

ESE 531: Digital Signal Processing

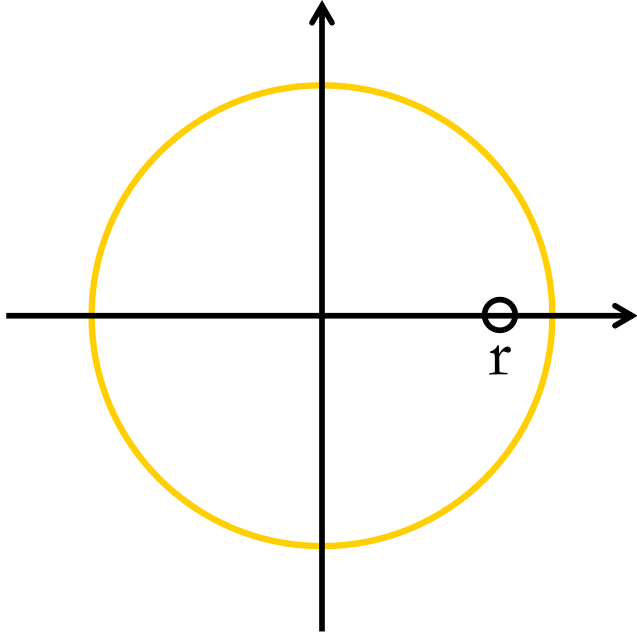
Lecture 14: March 1, 2022

All-Pass Systems, Min Phase Decomposition

Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

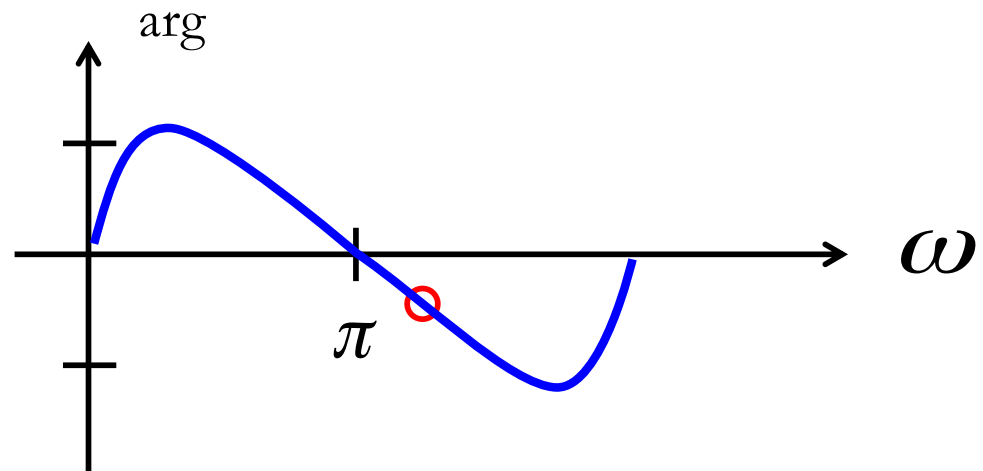
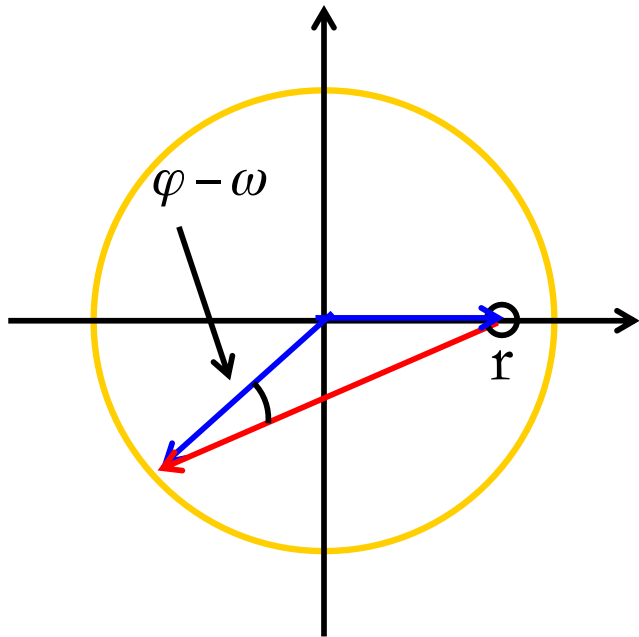
$$\arg[1 - re^{-j\omega}]$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

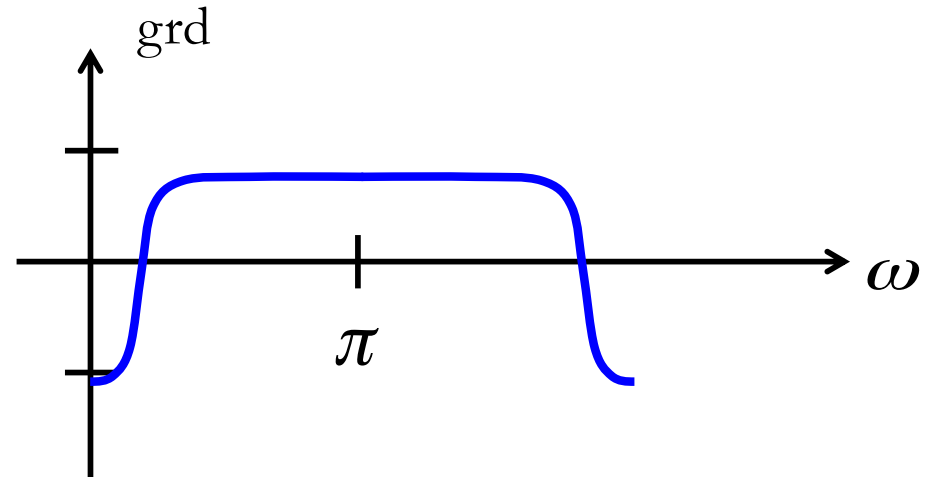
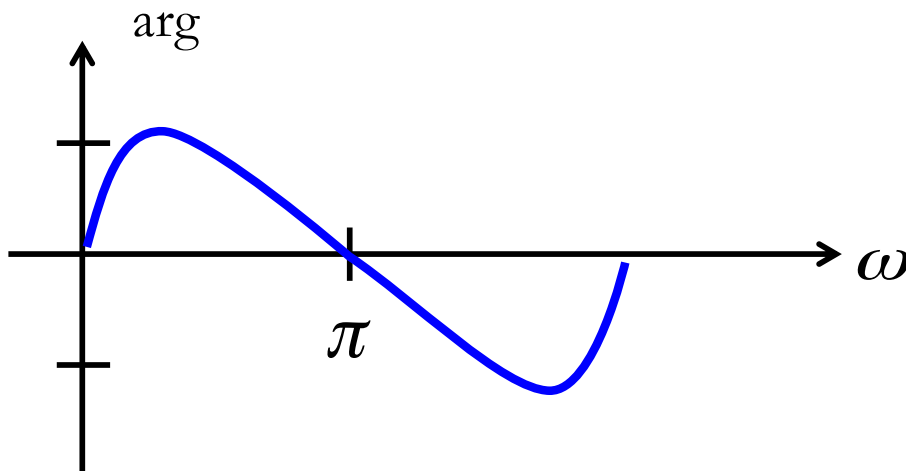
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



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Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

□ Look at each factor:

$\theta \neq 0?$

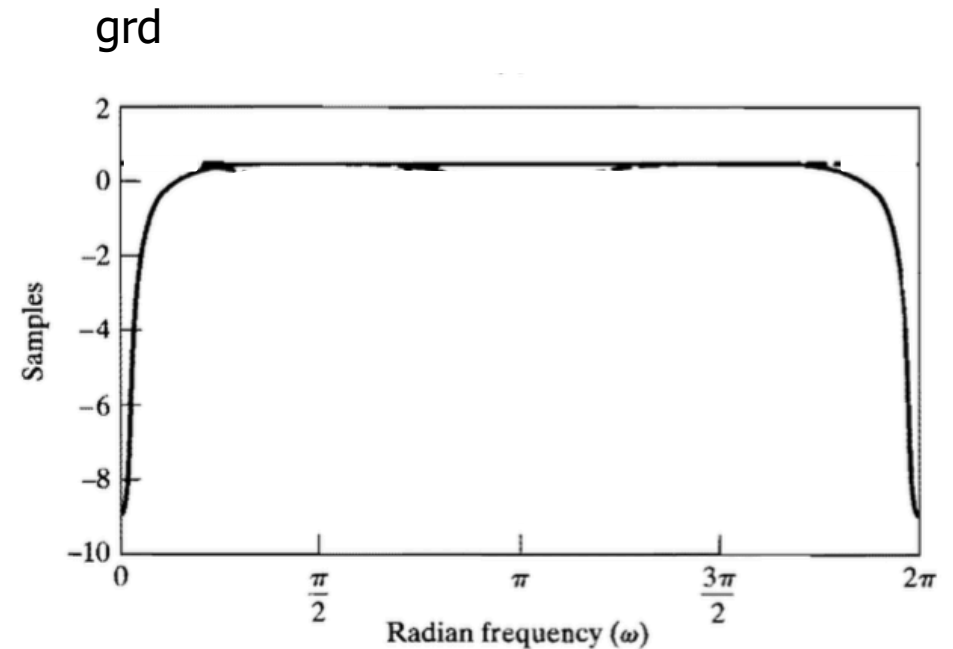
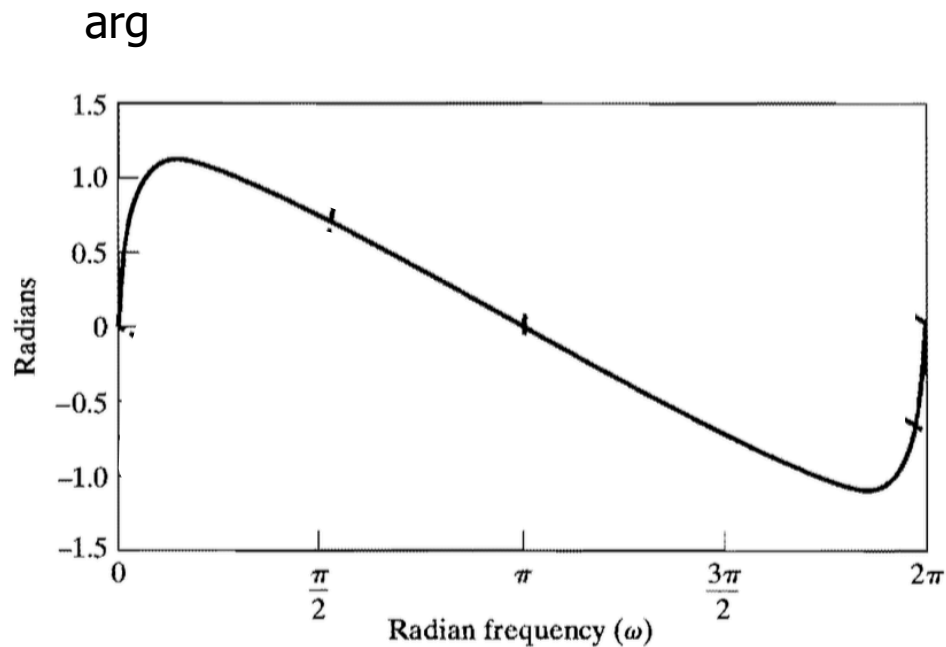
$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1} \left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

$$\text{grd}[1 - re^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta} e^{-j\omega}|^2}$$

Example: Zero on Real Axis

□ For $\theta=0$

— $\theta = 0$

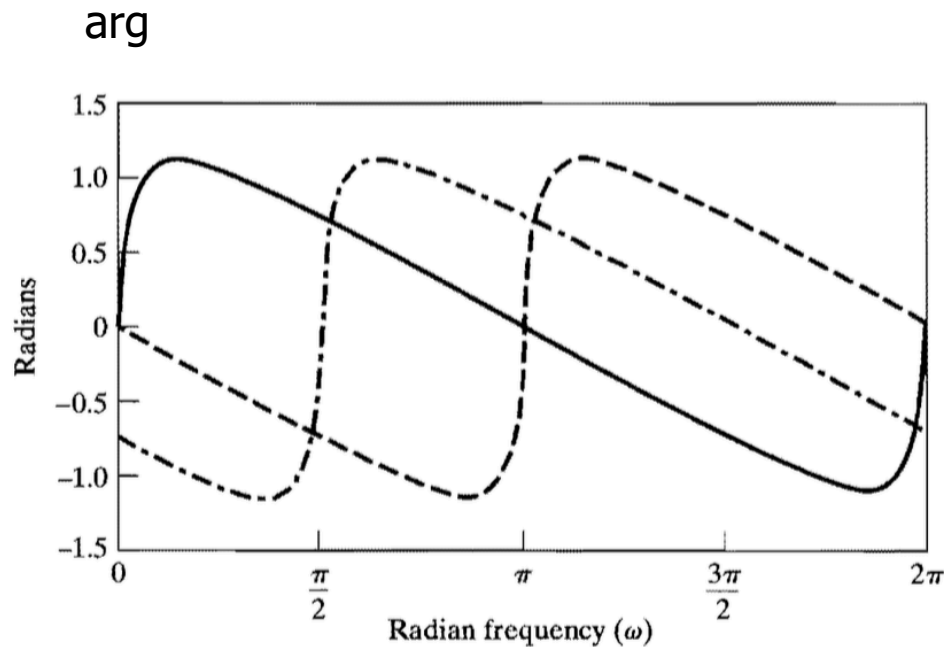


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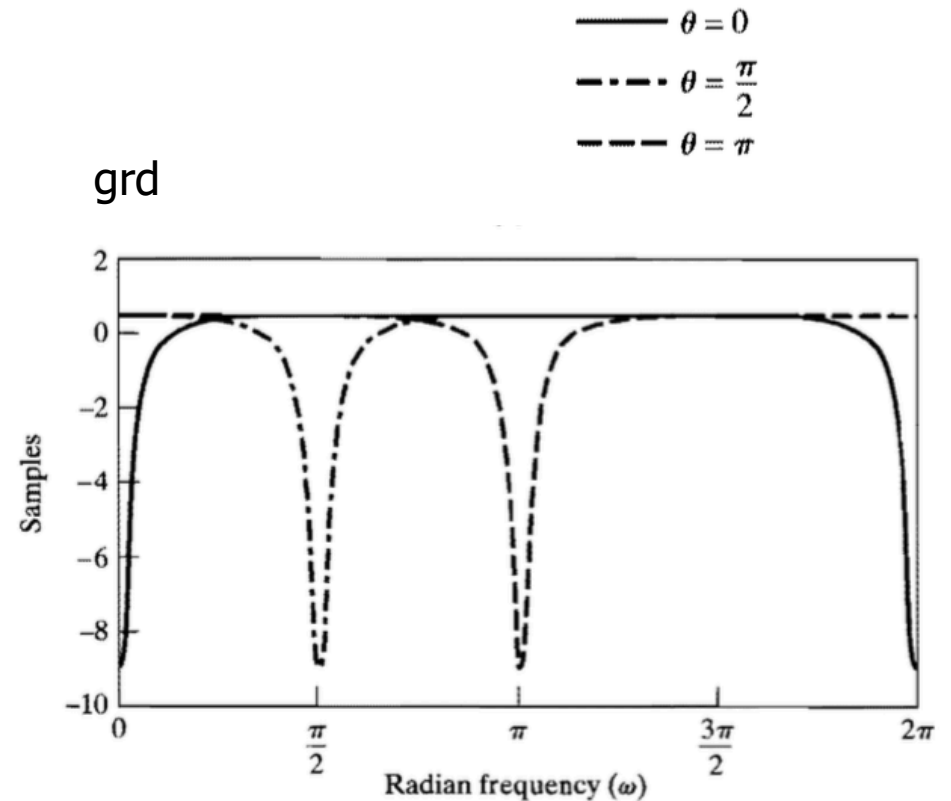
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Example: Zero on Real Axis

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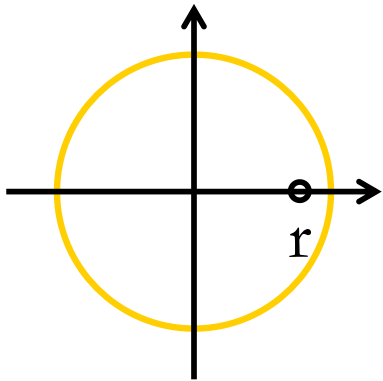
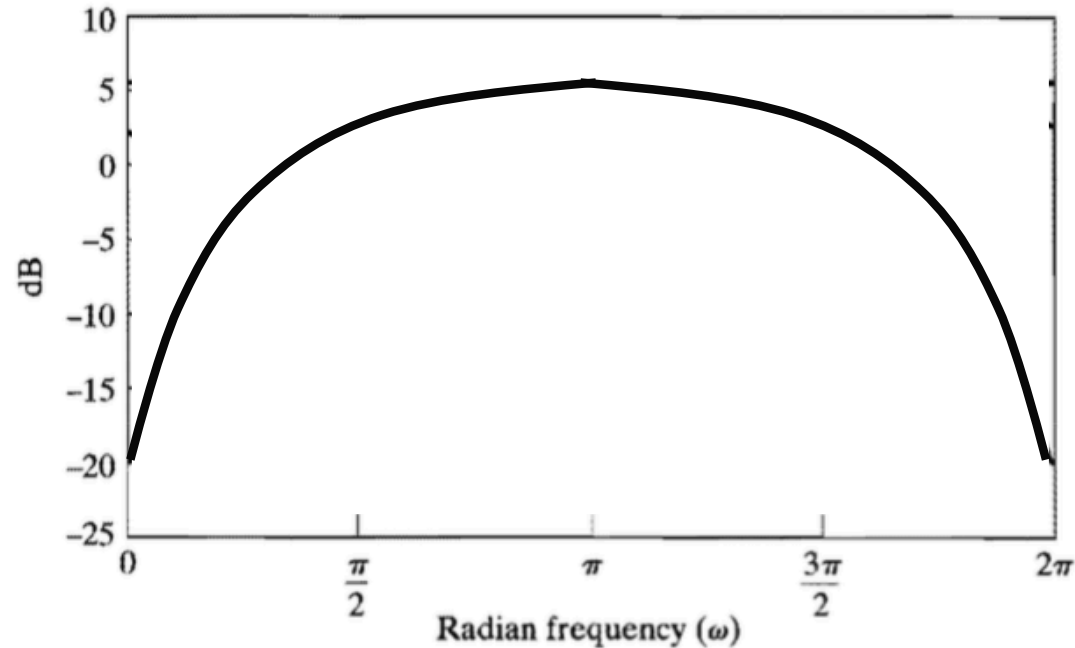
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Example: Zero on Real Axis

