ESE 531: Digital Signal Processing

Lecture 16: March 15, 2022

Design of IIR Filters

Linear Filter Design

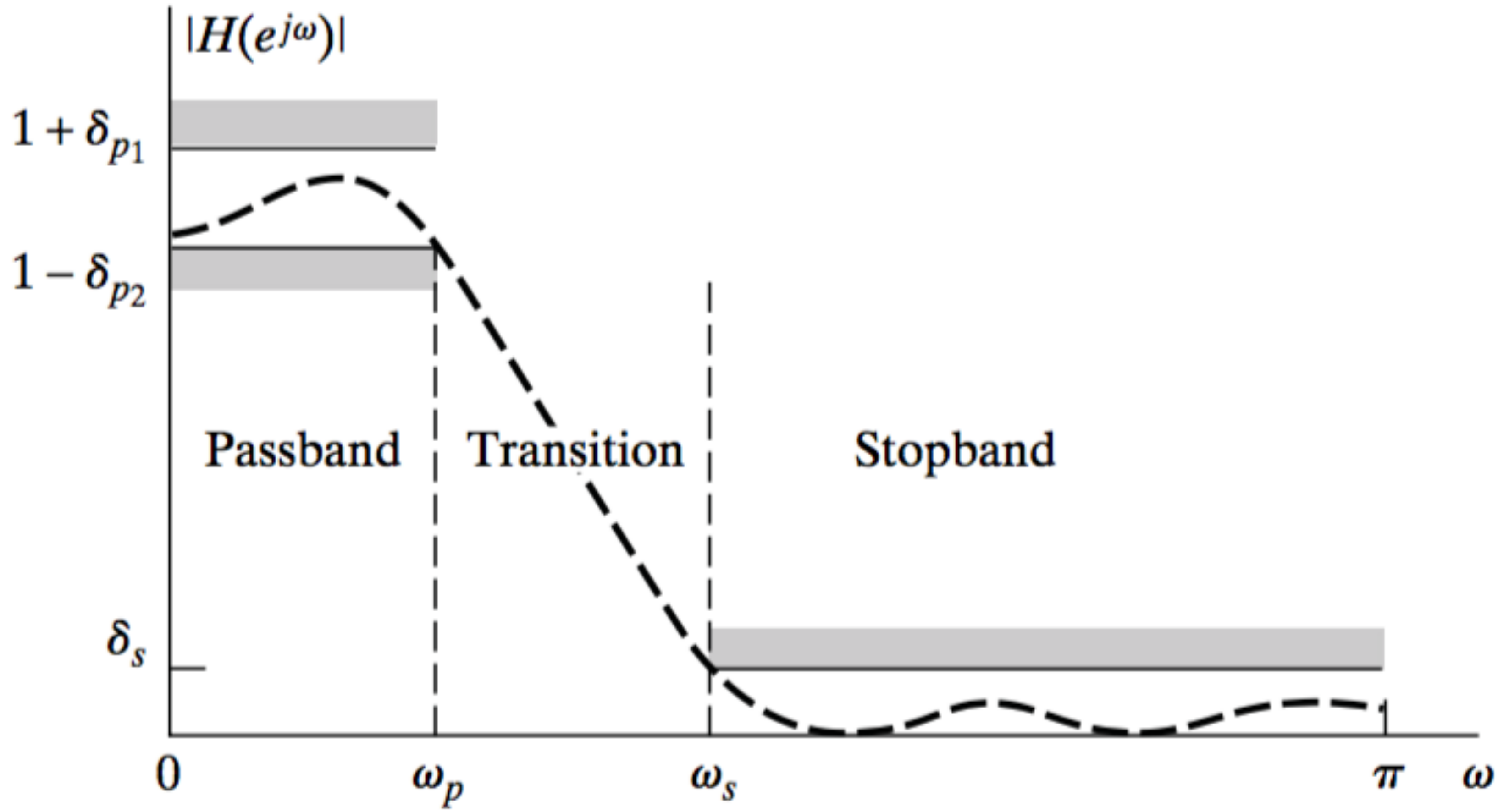
- Used to be an ambiguous process
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- Both classes use finite order of parameters for design
  - Filter order (ie. Length) restricts filter design

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- What does it mean to design a filter?
  - Determine the parameters of a transfer function or difference equation that approximates a desired impulse response ($h[n]$) or frequency response ($H(e^{j\omega})$).

