

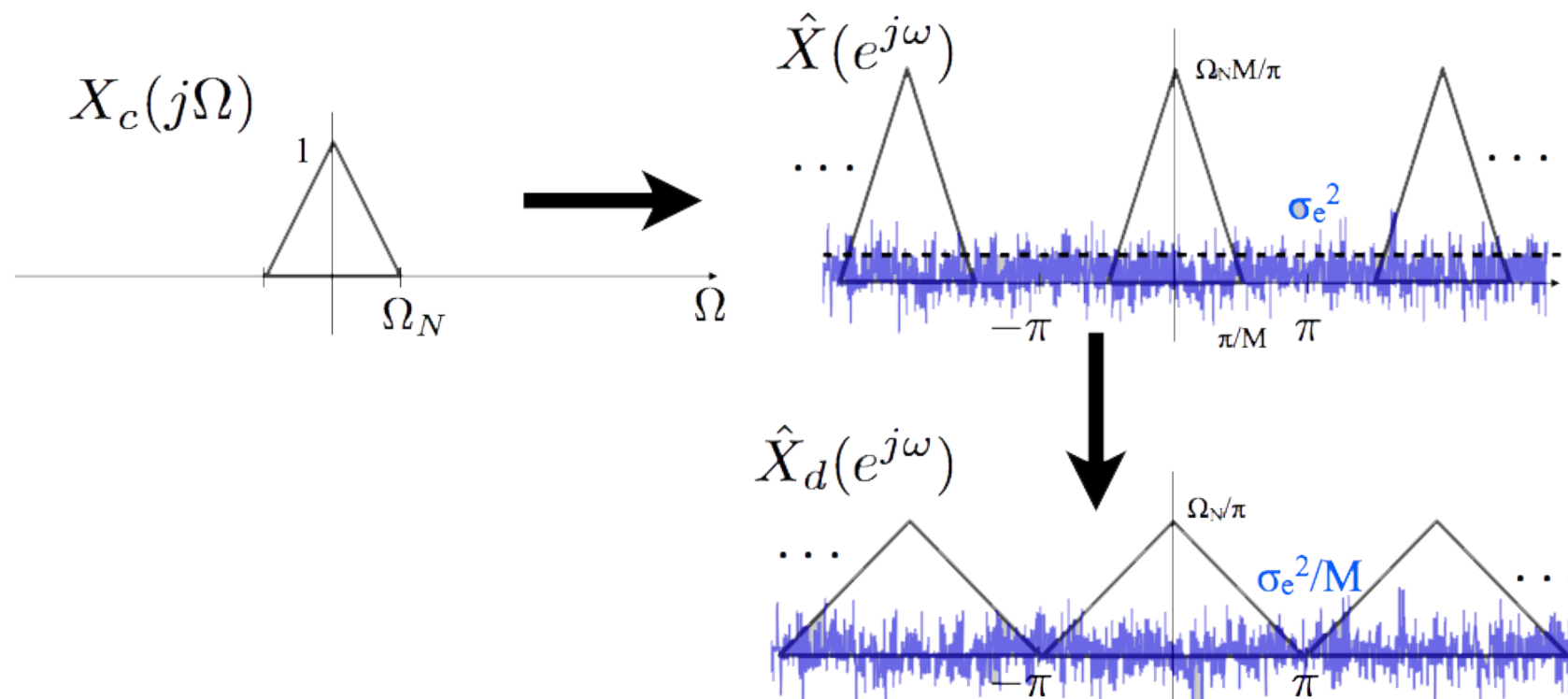
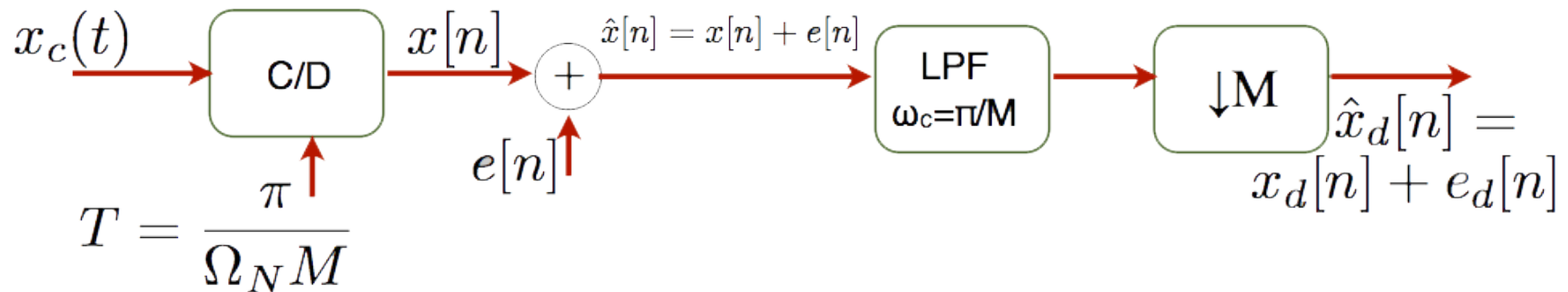
ESE 531: Digital Signal Processing

Lecture 13: February 24, 2022

Frequency Response of LTI Systems

Noise Shaping

Quantization Noise with Oversampling



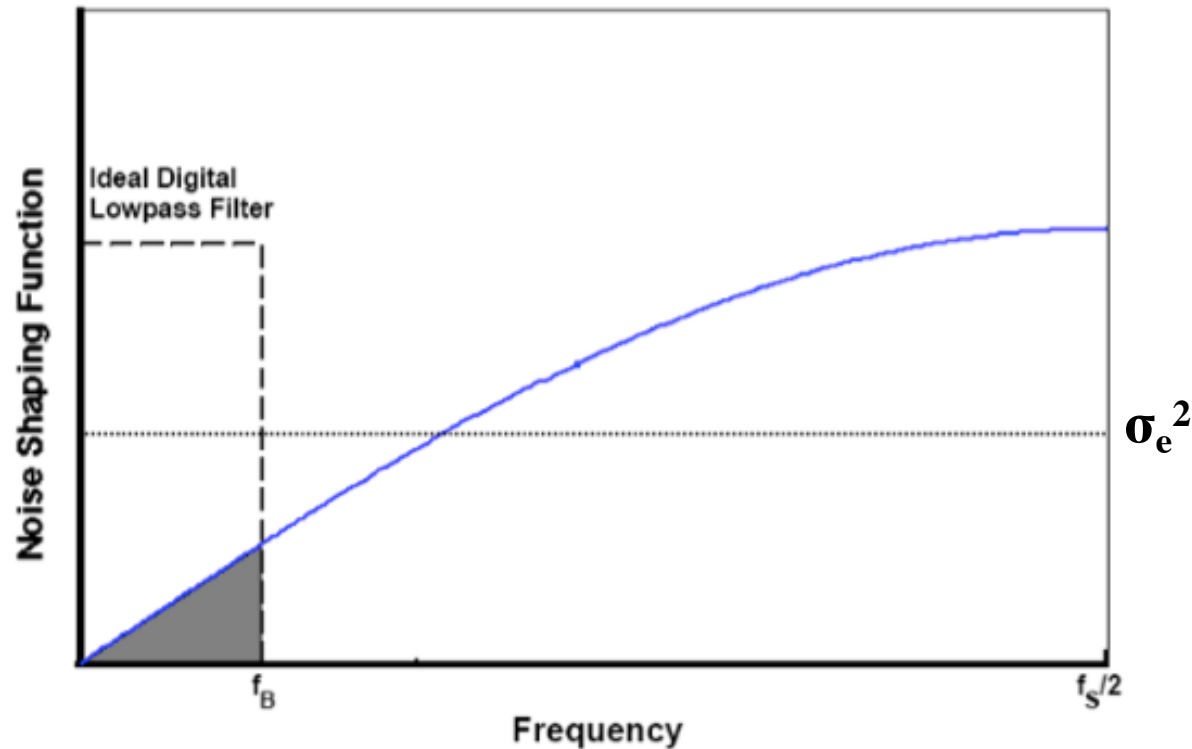
Quantization Noise with Oversampling

- ❑ Energy of $x_d[n]$ equals energy of $x[n]$
 - No filtering of signal!
- ❑ Noise variance is reduced by factor of M

$$\text{SNR}_Q = 6.02B + 10.8 - 20 \log_{10} \left(\frac{X_m}{\sigma_x} \right) + 10 \log_{10} M$$

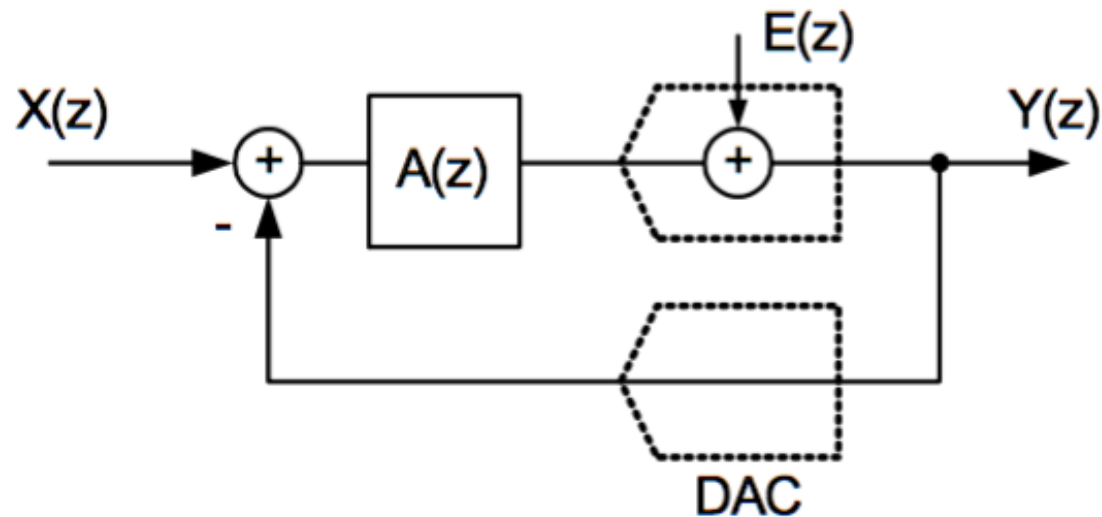
- ❑ For doubling of M we get 3dB improvement, which is the same as 1/2 a bit of accuracy
 - With oversampling of 16 with 8bit ADC we get the same quantization noise as 10bit ADC!

Noise Shaping



- ❑ Idea: "Somehow" build an ADC that has most of its quantization noise at high frequencies
- ❑ Key: Feedback

Noise Shaping Using Feedback



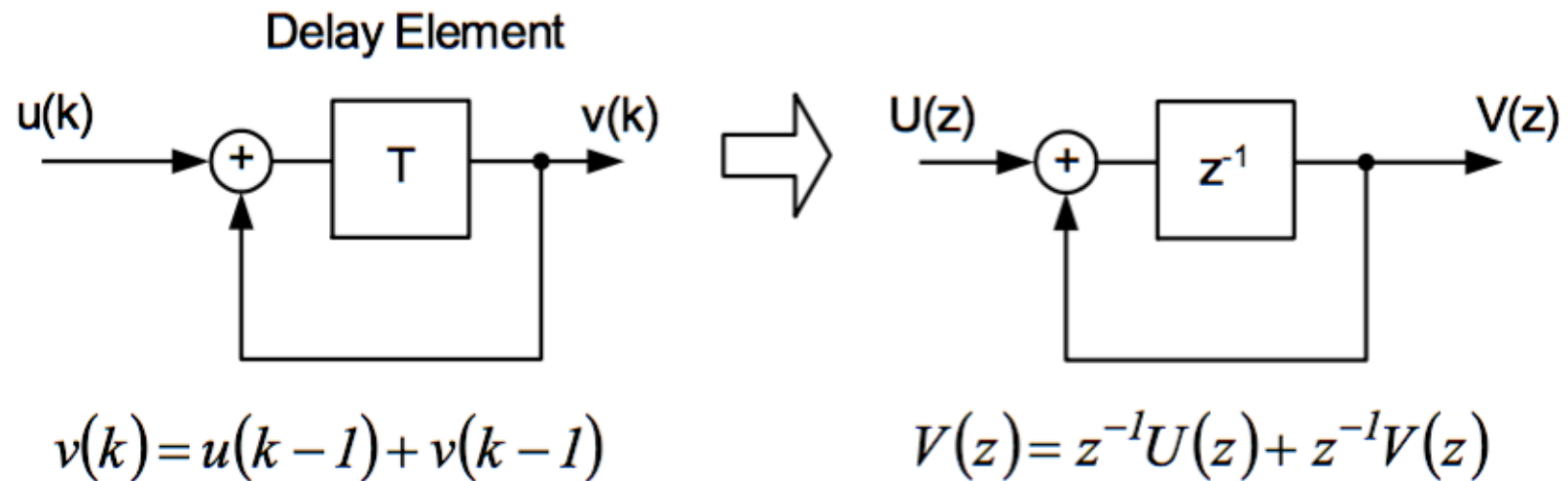
$$\begin{aligned} Y(z) &= E(z) + A(z)X(z) - A(z)Y(z) \\ &= E(z) \frac{1}{1 + A(z)} + X(z) \frac{A(z)}{1 + A(z)} \\ &= E(z) \underbrace{H_E(z)}_{\text{Noise Transfer Function}} + X(z) \underbrace{H_X(z)}_{\text{Signal Transfer Function}} \end{aligned}$$

Noise Shaping Using Feedback

$$Y(z) = E(z) \underbrace{\frac{1}{1 + A(z)}}_{\text{Noise Transfer Function}} + X(z) \underbrace{\frac{A(z)}{1 + A(z)}}_{\text{Signal Transfer Function}}$$

- ❑ Objective
 - Want to make STF unity in the signal frequency band
 - Want to make NTF "small" in the signal frequency band
- ❑ If the frequency band of interest is around DC ($0 \dots f_B$) we achieve this by making $|A(z)| \gg 1$ at low frequencies
 - Means that $\text{NTF} \ll 1$
 - Means that $\text{STF} \cong 1$

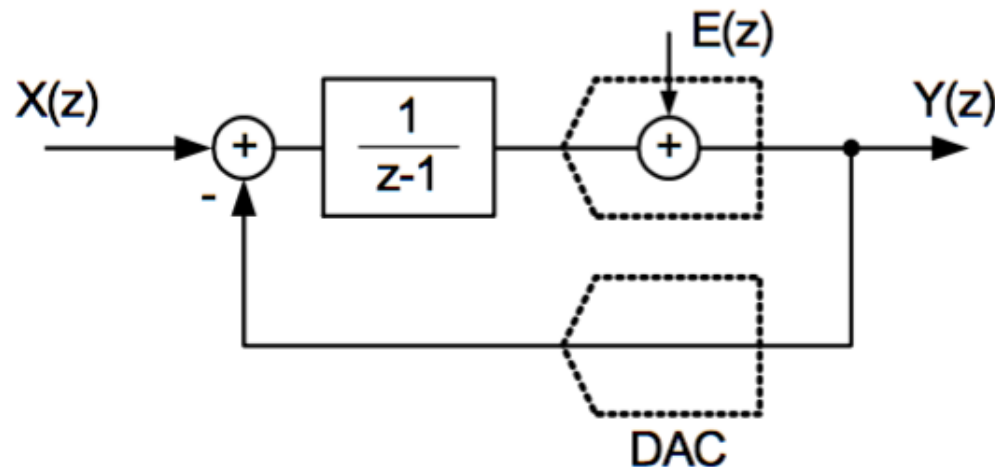
Discrete Time Integrator



$$\frac{V(z)}{U(z)} = \frac{z^{-1}}{1 - z^{-1}} = \frac{1}{z - 1} \quad z = e^{j\omega T}$$

❑ "Infinite gain" at DC ($\omega=0$, $z=1$)

First Order Sigma-Delta Modulator

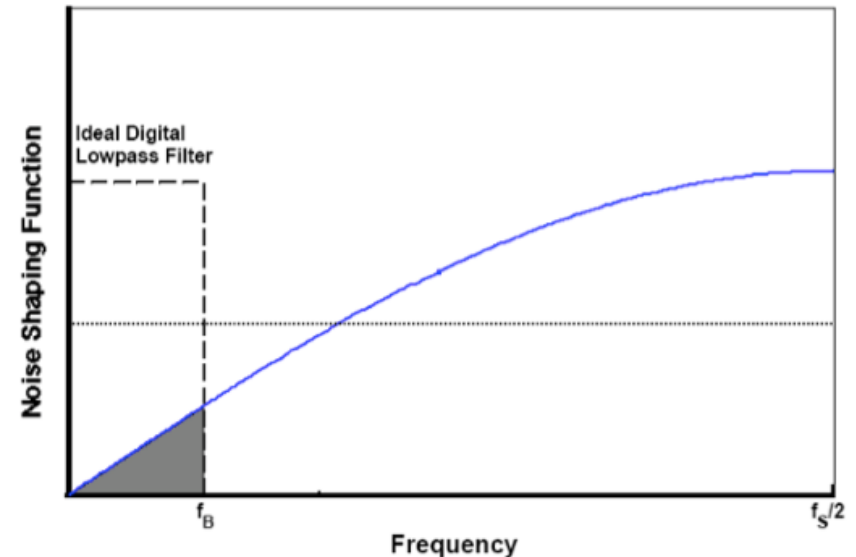


$$\begin{aligned} Y(z) &= E(z) \frac{1}{1 + \frac{1}{z-1}} + X(z) \frac{\frac{1}{z-1}}{1 + \frac{1}{z-1}} \\ &= E(z)(1 - z^{-1}) + X(z)z^{-1} \end{aligned}$$

- ❑ Output is equal to delayed input plus filtered quantization noise

NTF Frequency Domain Analysis

$$\begin{aligned}H_e(z) &= 1 - z^{-1} \\H_e(j\omega) &= (1 - e^{-j\omega T}) = 2e^{-j\omega T/2} \left(\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2} \right) \\&= 2e^{-j\frac{\omega T}{2}} \left(j \sin\left(\frac{\omega T}{2}\right) \right) = 2 \sin\left(\frac{\omega T}{2}\right) e^{-j\frac{\omega T - \pi}{2}} \\|H_e(f)| &= 2|\sin(\pi f T)| = 2\left| \sin\left(\pi \frac{f}{f_s}\right) \right|\end{aligned}$$

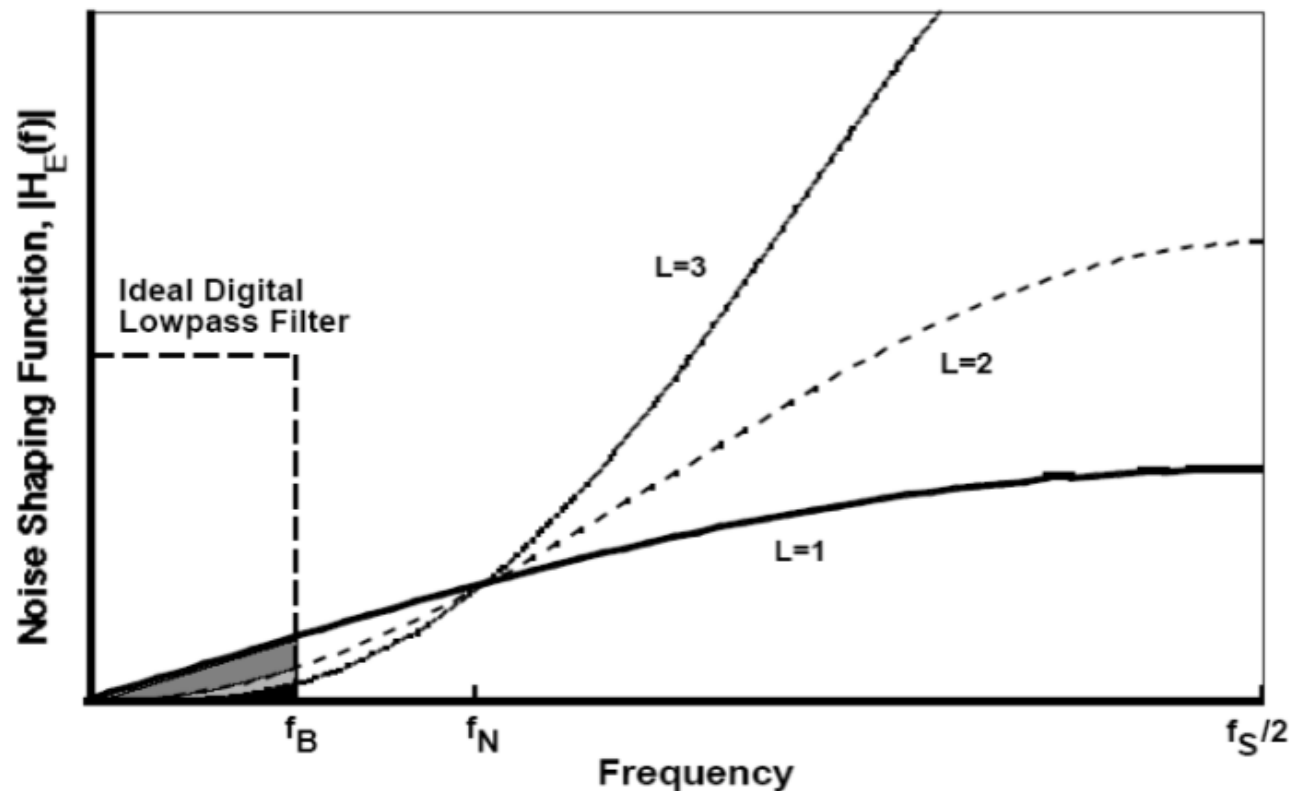


- ❑ "First order noise Shaping"
 - Quantization noise is attenuated at low frequencies, amplified at high frequencies

Higher Order Noise Shaping

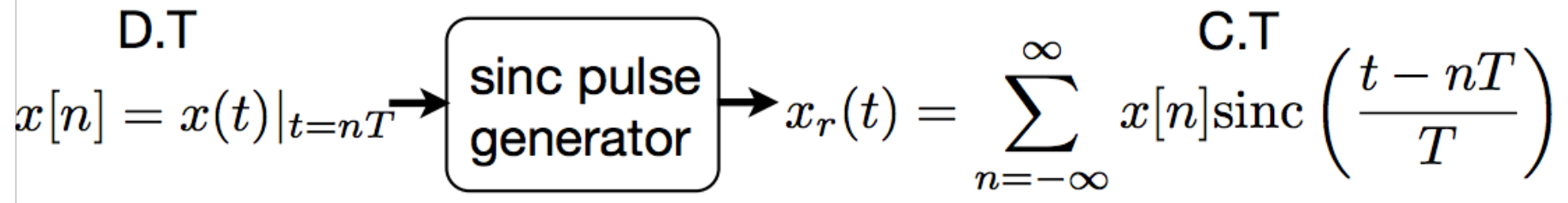
- L^{th} order noise transfer function

$$H_E(z) = (1 - z^{-1})^L$$



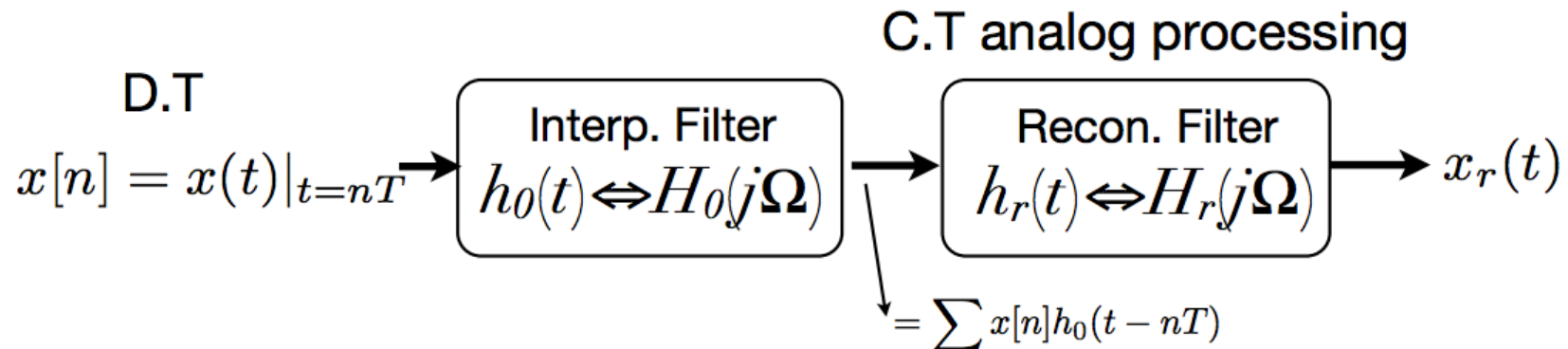
Practical DAC

Practical DAC

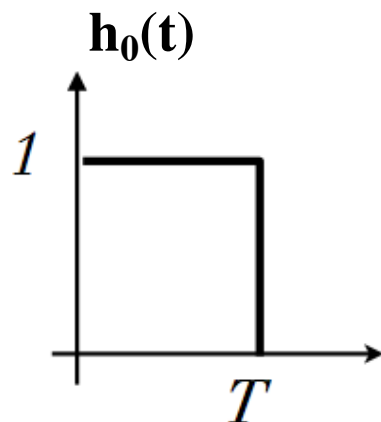


- ❑ Scaled train of sinc pulses
- ❑ Difficult to generate sinc $\rightarrow$ Too long!

