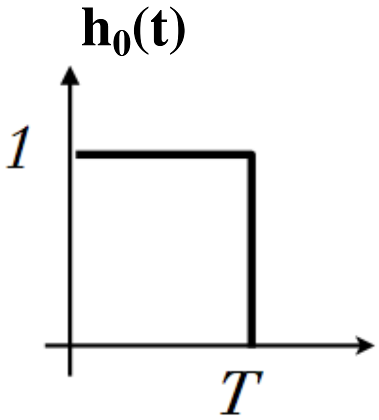
Recon. Filter

$h_r(t) \Leftrightarrow H_r(j\Omega)$

$x_r(t)$

$= \sum x[n]h_0(t-nT)$

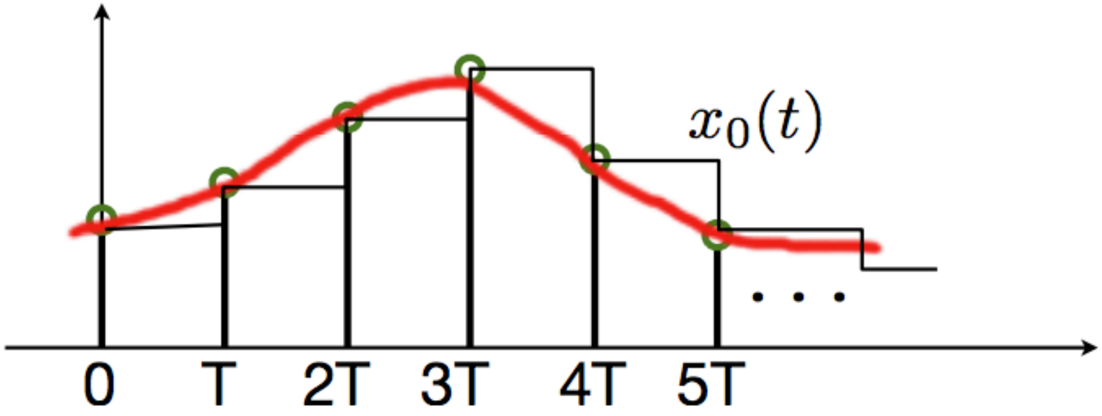
- $h_0(t)$ is finite length pulse → easy to implement
- For example: zero-order hold



$$H_0(j\Omega) = Te^{-j\Omega\frac{T}{2}}\text{sinc}(\frac{\Omega}{\Omega_s})$$

Practical DAC

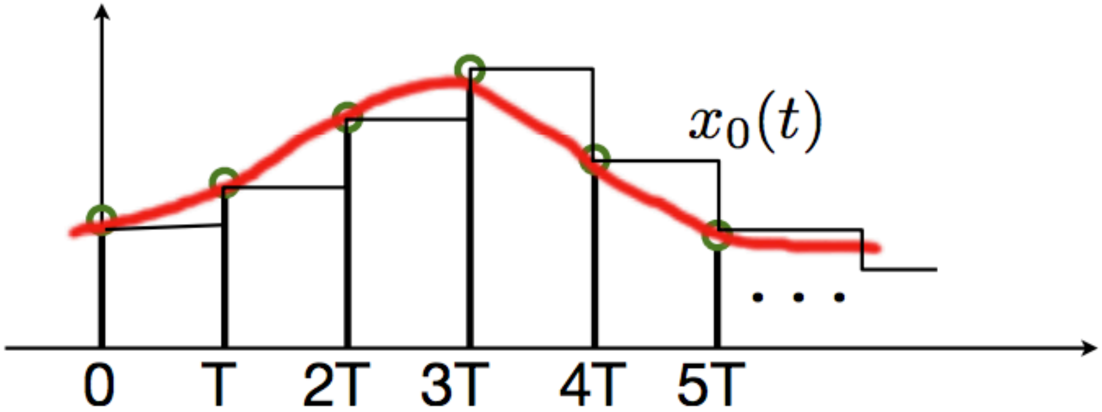
Zero-Order-Hold interpolation



$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT) = h_0(t) * x_s(t)$$

Practical DAC

Zero-Order-Hold interpolation



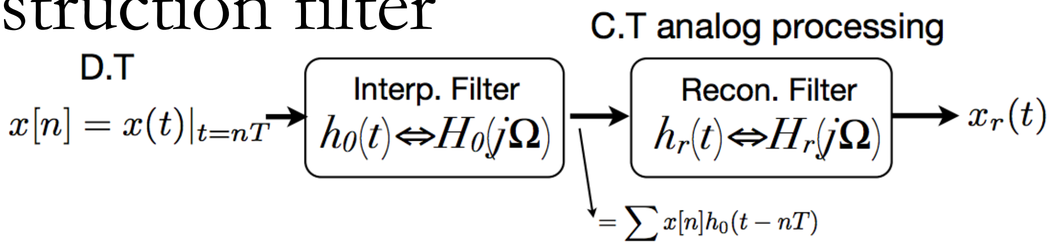
$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n] h_0(t - nT) = h_0(t) * x_s(t)$$

Taking a FT:

$$\begin{aligned} X_0(j\Omega) &= H_0(j\Omega) X_s(j\Omega) \\ &= H_0(j\Omega) \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s)) \end{aligned}$$