□ Magnitude Response

— $\theta = 0$

$$1 - re^{j\theta} e^{-j\omega} = 1 - re^{-j\omega}$$

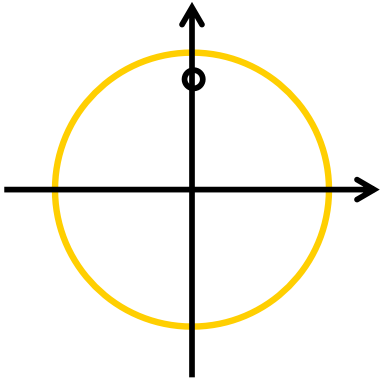
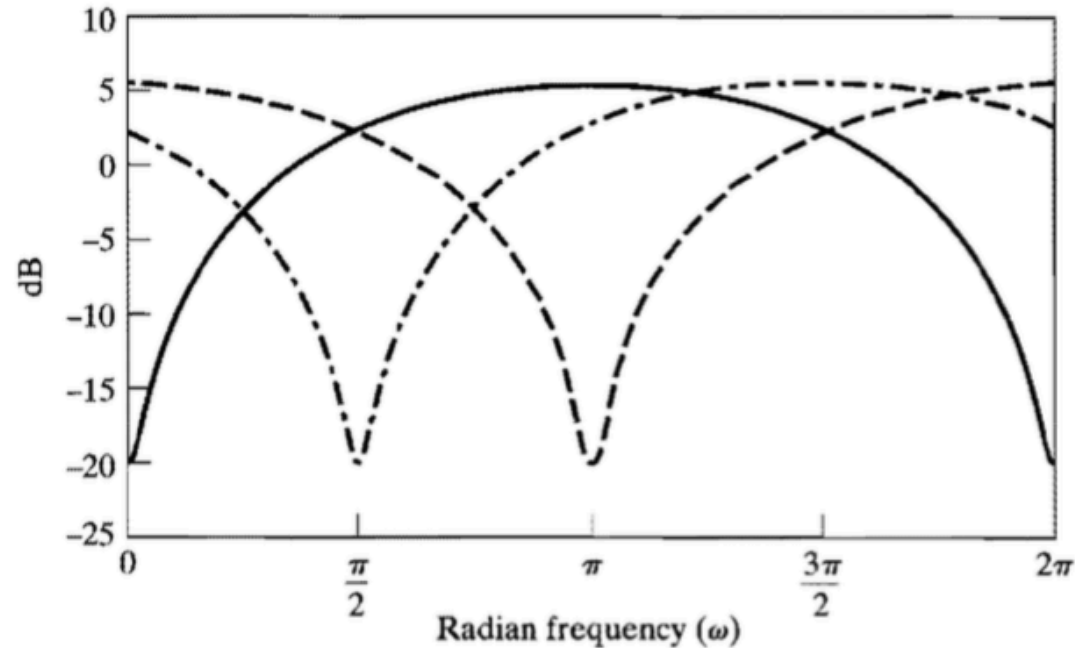


Example: Zero on Real Axis

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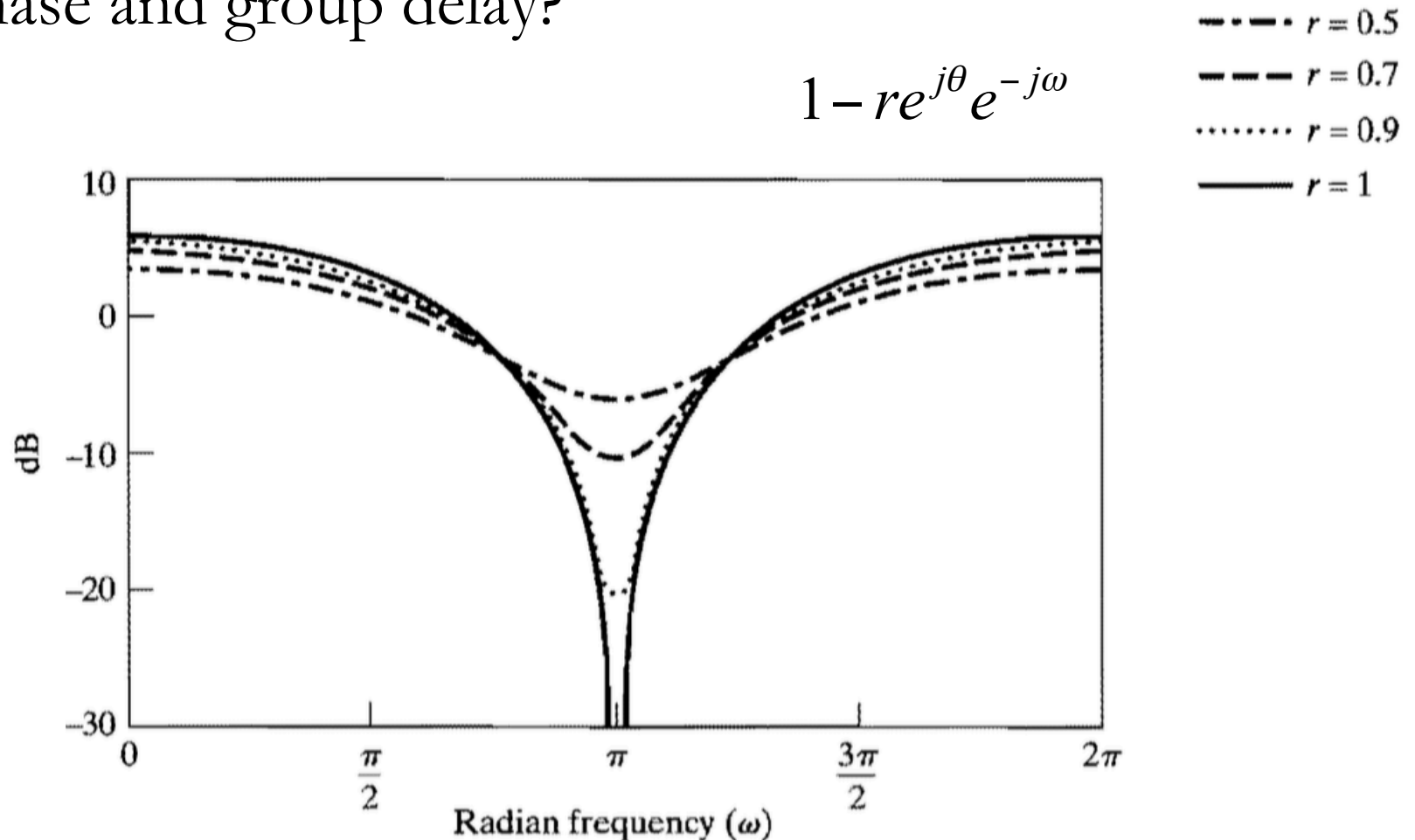
$$1 - re^{j\theta} e^{-j\omega}$$

— $\theta = 0$
- · - $\theta = \frac{\pi}{2}$
- - - $\theta = \pi$



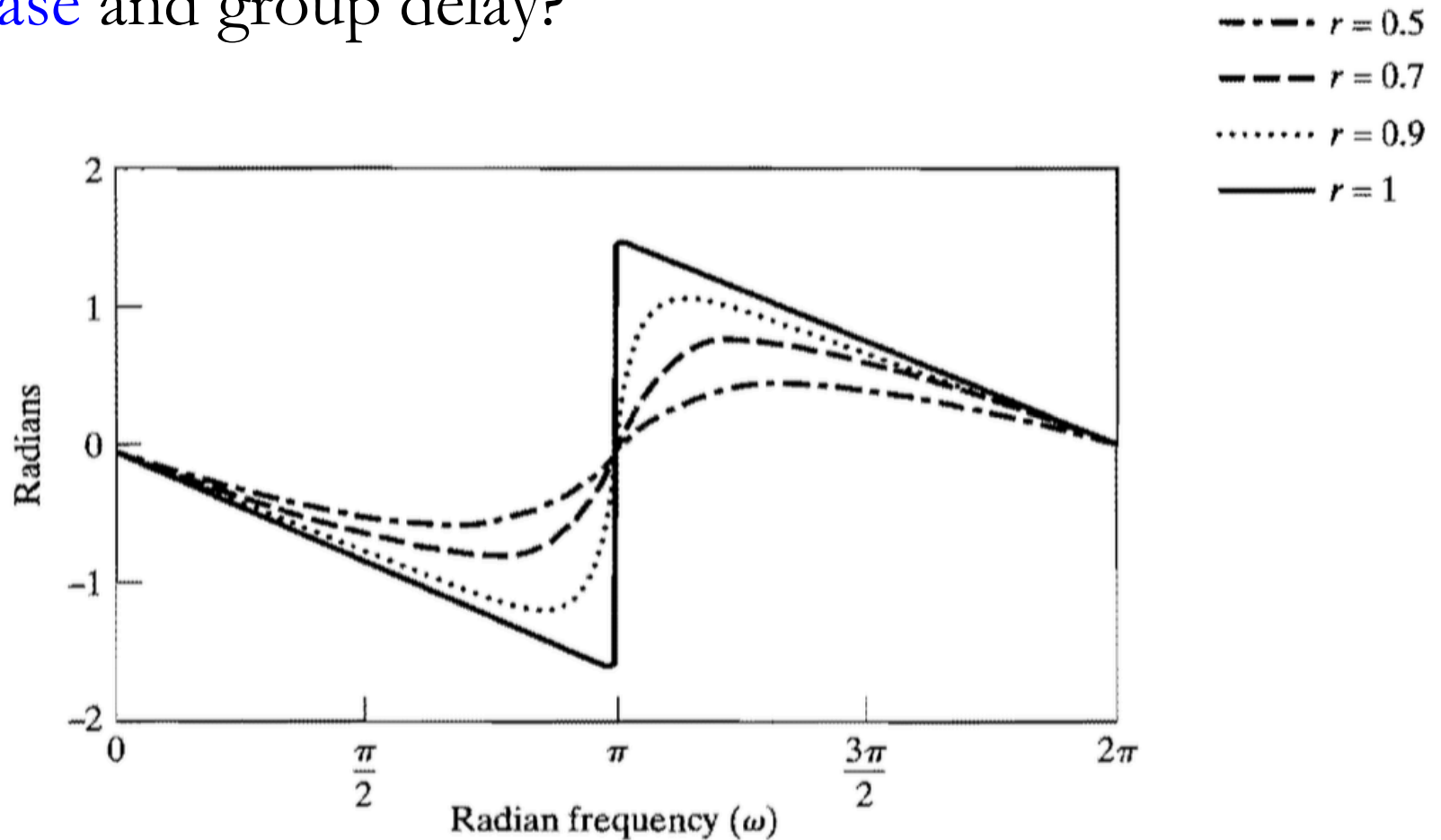
Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, phase and group delay?



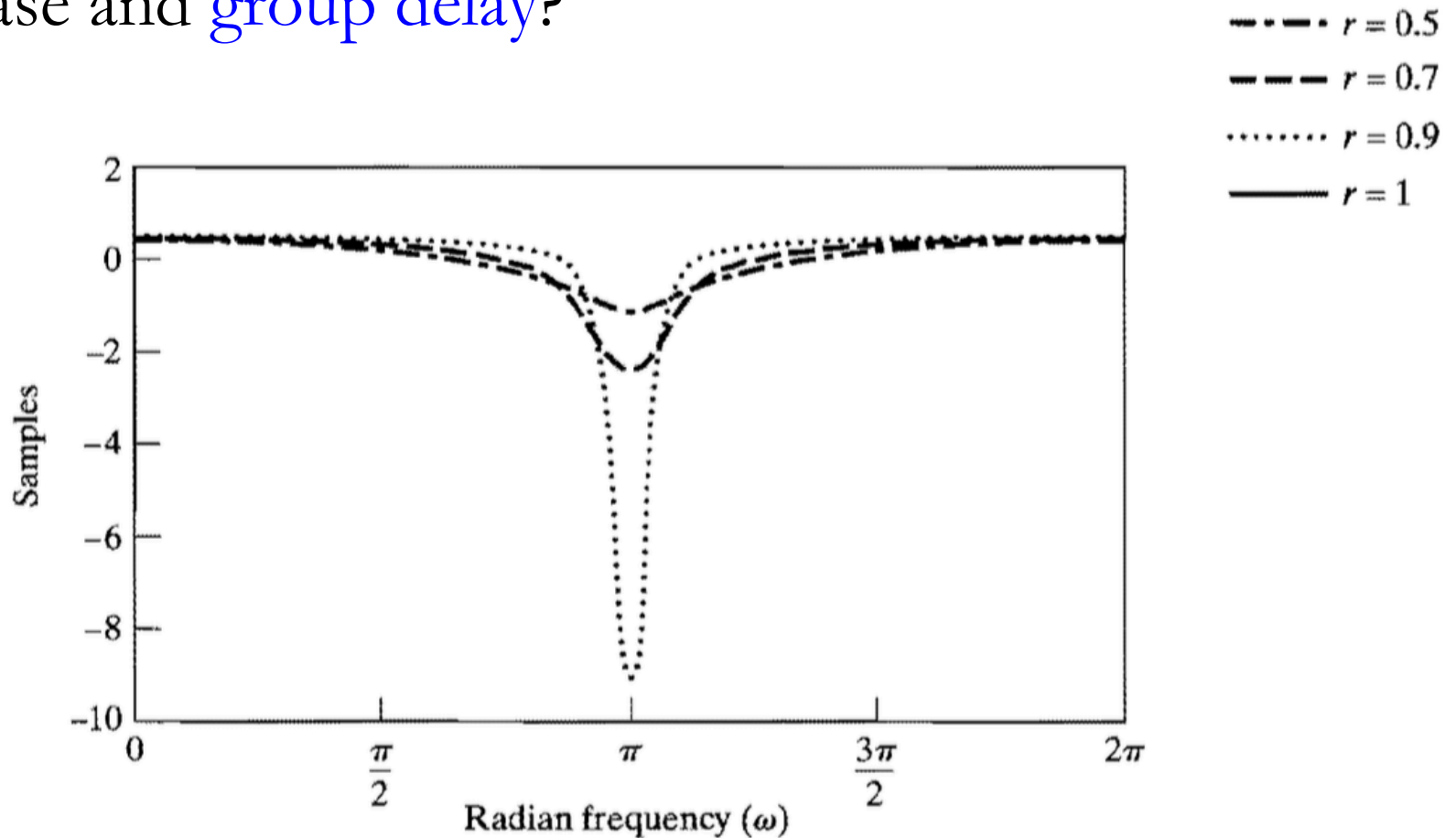
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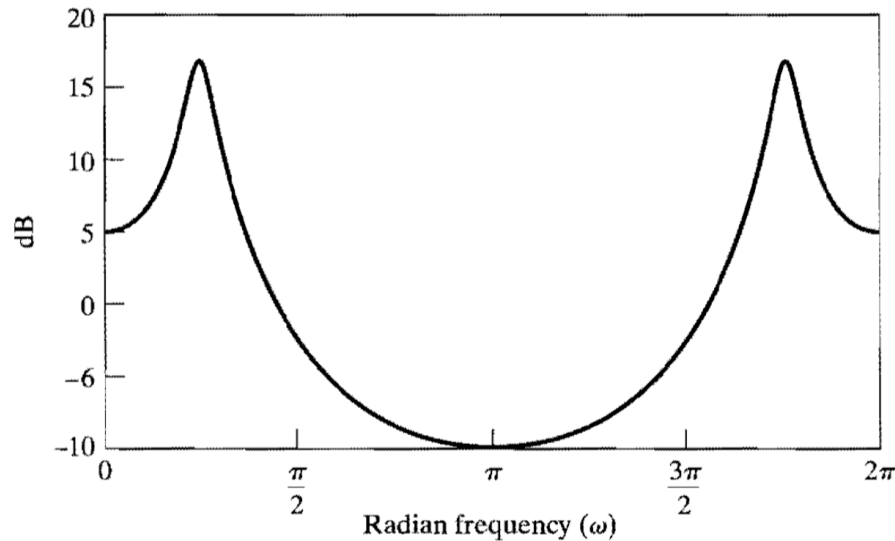


2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$$r=0.9, \theta=\pi/4$$

magnitude



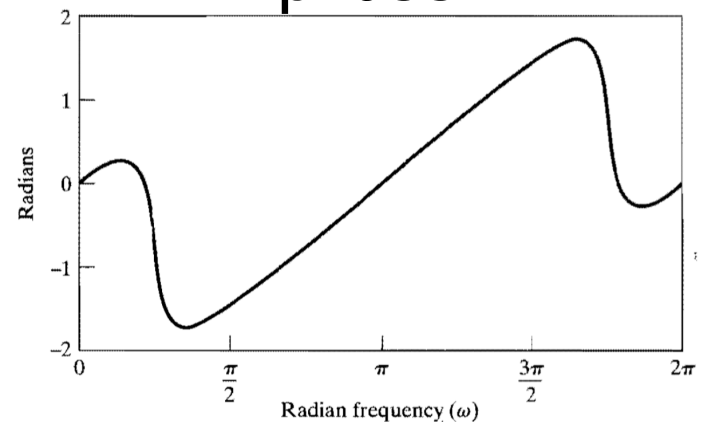
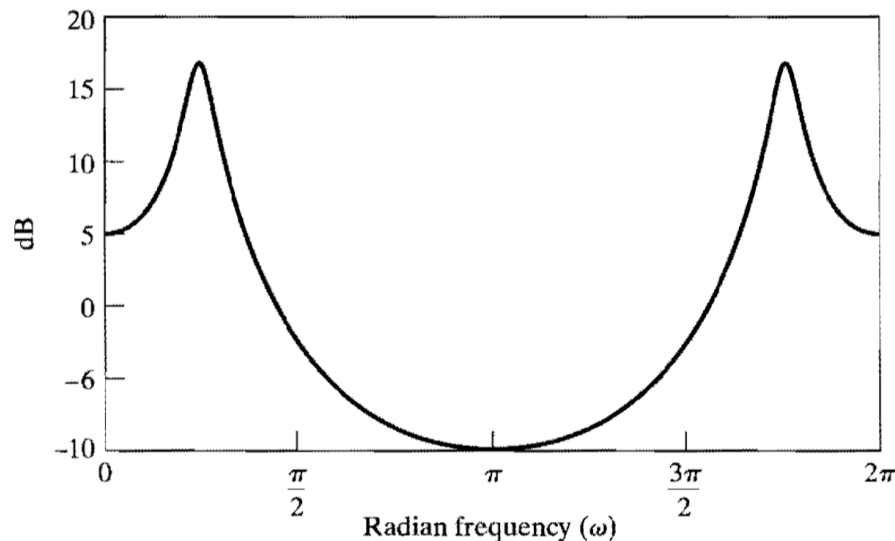
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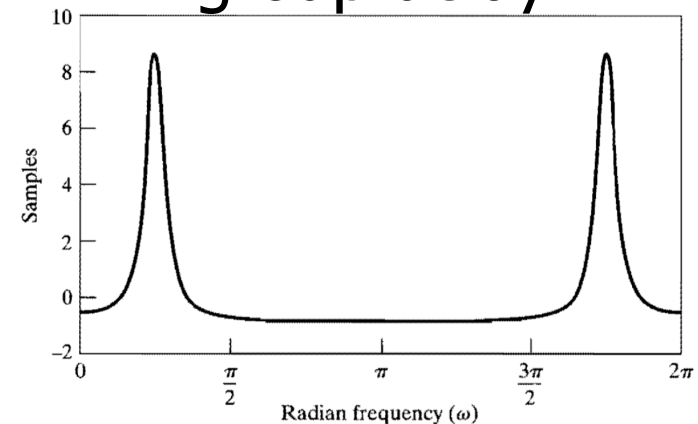
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phase