Practical DAC



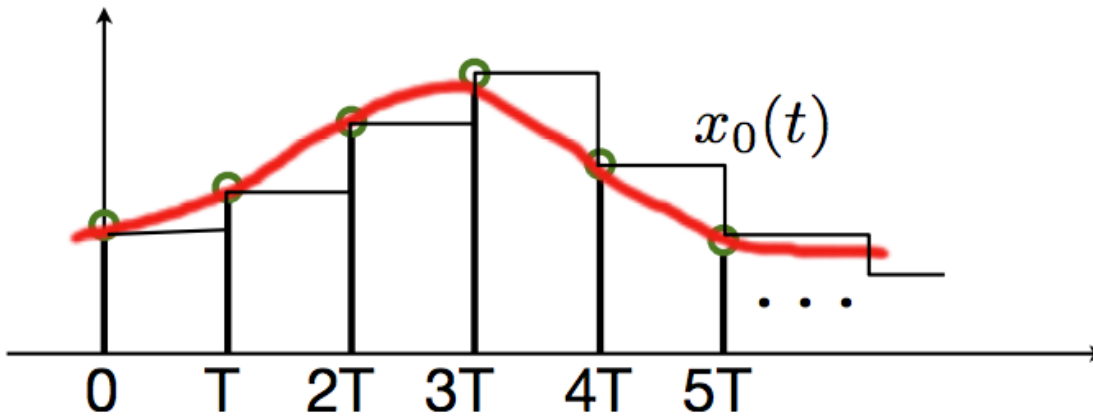
- ❑ $h_0(t)$ is finite length pulse $\rightarrow$ easy to implement
- ❑ For example: zero-order hold



$$H_0(j\Omega) = T e^{-j\Omega \frac{T}{2}} \text{sinc}\left(\frac{\Omega}{\Omega_s}\right)$$

Practical DAC

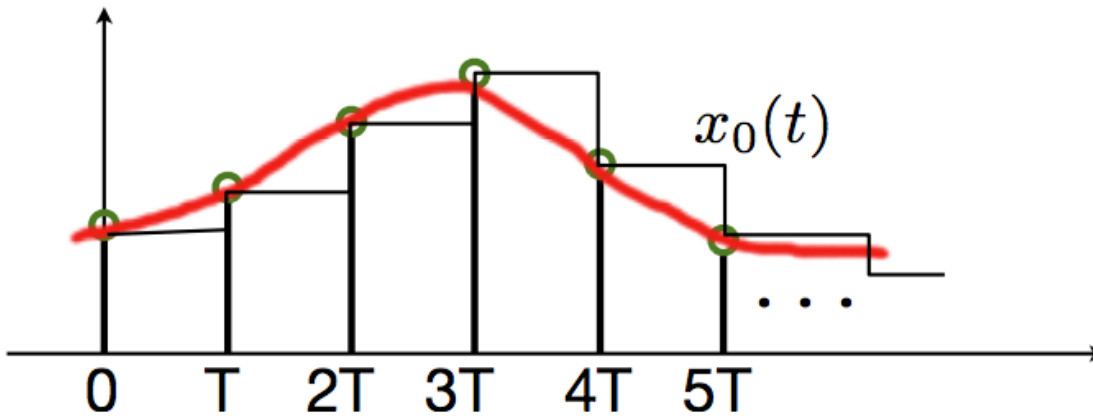
Zero-Order-Hold interpolation



$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t - nT) = h_0(t) * x_s(t)$$

Practical DAC

Zero-Order-Hold interpolation



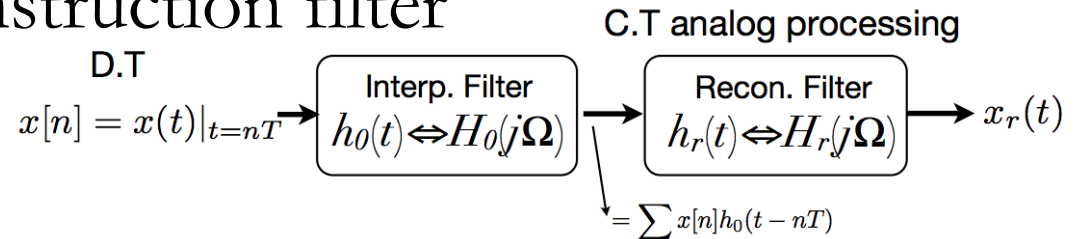
$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t - nT) = h_0(t) * x_s(t)$$

Taking a FT:

$$\begin{aligned} X_0(j\Omega) &= H_0(j\Omega)X_s(j\Omega) \\ &= H_0(j\Omega)\frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s)) \end{aligned}$$

Practical DAC

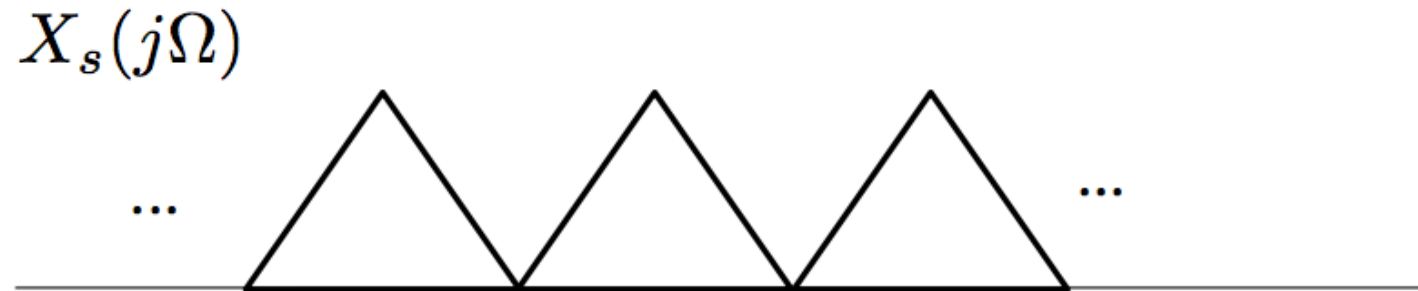
□ Output of the reconstruction filter



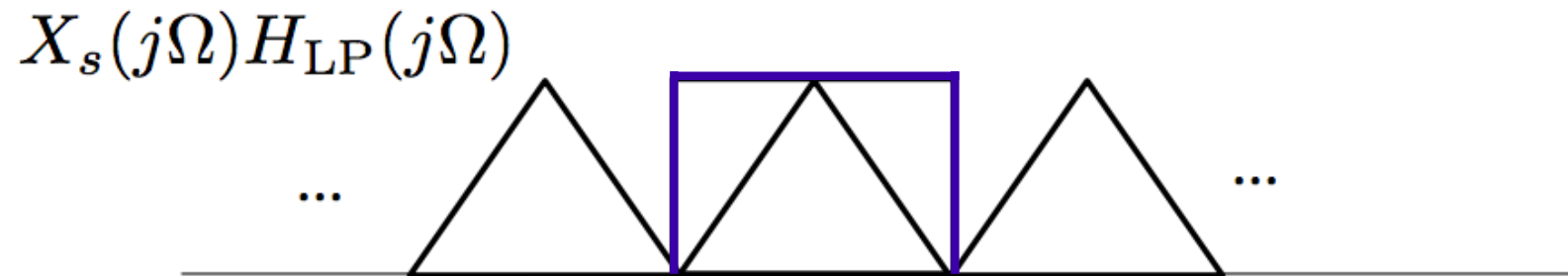
$$X_r(j\Omega) = H_r(j\Omega) \cdot H_0(j\Omega) \cdot X_s(j\Omega)$$



Practical DAC

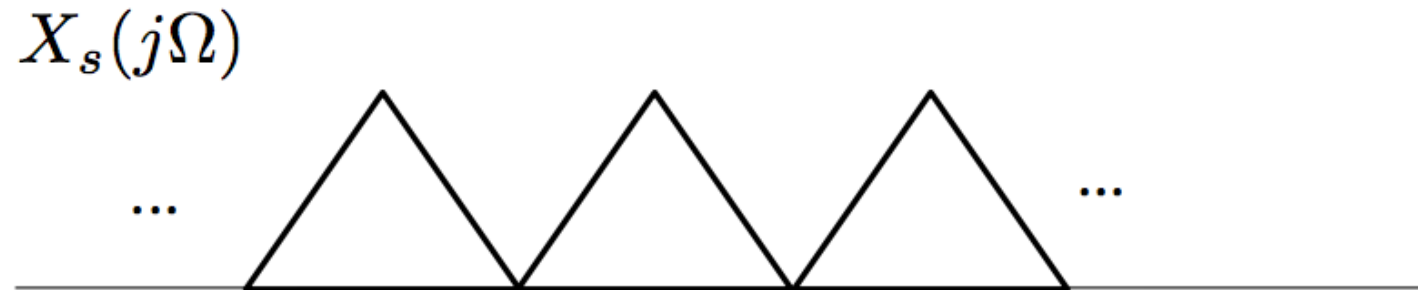


Ideally:





Practical DAC



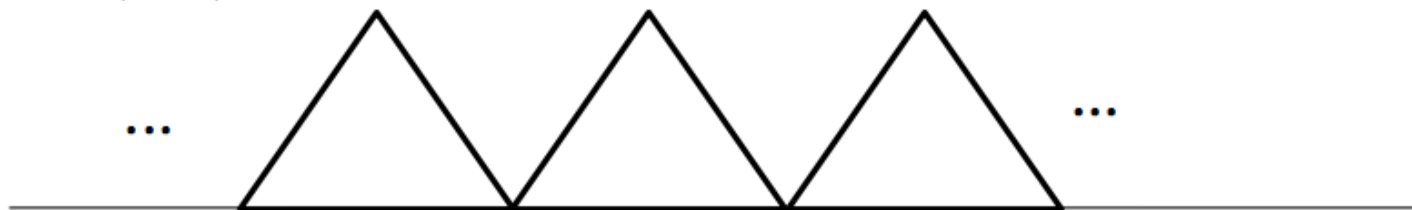
Practically:





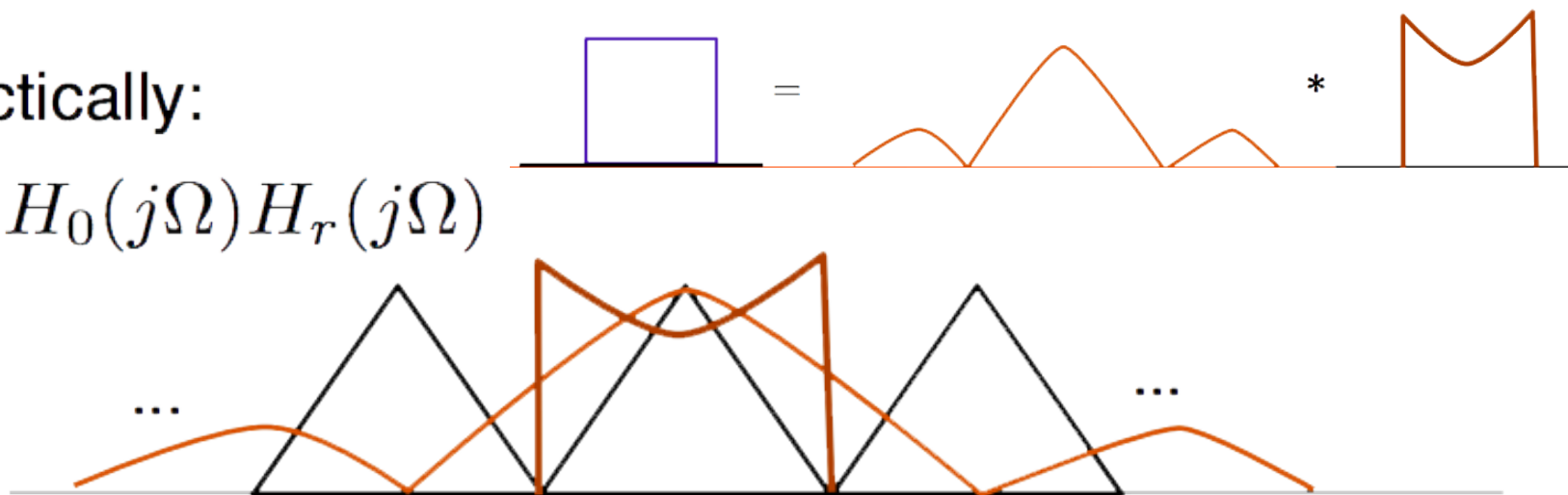
Practical DAC

$$X_s(j\Omega)$$



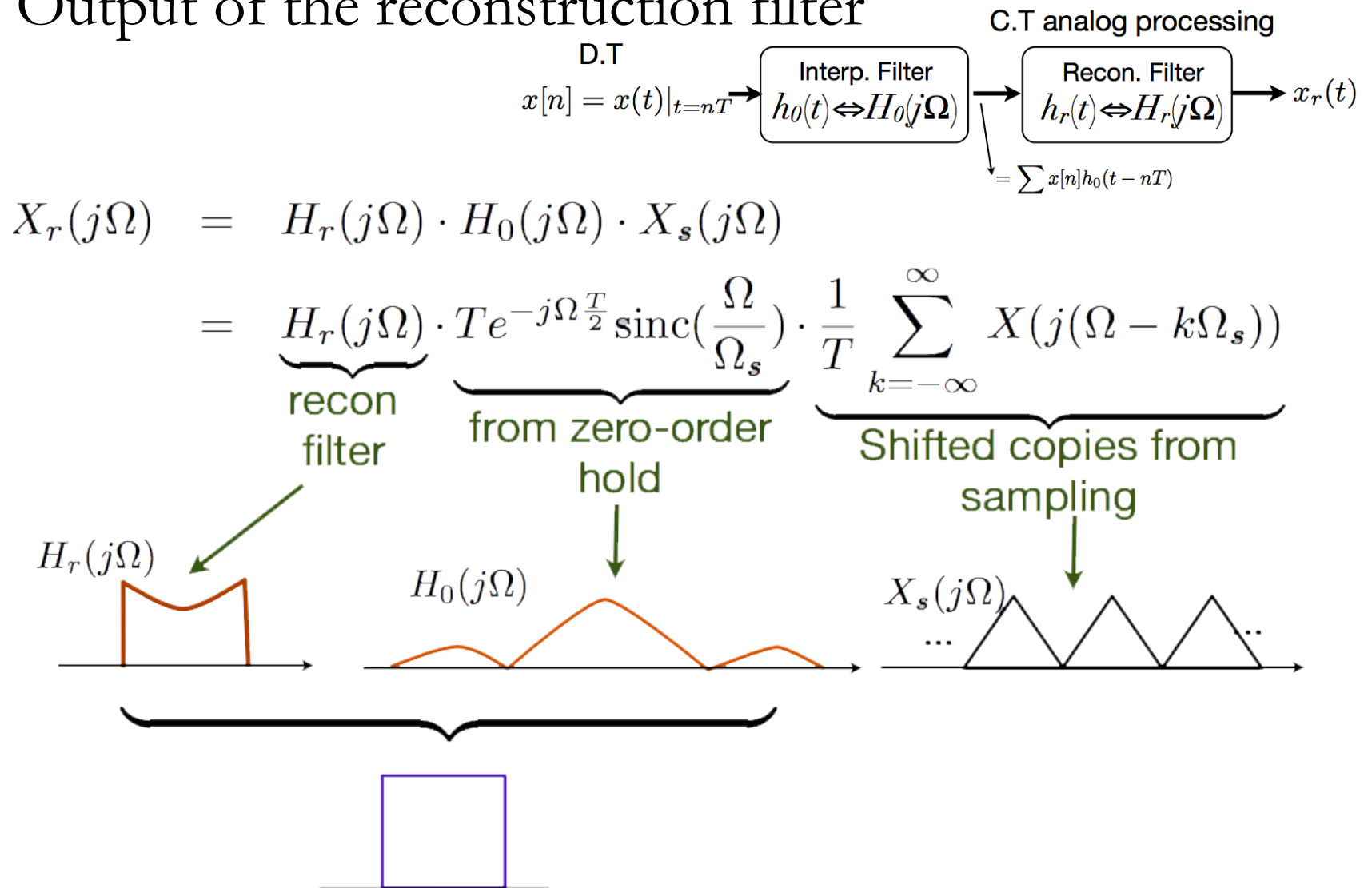
Practically:

$$X_s(j\Omega)H_0(j\Omega)H_r(j\Omega)$$

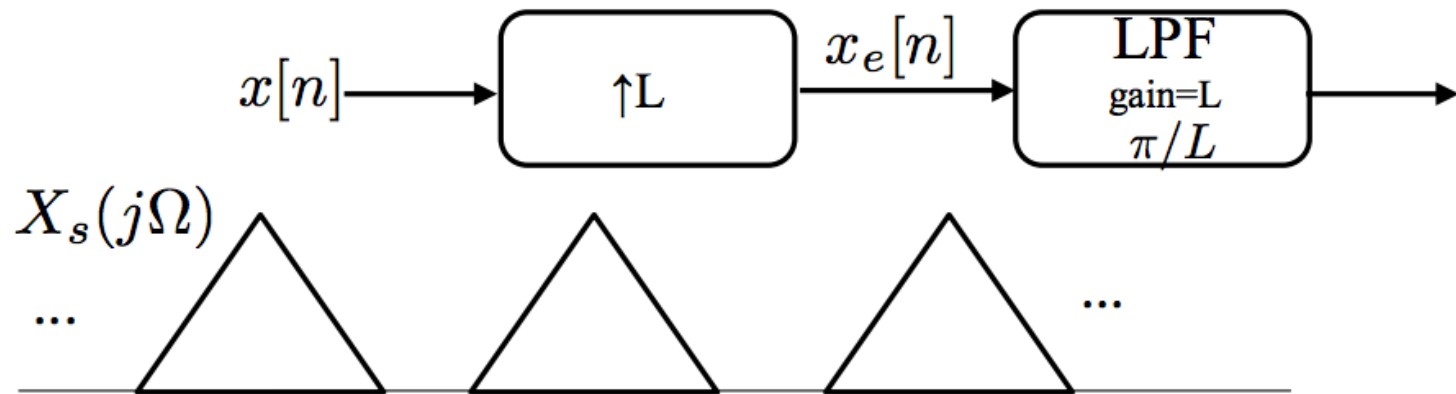


Practical DAC

□ Output of the reconstruction filter

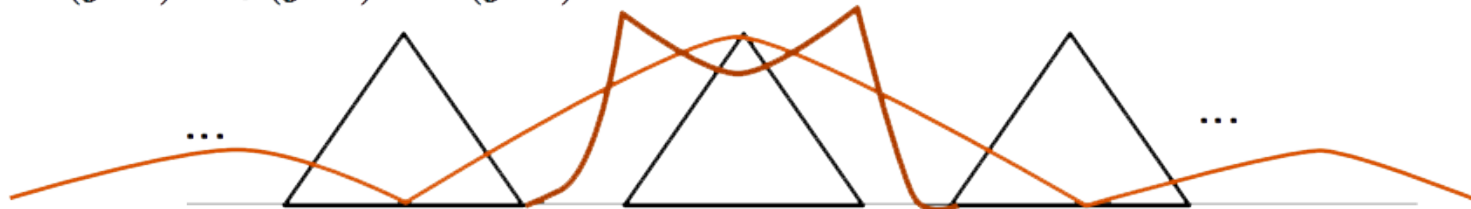


Practical DAC with Upsampling



Practically:

$$X_s(j\Omega) H_0(j\Omega) H_r(j\Omega)$$



Frequency Response of LTI Systems



Lecture Outline

- Frequency Response of LTI Systems
 - Magnitude Response
 - Simple Filters
 - Phase Response
 - Group Delay
 - Example: Zero on Real Axis
 - Example: Higher order systems



Frequency Response of LTI System

- ❑ LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[k] * h[k]$$

- ❑ We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- ❑ Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- ❑ $H(e^{j\omega})$ defines magnitude and phase change at each frequency



Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

- And a phase response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$



Phase Response

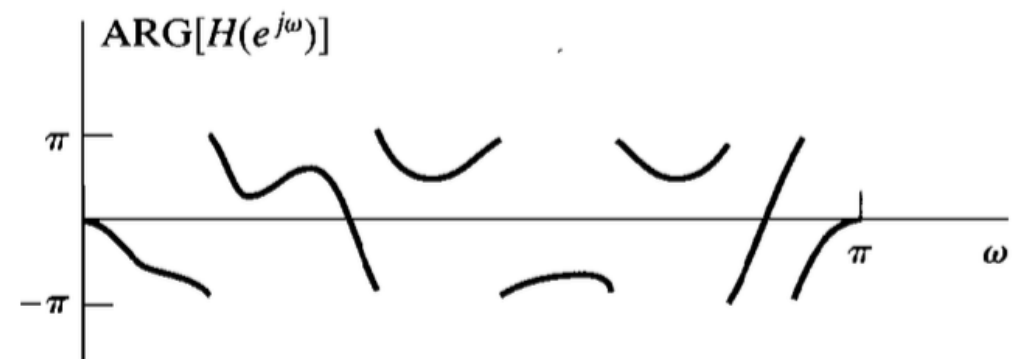
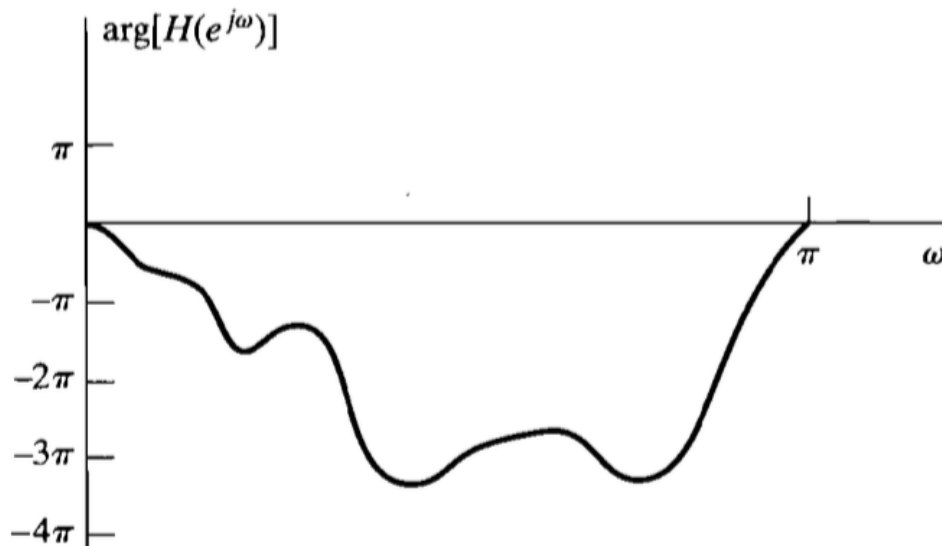
- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$

Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$



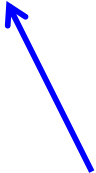
Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

- More later...