Practical DAC

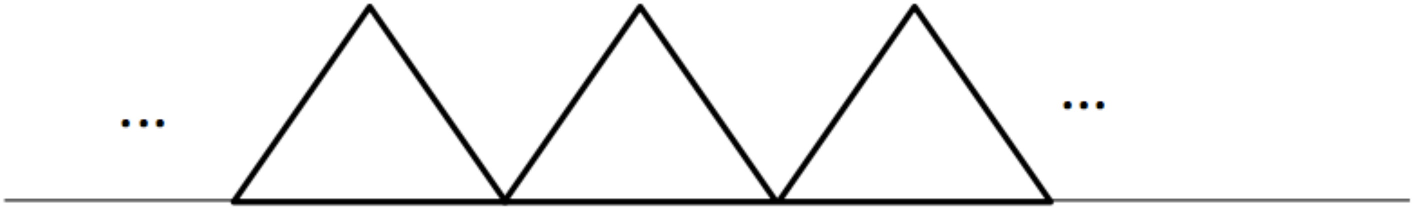
- Output of the reconstruction filter



$$X_r(j\Omega) \quad = \quad H_r(j\Omega) \cdot H_0(j\Omega) \cdot X_s(j\Omega)$$

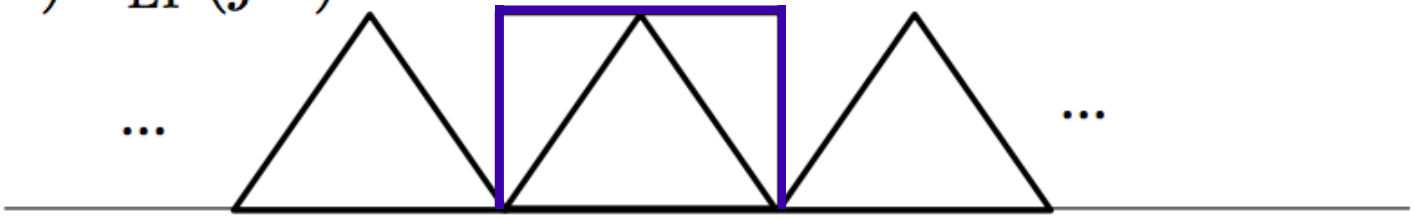
Practical DAC

$X_s(j\Omega)$

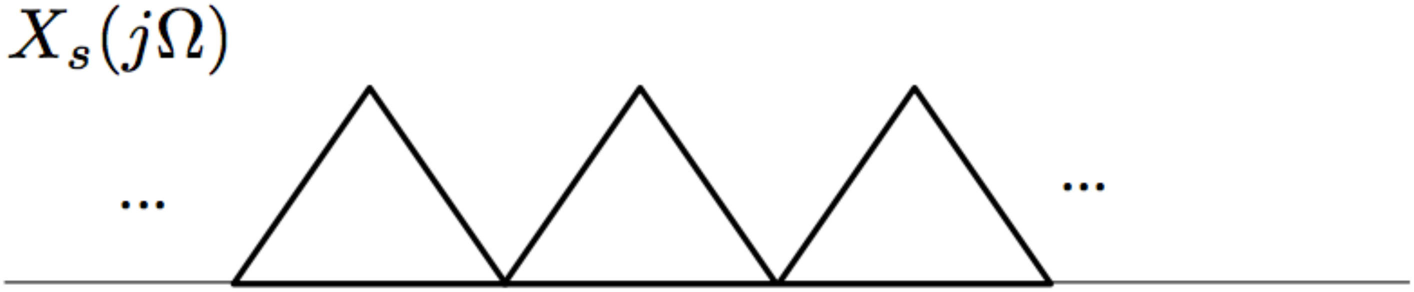


Ideally:

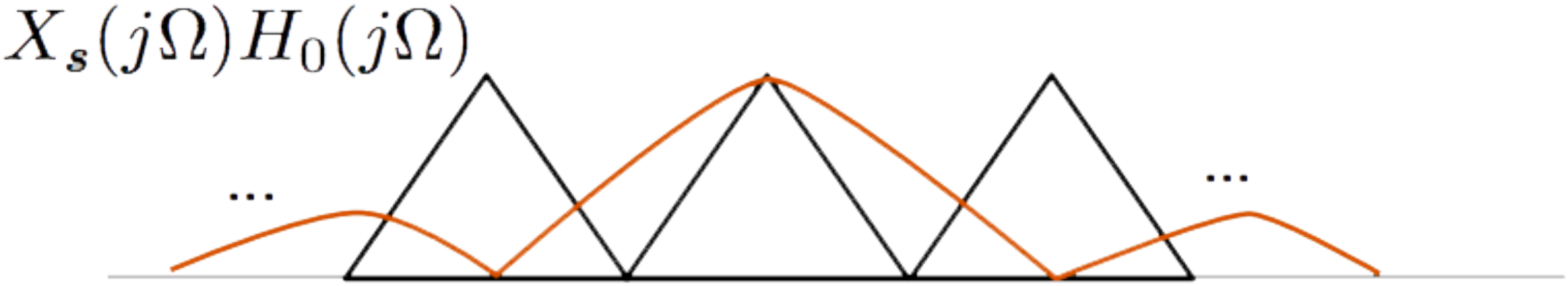
$X_s(j\Omega)H_{\mathrm{LP}}(j\Omega)$



Practical DAC

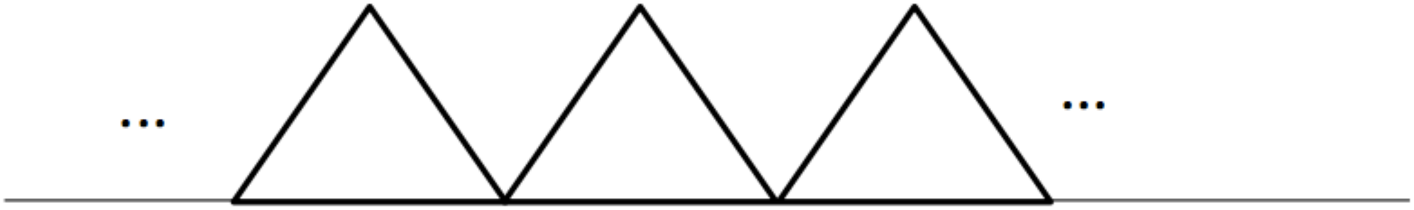


Practically:



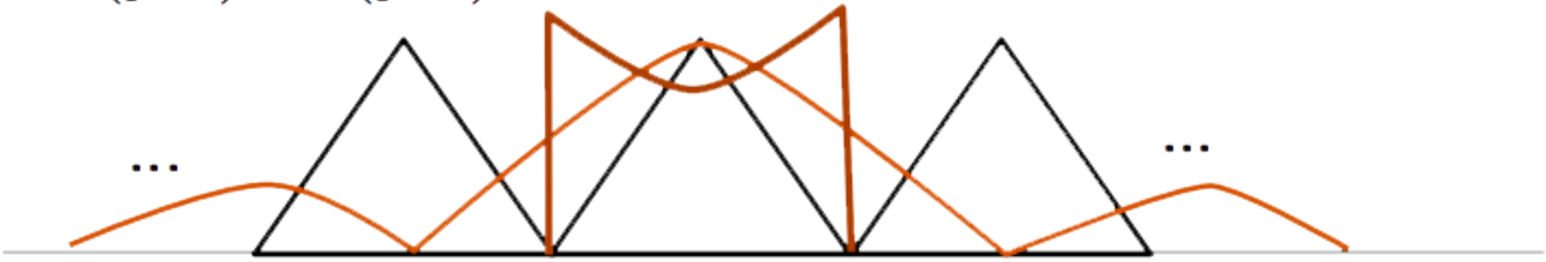
Practical DAC

$X_s(j\Omega)$



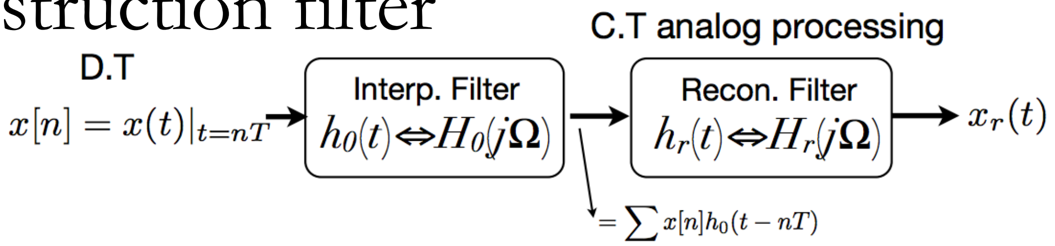
Practically:

$X_s(j\Omega)H_0(j\Omega)H_r(j\Omega)$



Practical DAC

- Output of the reconstruction filter



C.T analog processing

D.T

$x[n] = x(t)|_{t=nT}$

Interp. Filter

$h_0(t) \Leftrightarrow H_0(j\Omega)$

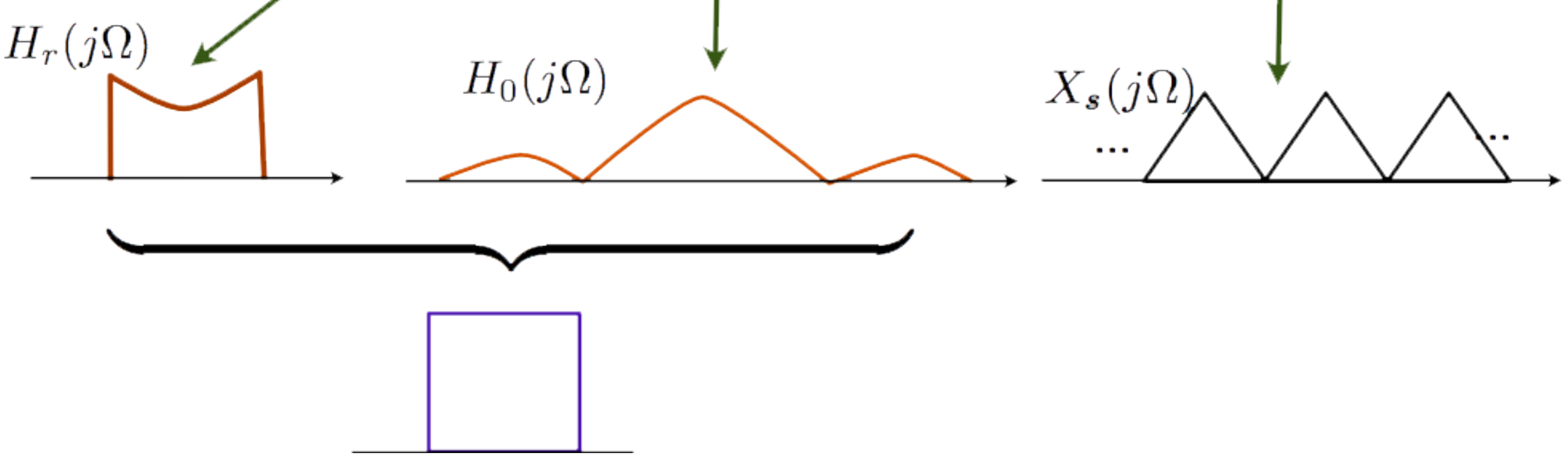
Recon. Filter

$h_r(t) \Leftrightarrow H_r(j\Omega)$

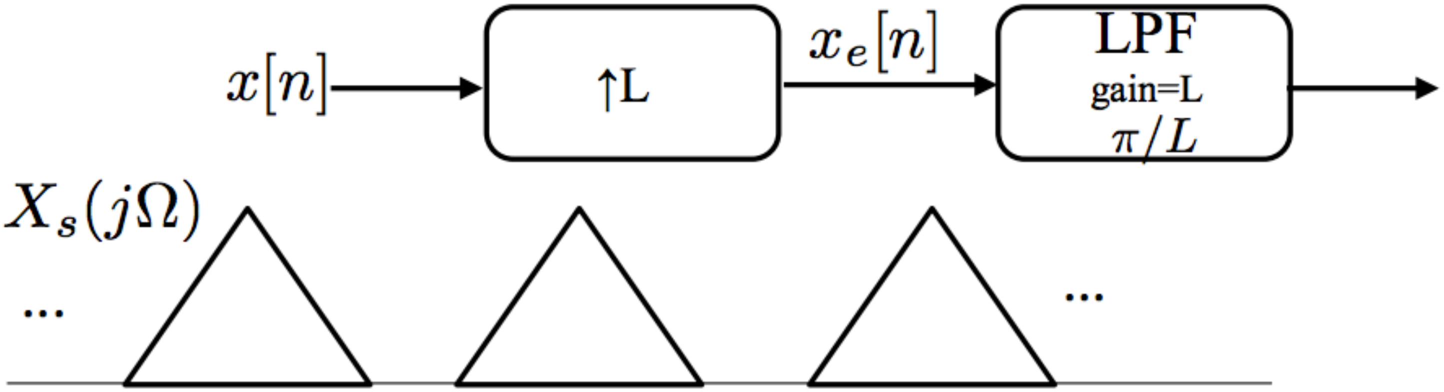
$x_r(t)$

$= \sum x[n]h_0(t-nT)$

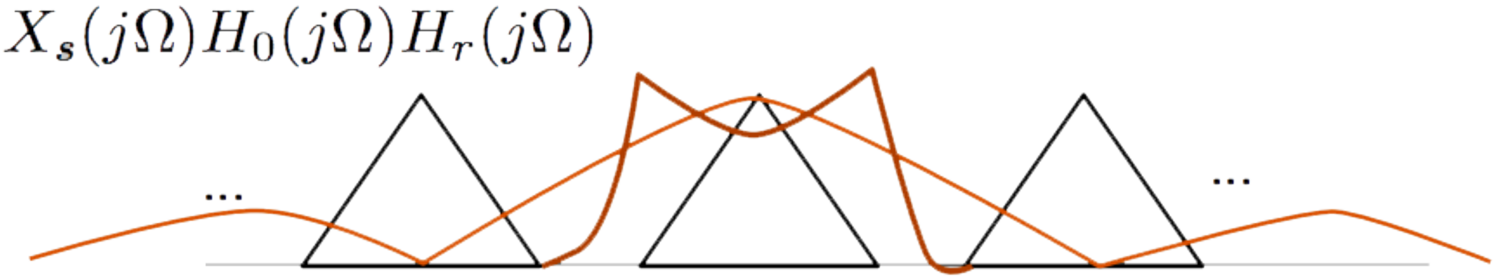
$$\begin{aligned} X_r(j\Omega) &= H_r(j\Omega)\cdot H_0(j\Omega)\cdot X_s(j\Omega)\\ &= \underbrace{H_r(j\Omega)}_{\text{recon filter}} \cdot \underbrace{Te^{-j\Omega\frac{T}{2}}\text{sinc}(\frac{\Omega}{\Omega_s})}_{\text{from zero-order hold}} \cdot \underbrace{\frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega-k\Omega_s))}_{\text{Shifted copies from sampling}} \end{aligned}$$