magnitude



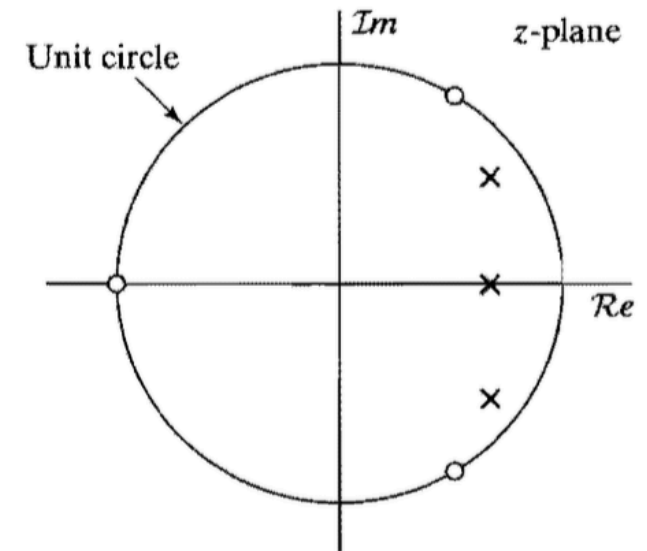
group delay





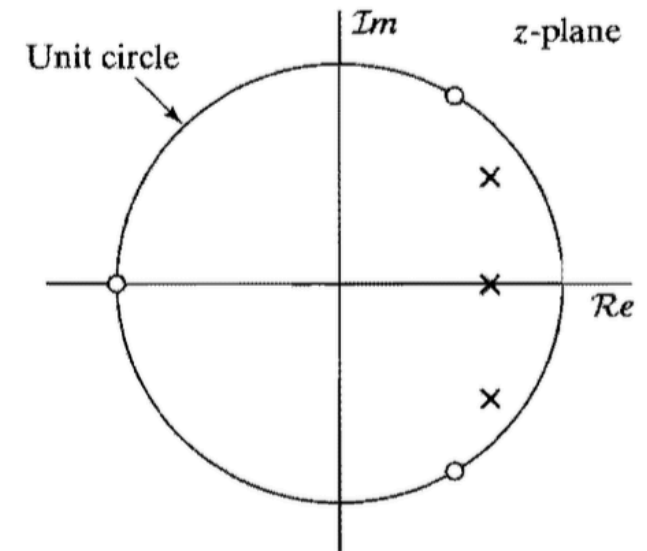
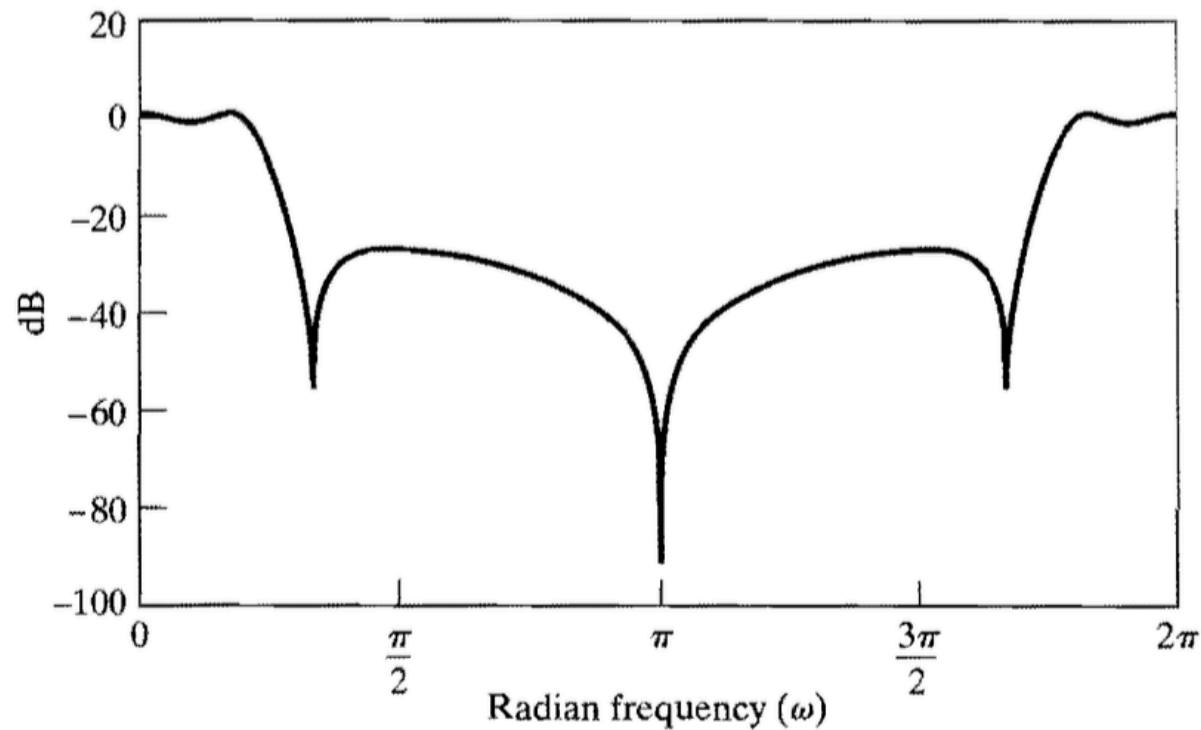
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



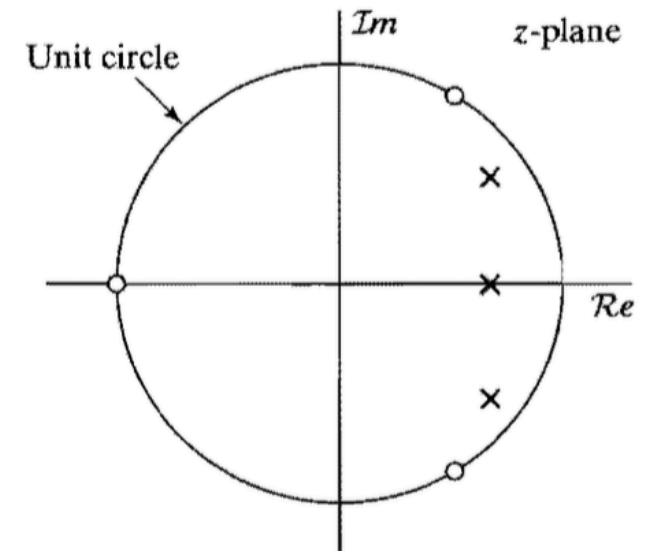
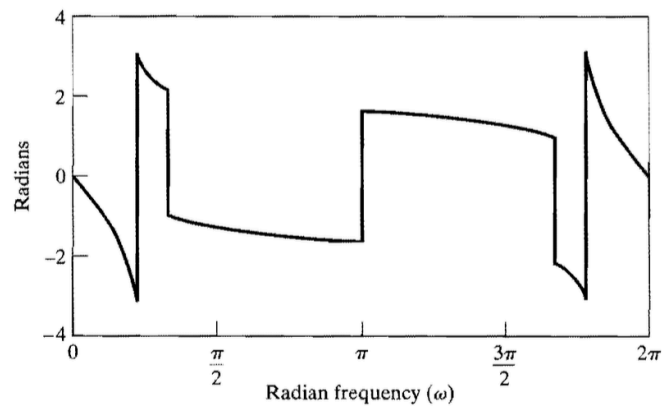
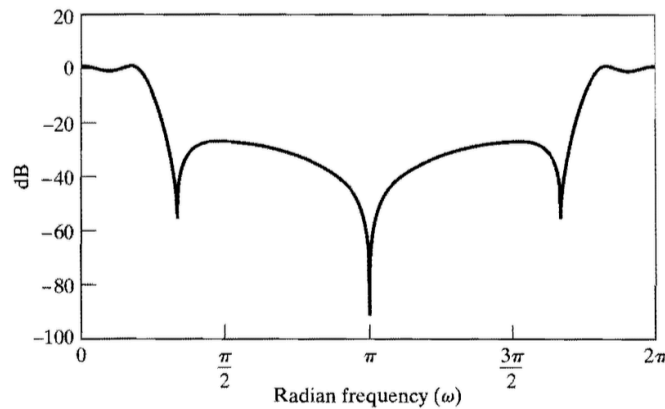
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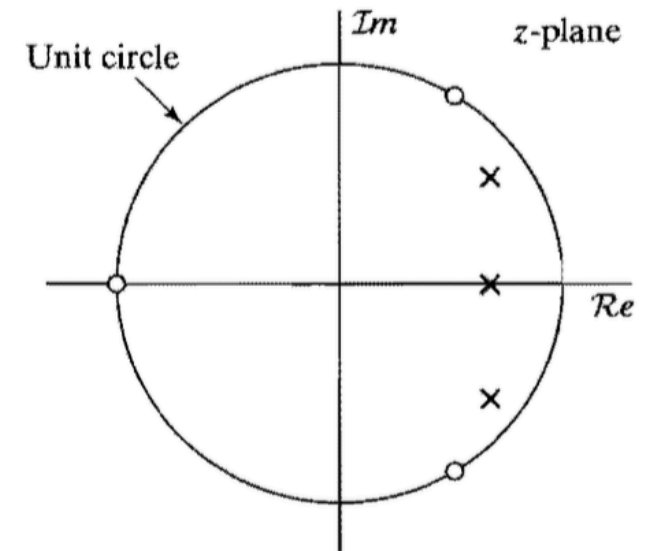
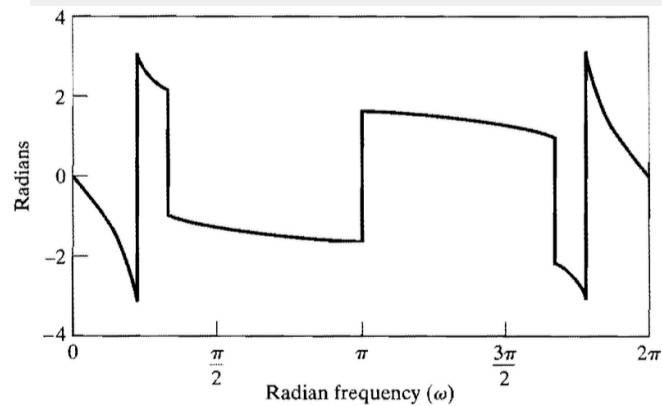
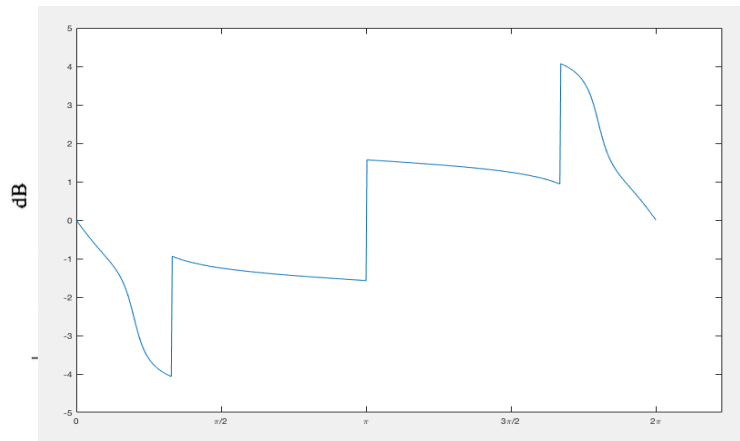
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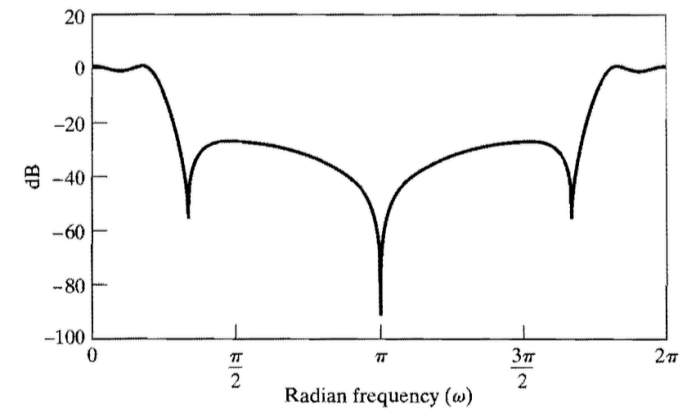
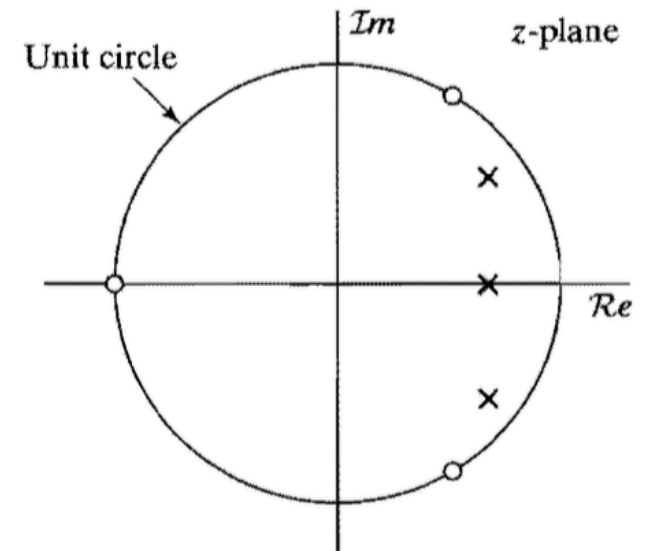
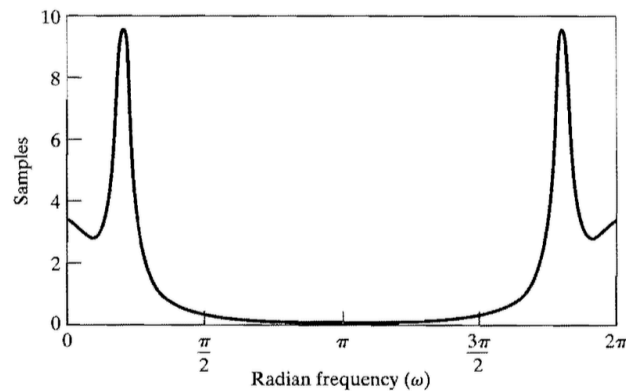
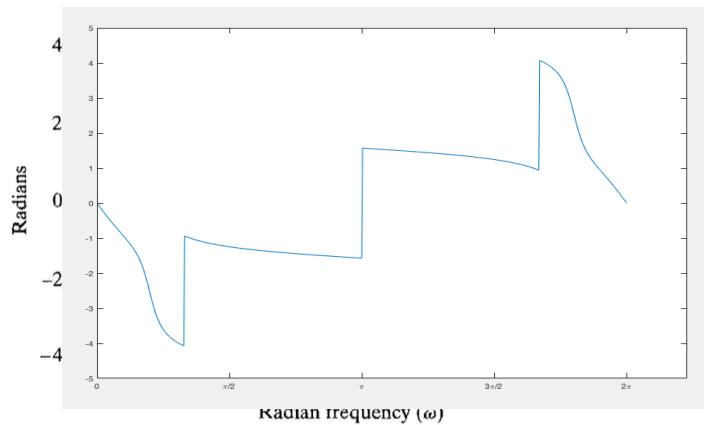
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All Pass Systems

Stability and Causality



LTI System

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

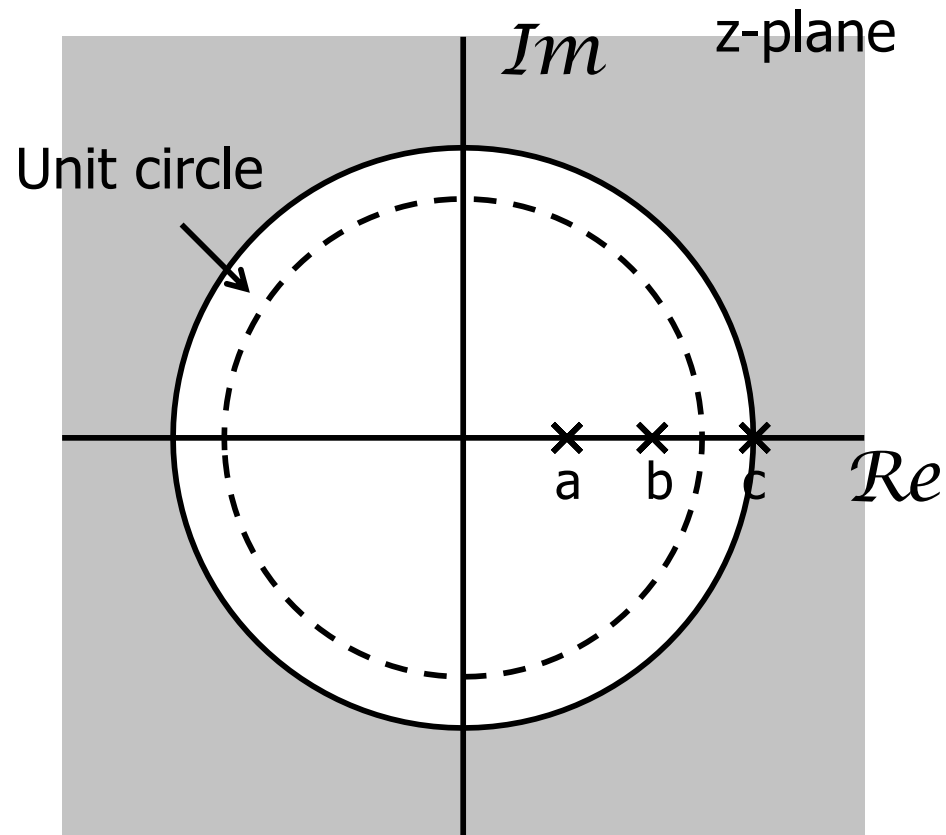
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

- ❑ Transfer function is not unique without ROC
 - If diff. eq represents LTI and causal system, ROC is region outside all singularities (i.e poles)
 - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

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Example: $y[n] = x[n] + 0.1y[n-1]$

If stable and
causal, all poles
inside unit circle

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All-Pass Systems



All-Pass Filters

- A system is an all-pass system if

$$|H(e^{j\omega})| = 1, \text{ all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial



First Order All-Pass Filter (a real)

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

First Order All-Pass Filter (a real)

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$$\begin{aligned} |H(e^{j\omega})| &= \frac{|e^{-j\omega} - a^*|}{|1 - ae^{-j\omega}|} \\ &= \frac{|e^{-j\omega}(1 - a^*e^{j\omega})|}{|1 - ae^{-j\omega}|} \\ &= \frac{|1 - a^*e^{j\omega}|}{|1 - ae^{-j\omega}|} = 1 \end{aligned}$$