Filter Specifications



What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

Today

- IIR Filter Design
  - Impulse Invariance
  - Bilinear Transformation
- Transformation of DT Filters

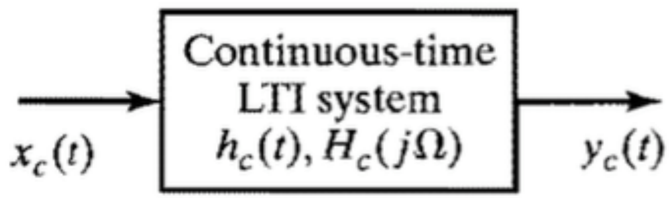
IIR Filter Design

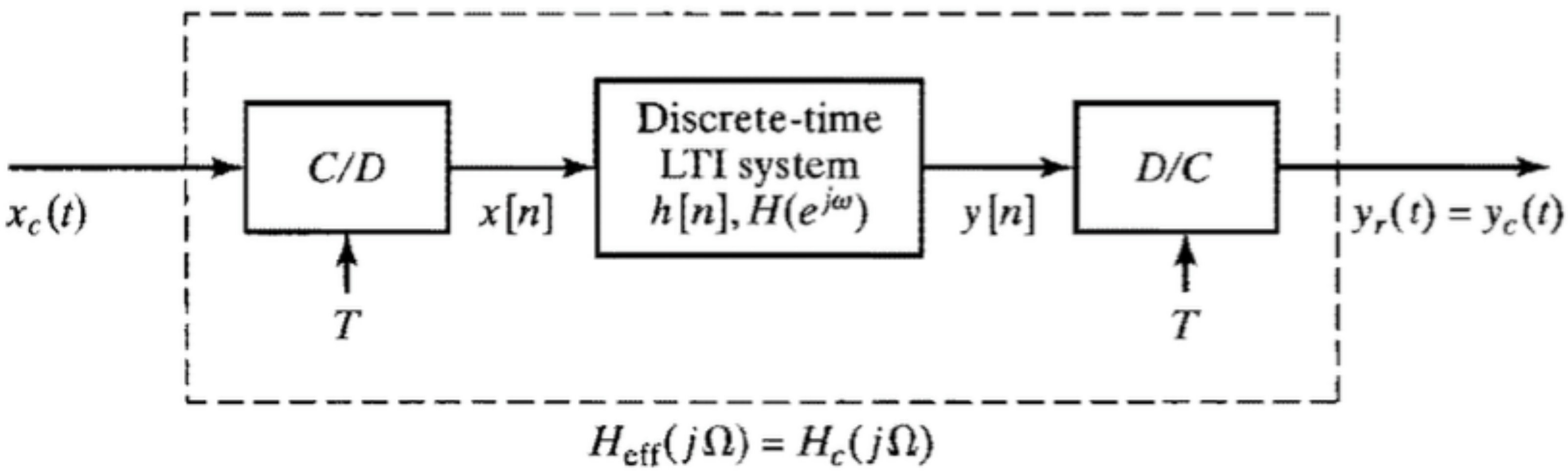
- Transform continuous-time filter into a discrete-time filter meeting specs
  - Pick suitable transformation from s (Laplace variable) to z (or t to n)
  - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$