Unwrapped phase





Linear Difference Equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$



Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$



Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

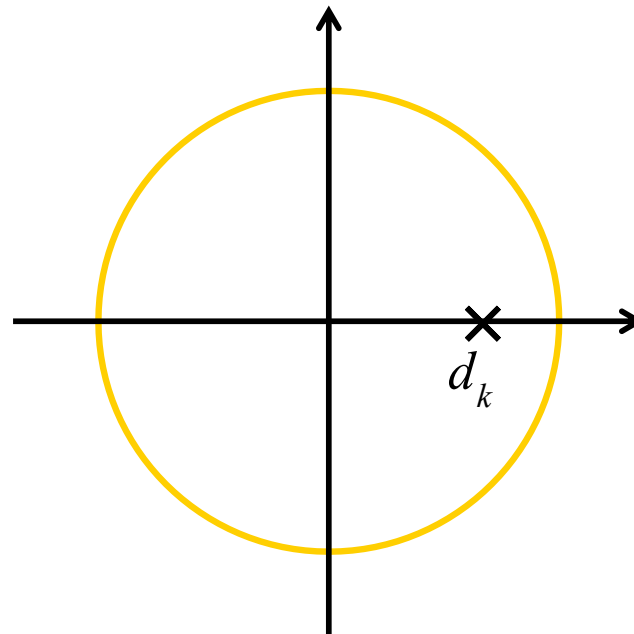
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



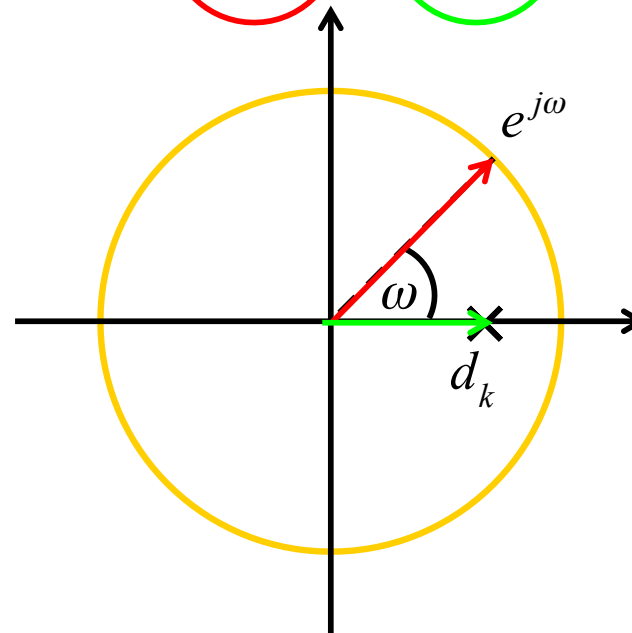
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



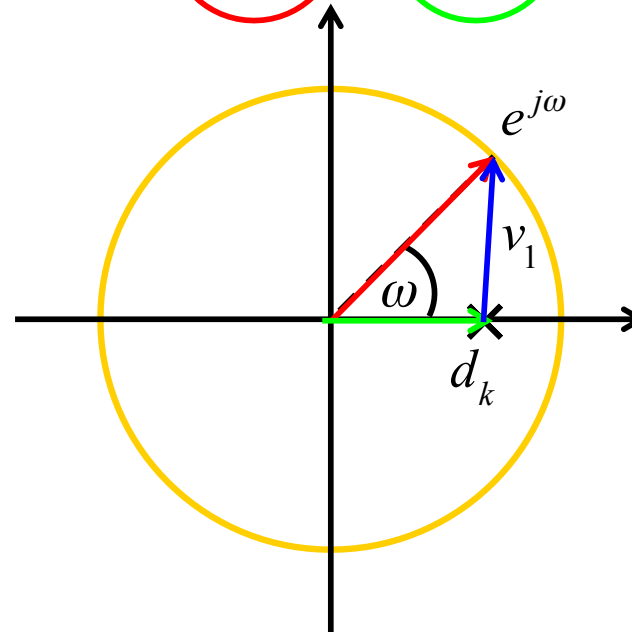
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



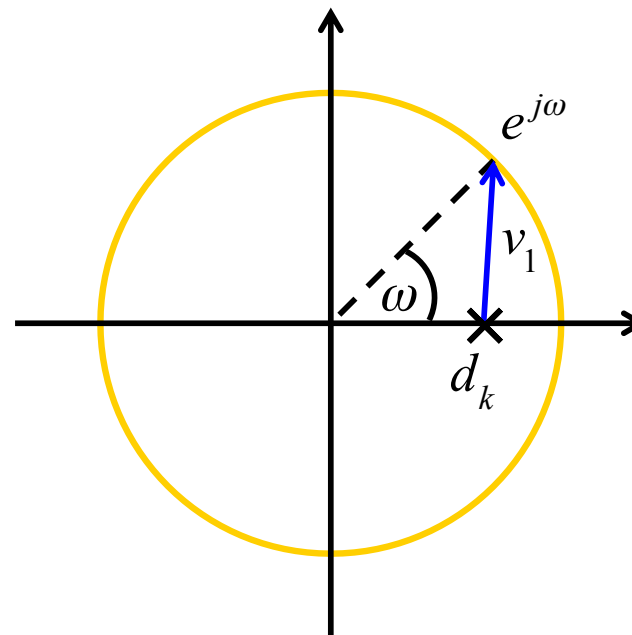
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



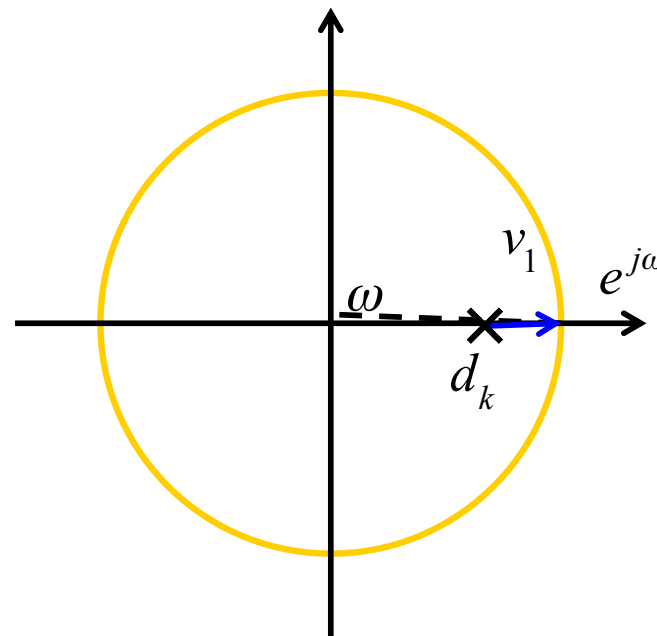
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



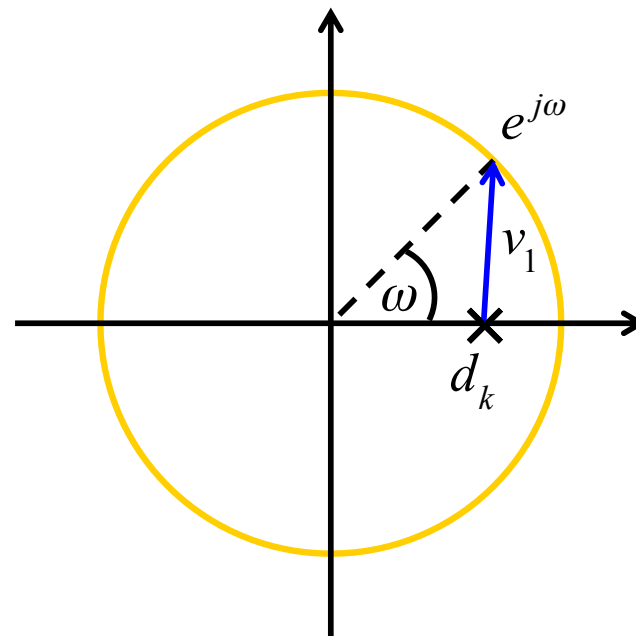
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



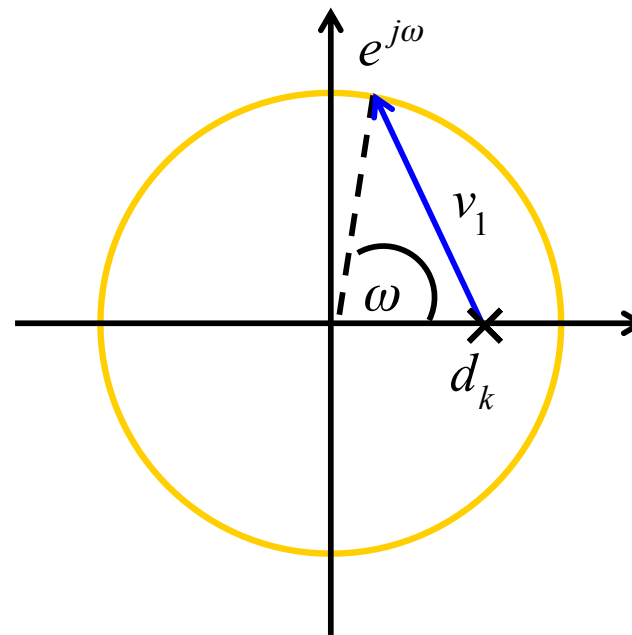
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



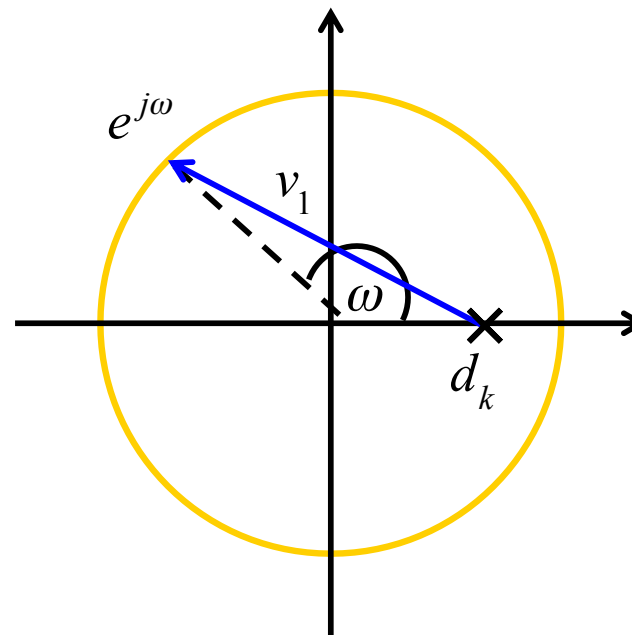
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



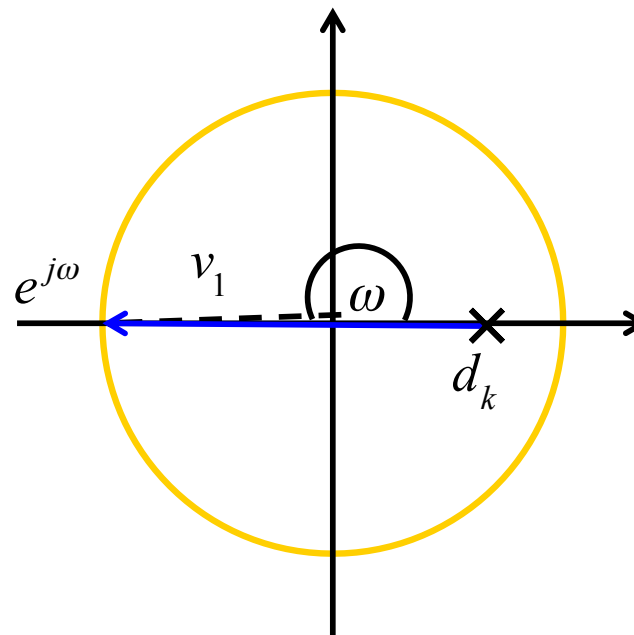
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



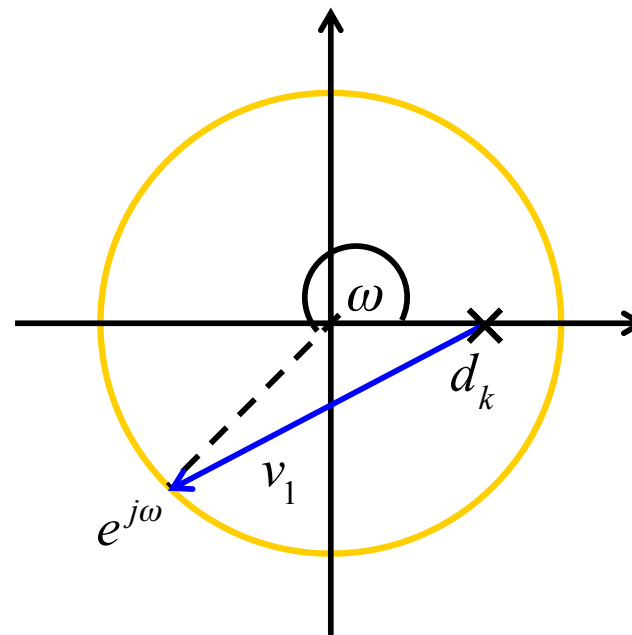
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



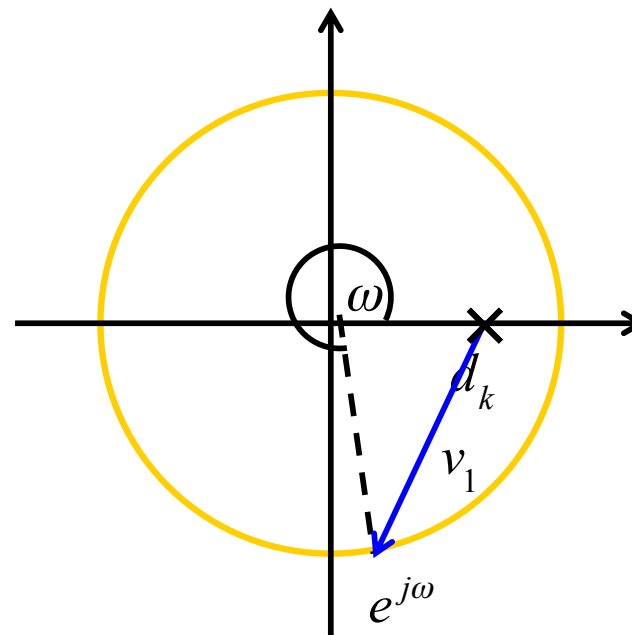
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



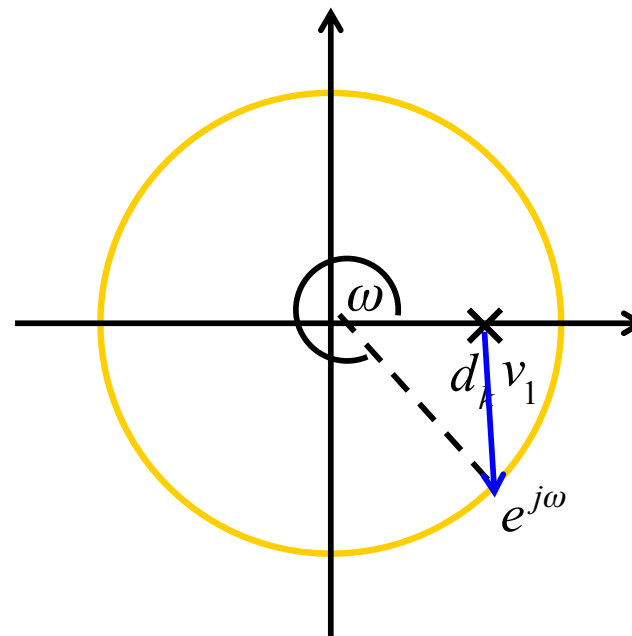
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$





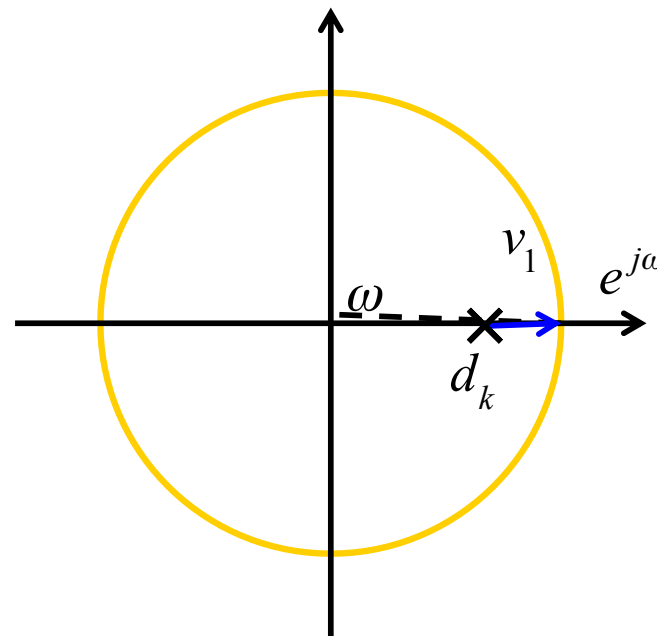
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

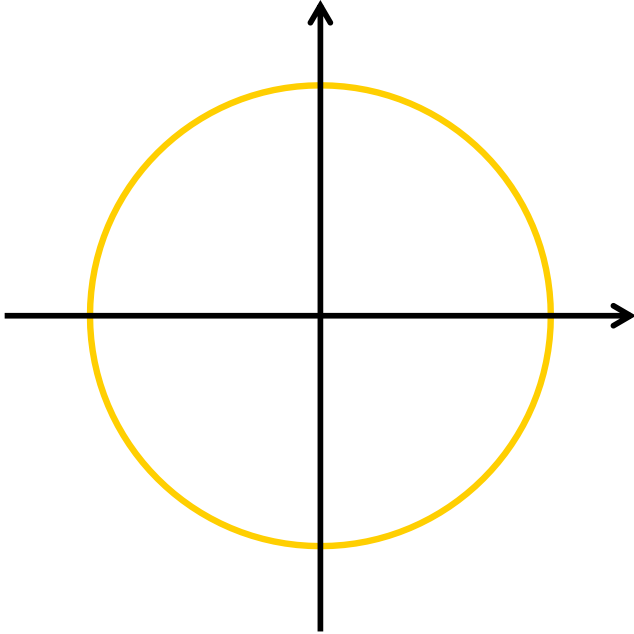




Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

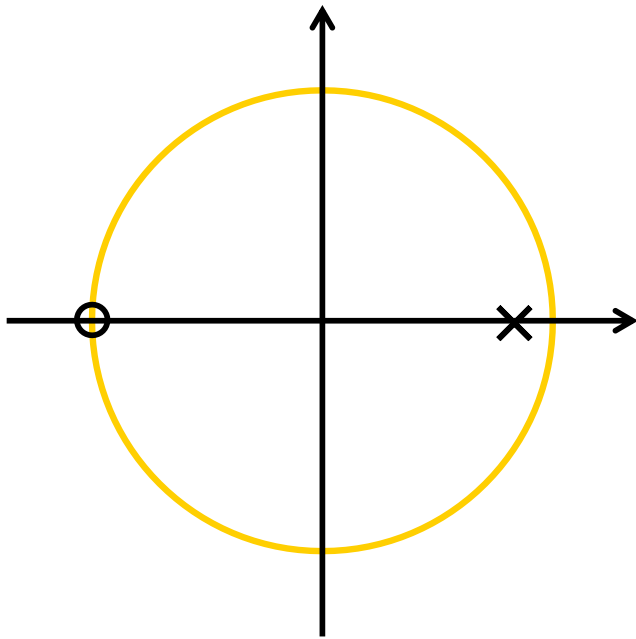
$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$

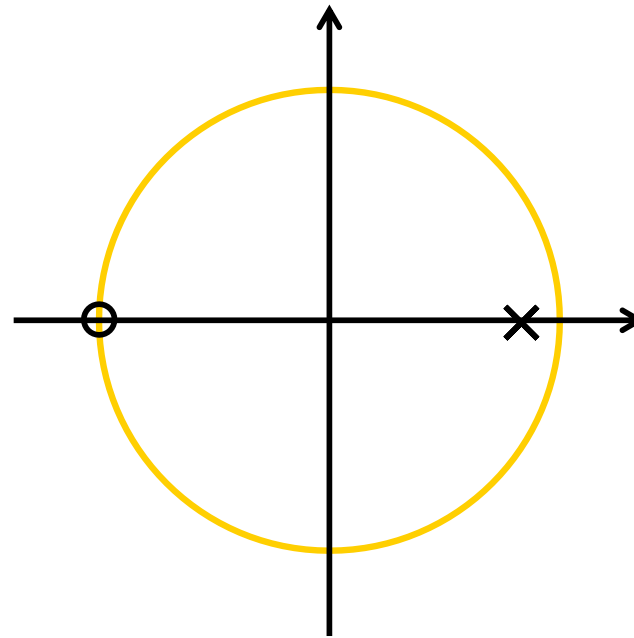
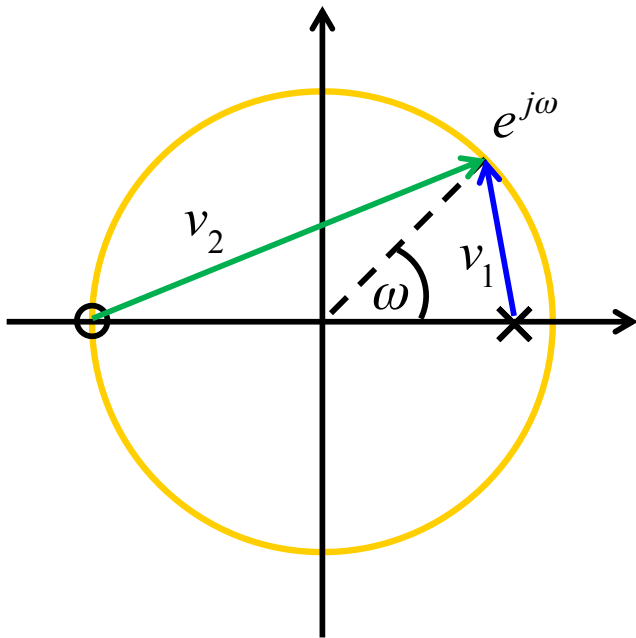




Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

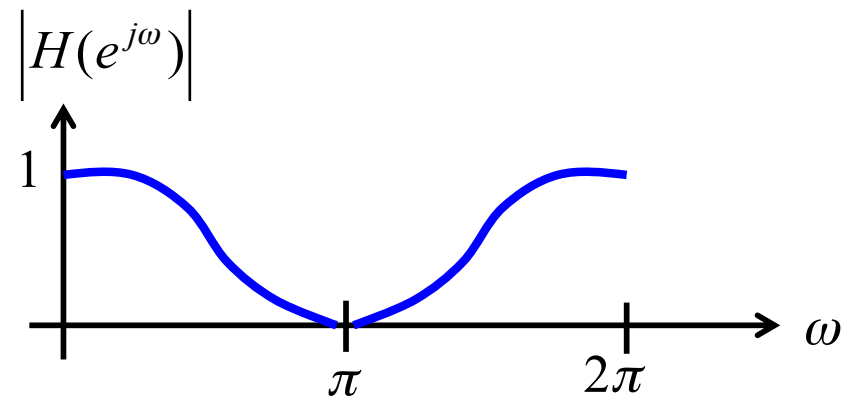
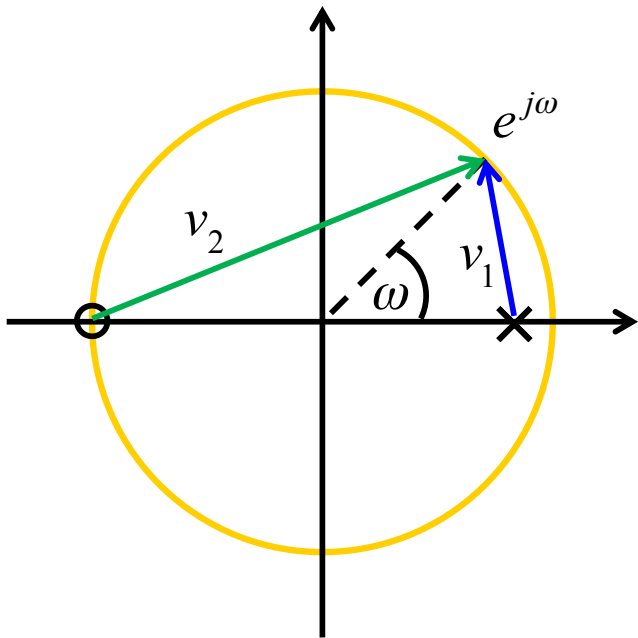
$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



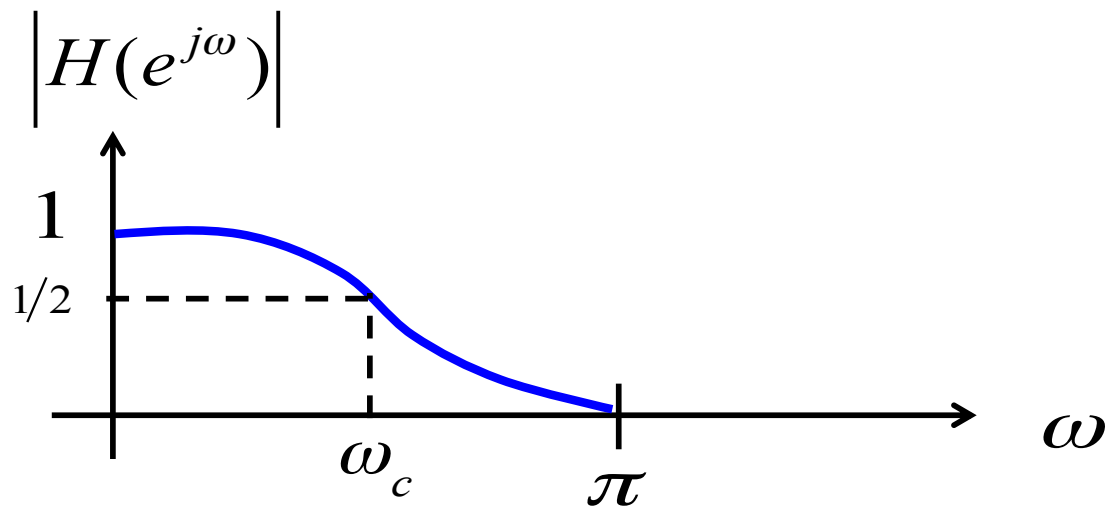


Simple Low Pass Filter

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$

Simple Low Pass Filter

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$



ω_c is the 3dB cutoff frequency

$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$



Simple Low Pass Filter $\rightarrow$ High Pass

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$



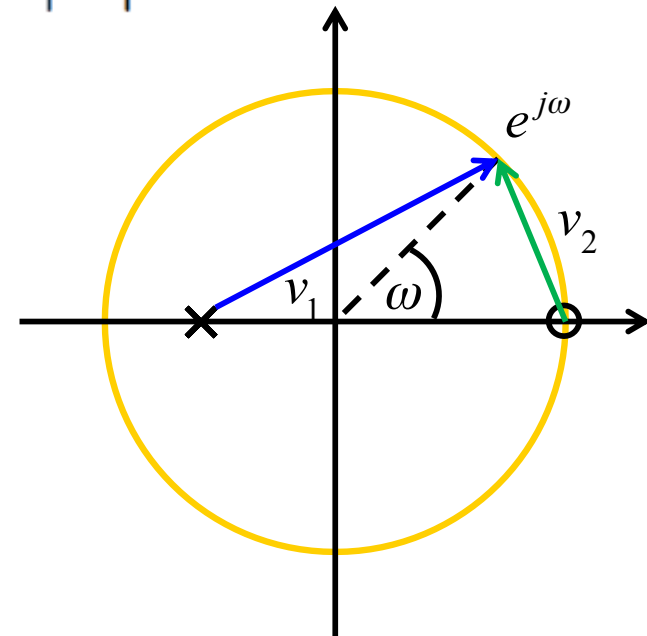
Simple High Pass Filter

$$H_{HP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-1}}{1 + \alpha z^{-1}} \quad |\alpha| < 1$$

Simple High Pass Filter

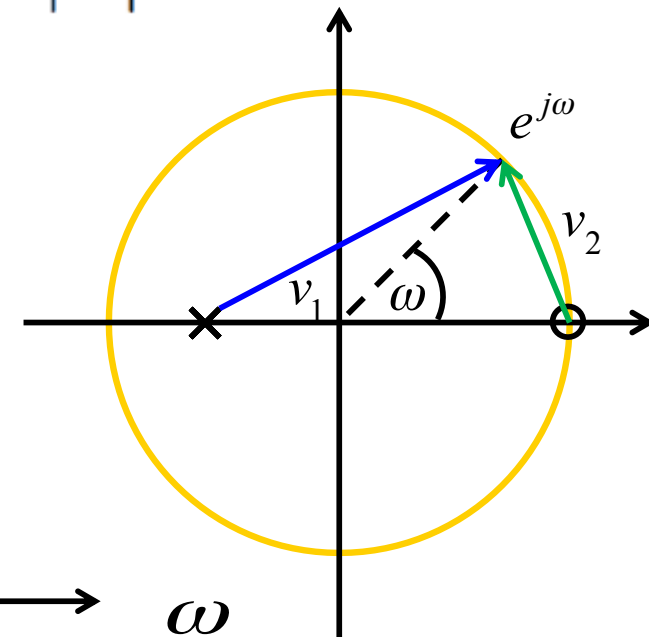
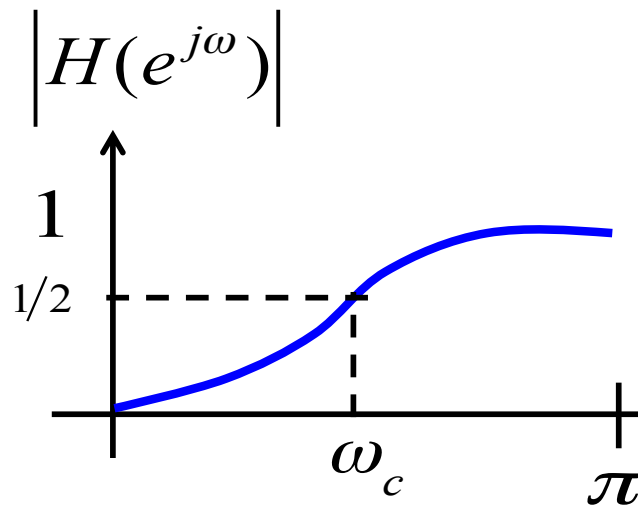
$$H_{HP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-1}}{1 + \alpha z^{-1}}$$

$$|\alpha| < 1$$



Simple High Pass Filter

$$H_{HP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-1}}{1 + \alpha z^{-1}} \quad |\alpha| < 1$$



ω_c is the 3dB cutoff frequency

$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$

Simple Band-Stop (Notch) Filter

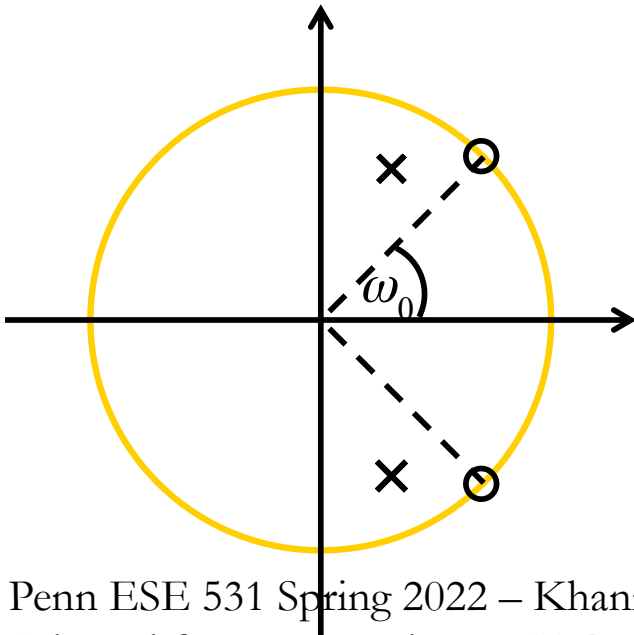
$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$
 $\cos(\omega_0) = \beta$

Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

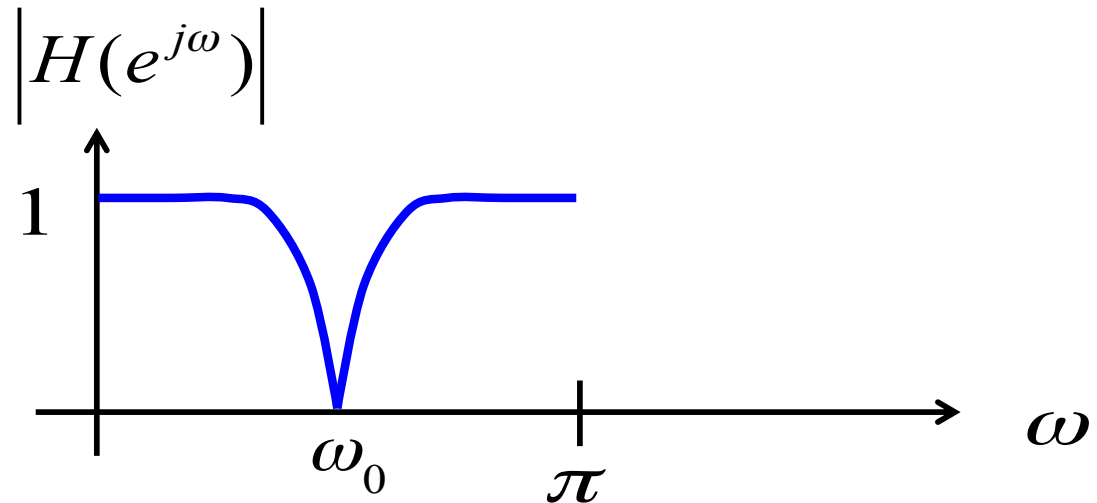
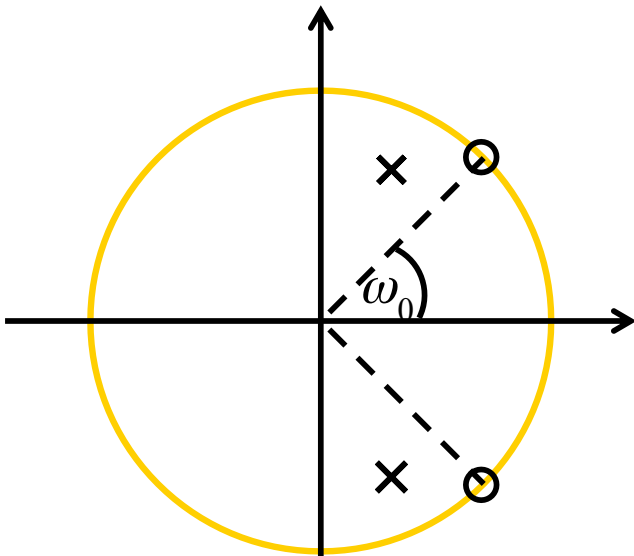
Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$
 $\cos(\omega_0) = \beta$



Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$
 $\cos(\omega_0) = \beta$

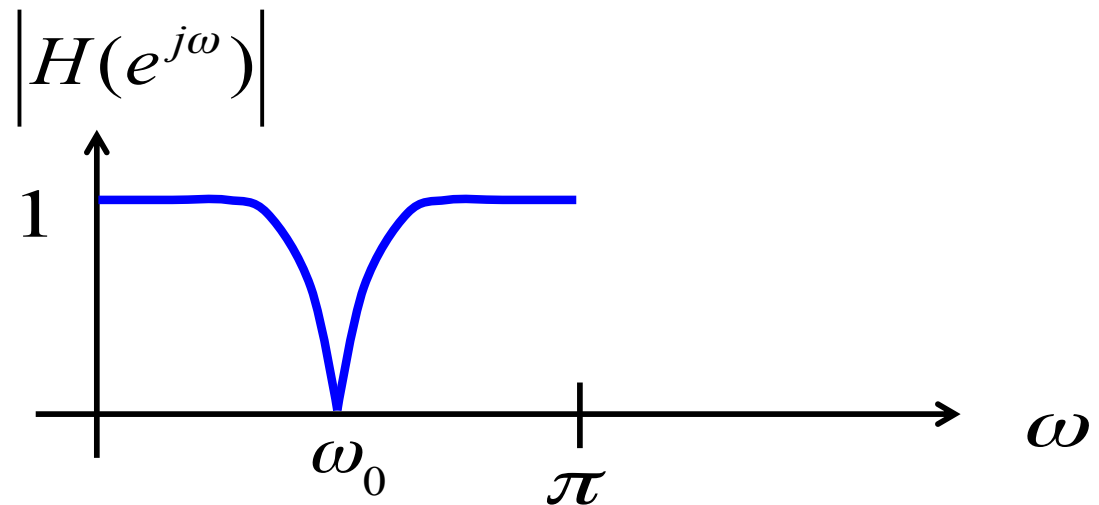
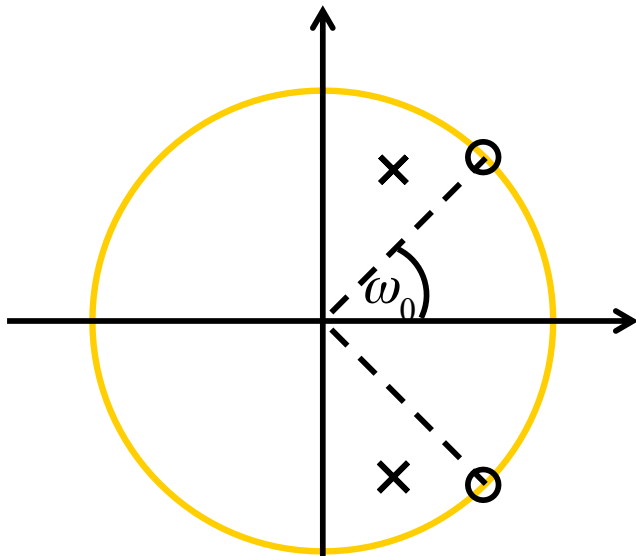


Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note:

$$H_{BS}(\mp 1) = \frac{1 + \alpha}{2} \frac{2 \pm 2\beta}{(1 + \alpha)(1 \pm \beta)} = 1$$

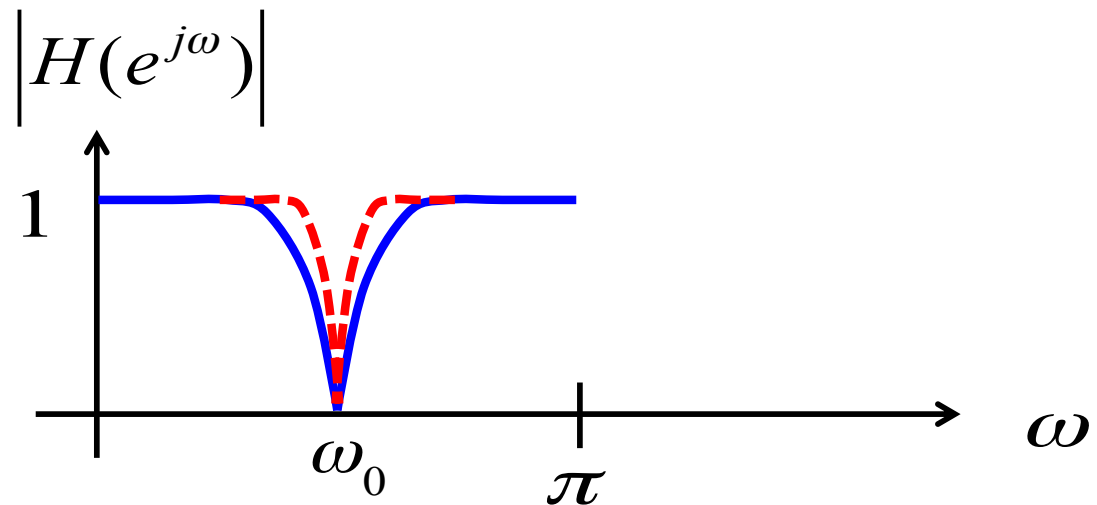
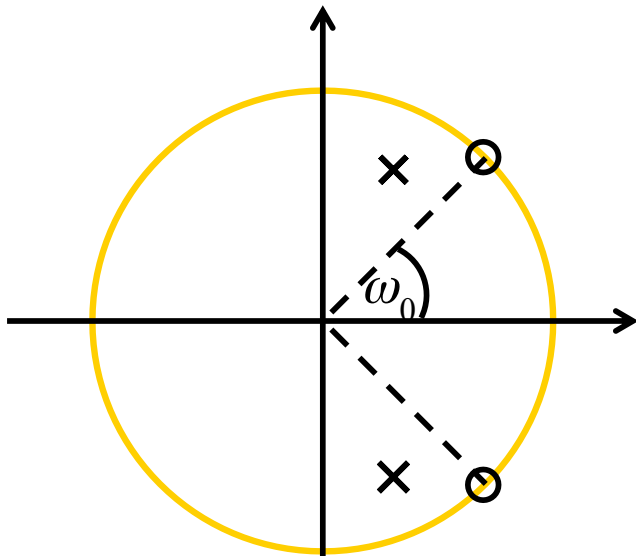


Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: As $\alpha \rightarrow 1$ poles approach zeros

$$H_{BS}(\mp 1) = \frac{1 + \alpha}{2} \frac{2 \pm 2\beta}{(1 + \alpha)(1 \pm \beta)} = 1$$



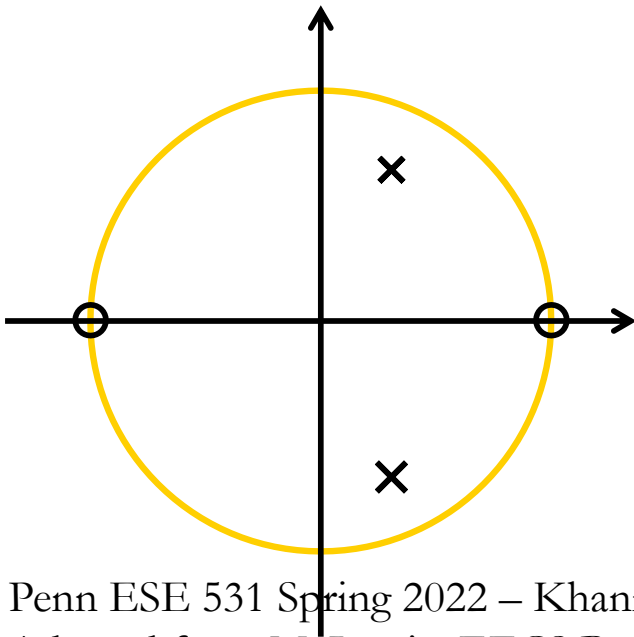


Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

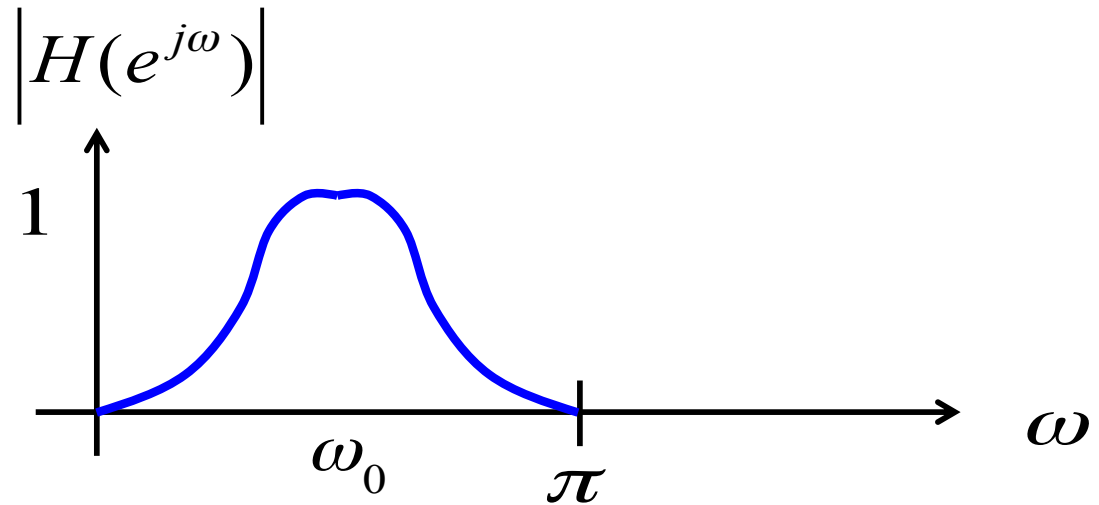
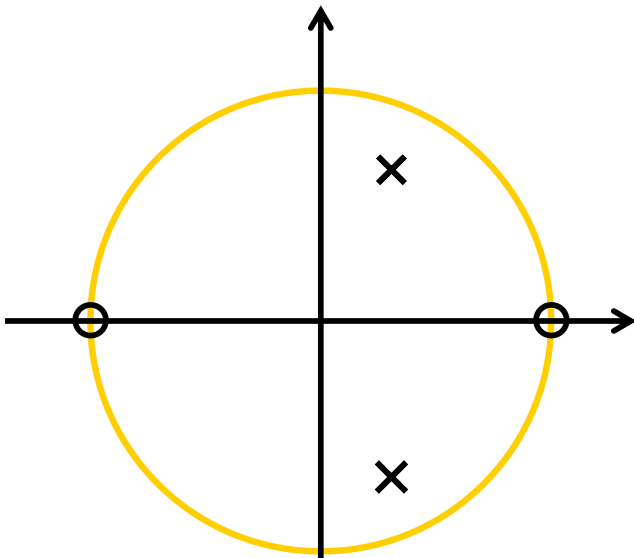
Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$