General All-Pass Filter

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

General All-Pass Filter

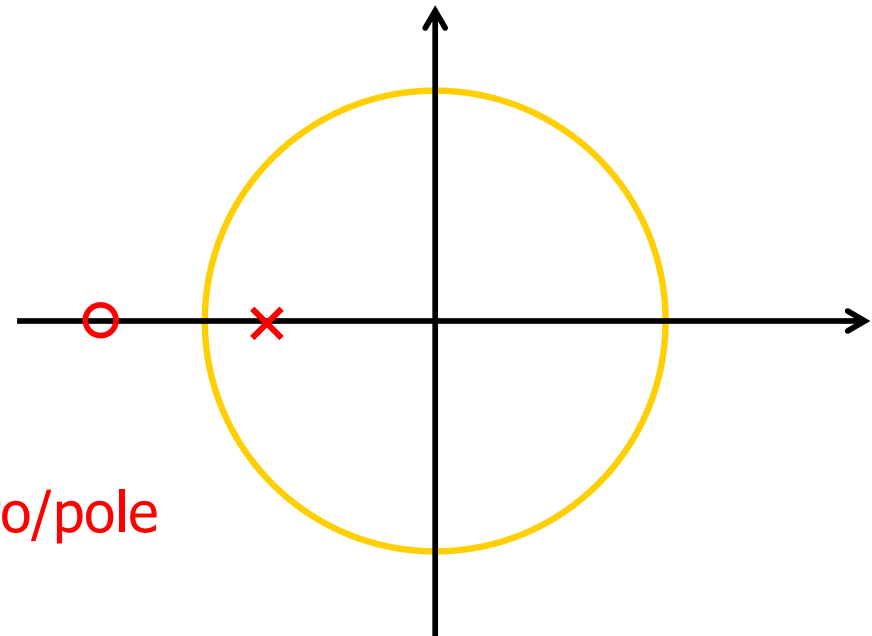
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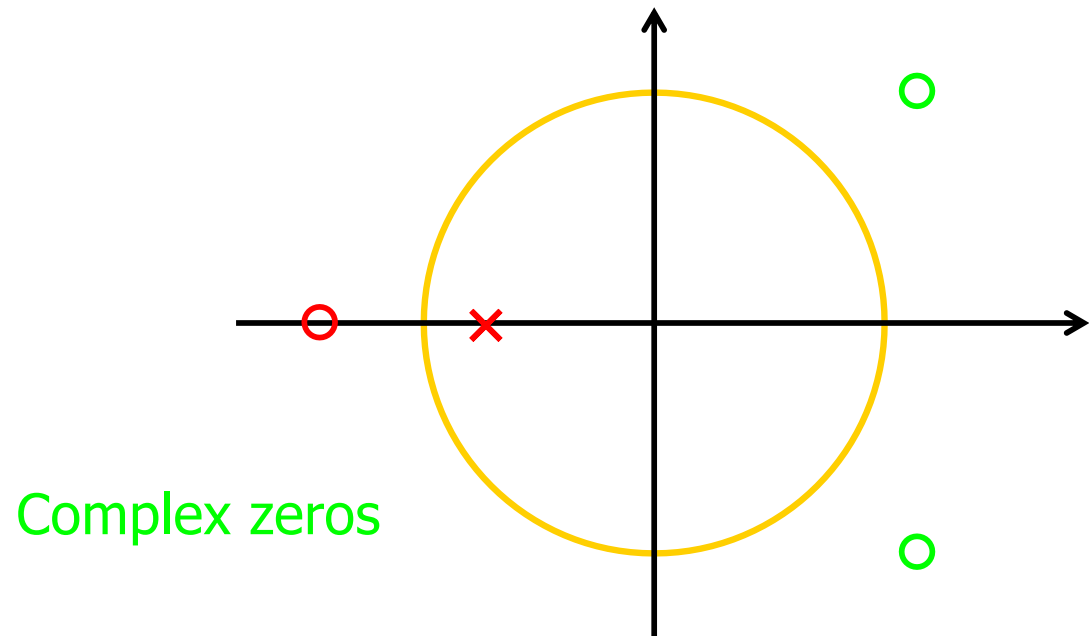
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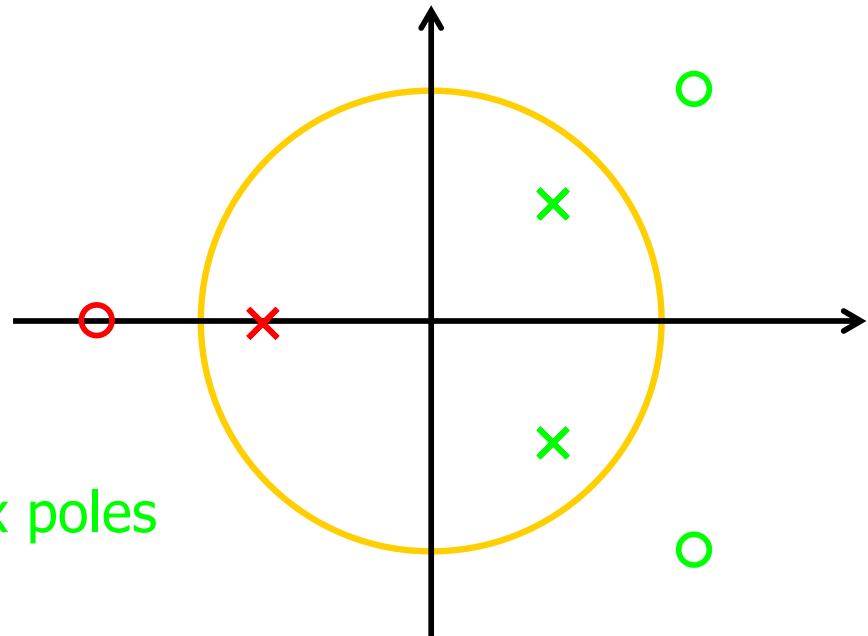
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Complex poles



All Pass Filter Phase Response

□ First order system

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$
$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

□ phase



All Pass Filter Phase Response

□ phase $\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$

All Pass Filter Phase Response

□ phase

$$\begin{aligned} & \arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right) \\ &= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right) \\ &= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega}) \\ &= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega}) \\ &= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega}) \end{aligned}$$

Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

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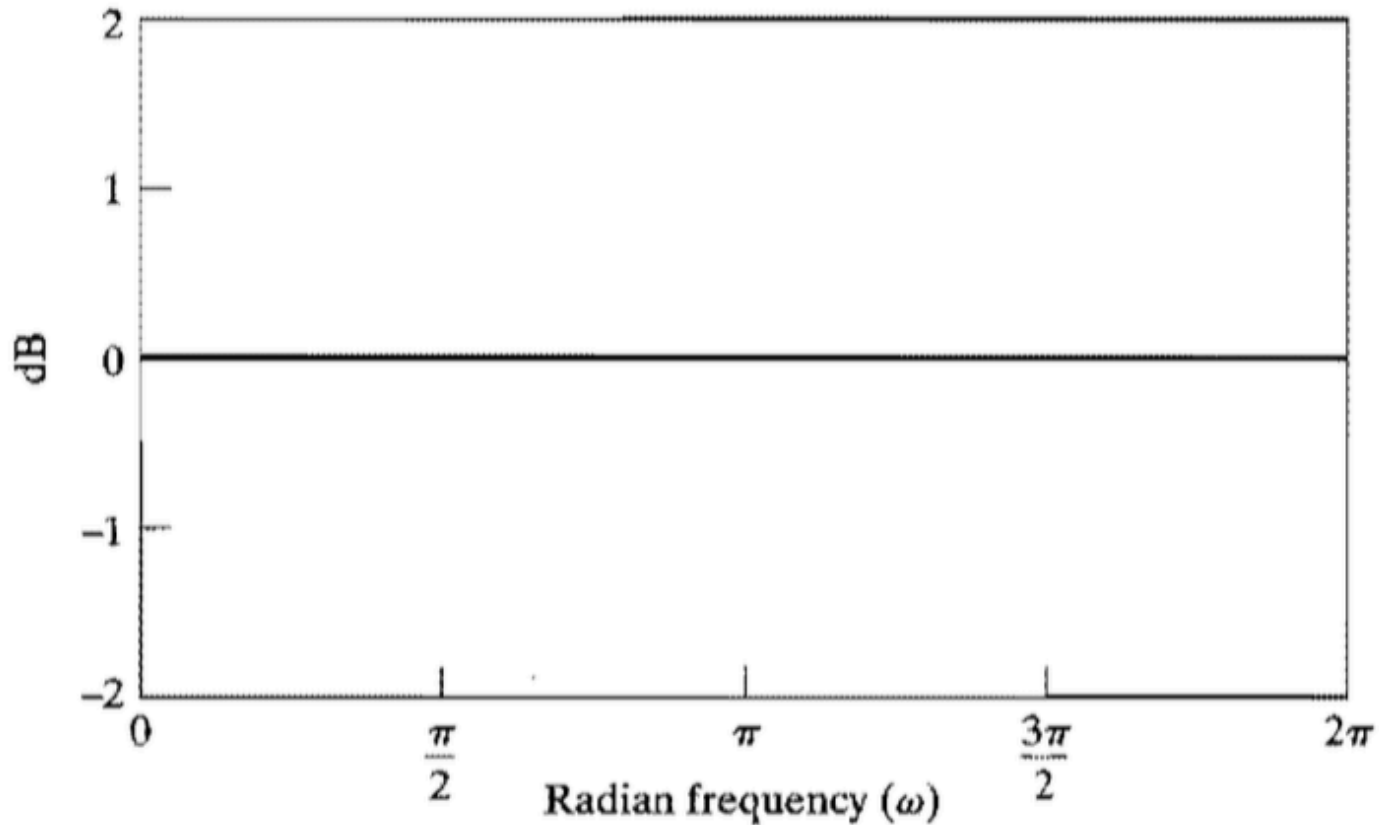
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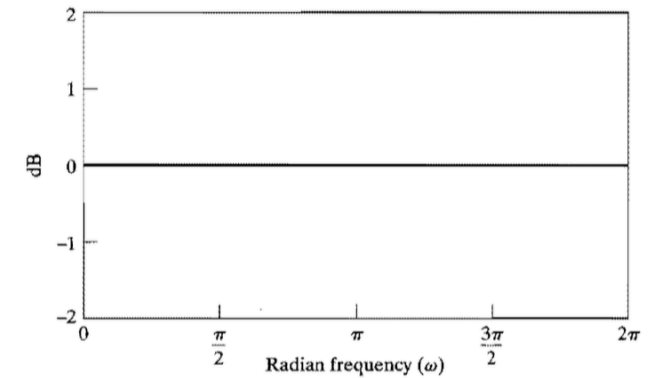
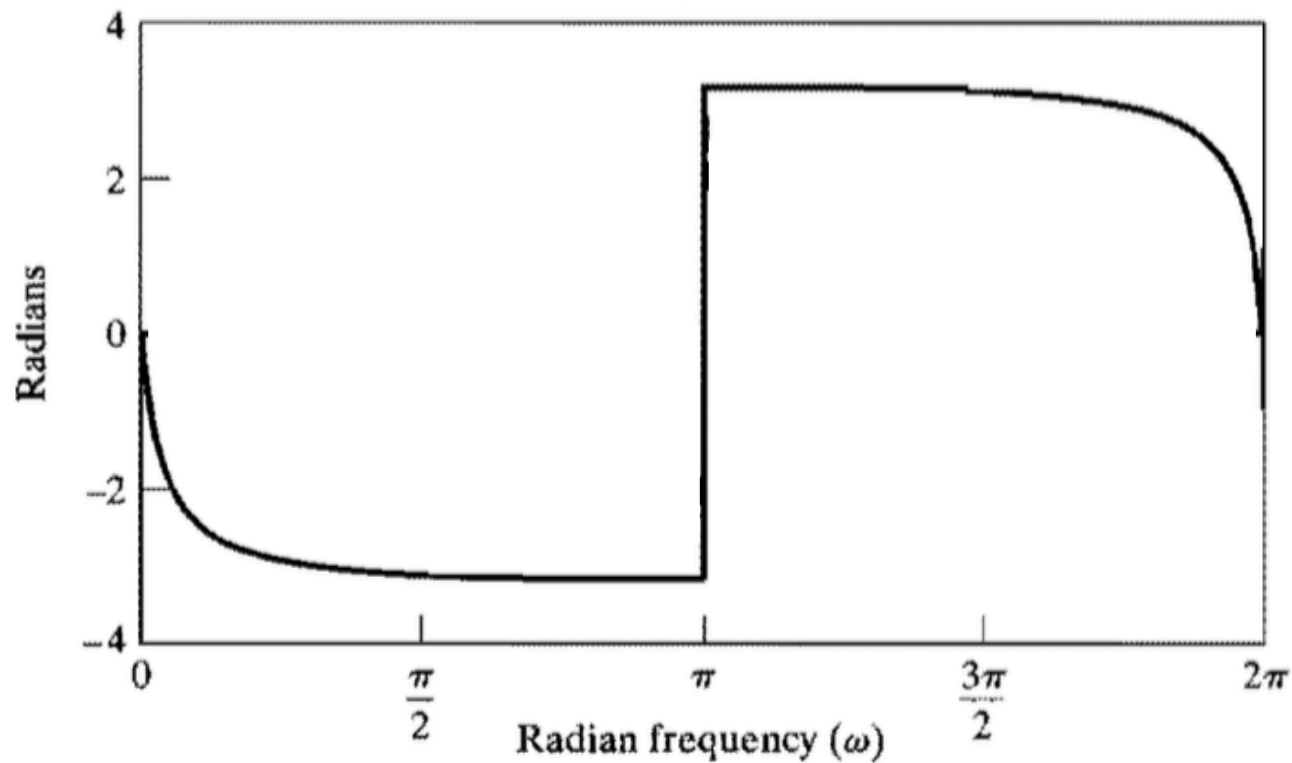
First Order Example

□ Magnitude:



First Order Example

□ Phase:

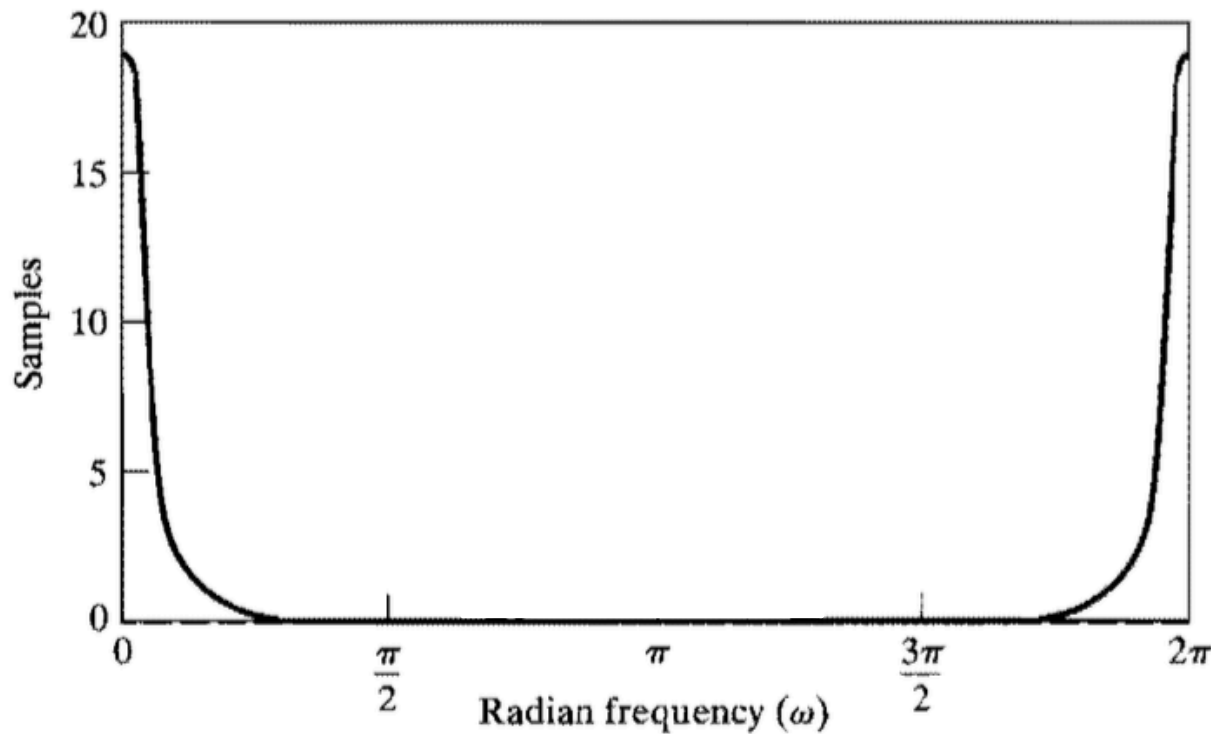
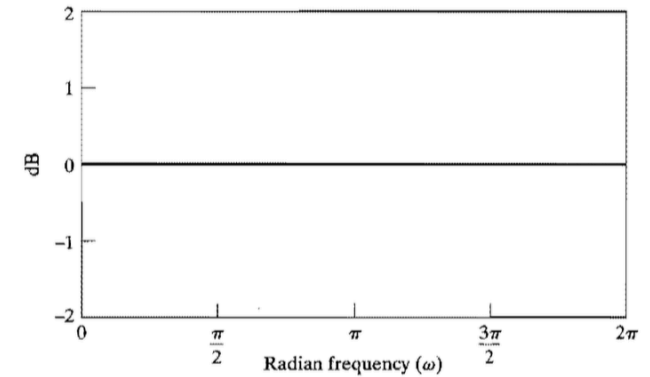
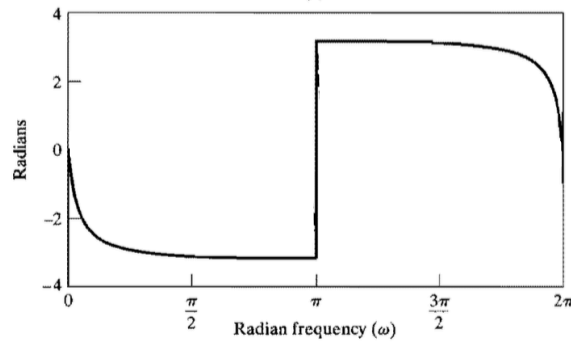


— $z = 0.9$

— $\theta = 0$

First Order Example

□ Group Delay:



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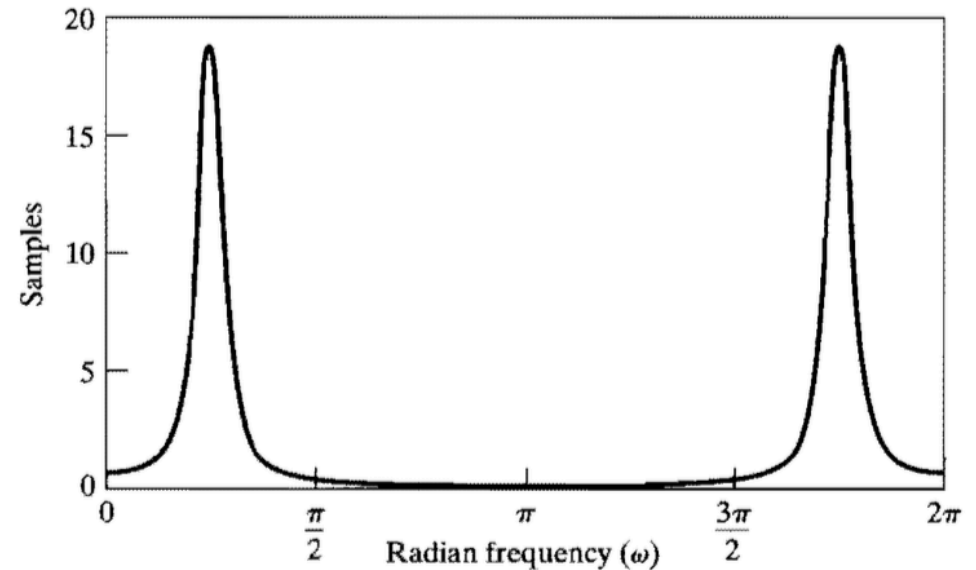
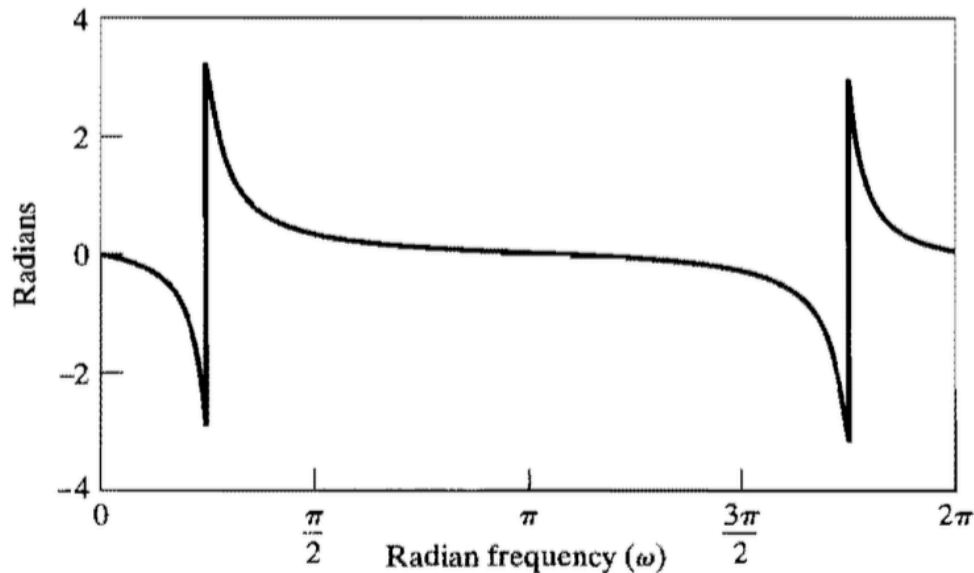
All Pass Filter Phase Response

- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

$$\angle \left[\frac{(e^{-j\omega} - re^{-j\theta})(e^{-j\omega} - re^{j\theta})}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})} \right] = -2\omega - 2 \arctan \left[\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right] \\ - 2 \arctan \left[\frac{r \sin(\omega + \theta)}{1 - r \cos(\omega + \theta)} \right].$$

Second Order Example

□ Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)



All-pass Example

- d_k =real pole, e_k =complex poles paired w/ conjugate, e_k^*

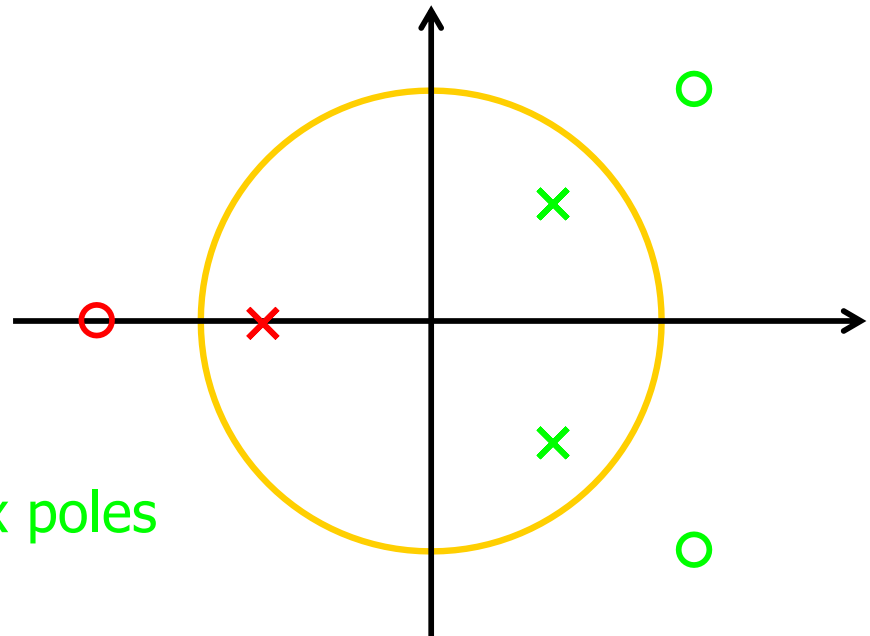
$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

- Example:

$$d_k = -\frac{3}{4}$$

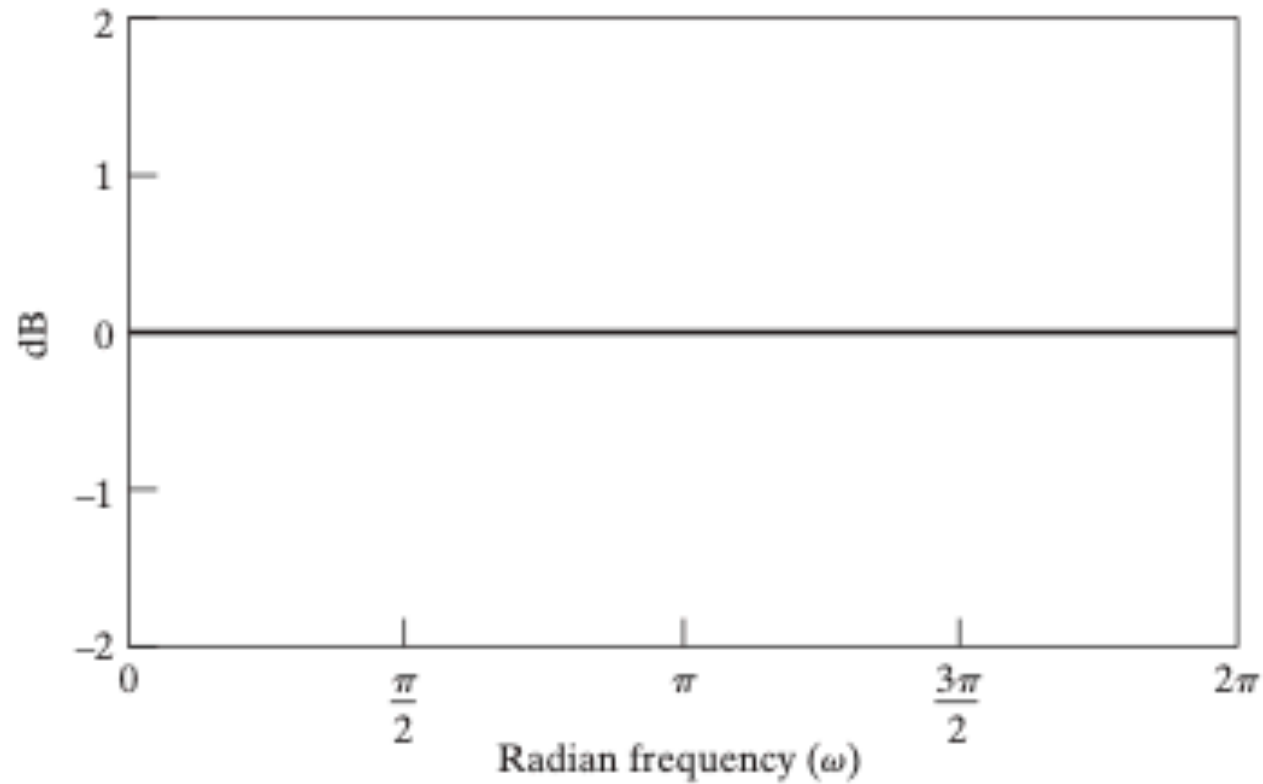
$$e_k = 0.8e^{j\pi/4}$$

Complex poles

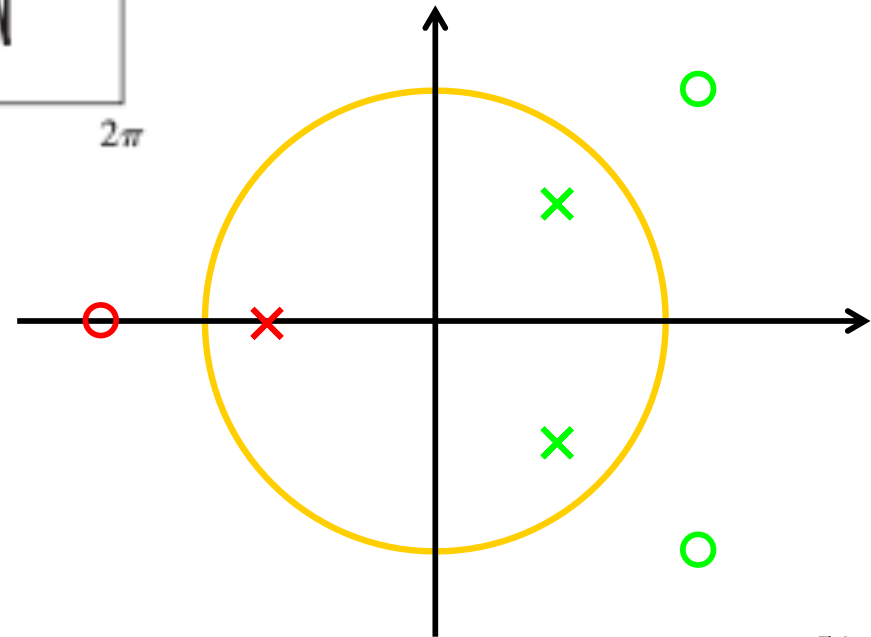
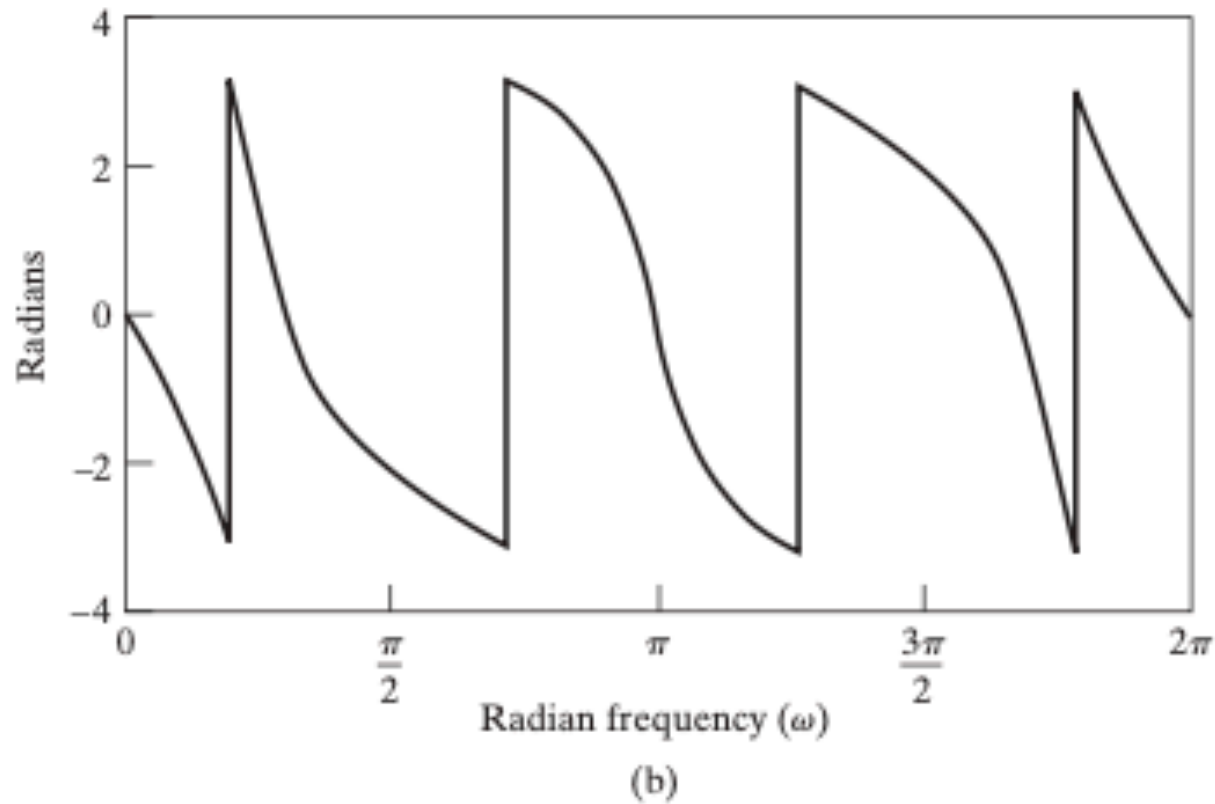




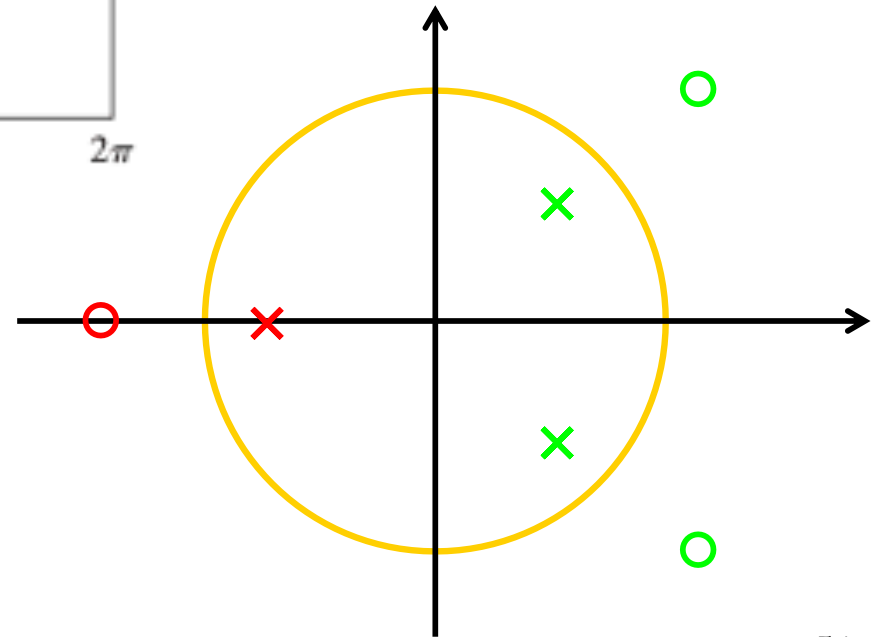
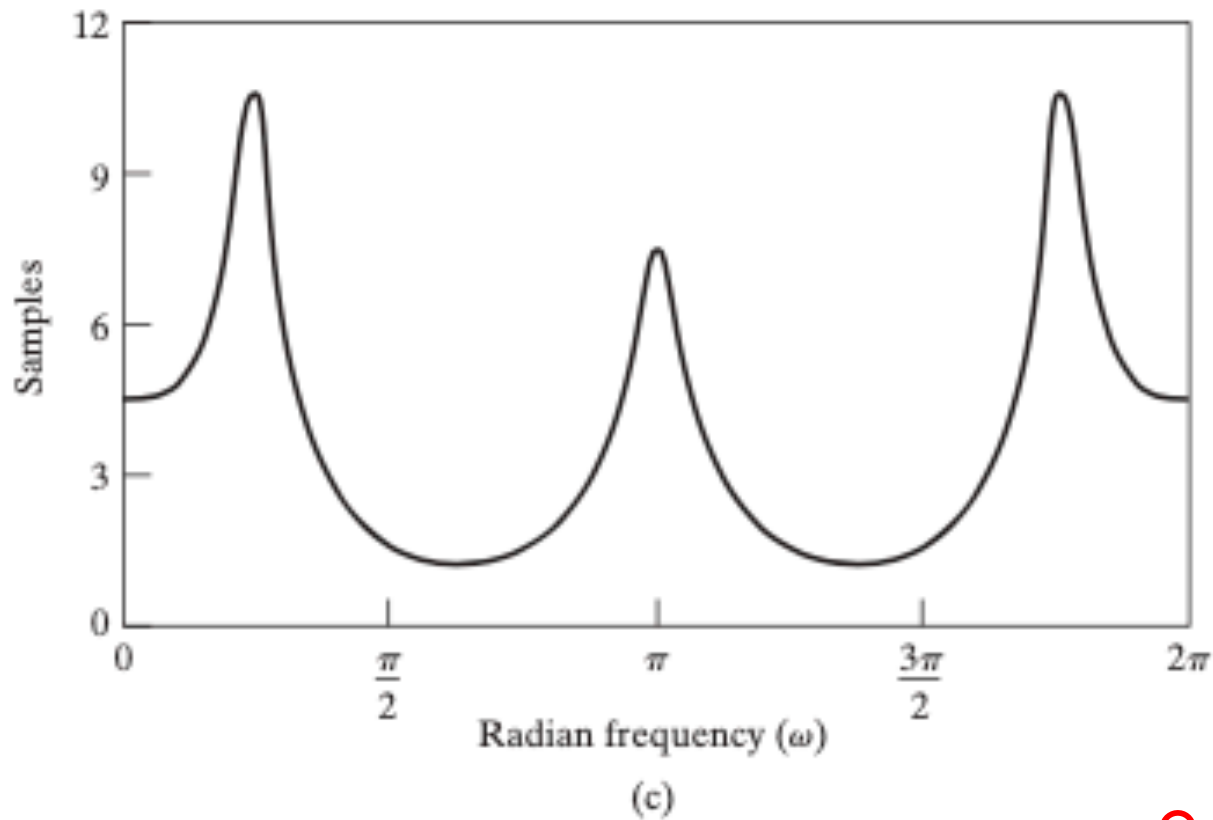
All-pass Example: Magnitude



All-pass Example: Phase



All-pass Example: Group Delay





All-Pass Properties

□ Claim: For a stable, causal ($r < 1$) all-pass system:

- $\arg[H_{ap}(e^{j\omega})] \leq 0$

- Unwrapped phase always non-positive and decreasing

- $\text{grd}[H_{ap}(e^{j\omega})] > 0$

- Group delay always positive

Pg 309 in text book

- Intuition

- delay is positive $\rightarrow$ system is causal

- Phase negative $\rightarrow$ phase lag

Minimum Phase Systems

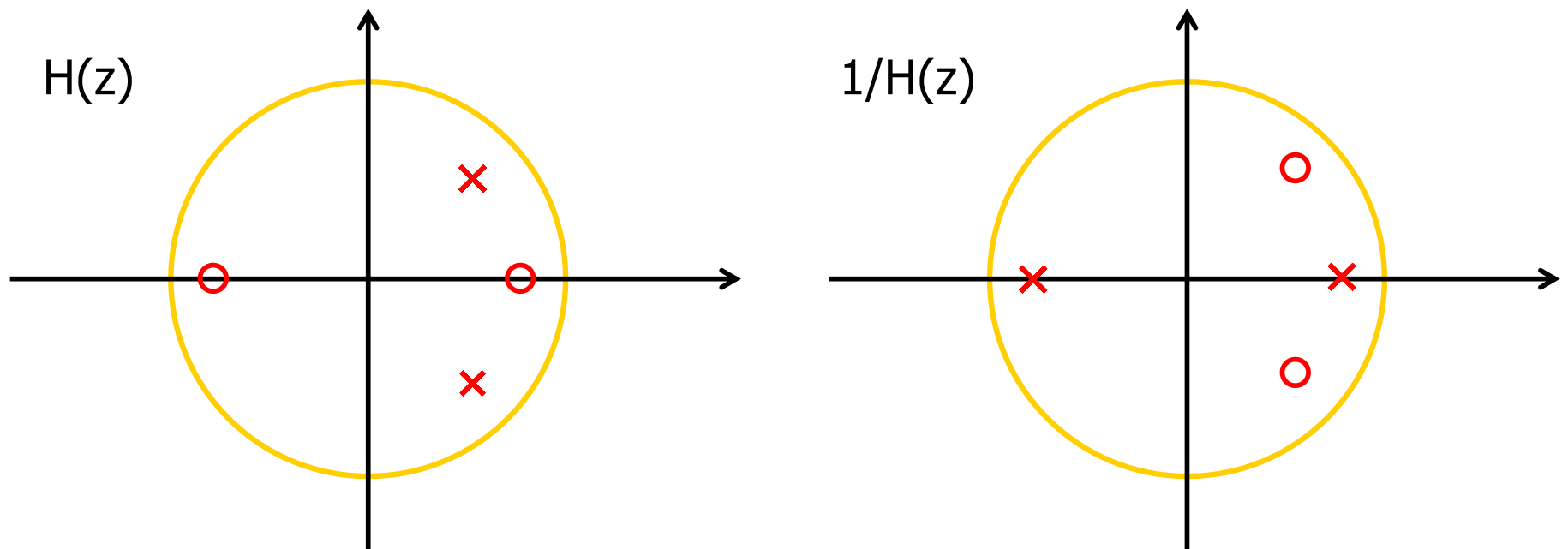


Minimum-Phase Systems

- Definition: A stable and causal system $H(z)$ (i.e. poles inside unit circle) whose inverse $1/H(z)$ is also stable and causal (i.e. zeros inside unit circle)

Minimum-Phase Systems

- Definition: A stable and causal system $H(z)$ (i.e. poles inside unit circle) whose inverse $1/H(z)$ is also stable and causal (i.e. zeros inside unit circle)
 - All poles and zeros inside unit circle





All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

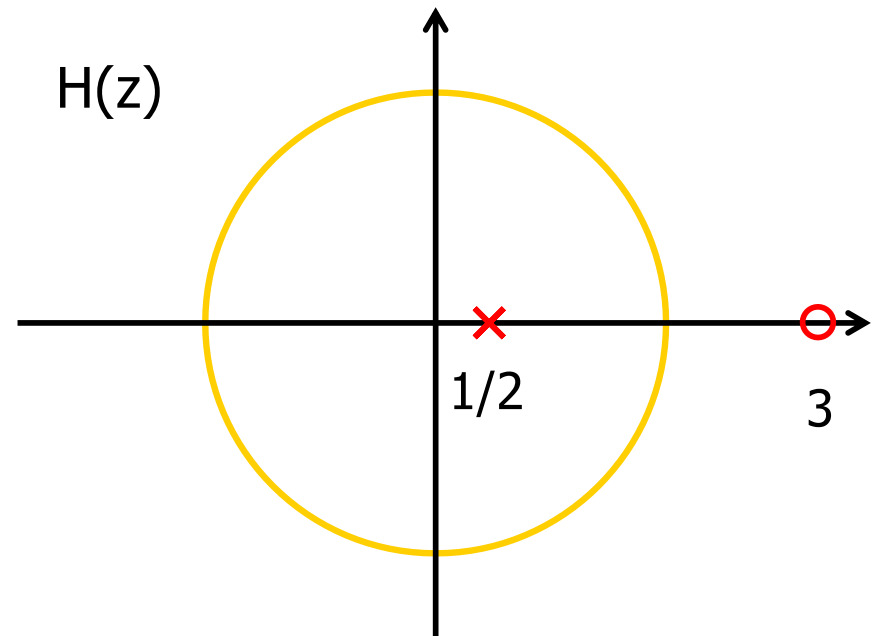
$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
 - (1) First construct H_{ap} with all zeros outside unit circle
 - (2) Compute

$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Min-Phase Decomposition Example

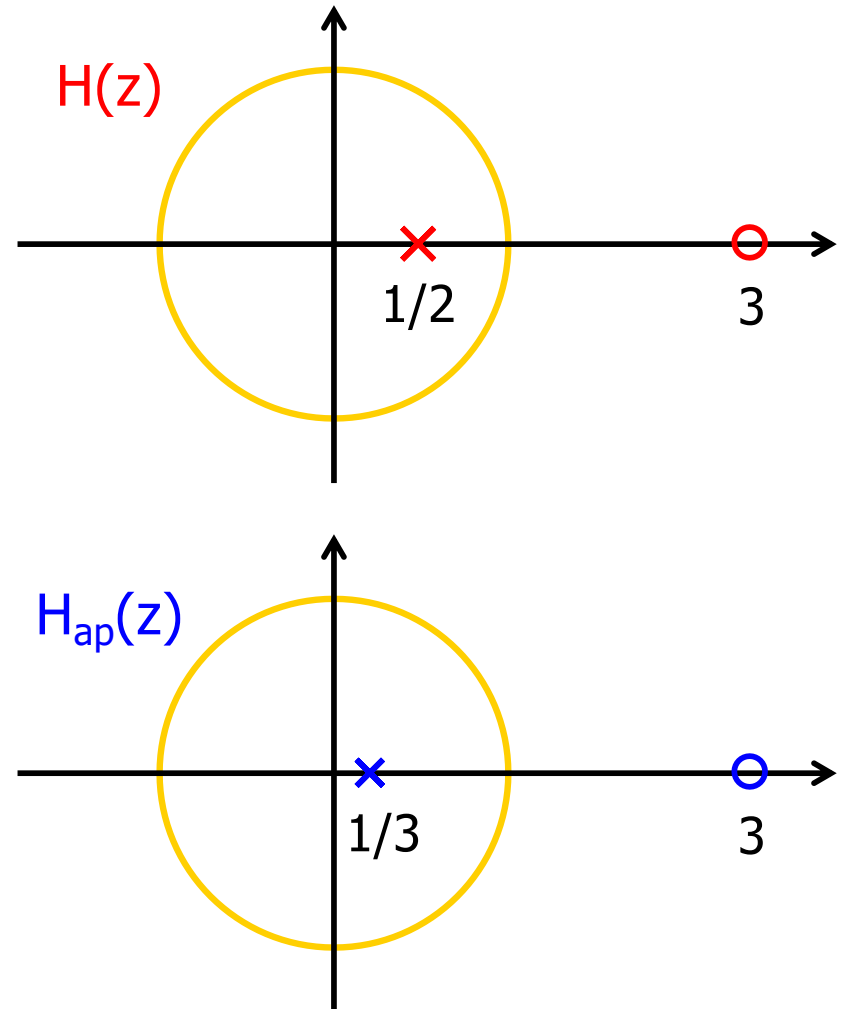
$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

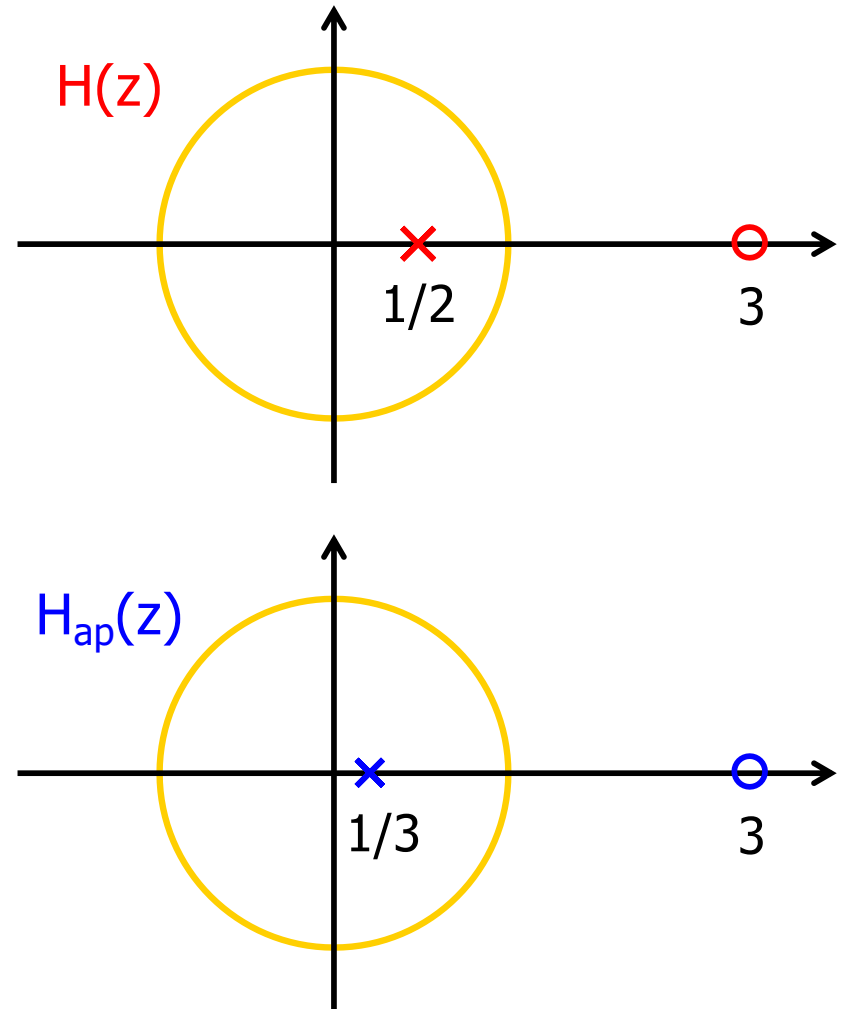
□ Set $H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$



Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

□ Set $H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$

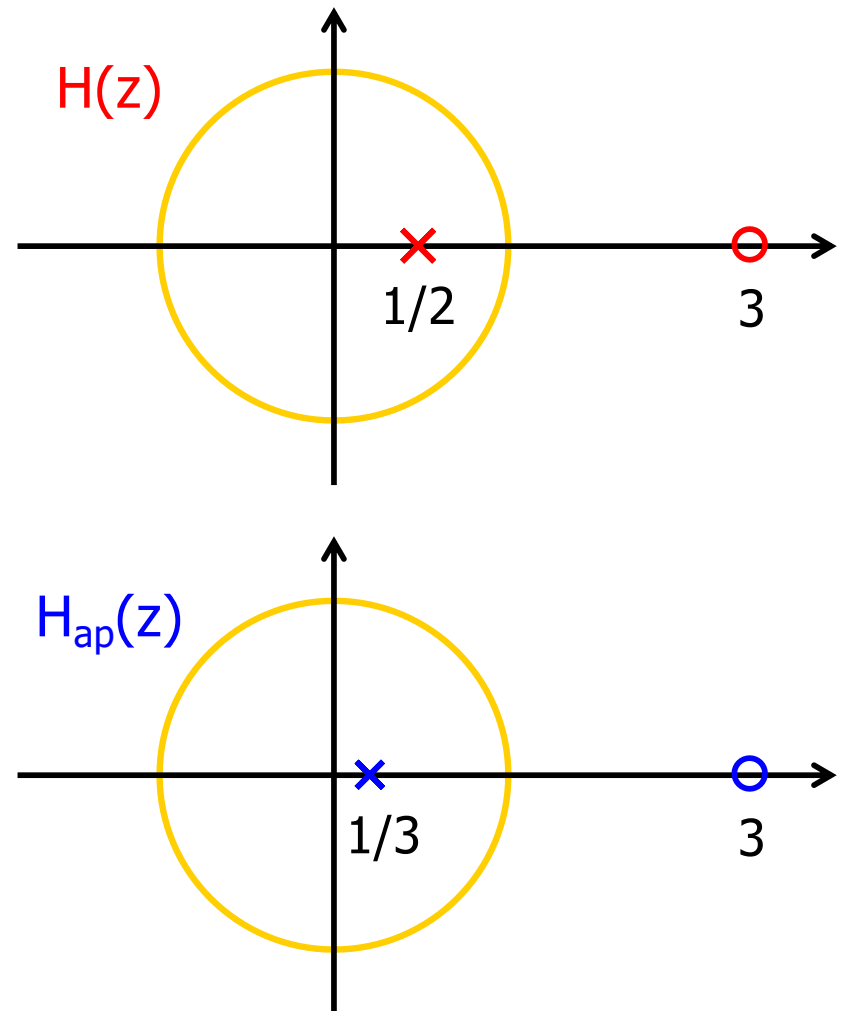


Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

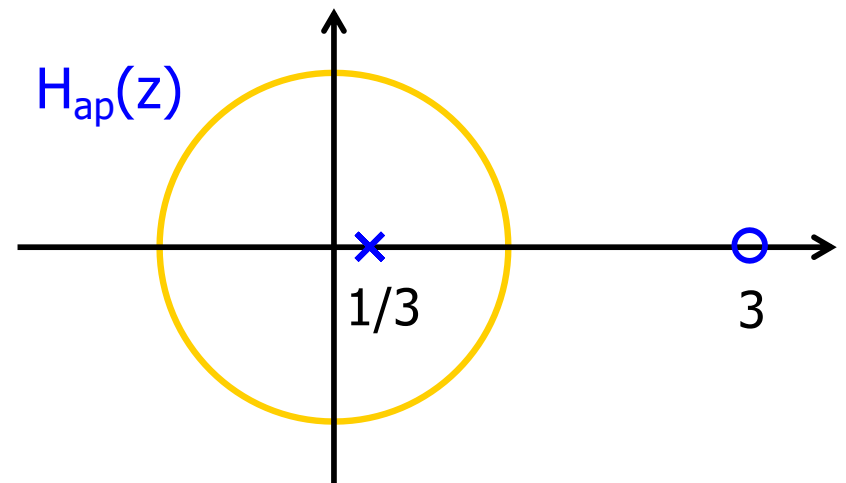
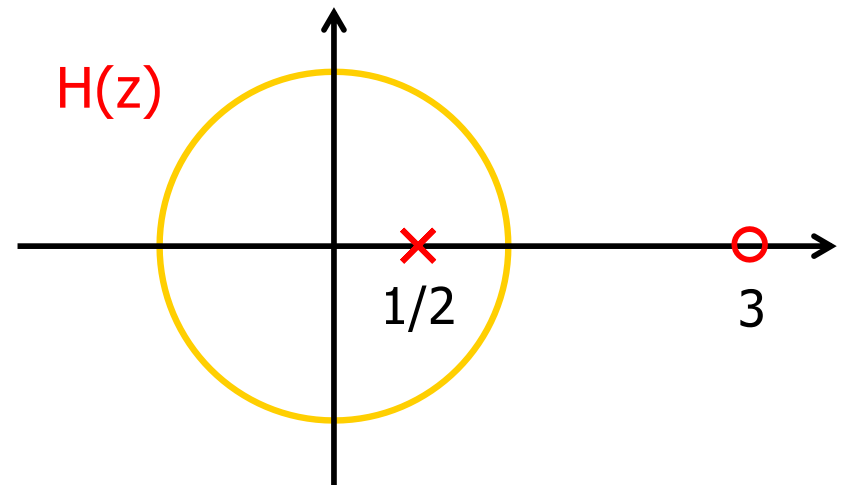
□ Set $H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$

$$H_{\min}(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}} \left(\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}} \right)^{-1}$$



Min-Phase Decomposition Example

$$H_{\min}(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}} \left(\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}} \right)^{-1}$$