- We've seen this before… impulse invariance

Impulse Invariance

- Want to implement continuous-time system in discrete-time





Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

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$$\boxed{h[n] = Th_c(nT)}$$

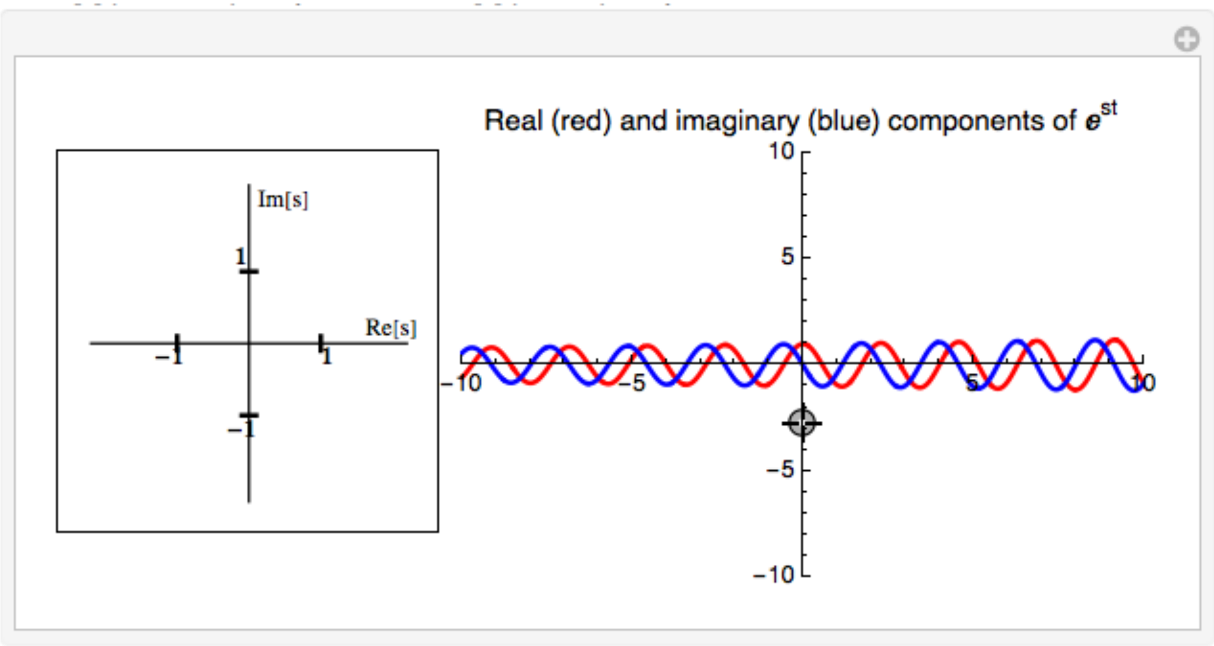
Laplace Transform

- The Laplace transform takes a function of time, t, and transforms it to a function of a complex variable, s.
- Because the transform is invertible, no information is lost and it is reasonable to think of a function f(t) and its Laplace transform F(s) as two views of the same phenomenon.
- Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