Simple Band-Pass Filter

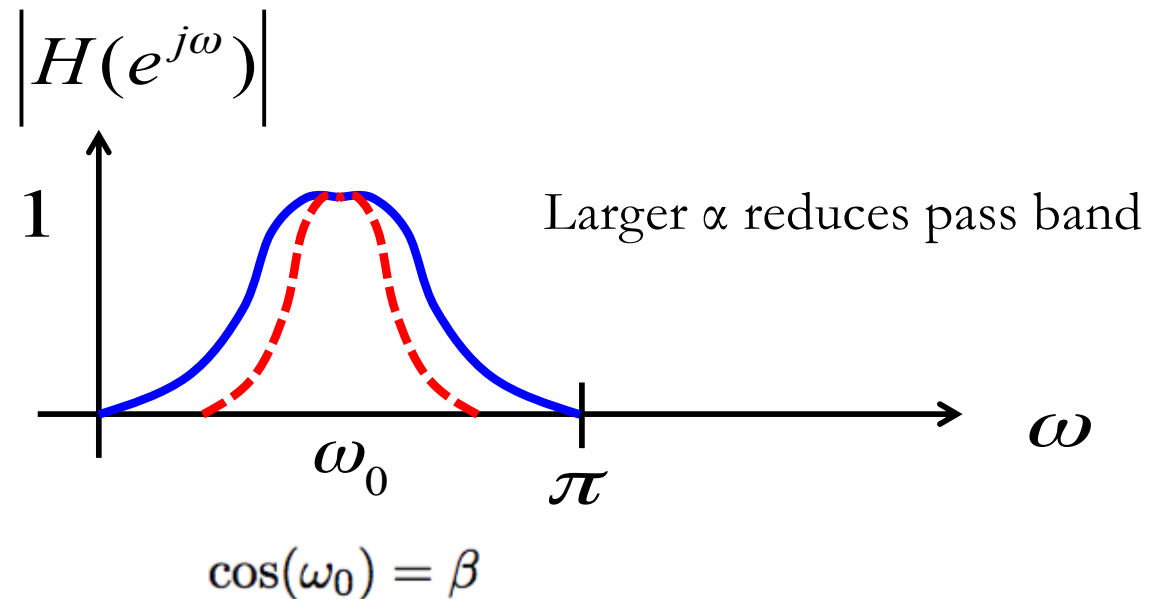
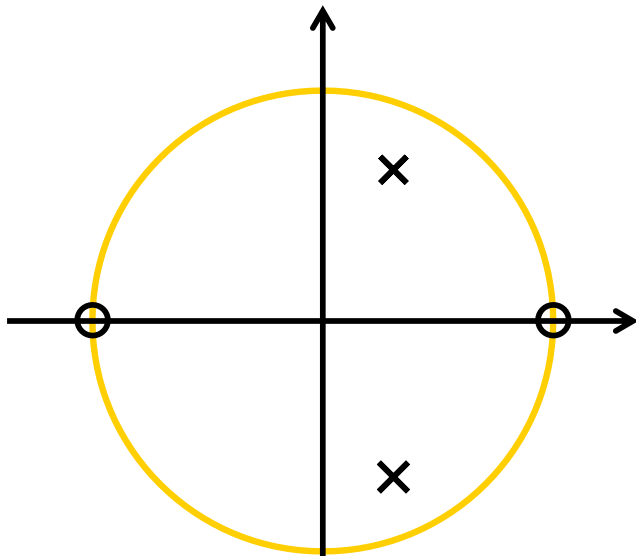
$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$



$$\cos(\omega_0) = \beta$$

Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

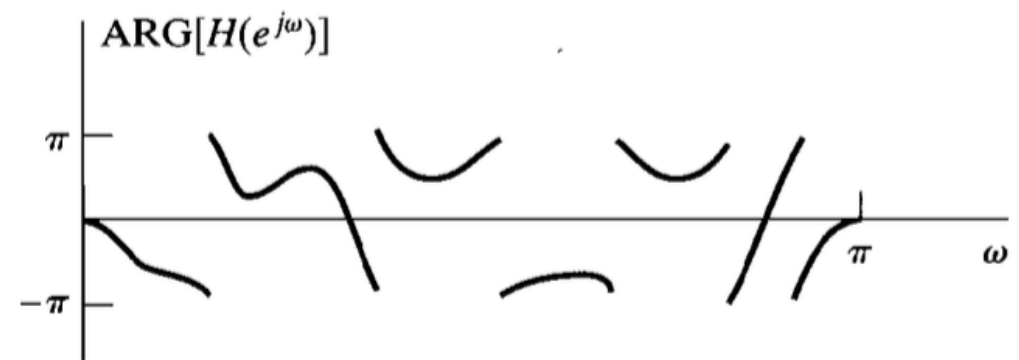
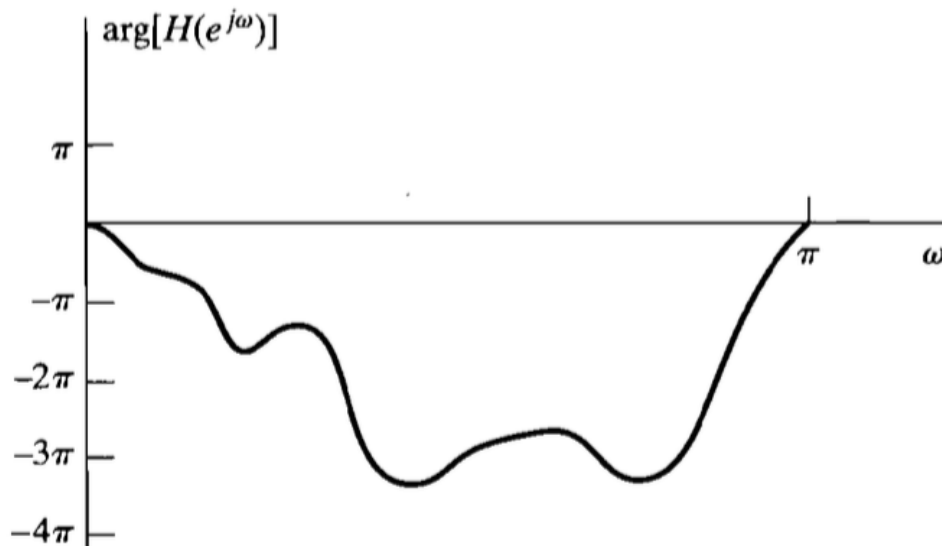


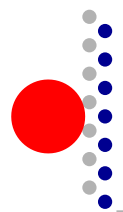


Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$





Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$



Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase

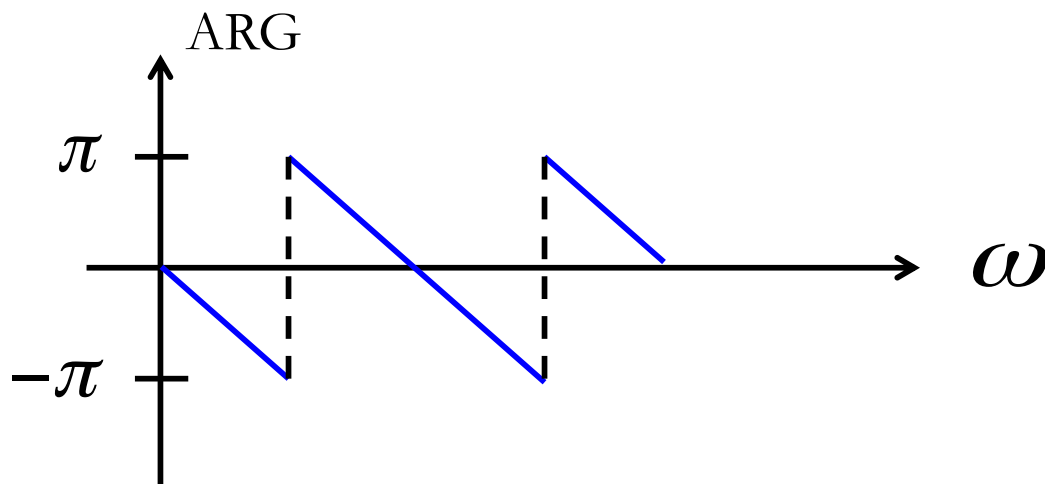
Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

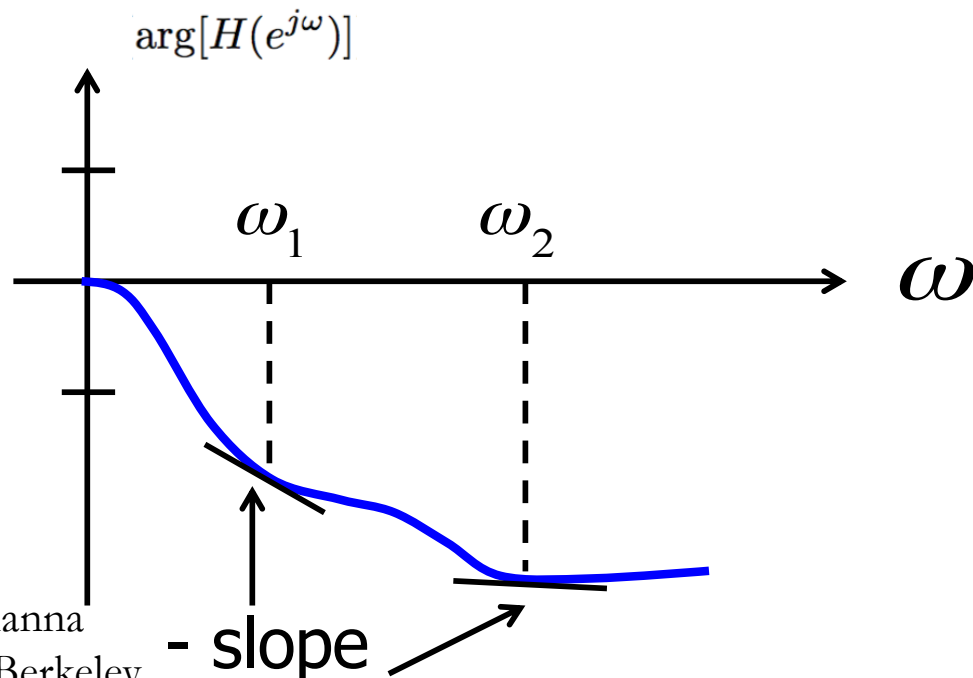
ARG is the wrapped phase
arg is the unwrapped phase



Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$



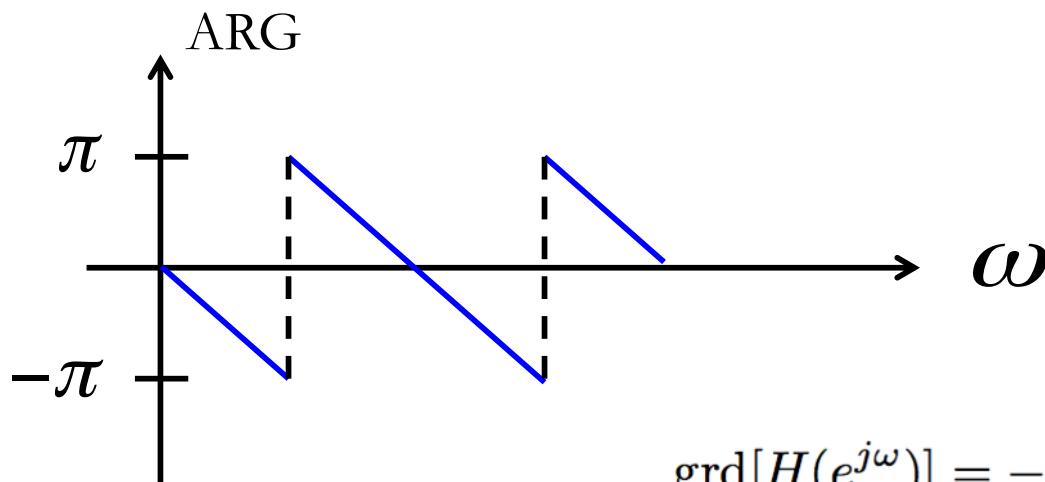
Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase



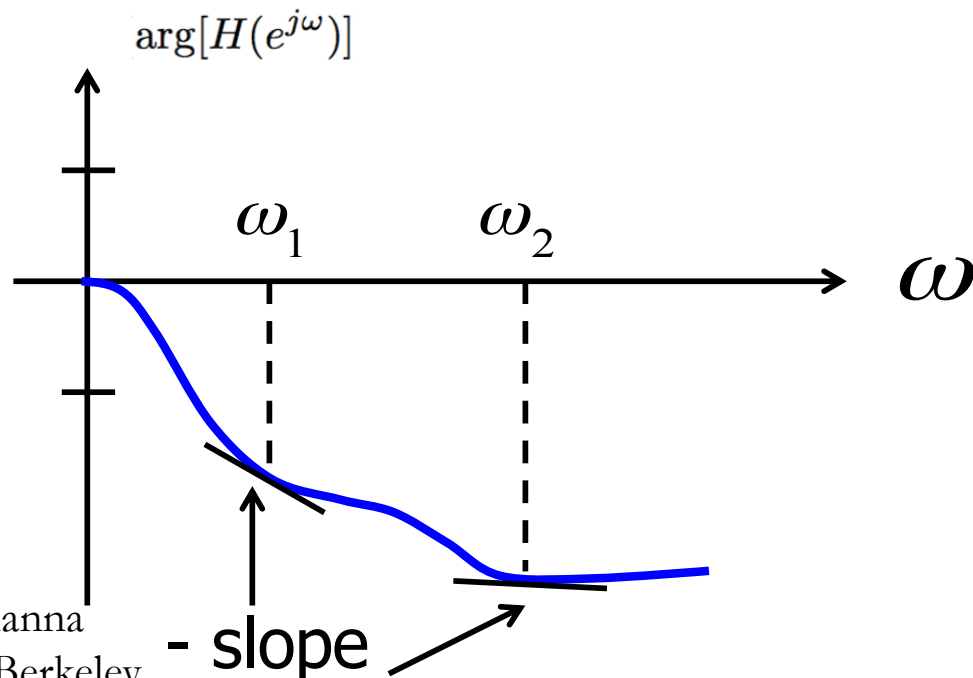
$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

For linear phase system, group delay is n_d

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

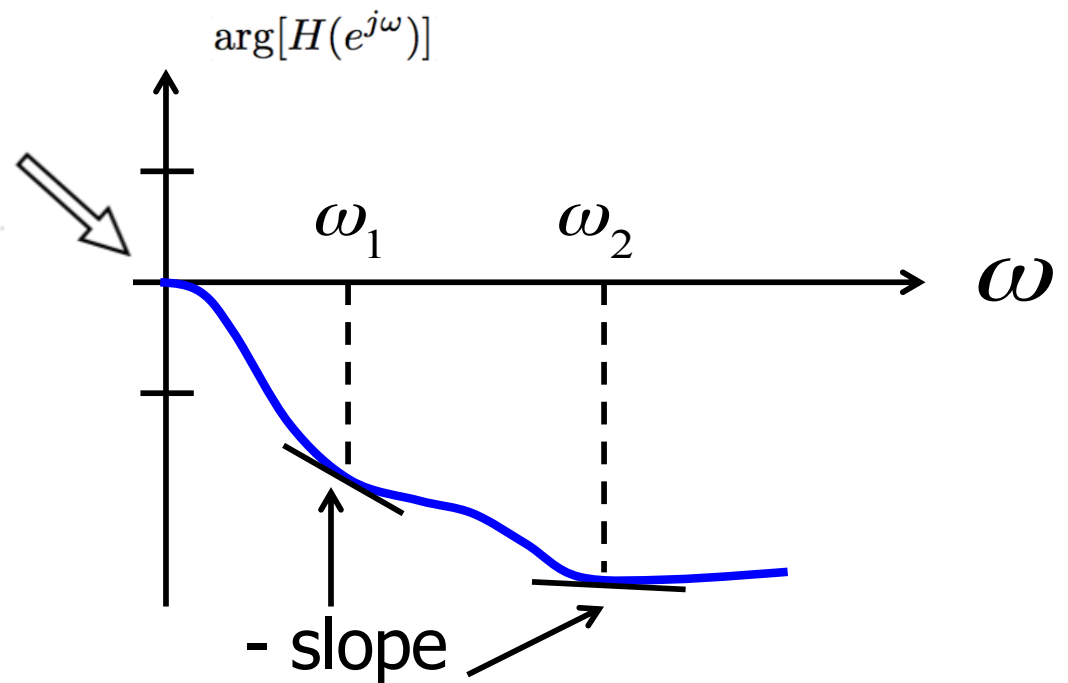
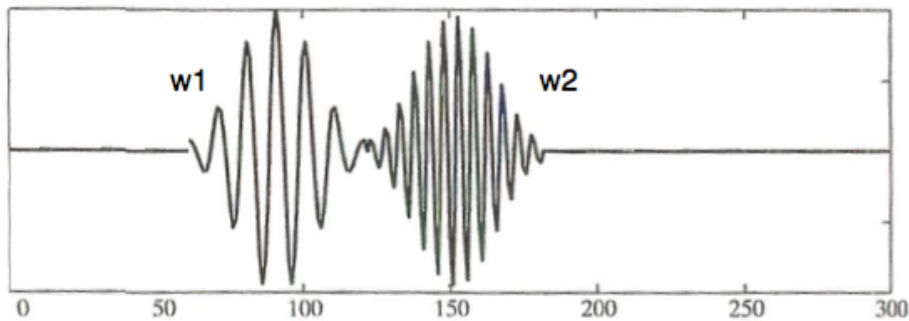
$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$



Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

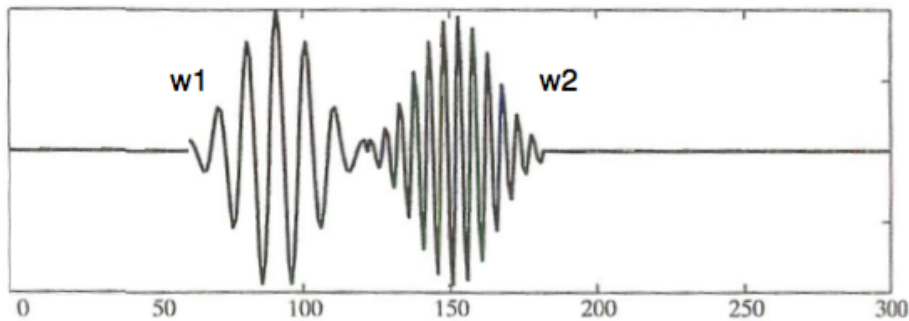
Input



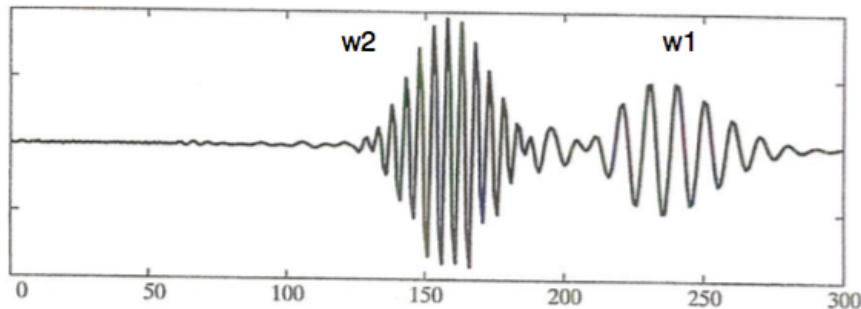
Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

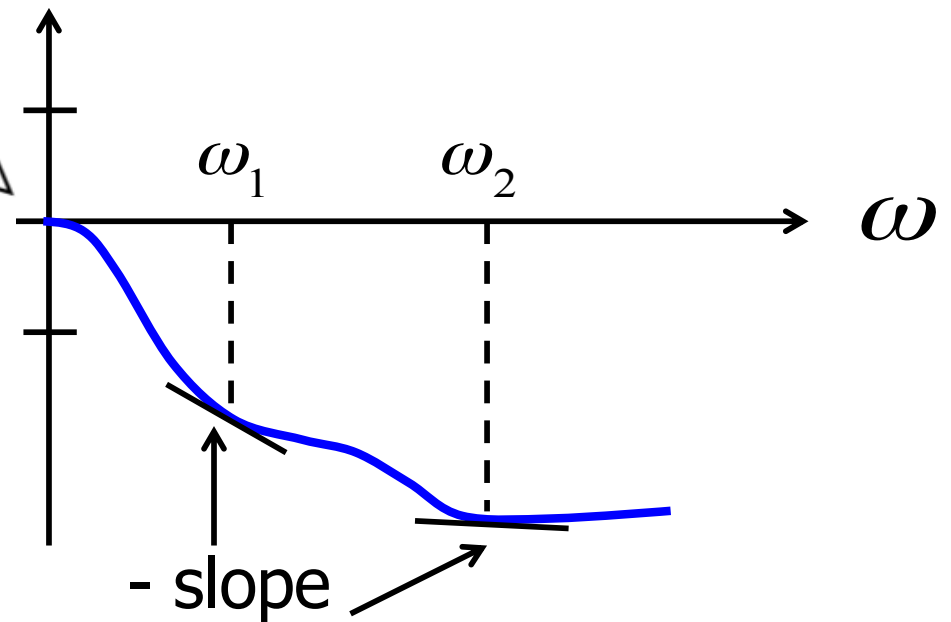
Input



Output



$\arg[H(e^{j\omega})]$





Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^M \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \arg[1 - d_k e^{-j\omega}]$$

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$



Group Delay Math

- Look at each factor:

$$[1 - re^{j\theta} e^{-j\omega}] =$$



Group Delay Math

□ Look at each factor:

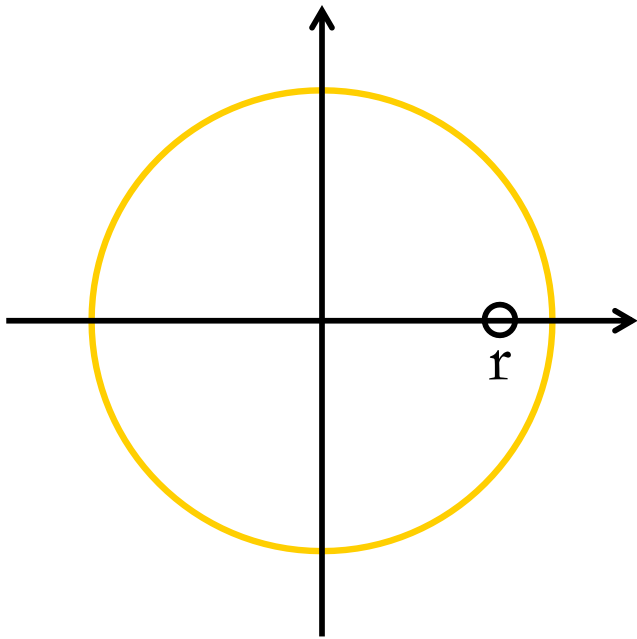
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