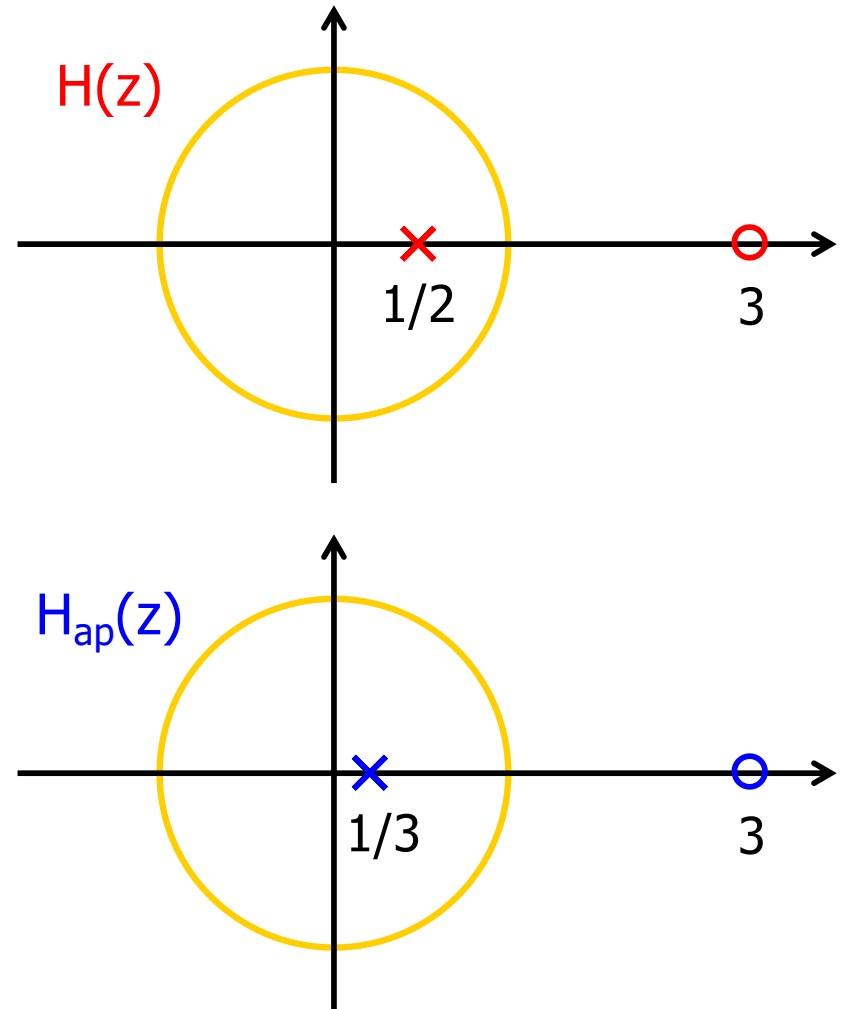


Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

□ Set $H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$

$$H_{\min}(z) = -3 \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

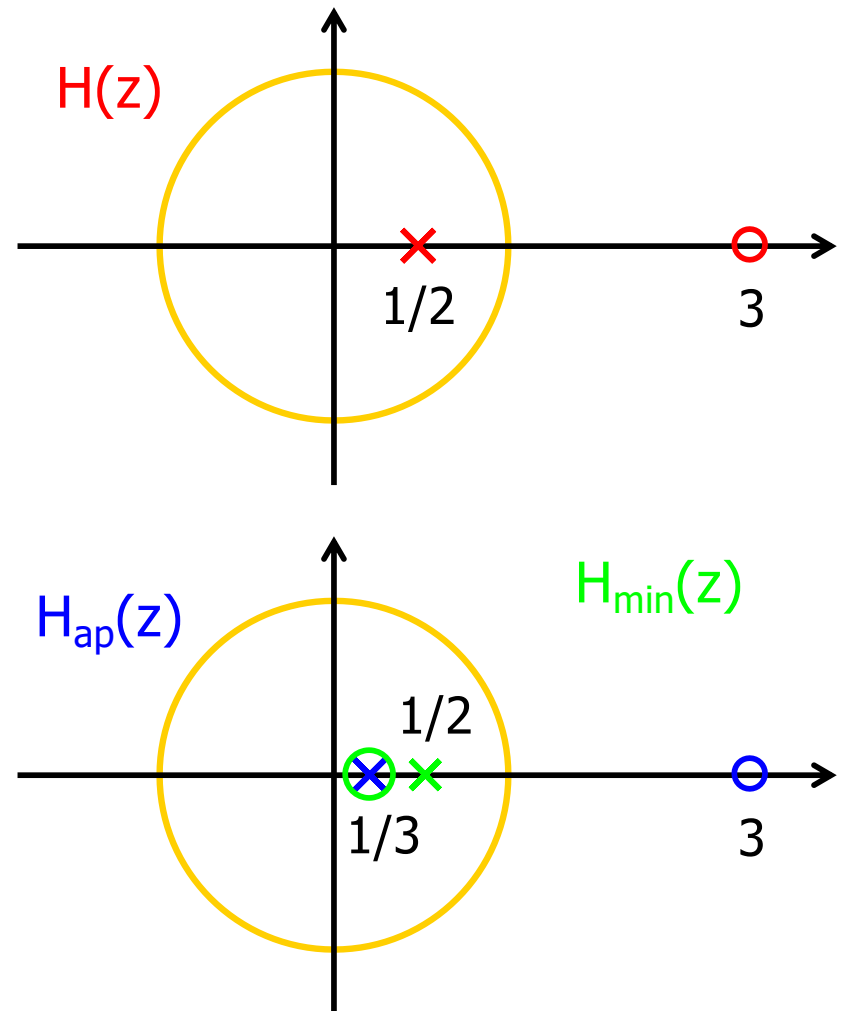


Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

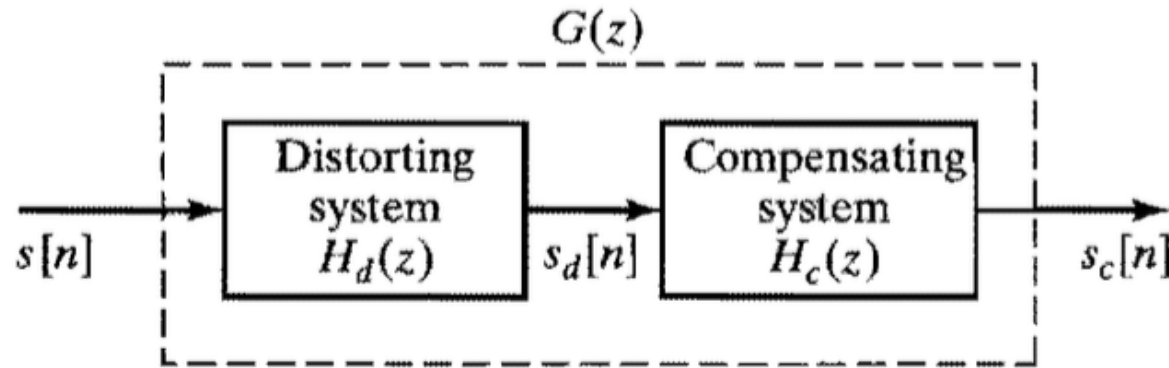
□ Set $H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$

$$H_{\min}(z) = -3 \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



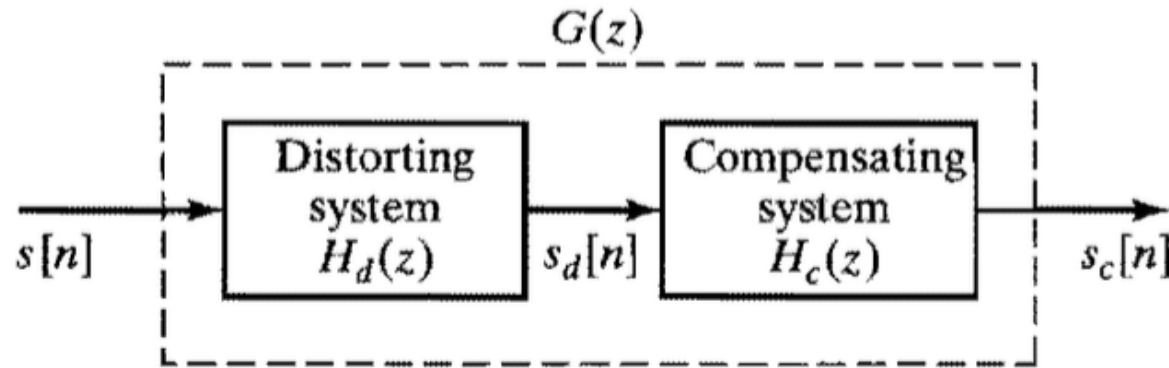
Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



Min-Phase Decomposition Purpose

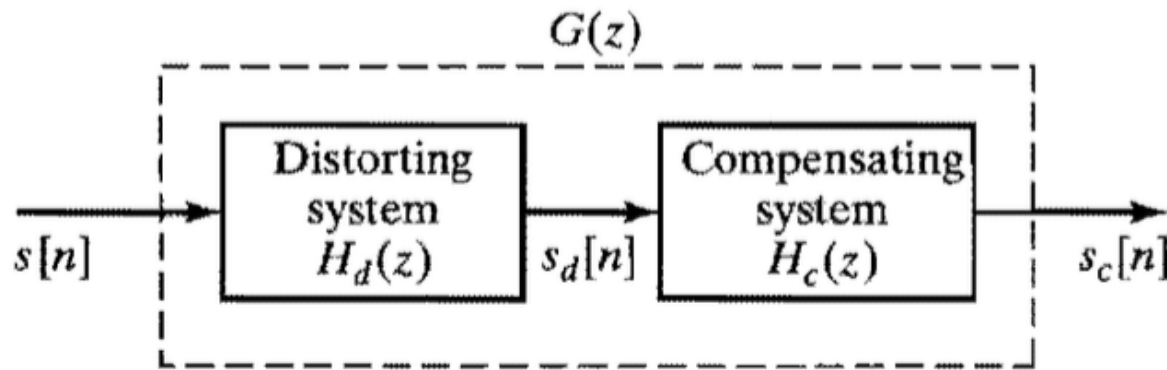
- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
 - $H_c(z) = 1/H_d(z)$ ← also stable and causal

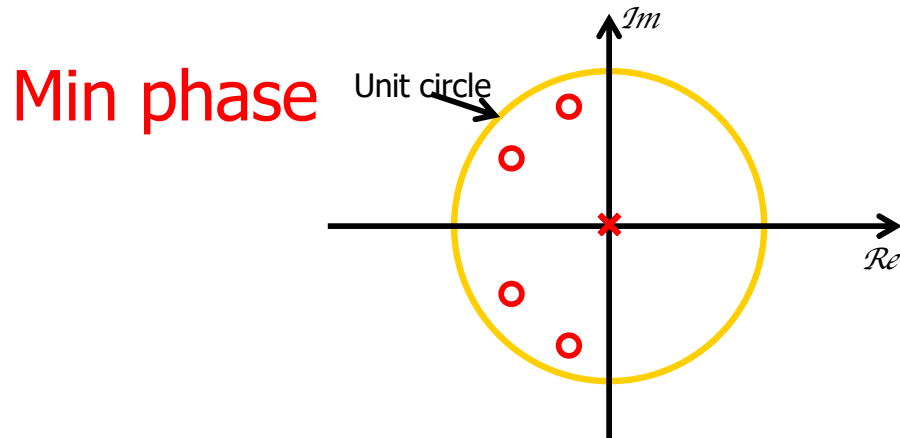
Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:

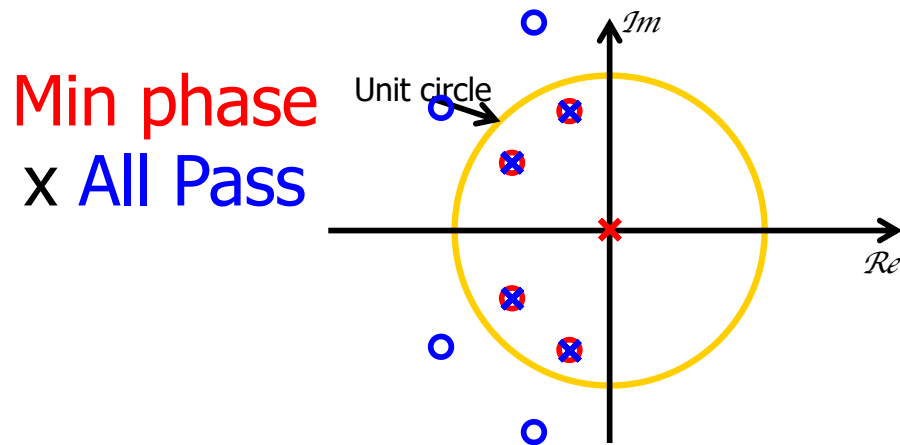


- If $H_d(z)$ is min phase, easy:
 - $H_c(z) = 1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z) = H_{d,\min}(z) H_{d,\text{ap}}(z)$
 - $H_c(z) = 1/H_{d,\min}(z) \rightarrow H_d(z)H_c(z) = H_{d,\text{ap}}(z)$
 - Compensate for magnitude distortion

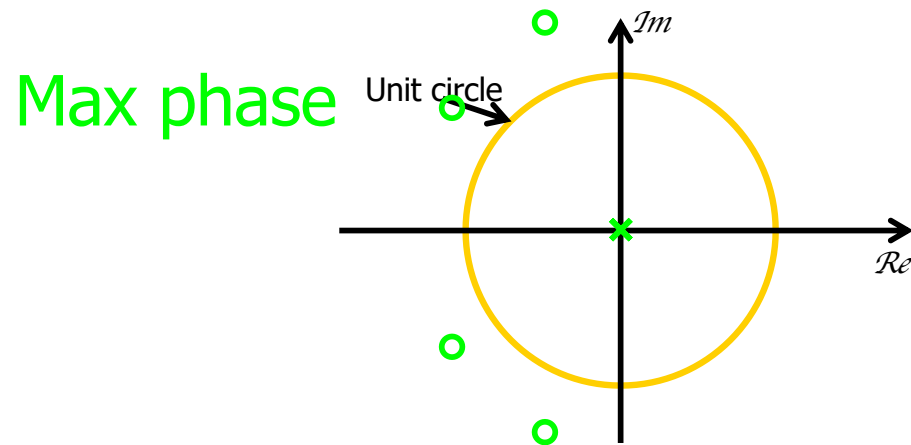
Minimum Phase to Max Phase



Minimum Phase to Max Phase

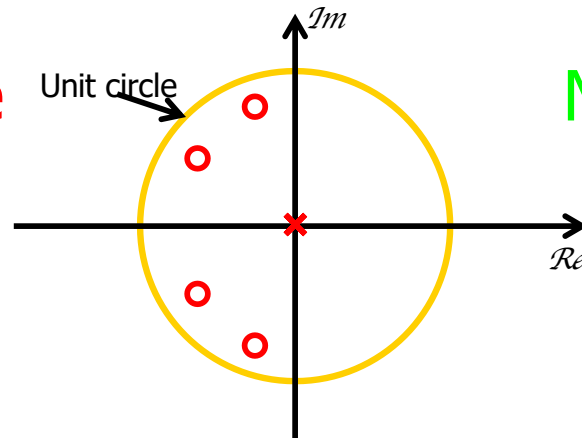


Minimum Phase to Max Phase

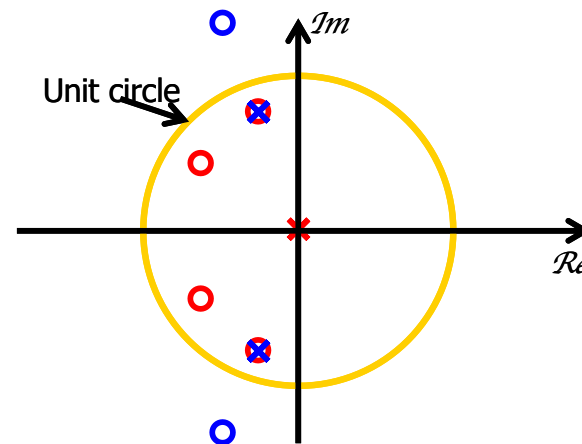
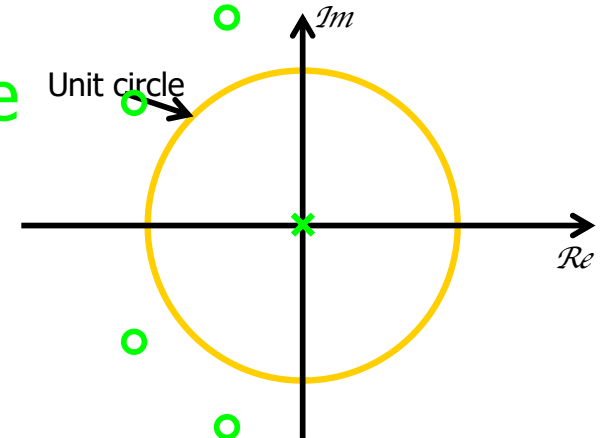


Minimum Phase

Min phase

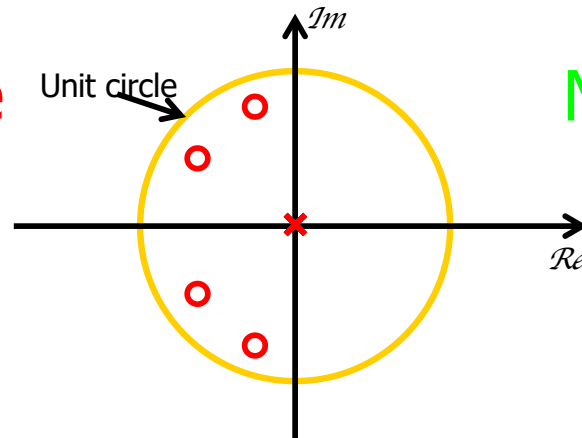


Max phase

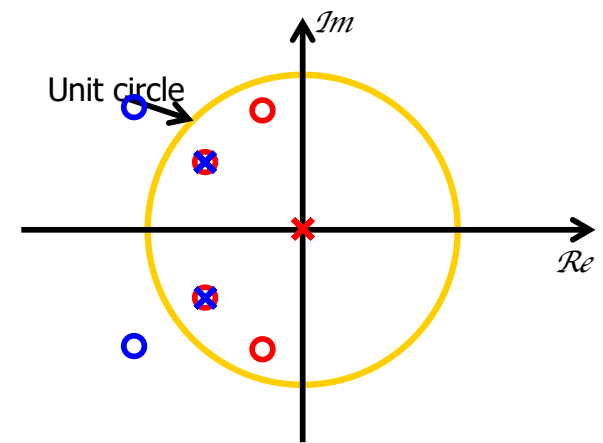
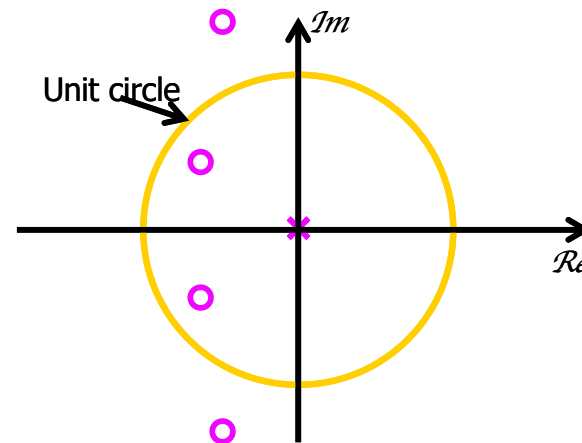
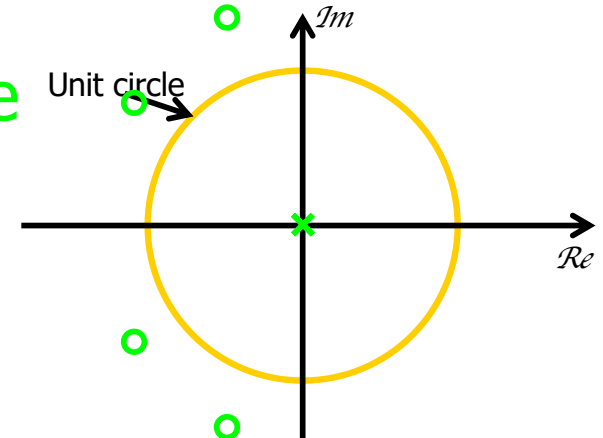