S-Plane

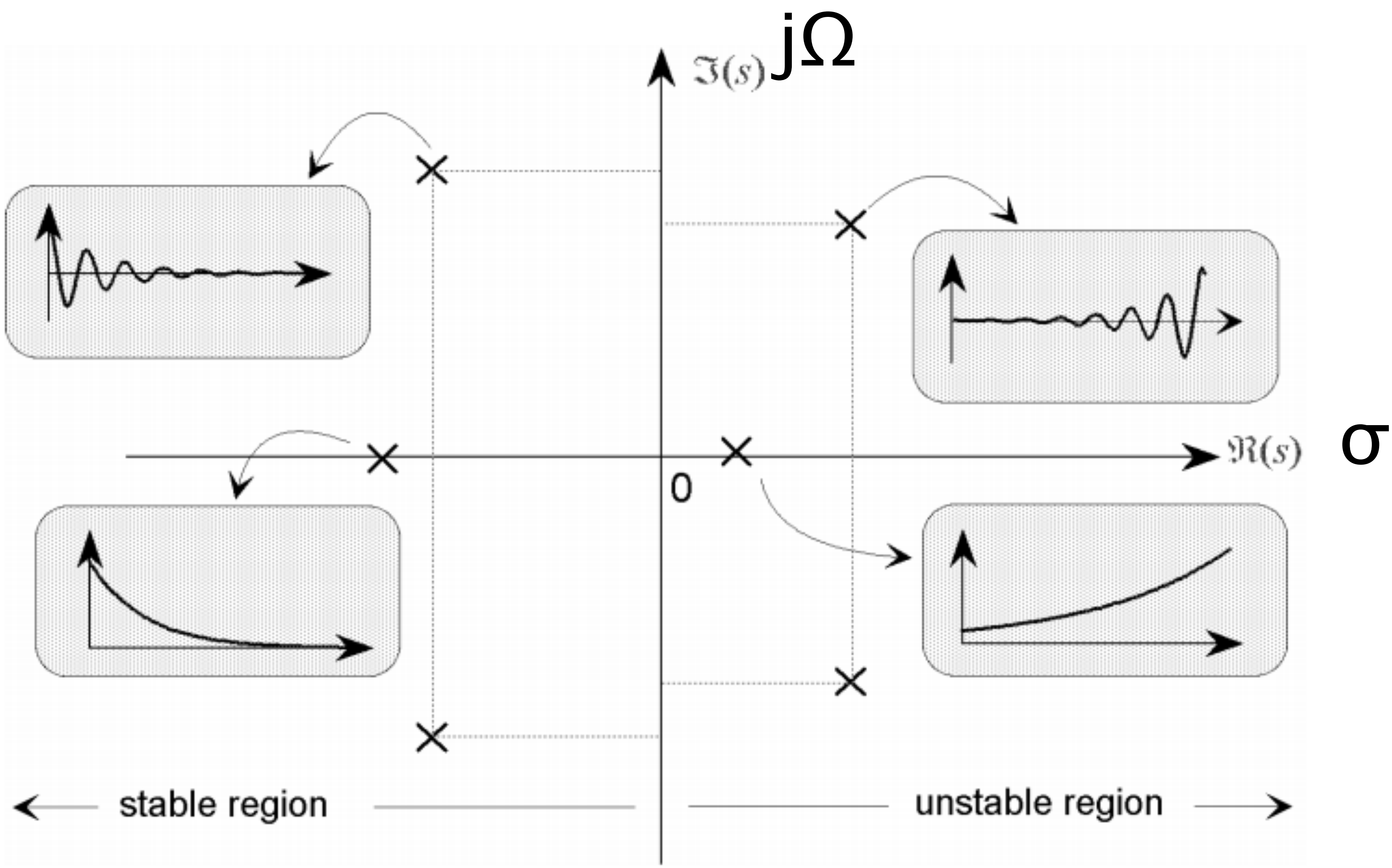
- s=σ+jΩ

- Wolfram Demo

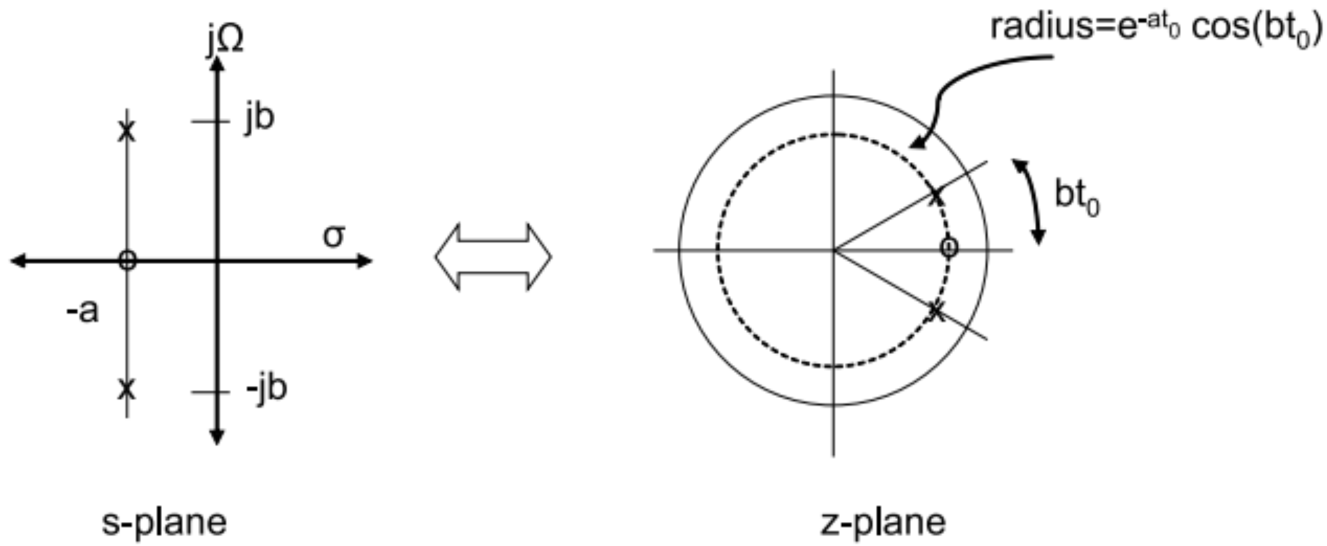


- http://pilot.cnxproject.org/content/collection/col10064/latest/module/m10060/latest