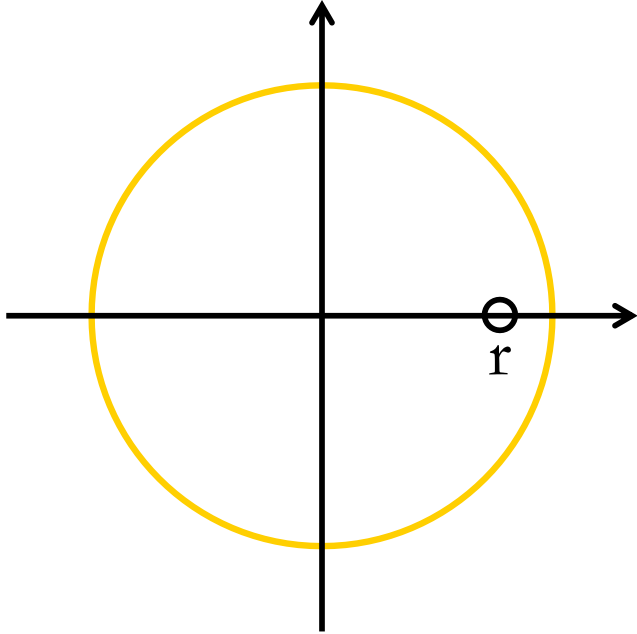
$$\arg[1 - re^{-j\omega}]$$



Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

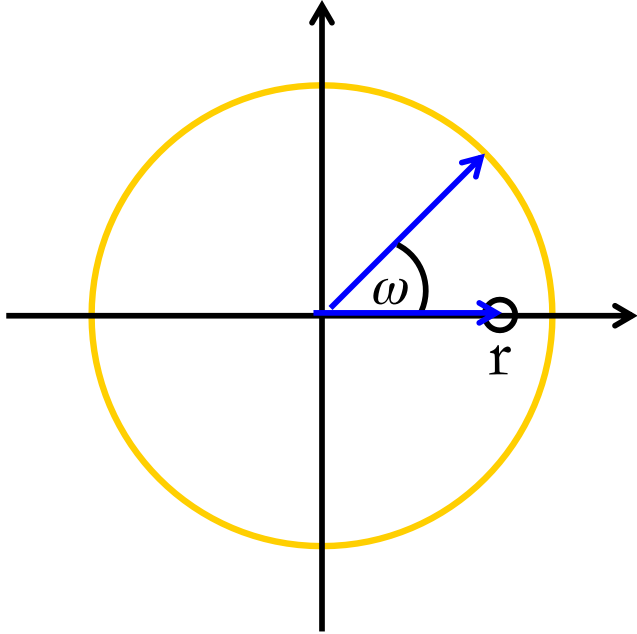




Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

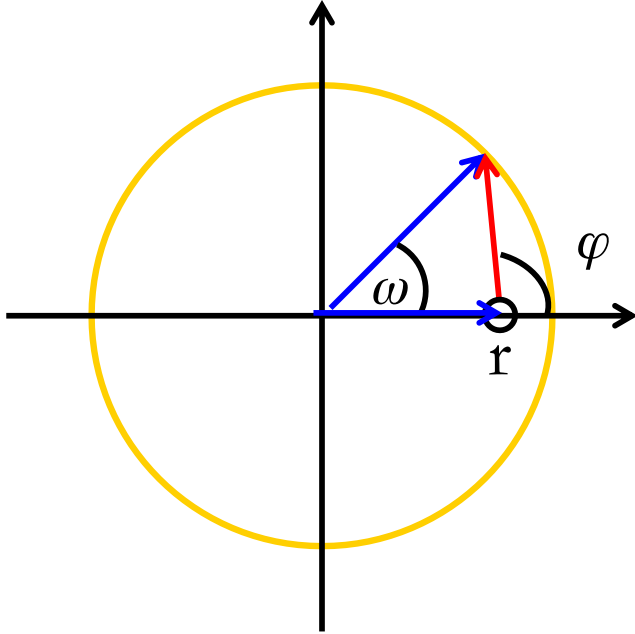




Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

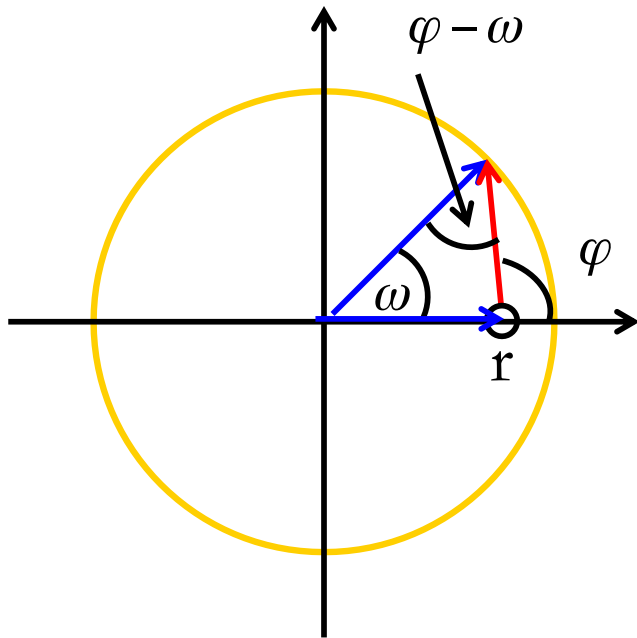
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

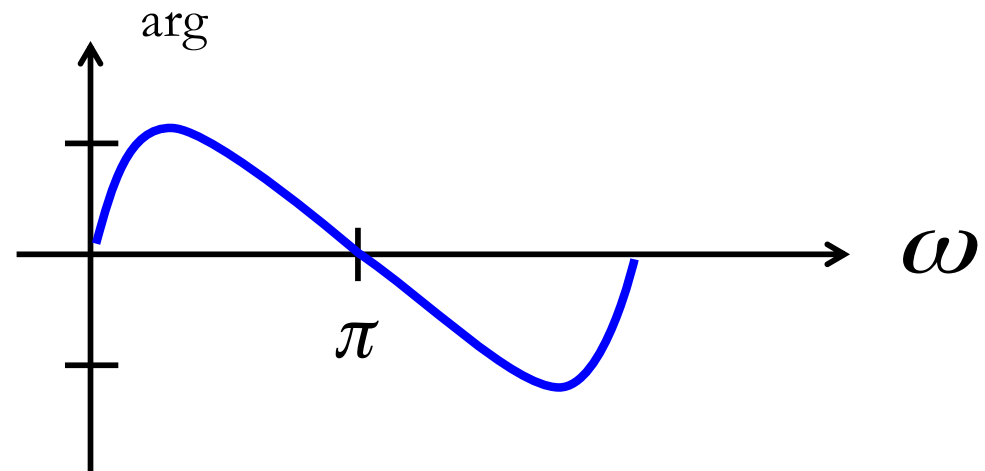
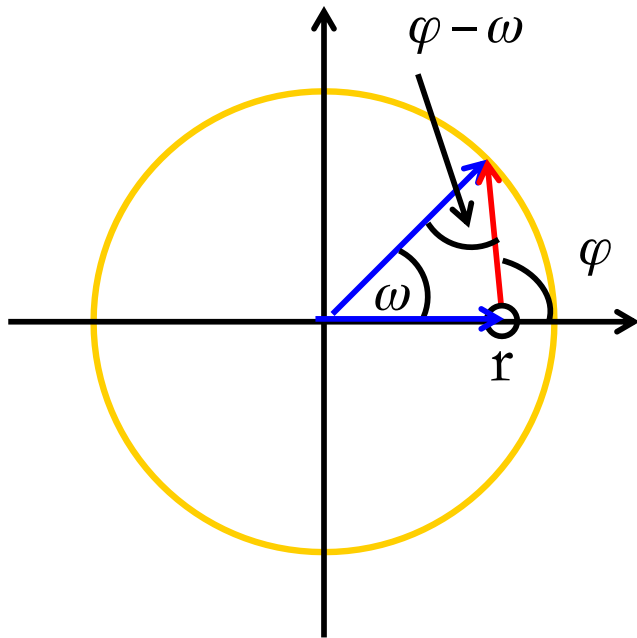
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

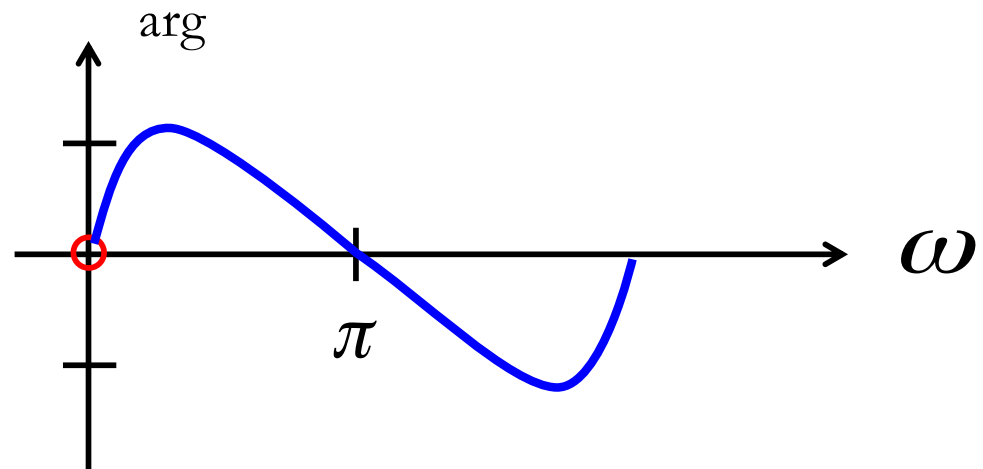
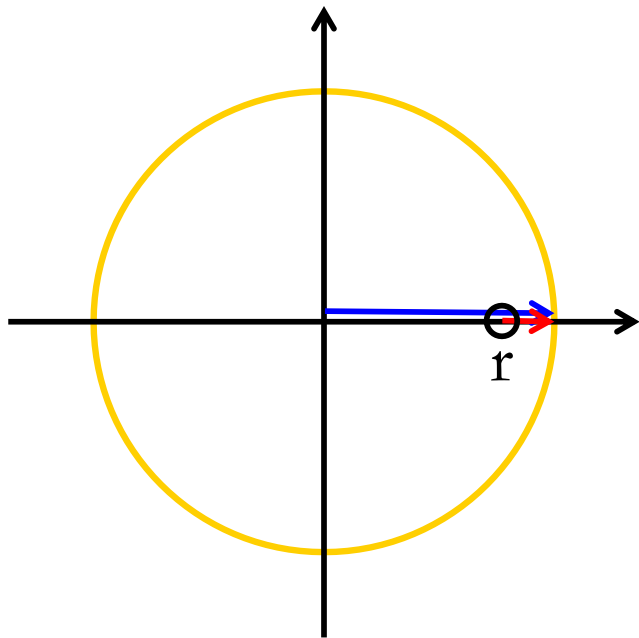


Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

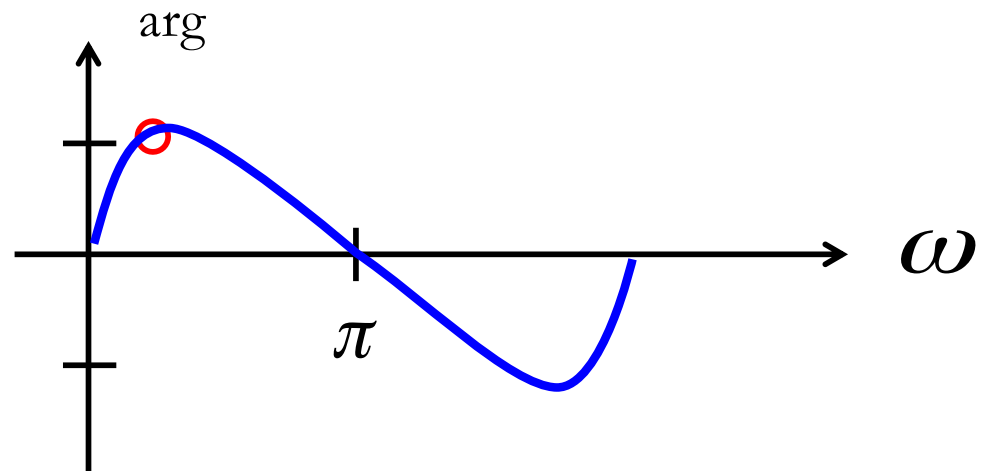
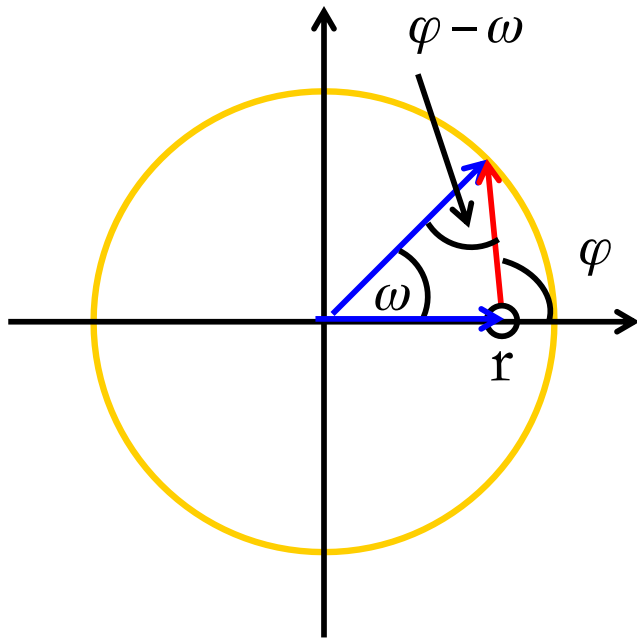
$\omega = 0$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

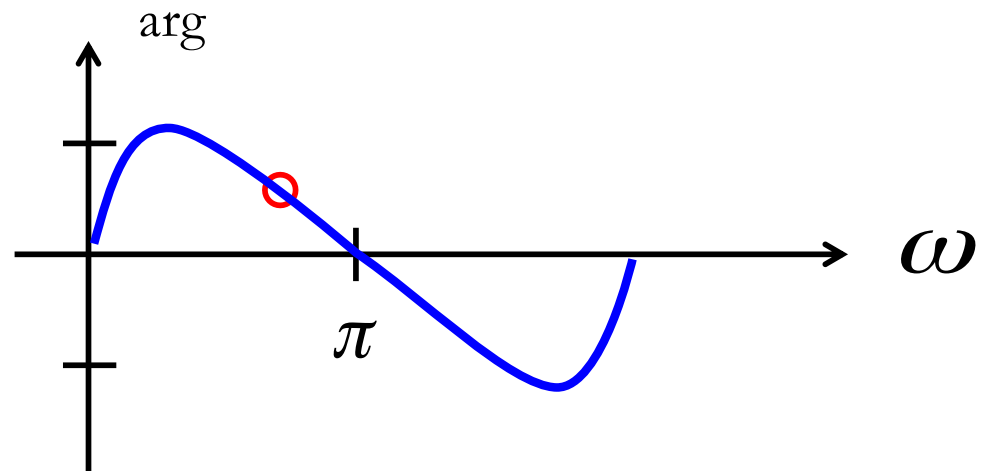
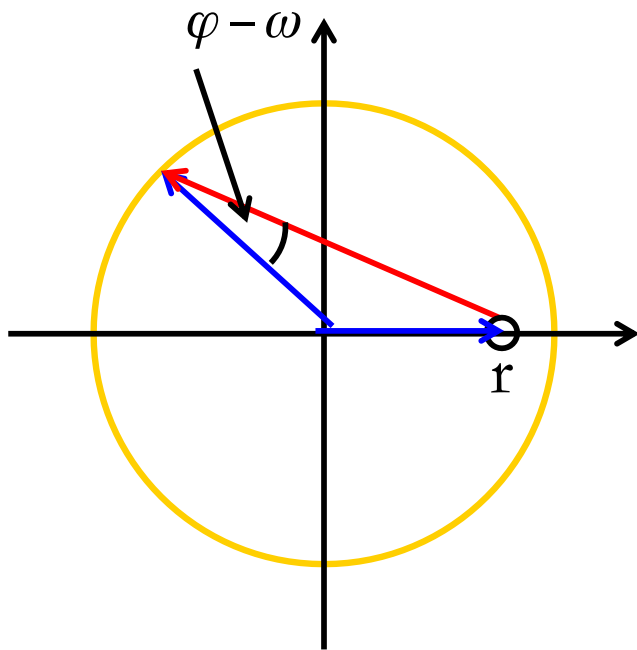
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

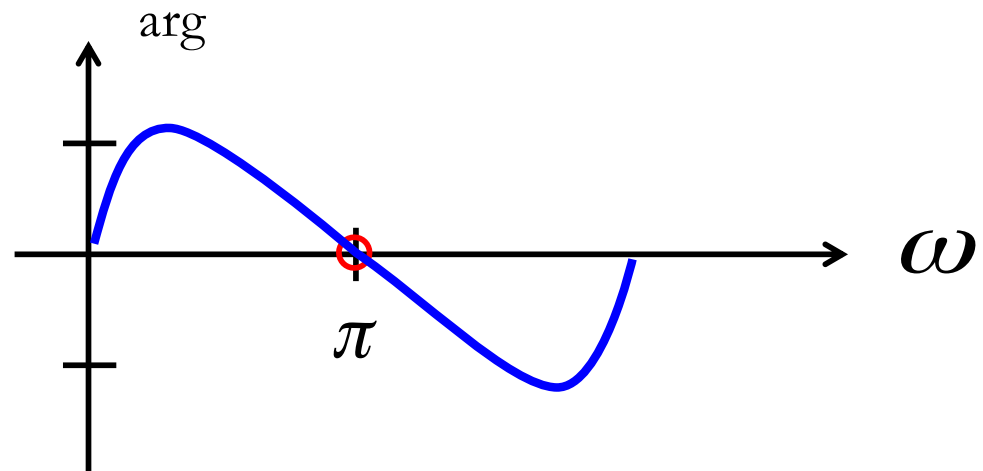
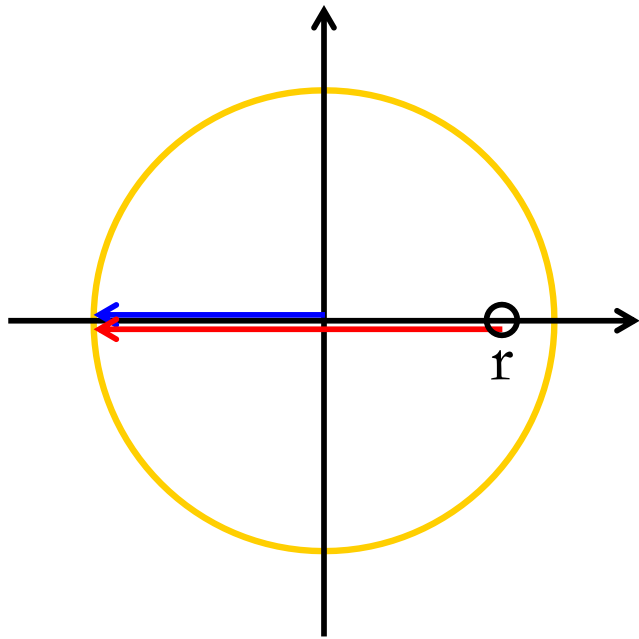


Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

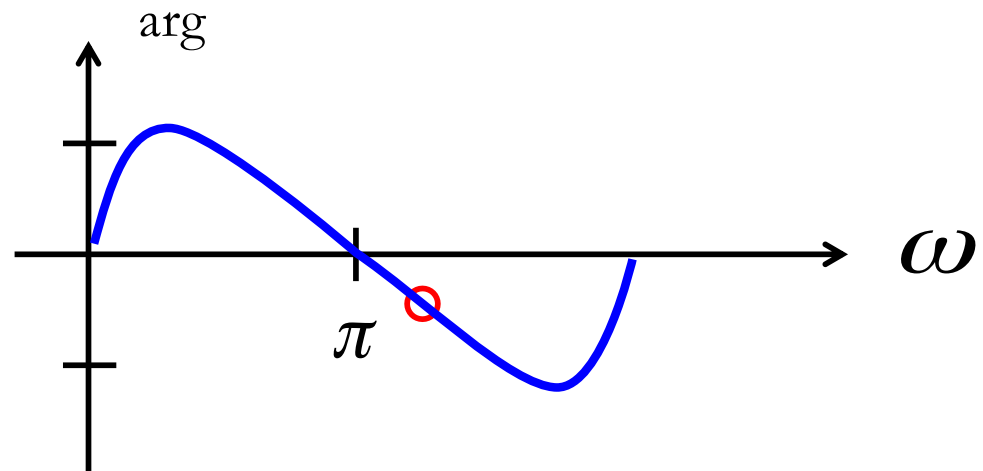
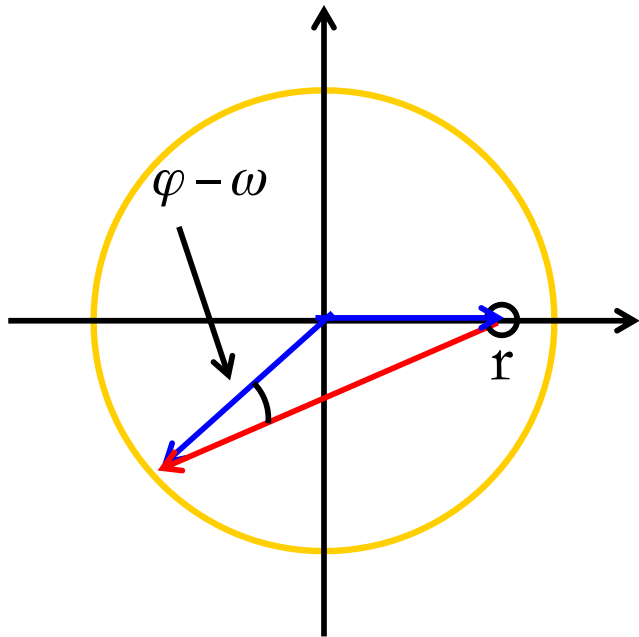
$$\omega = \pi$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

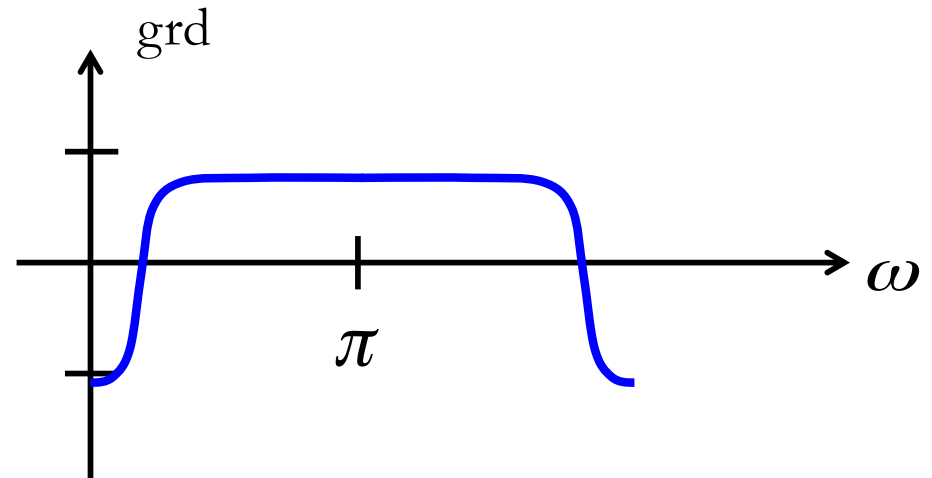
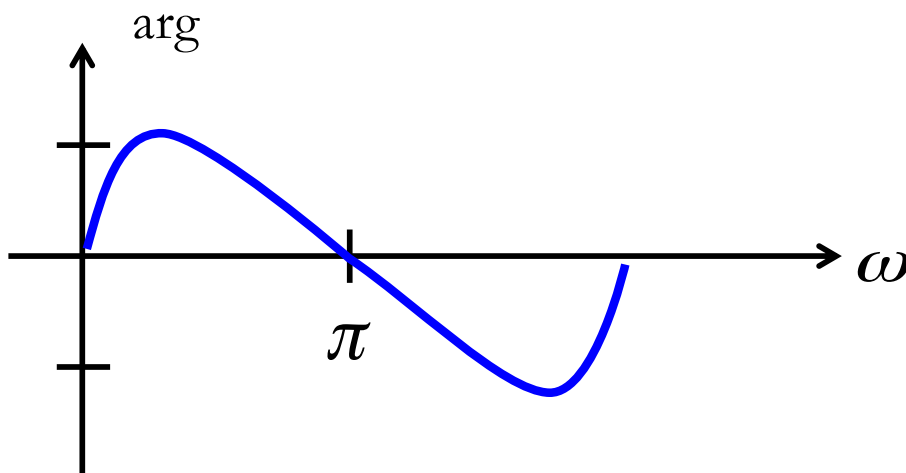
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