Minimum Phase

Min phase

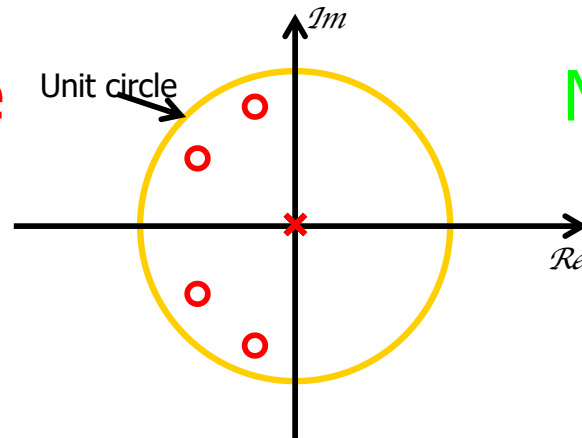


Max phase

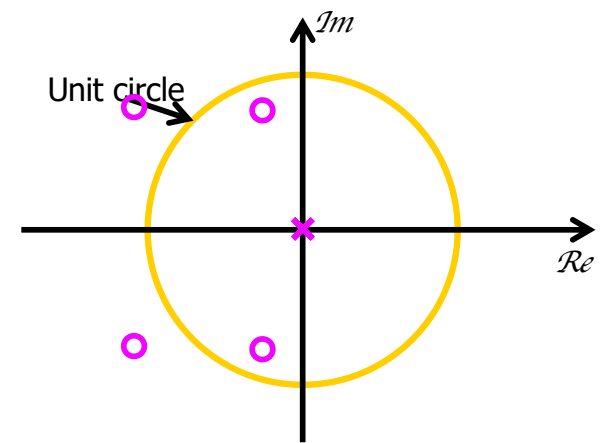
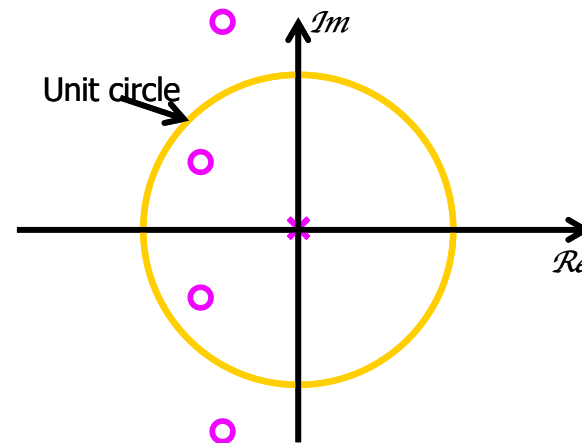
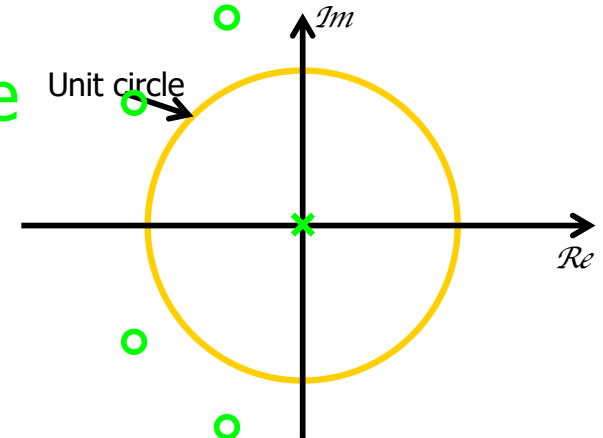


Minimum Phase

Min phase



Max phase





Minimum Phase Lag Property

- All pass properties
 - $\arg[H_{ap}(e^{j\omega})] \leq 0$
 - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

Minimum Phase Lag Property

□ All pass properties

■ $\arg[H_{ap}(e^{j\omega})] \leq 0$

■ $\text{grd}[H_{ap}(e^{j\omega})] > 0$

□
$$\begin{aligned}\arg[H_{\max}(e^{j\omega})] &= \arg[H_{\min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})] \\ &= \arg[H_{\min}(e^{j\omega})] + \leq 0\end{aligned}$$

Minimum Group Delay Property

□ All pass properties

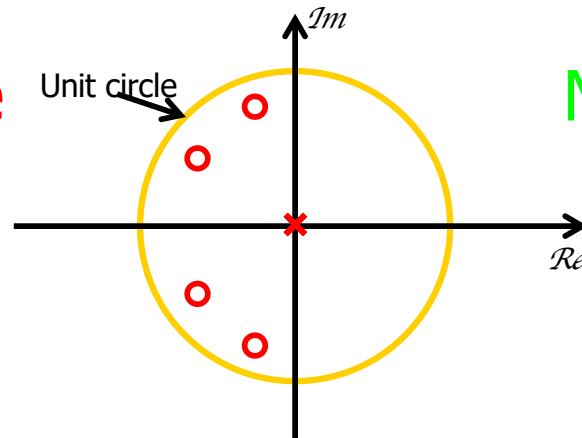
■ $\arg[H_{ap}(e^{j\omega})] \leq 0$

■ $\text{grd}[H_{ap}(e^{j\omega})] > 0$

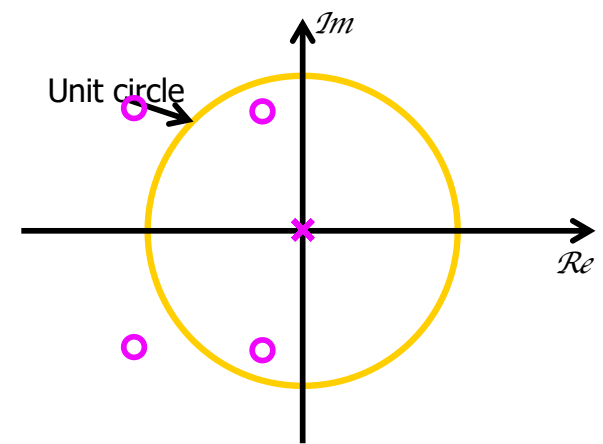
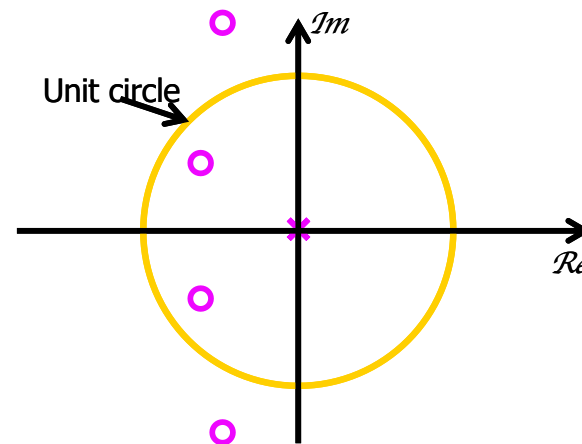
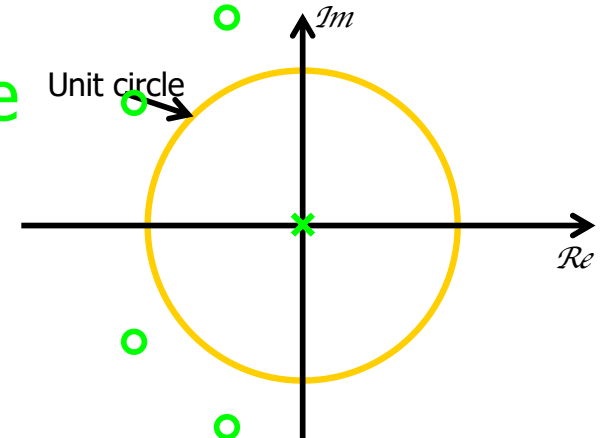
□
$$\begin{aligned}\text{grd}[H_{\max}(e^{j\omega})] &= \text{grd}[H_{\min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \text{grd}[H_{\min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})] \\ &= \text{grd}[H_{\min}(e^{j\omega})] + \geq 0\end{aligned}$$

Minimum Phase

Min phase

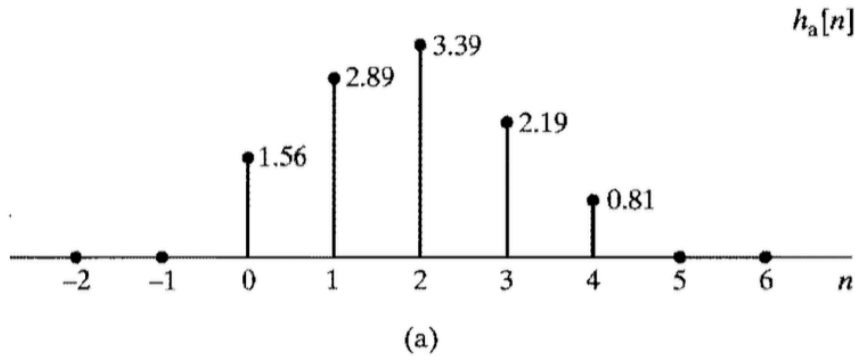


Max phase

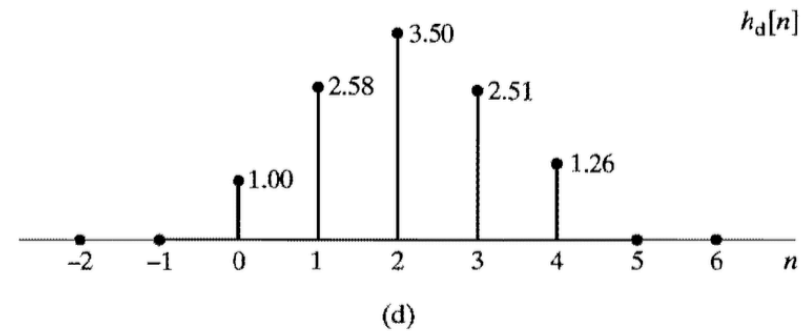
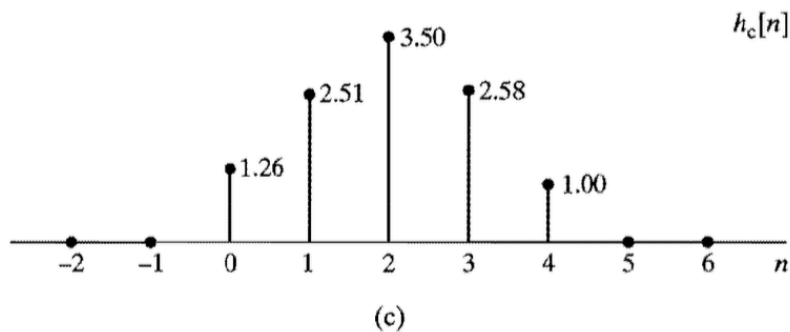
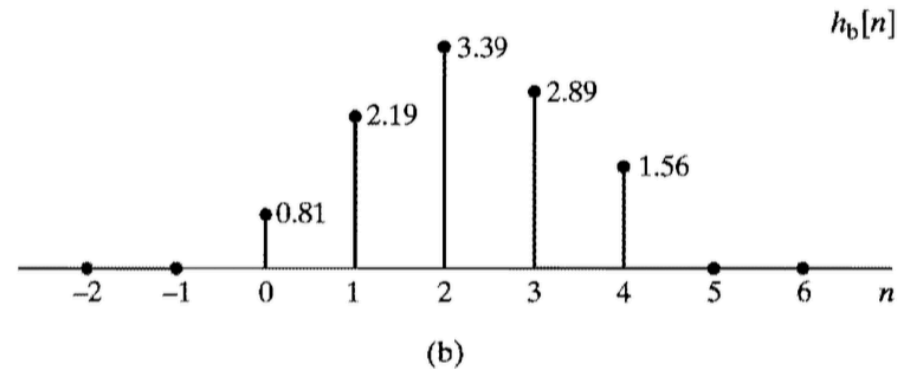


Minimum Energy-Delay Property

Min phase



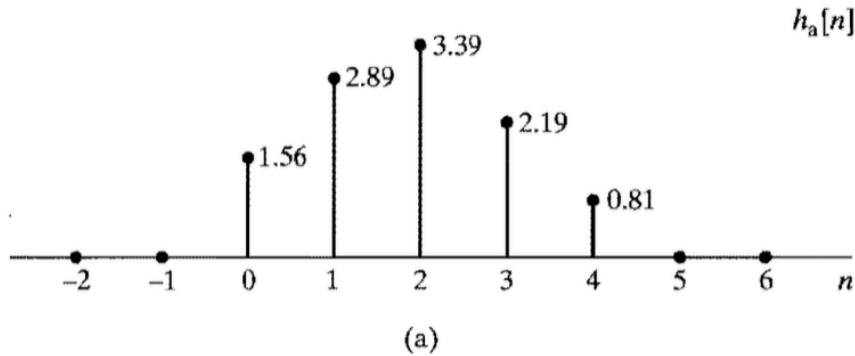
Max phase



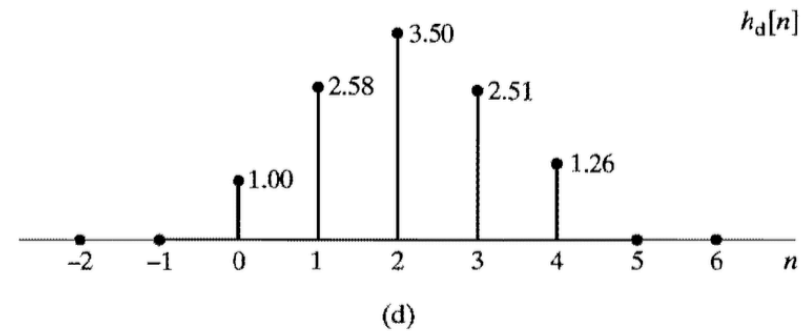
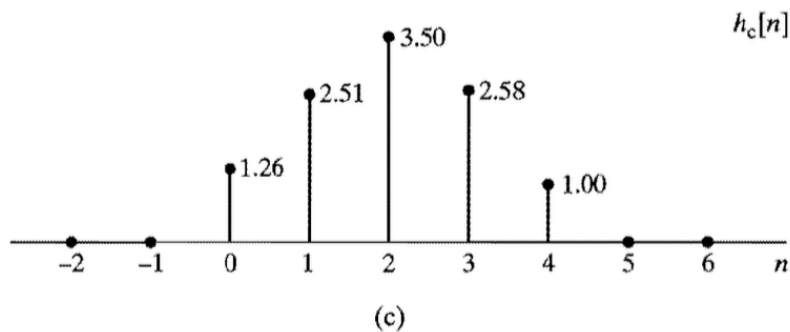
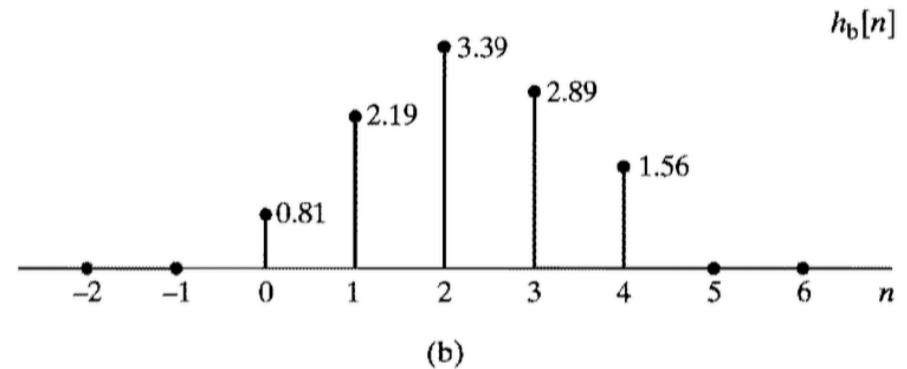
Total energy: $E = \sum_{n=0}^{\infty} |h[n]|^2$

Minimum Energy-Delay Property

Min phase

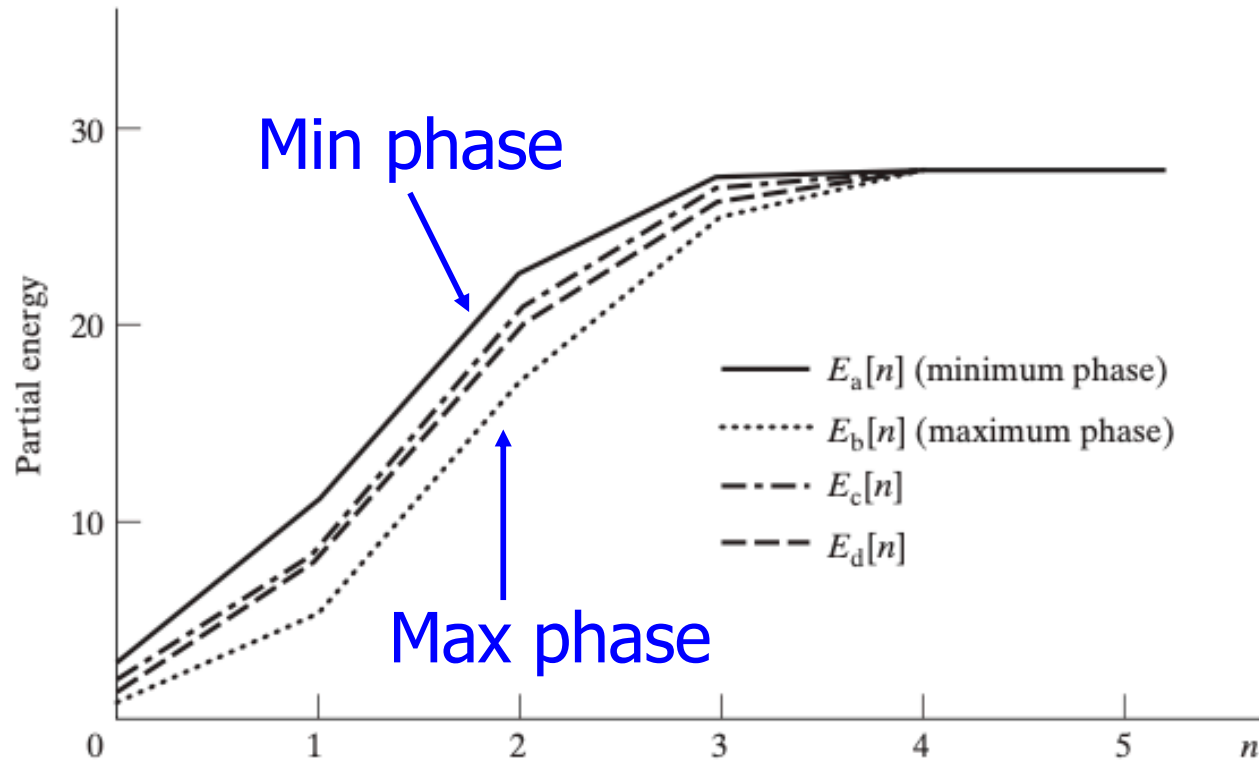


Max phase



Partial energy: $E[n] = \sum_{m=0}^n |h[m]|^2$

Minimum Energy-Delay Property



Partial energy: $E[n] = \sum_{m=0}^n |h[m]|^2$



Big Ideas

- ❑ Frequency Response of LTI Systems
 - Magnitude Response, Phase Response, Group Delay
- ❑ Review: LTI Stability and Causality
 - If all poles inside unit circle
- ❑ All Pass Systems
 - Used for delay compensation
- ❑ Minimum Phase Systems
 - Can compensate for magnitude distortion
 - Minimum energy-delay property



Admin

- HW 6 (UPDATE)
 - Due Sunday 3/13 (after spring break now)