Practical DAC with Upsampling

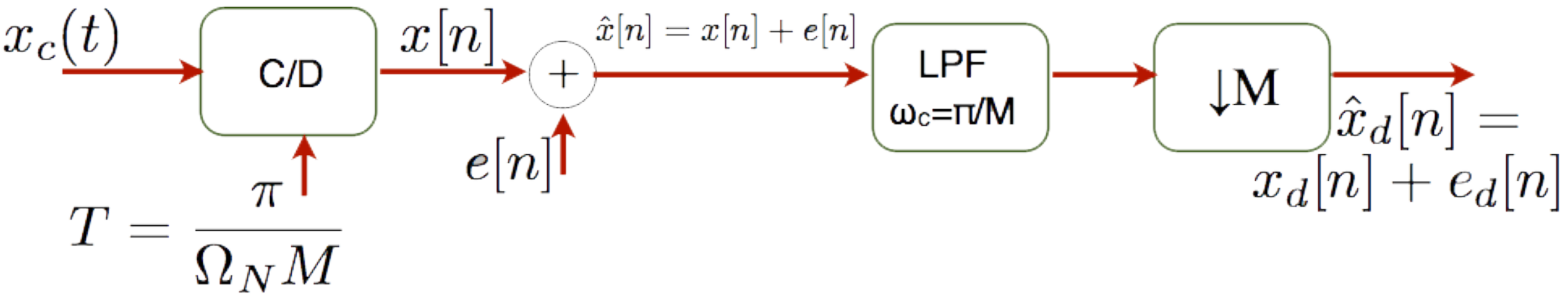


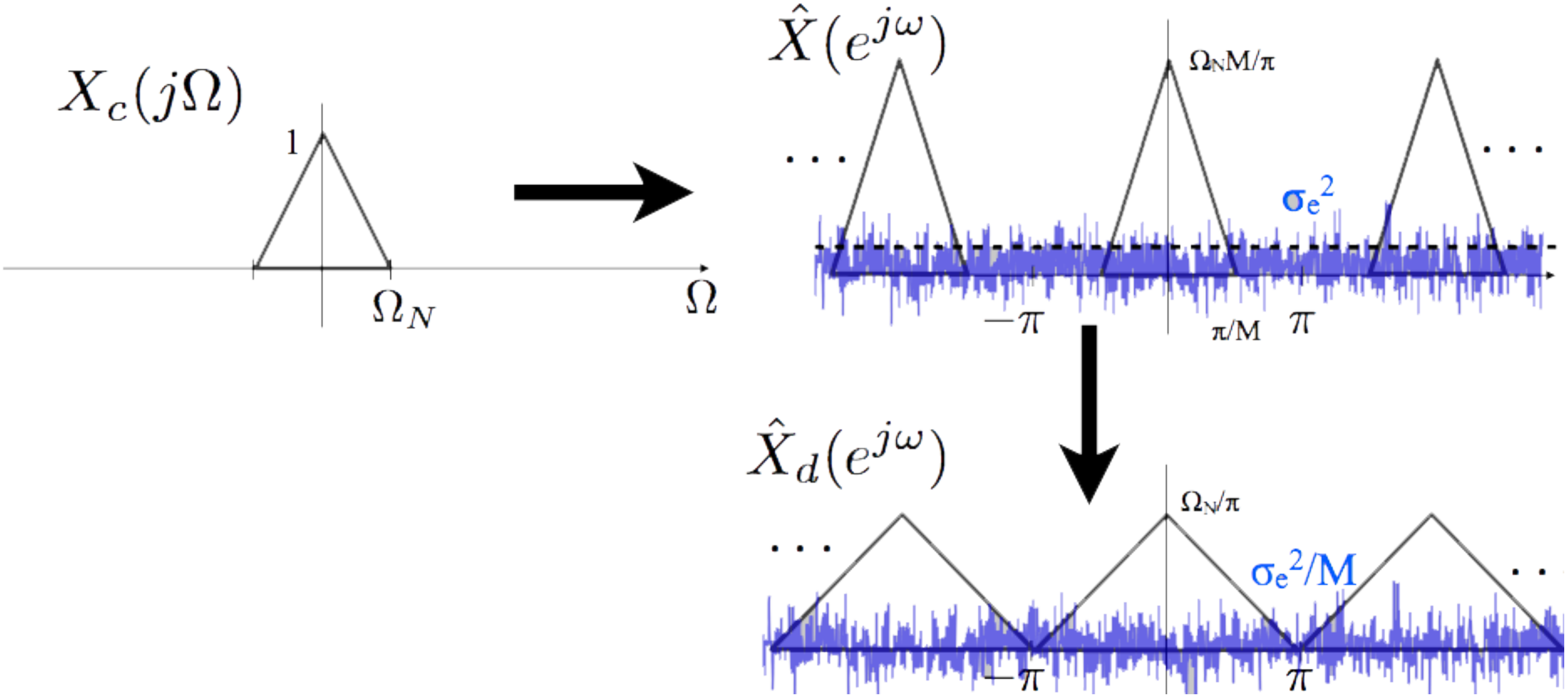
Practically:



Noise Shaping

Quantization Noise with Oversampling





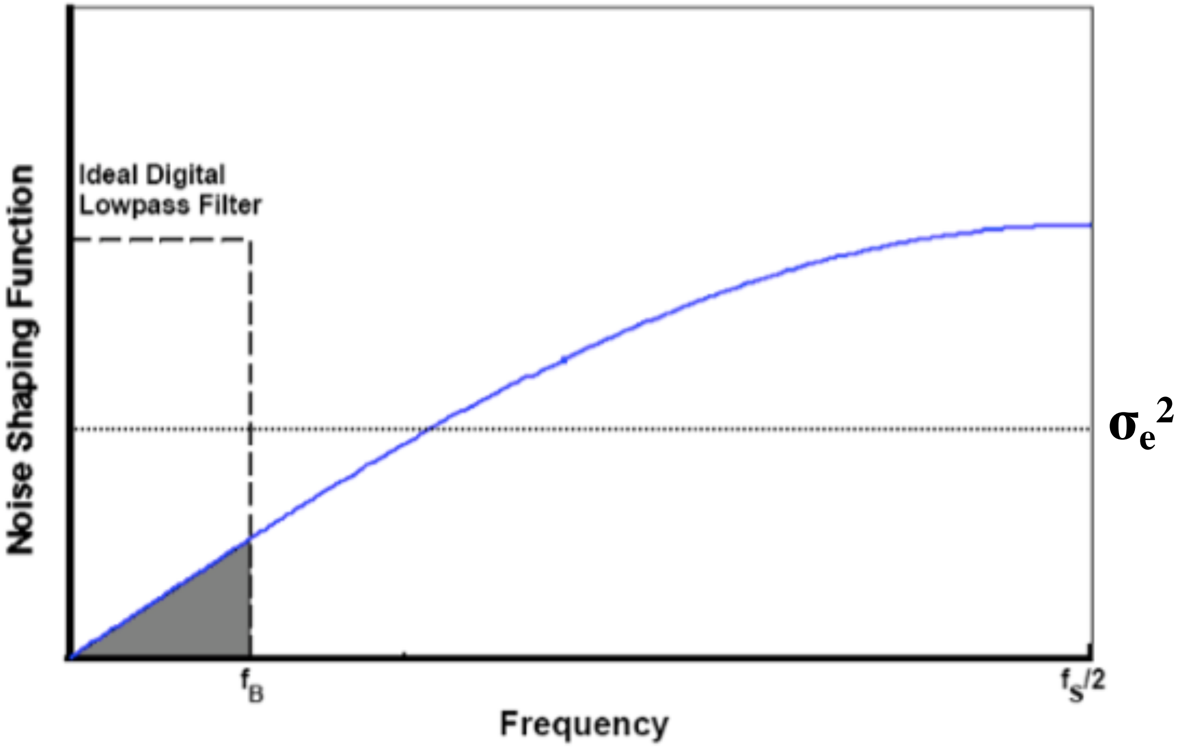
Quantization Noise with Oversampling

- Energy of $x_d[n]$ equals energy of x[n]
  - No filtering of signal!
- Noise variance is reduced by factor of M