□ Look at each factor:

$\theta \neq 0?$

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1} \left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

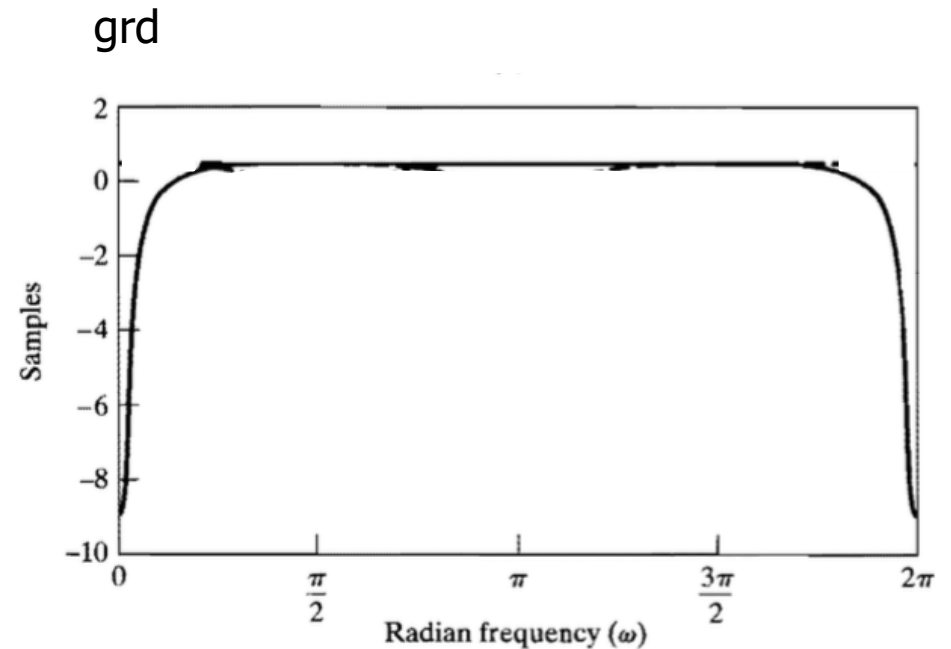
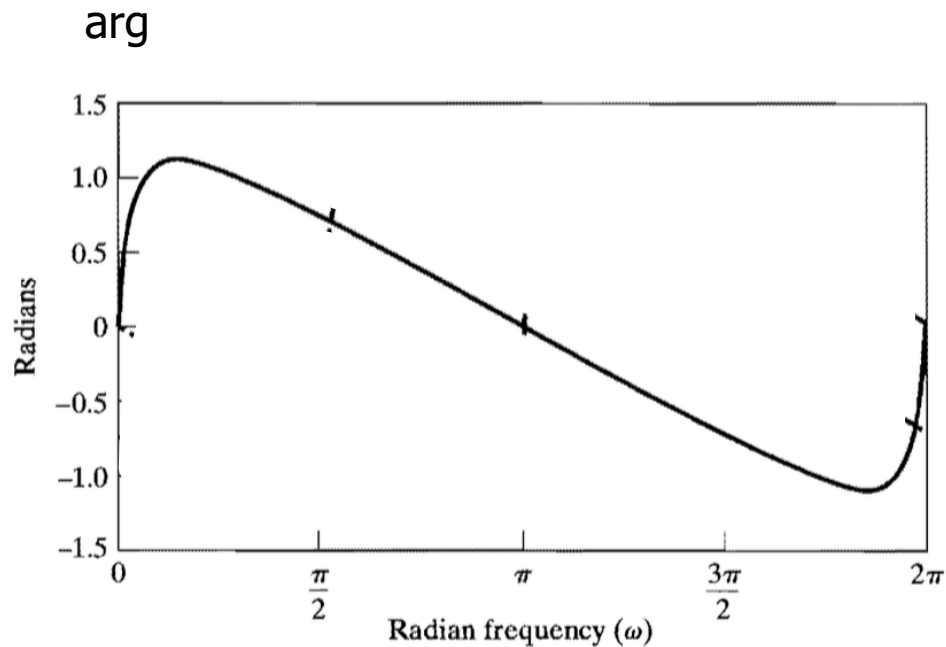
$$\text{grd}[1 - re^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta} e^{-j\omega}|^2}$$



Example: Zero on Real Axis

□ For $\theta=0$

— $\theta = 0$



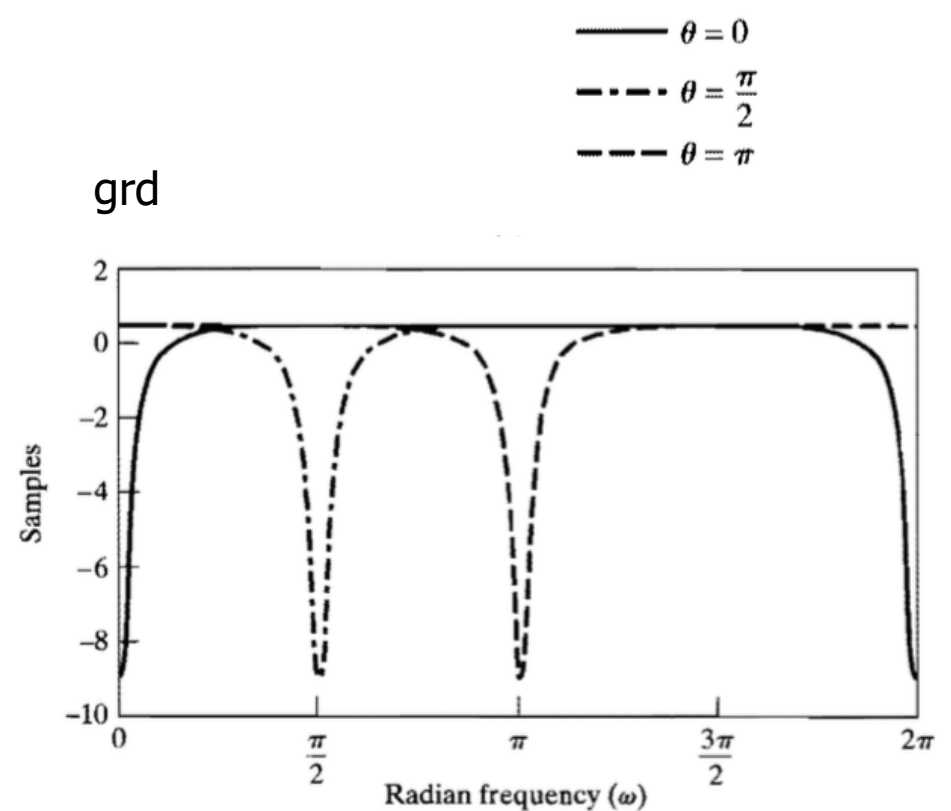
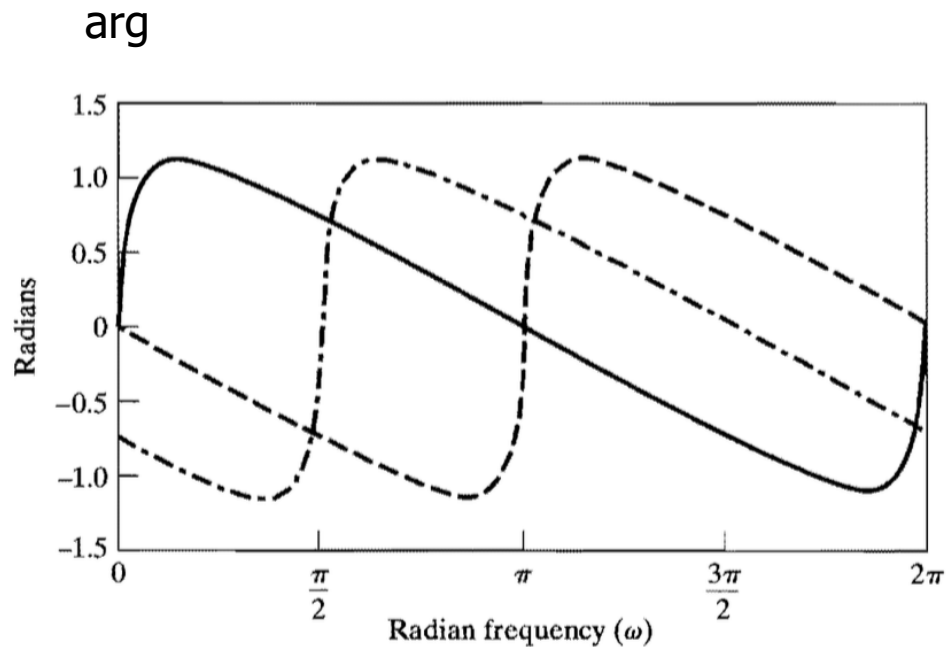
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$



Example: Zero on Real Axis

□ For $\theta \neq 0$

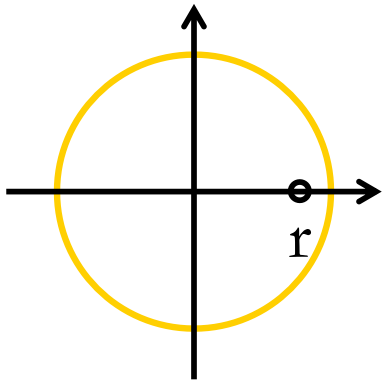
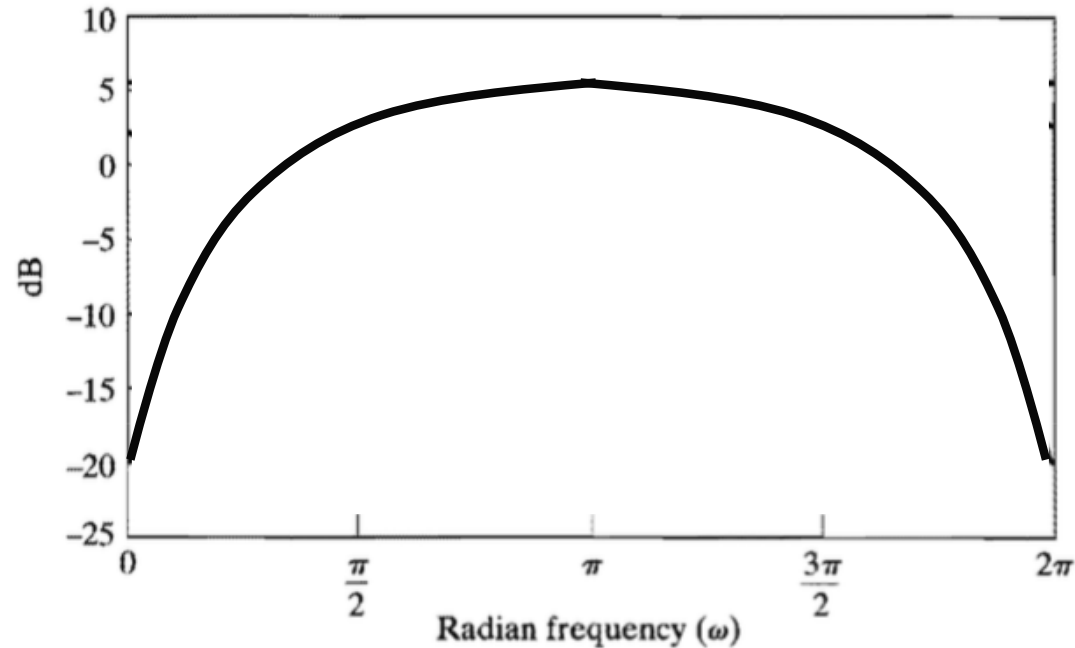


Example: Zero on Real Axis

□ Magnitude Response

— $\theta = 0$

$$1 - re^{j\theta} e^{-j\omega} = 1 - re^{-j\omega}$$

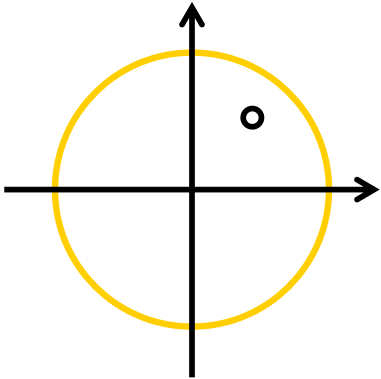
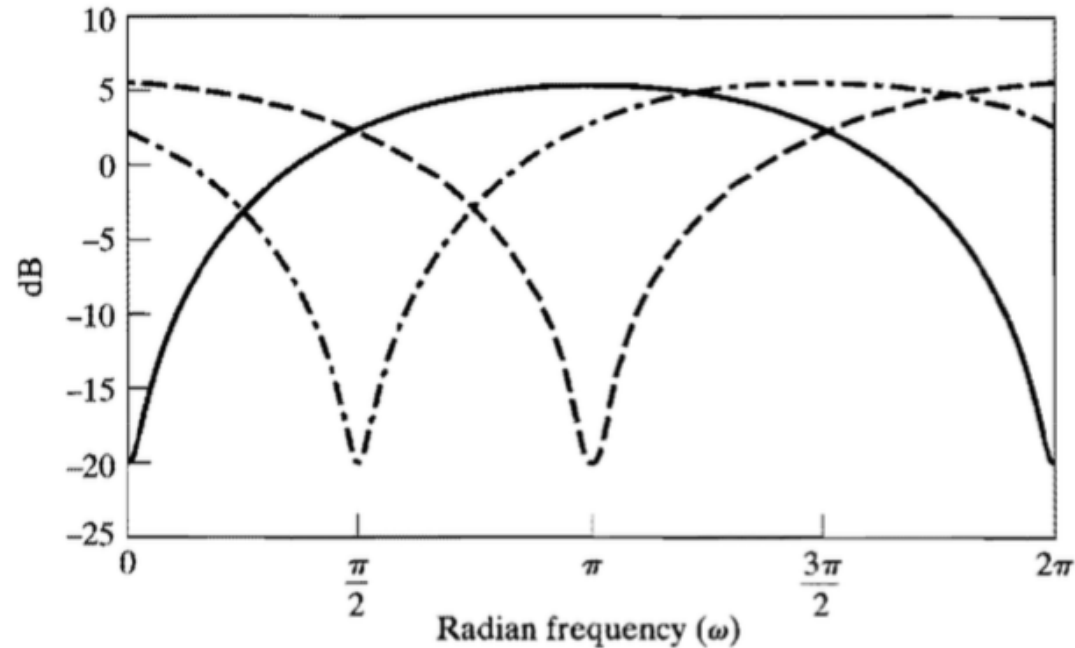


Example: Zero on Real Axis

□ Magnitude Response

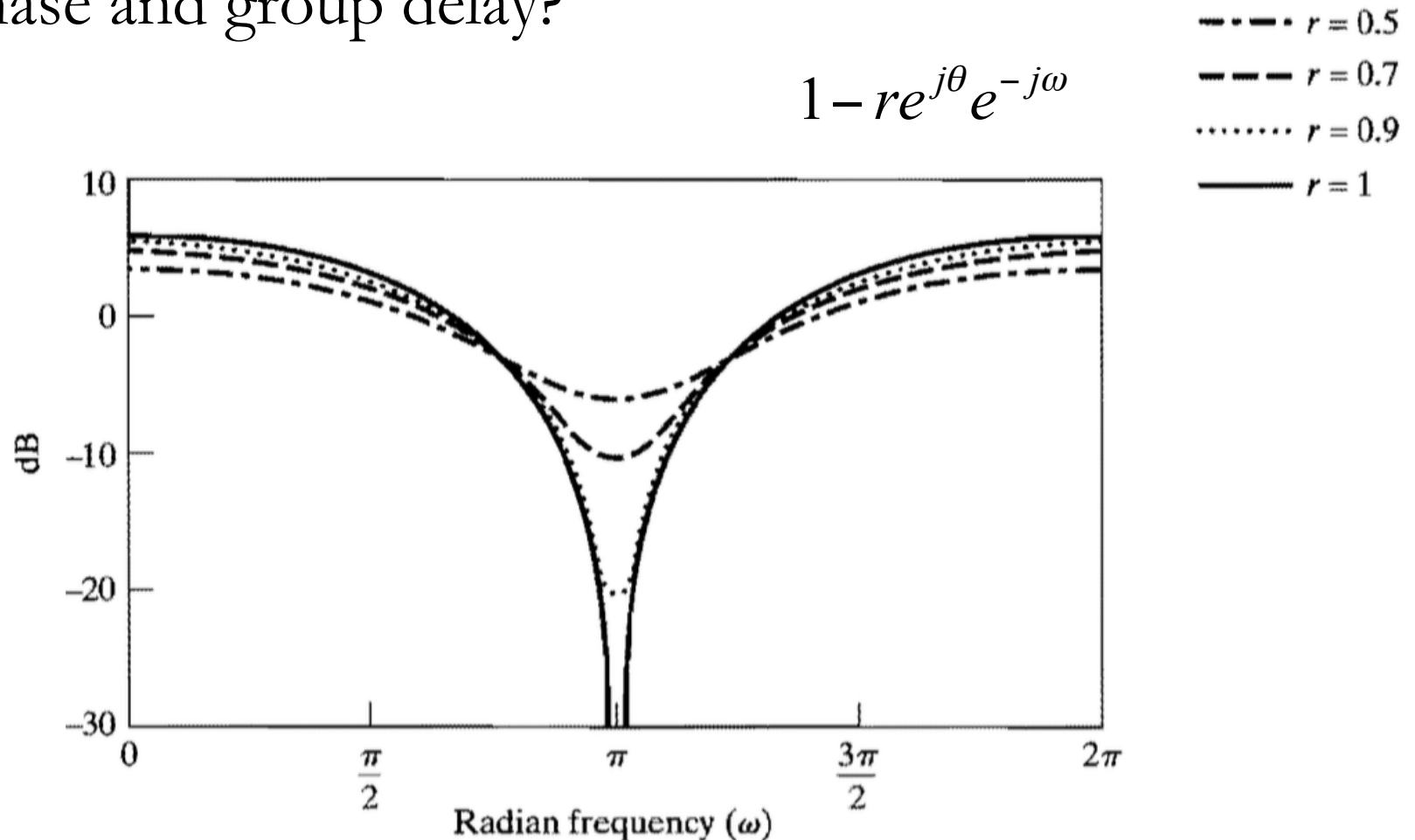
$$1 - re^{j\theta}e^{-j\omega}$$

— $\theta = 0$
- · - $\theta = \frac{\pi}{2}$
- - - $\theta = \pi$



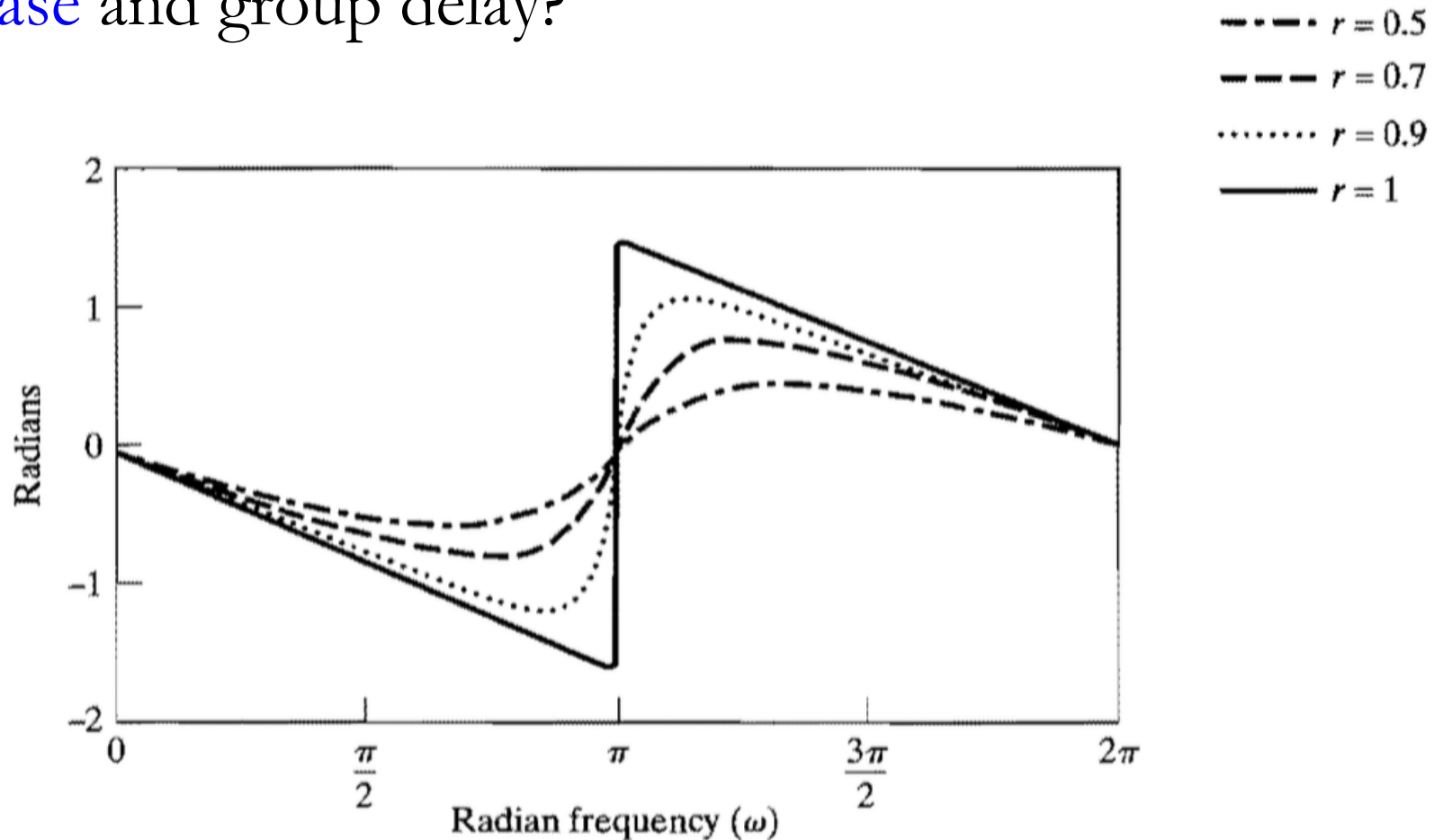
Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, phase and group delay?