S-plane and stability



S-Plane Mapping to Z-Plane



Example

Example: If $\boxed{H_c(s) = \dfrac{A_k}{s - p_k}}$

Laplace: $$e^{at} \overset{L}{\longleftrightarrow} \frac{1}{s-a}$$

Z-transform: $$a^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$

Example

Example: If $\boxed{H_c(s) = \dfrac{A_k}{s - p_k}}$

$$e^{at} \xleftrightarrow{L} \frac{1}{s-a}$$

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}$$

Example

$$Example: \text{ If } \boxed{H_c(s) = \frac{A_k}{s - p_k}} \quad \text{(e.g. one term in PF expansion)}$$

$$h_c(t) = A_k e^{p_k t}, \quad t \geq 0; \qquad h[n] = T_d A_k e^{p_k T_d n} = T_d A_k \left(e^{p_k T_d}\right)^n$$

$$\therefore \quad \boxed{H(z) = T_d A_k \frac{1}{1 - e^{p_k T_d} z^{-1}}} \qquad \text{Pole mapping is } z \leftarrow e^{sT_d}$$

Zeros do *not* map the same way; *not* the general mapping of s to z

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$\therefore \quad \boxed{H(z) = T_d A_k \frac{1}{1 - e^{p_k T_d} z^{-1}}}$ Pole mapping is $z \leftarrow e^{sT_d}$

Zeros do *not* map the same way; *not* the general mapping of s to z

- Stability, causality, preserved.
- $j\Omega$ axis mapped linearly to unit-circle, with aliasing
- No control of zeros or of phase

Impulse Invariance

- Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
  - $z = e^{sTd} = r\ e^{j\omega}$
- The entire $\boldsymbol{\Omega}$ axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times

Impulse Invariance

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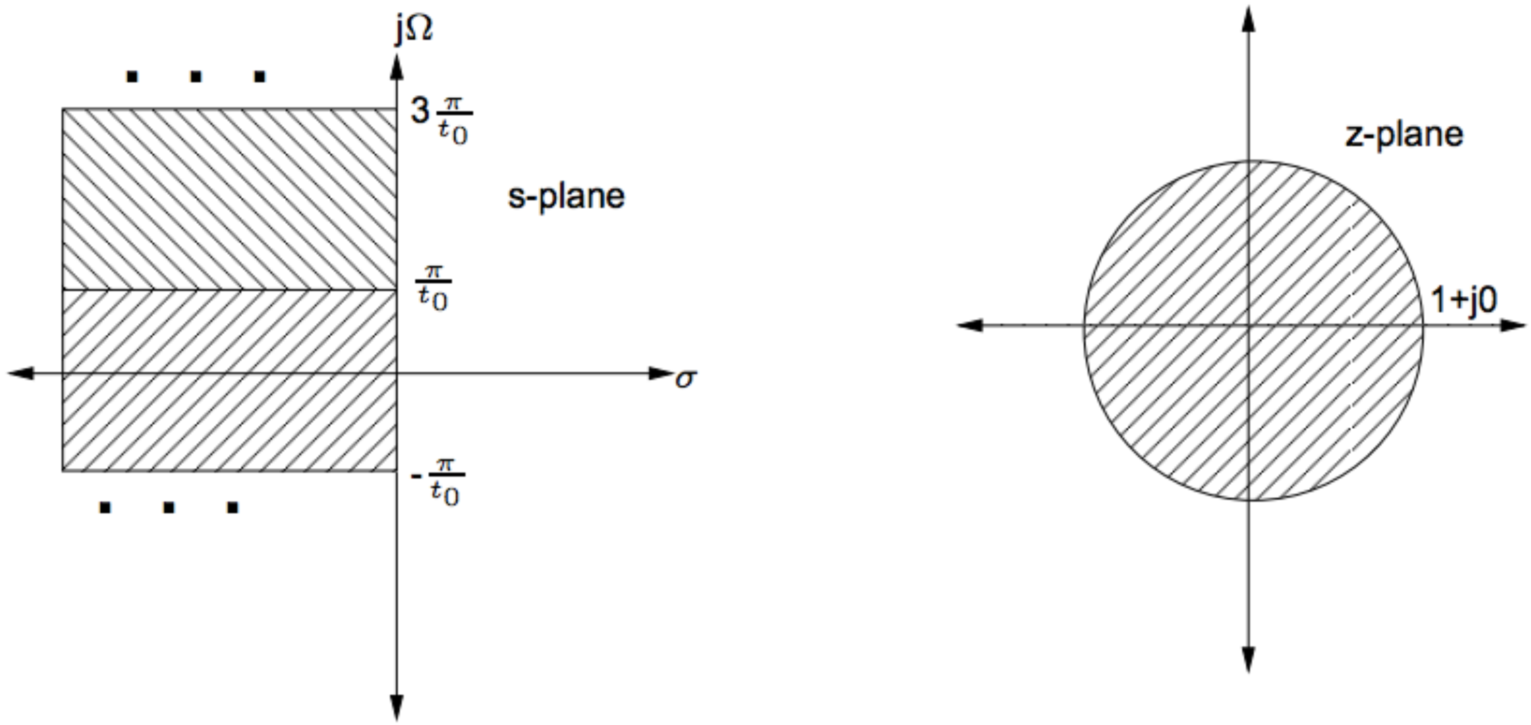
- The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior

Impulse Invariance Mapping

Mapping

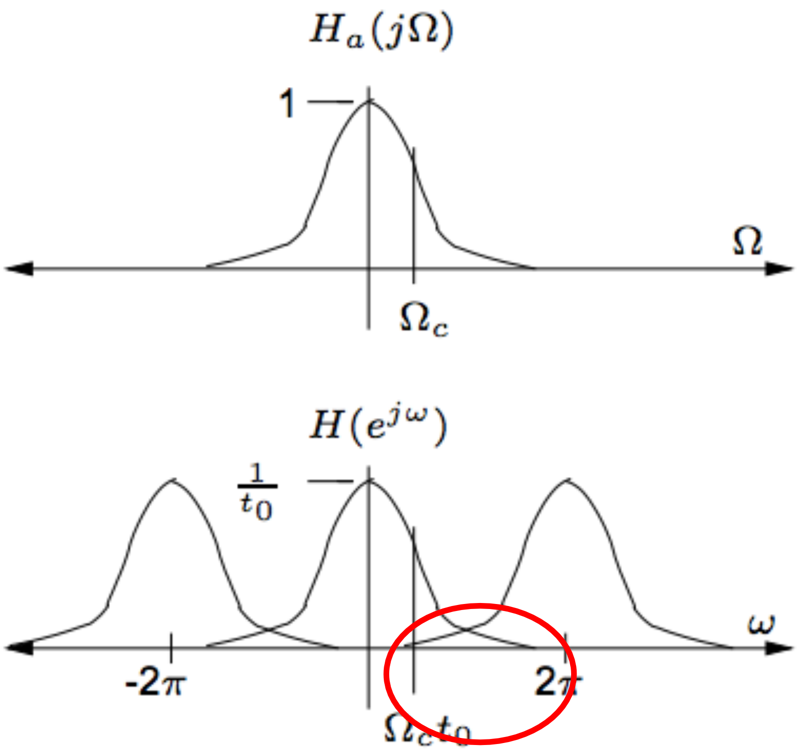


Impulse Invariance

- Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
  - $z = e^{sT_d} = r\,e^{j\omega}$
- The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times
- The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior
- This means stable analog filters (poles in LHP) will transform to stable digital filters (poles inside unit circle)
- This is a many-to-one mapping of strips of the s-plane to regions of the z-plane
  - Not a conformal mapping
  - The poles map according to $z = e^{sT_d}$, but the zeros do not always

Impulse Invariance

- Limitation of Impulse Invariance: overlap of images of the frequency response. This prevents it from being used for high-pass filter design



Bilinear Transformation

- The technique uses an algebraic transformation between the variables *s* and *z* that maps the entire jΩ-axis in the s-plane to one revolution of the unit circle in the z-plane.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right).$$

Bilinear Transformation

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

- Substituting $s = \sigma + j\,\Omega$ and $z = e^{j\omega}$

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- Substituting $s = \sigma + j\,\Omega$ and $z = e^{j\omega}$

$$s = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right),$$

$$s = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j\sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = \frac{2j}{T_d}\tan(\omega/2).$$