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) + \boxed{10\log_{10} M}$$

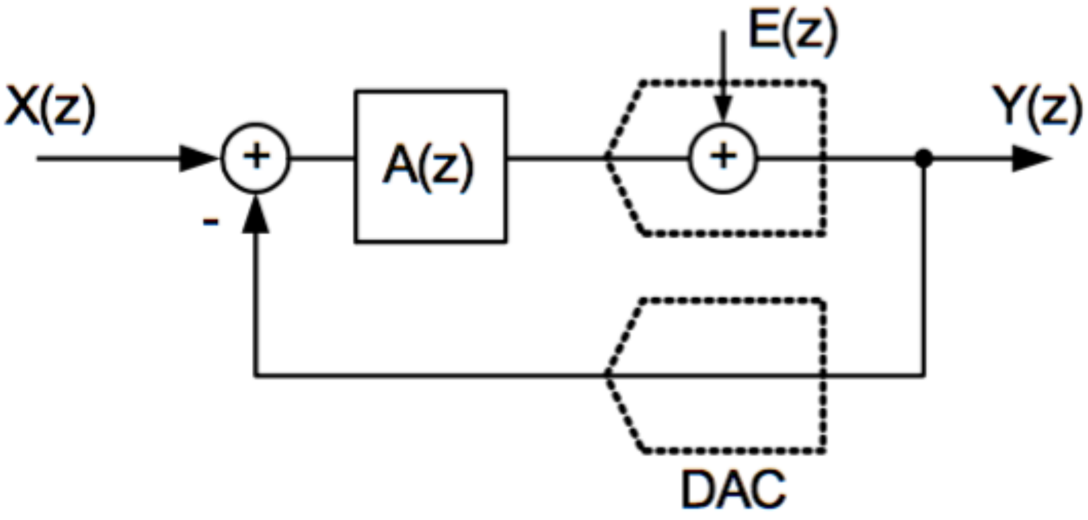
- For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
  - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Noise Shaping

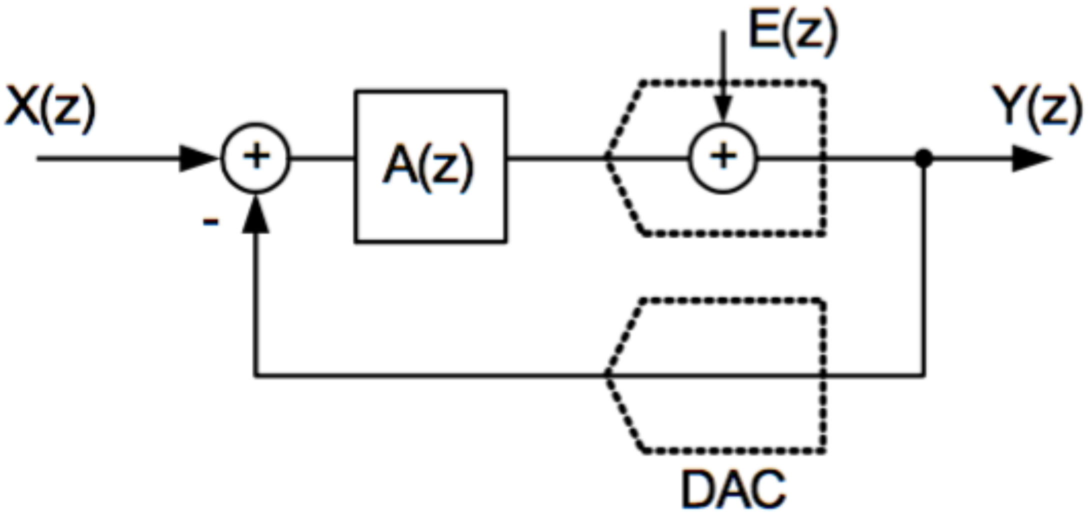


- Idea: "Somehow" build an ADC that has most of its quantization noise at high frequencies
- Key: Feedback

Noise Shaping Using Feedback



Noise Shaping Using Feedback



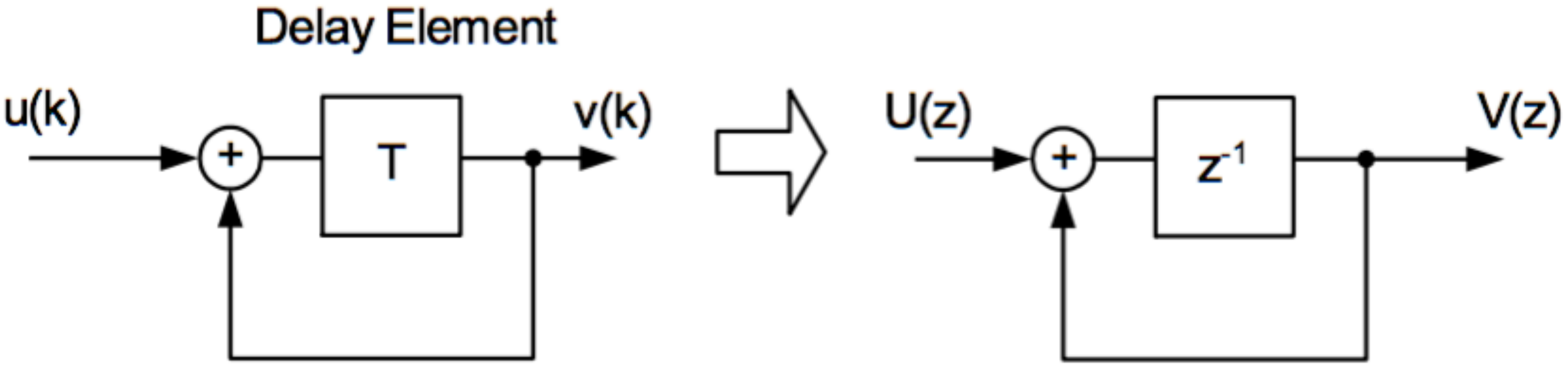
$$\begin{aligned}Y(z) &= E(z) + A(z)X(z) - A(z)Y(z) \\ &= E(z)\frac{1}{1+A(z)} + X(z)\frac{A(z)}{1+A(z)} \\ &= E(z)\underbrace{H_E(z)}_{\substack{\text{Noise}\\\text{Transfer}\\\text{Function}}} + X(z)\underbrace{H_X(z)}_{\substack{\text{Signal}\\\text{Transfer}\\\text{Function}}}\end{aligned}$$