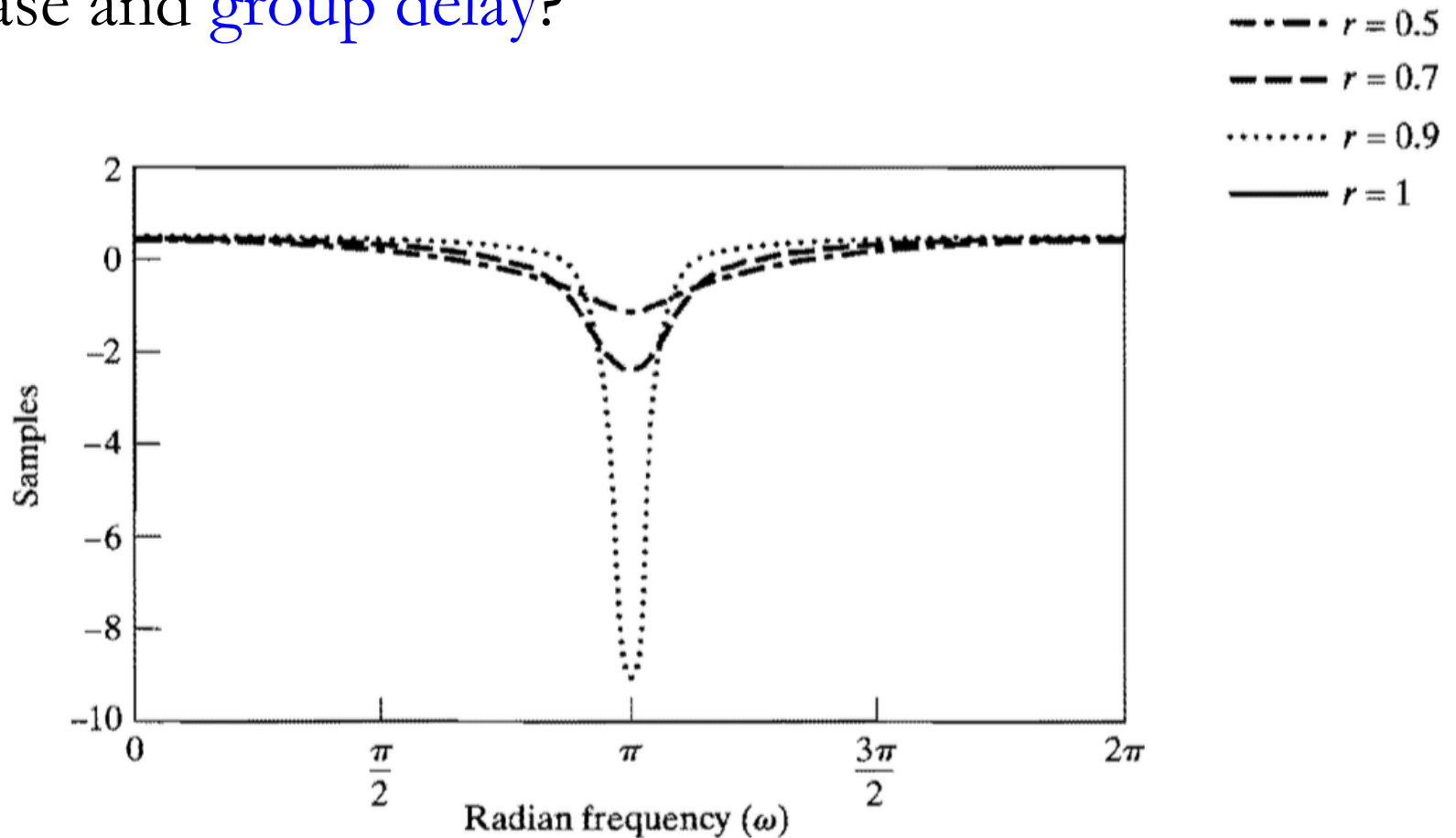
Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, **phase** and group delay?



Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, phase and group delay?

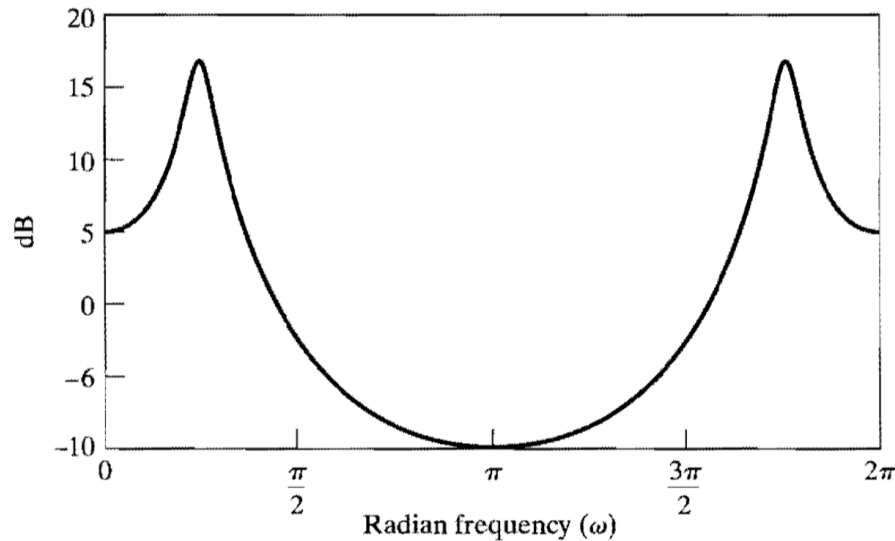


2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$$r=0.9, \theta=\pi/4$$

magnitude



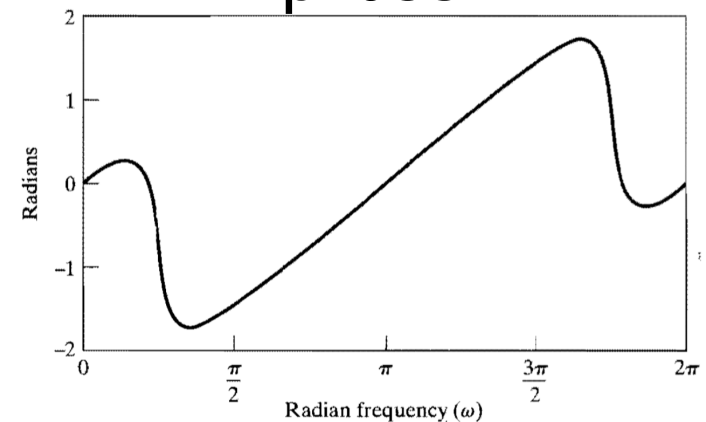
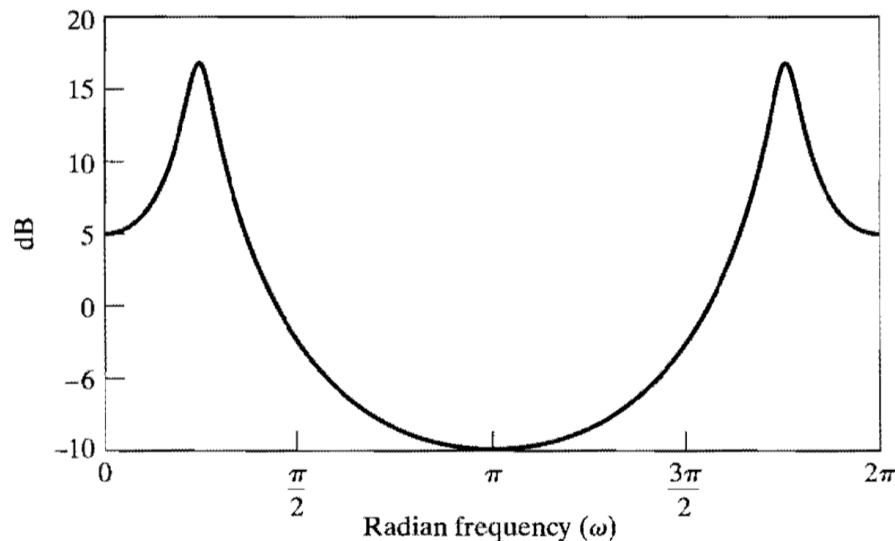
2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

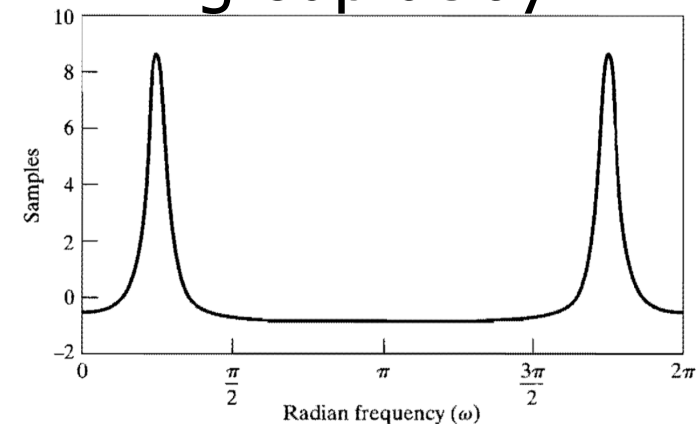
$$r=0.9, \theta=\pi/4$$

phase

magnitude



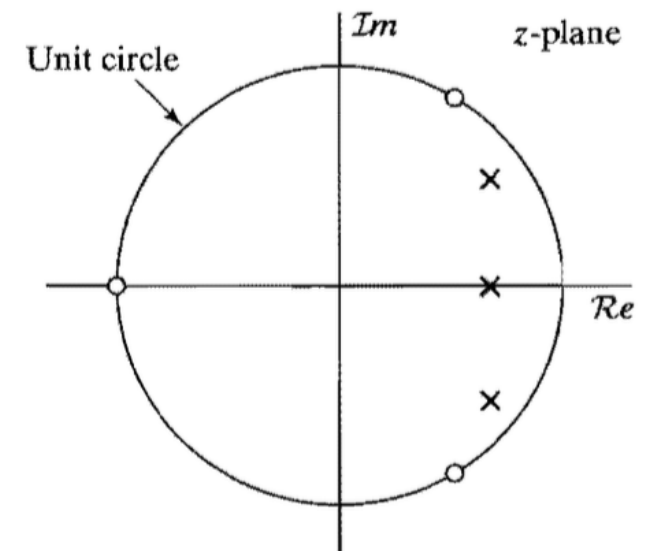
group delay





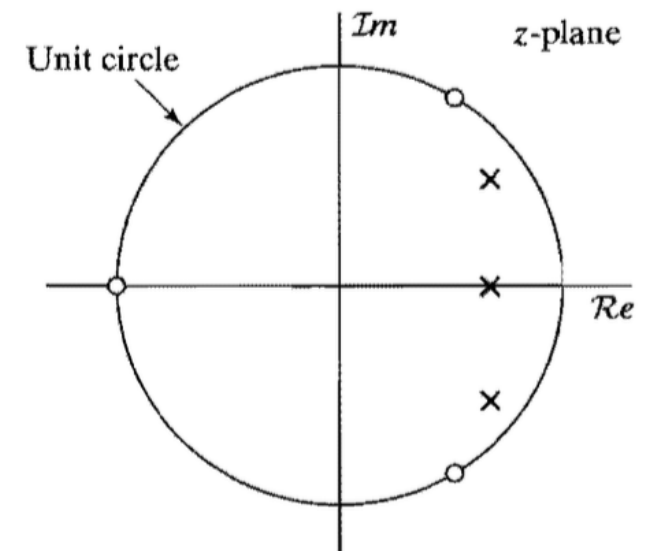
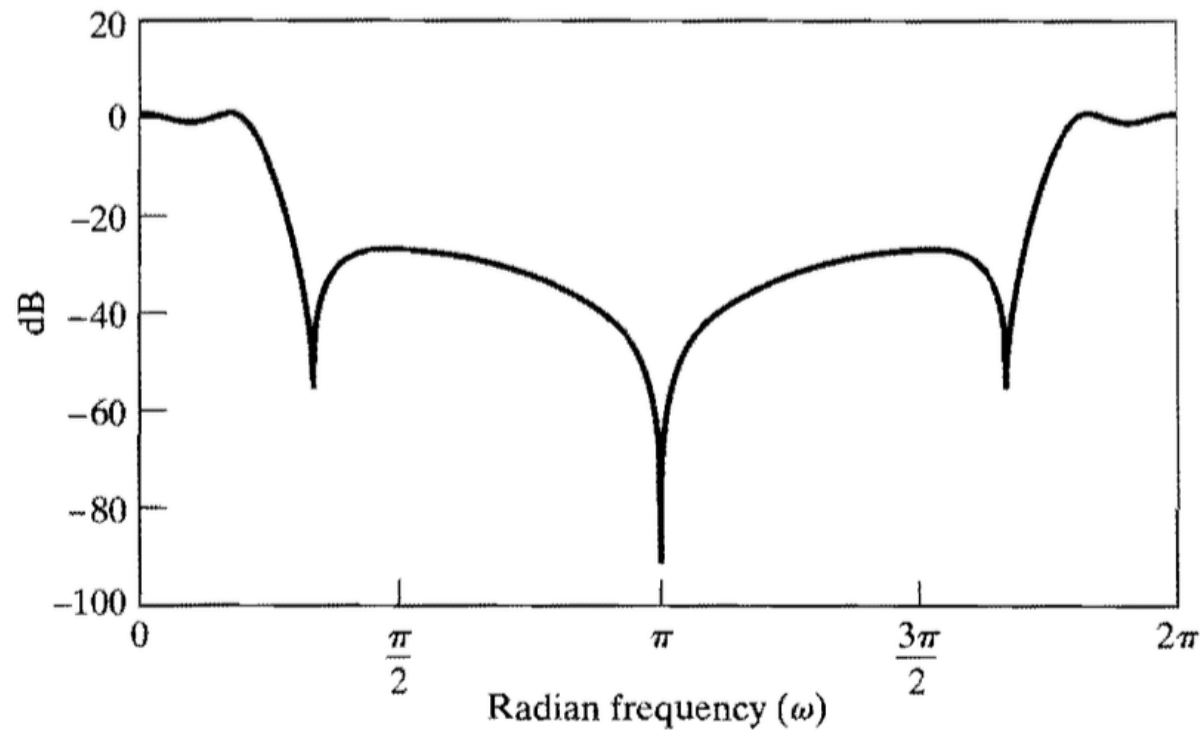
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



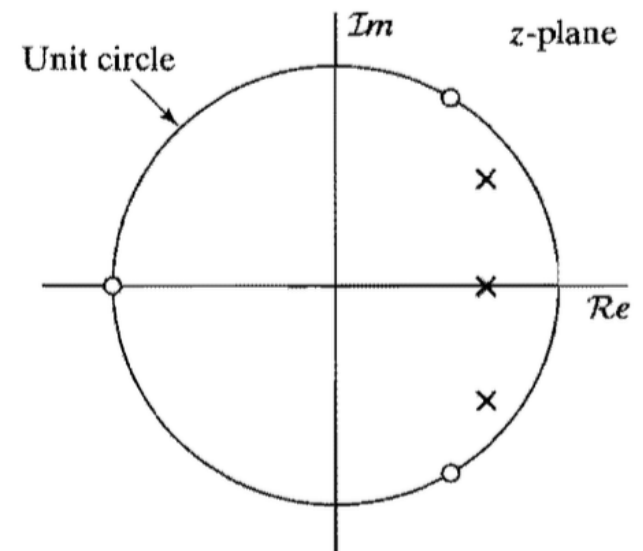
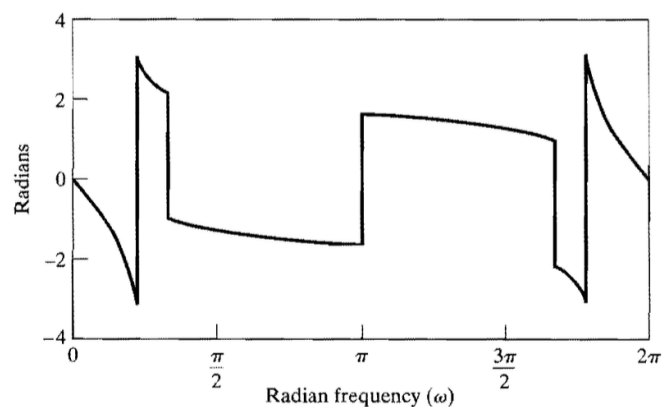
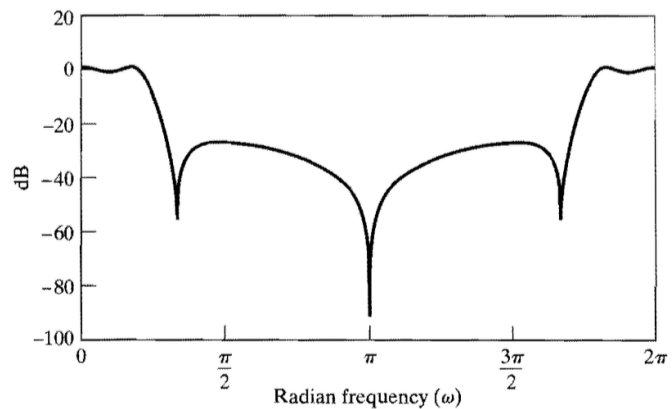
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



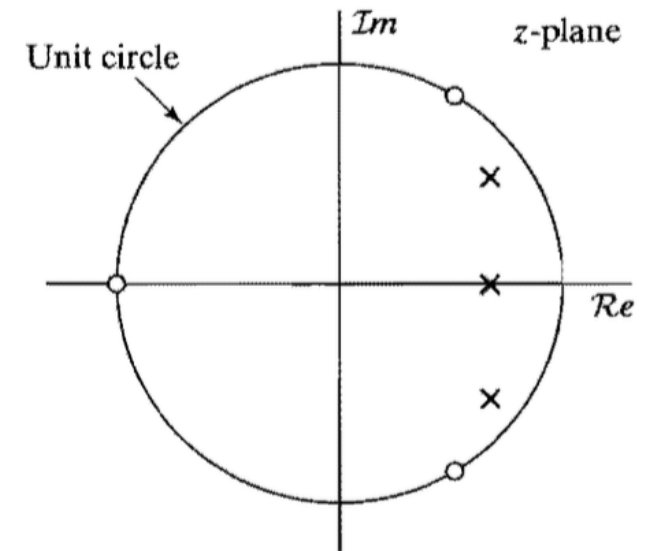
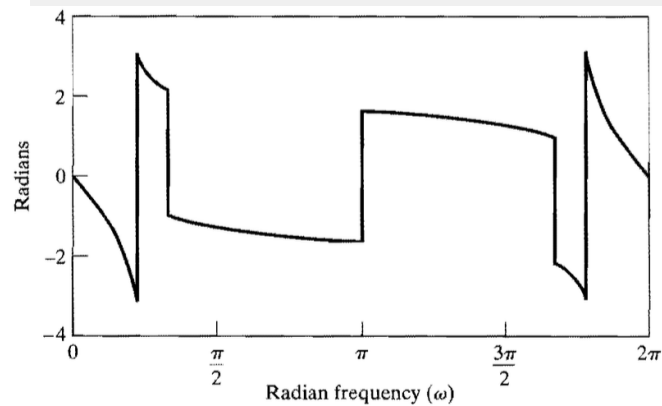
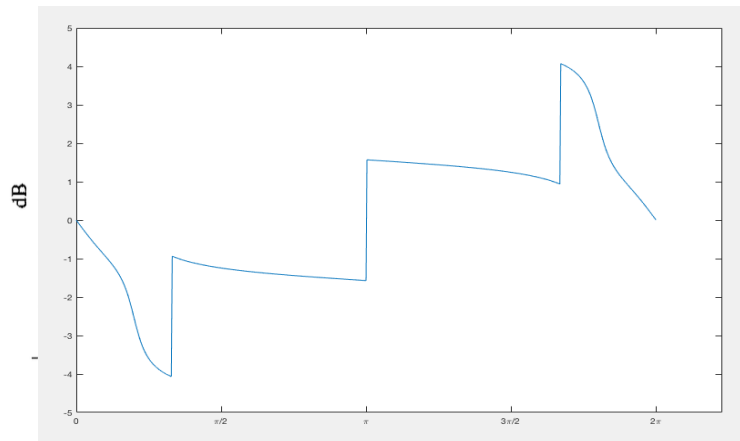
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



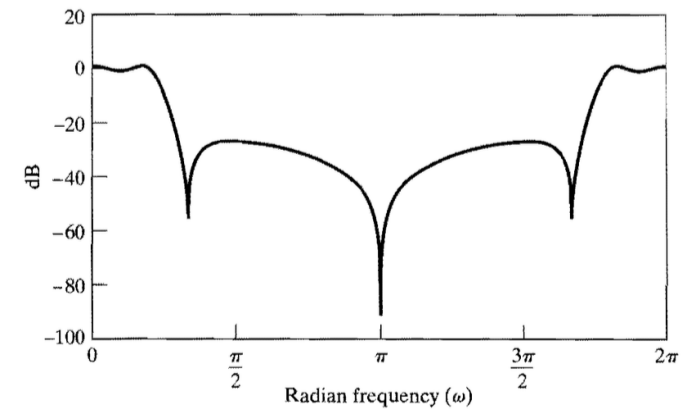
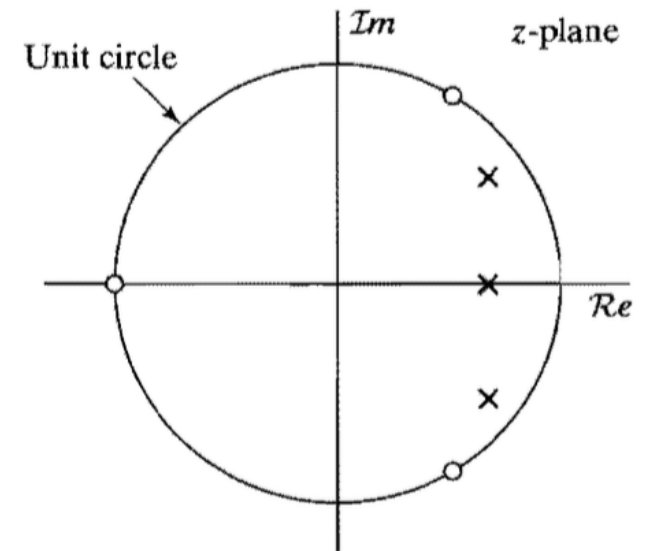
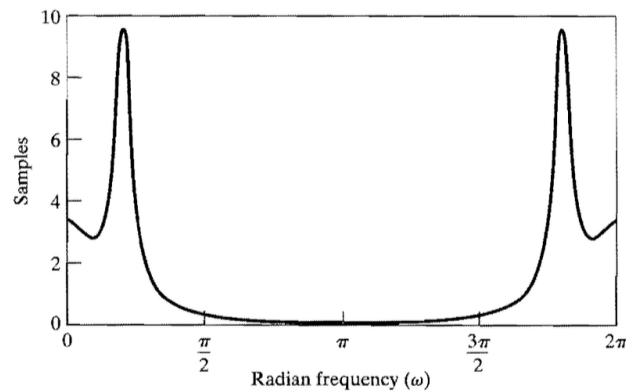
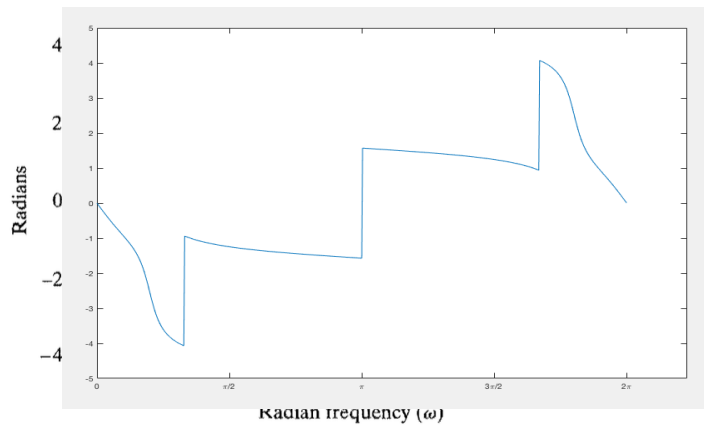
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$





Big Ideas

- Frequency Response of LTI Systems
 - Magnitude Response
 - Simple Filters
 - Phase Response
 - Group Delay
 - Example: Zero on Real Axis
 - Example: Higher order systems



Admin

- ❑ HW 5 due Sunday 2/27