Noise Shaping Using Feedback

$$Y(z) = E(z)\underbrace{\frac{1}{1+A(z)}}_{\substack{\text{Noise}\\ \text{Transfer}\\ \text{Function}}} + X(z)\underbrace{\frac{A(z)}{1+A(z)}}_{\substack{\text{Signal}\\ \text{Transfer}\\ \text{Function}}}$$

- Objective
  - Want to make STF unity in the signal frequency band
  - Want to make NTF "small" in the signal frequency band
- If the frequency band of interest is around DC ($0 \ldots f_B$) we achieve this by making $|A(z)| >> 1$ at low frequencies
  - Means that NTF << 1
  - Means that STF ≅ 1

Discrete Time Integrator



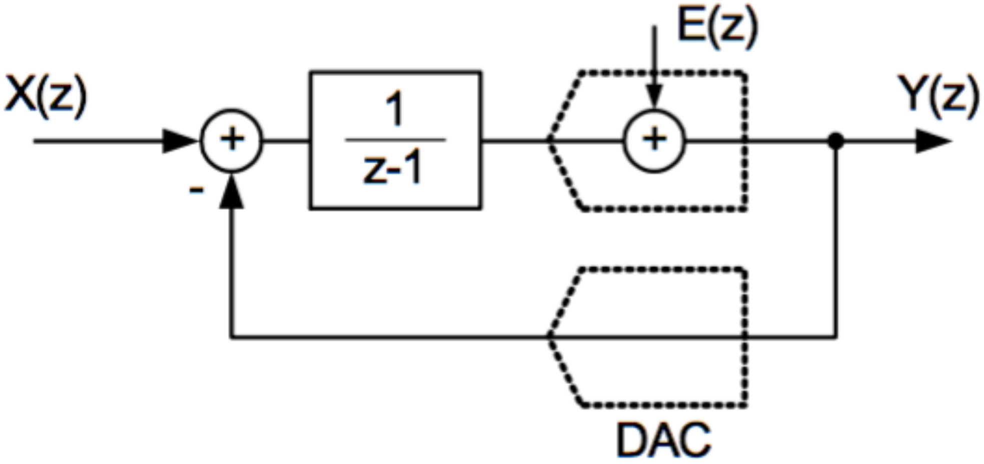
$$v(k)=u(k-1)+v(k-1)$$

$$V(z)=z^{-1}U(z)+z^{-1}V(z)$$

$$\frac{V(z)}{U(z)}=\frac{z^{-1}}{1-z^{-1}}=\frac{1}{z-1} \qquad z=e^{j\omega T}$$

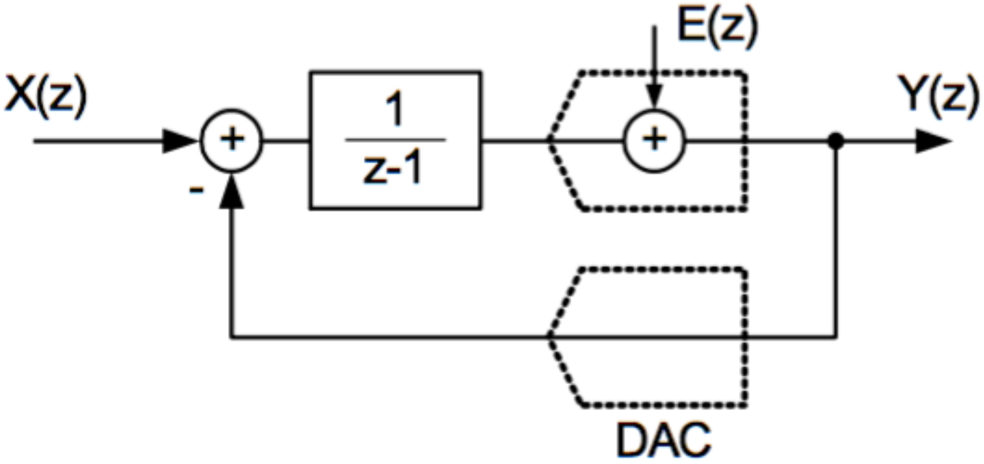
- "Infinite gain" at DC ($\omega=0$, $z=1$)

First Order Sigma-Delta Modulator



$$Y(z) = E(z)\frac{1}{1+\frac{1}{z-1}} + X(z)\frac{\frac{1}{z-1}}{1+\frac{1}{z-1}}$$

First Order Sigma-Delta Modulator



$$Y(z) = E(z)\frac{1}{1+\frac{1}{z-1}} + X(z)\frac{\frac{1}{z-1}}{1+\frac{1}{z-1}}$$

$$= E(z)\left(1-z^{-1}\right) + X(z)z^{-1}$$

- Output is equal to delayed input plus filtered quantization noise

NTF Frequency Domain Analysis

$$H_e(z) = 1 - z^{-1}$$

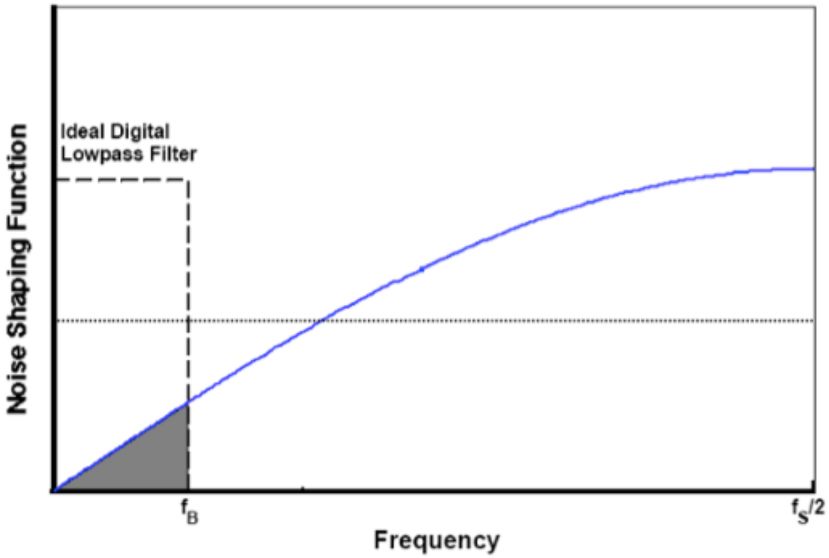
NTF Frequency Domain Analysis

$$H_e(z) = 1 - z^{-1}$$

$$H_e(j\omega) = \left(1 - e^{-j\omega T}\right) = 2e^{-j\omega T/2}\left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2}\right)$$

$$= 2e^{-j\frac{\omega T}{2}}\left(j\,sin\left(\frac{\omega T}{2}\right)\right) = 2\,sin\left(\frac{\omega T}{2}\right)e^{-j\frac{\omega T - \pi}{2}}$$

$$\left|H_e(f)\right| = 2\left|sin(\pi f T)\right| = 2\left|sin\left(\pi\frac{f}{f_s}\right)\right|$$

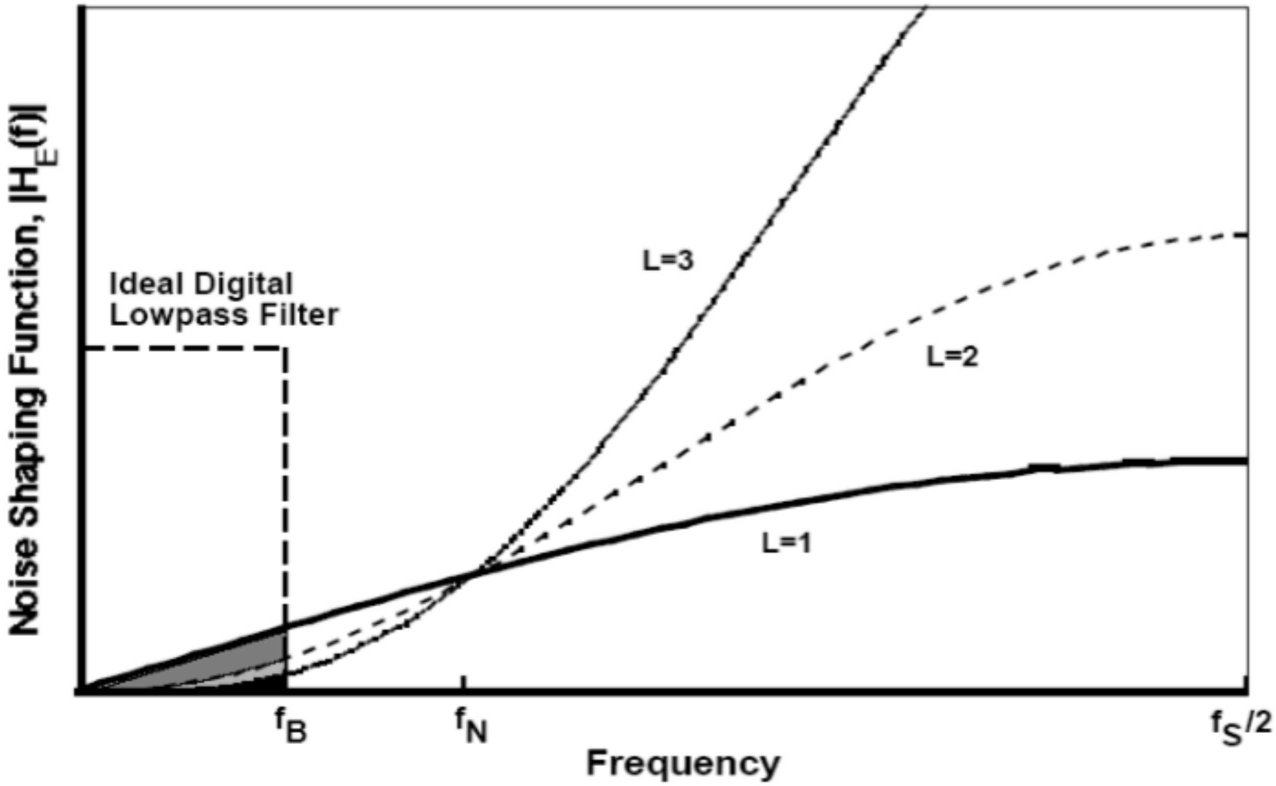


- "First order noise Shaping"
  - Quantization noise is attenuated at low frequencies, amplified at high frequencies

Higher Order Noise Shaping

- L^{th} order noise transfer function

$$H_E(z) = \left(1 - z^{-1}\right)^L$$



Big Ideas

- Quantizers
    - Introduces quantization noise
- Data Converters
    - Oversampling to reduce interference and quantization noise → increase ENOB (effective number of bits)
    - Practical DACs use practical interpolation and reconstruction filters with oversampling
- Noise Shaping
    - Use feedback to reduce oversampling factor