Bilinear Transformation

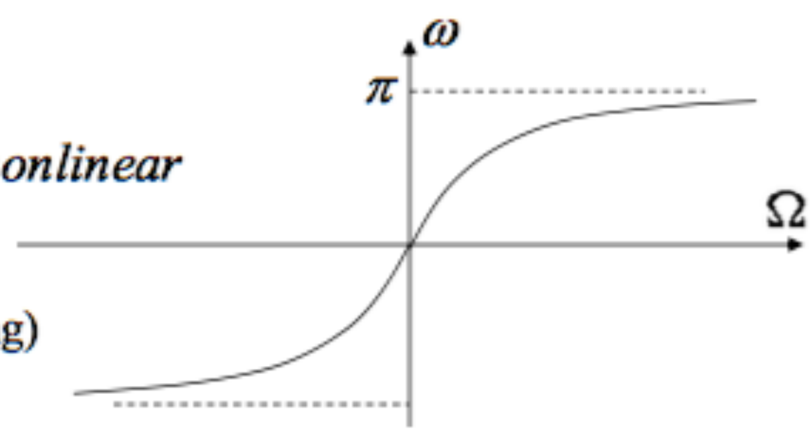
$$\Omega = \frac{2}{T_d}\tan(\omega/2),$$

$$\omega = 2\arctan(\Omega T_d/2).$$

No aliasing, but mapping nonlinear

(Impulse invariance:

linear mapping, but with aliasing)



Example: Notch Filter

- The continuous time filter with:

$$H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + Bs + \Omega_0^2}$$

$$\Omega = \frac{2}{T_d} \tan(\omega/2).$$

$$\omega = 2 \arctan(\Omega T_d/2).$$

Example: Notch Filter

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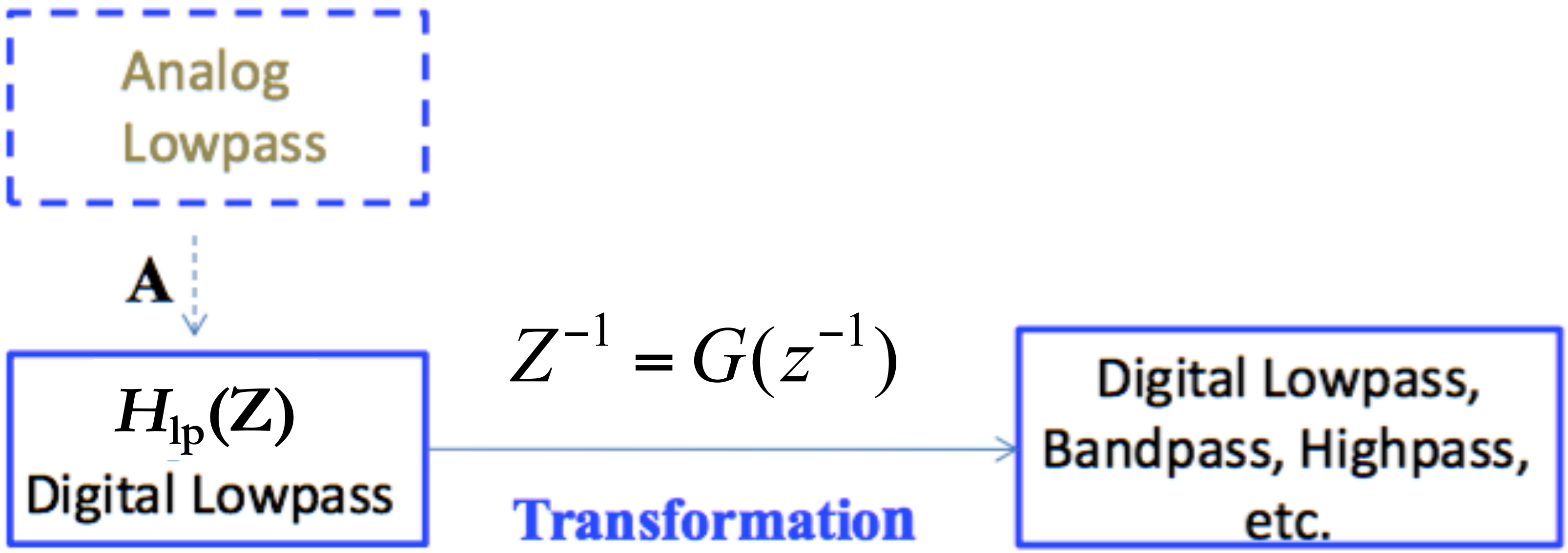
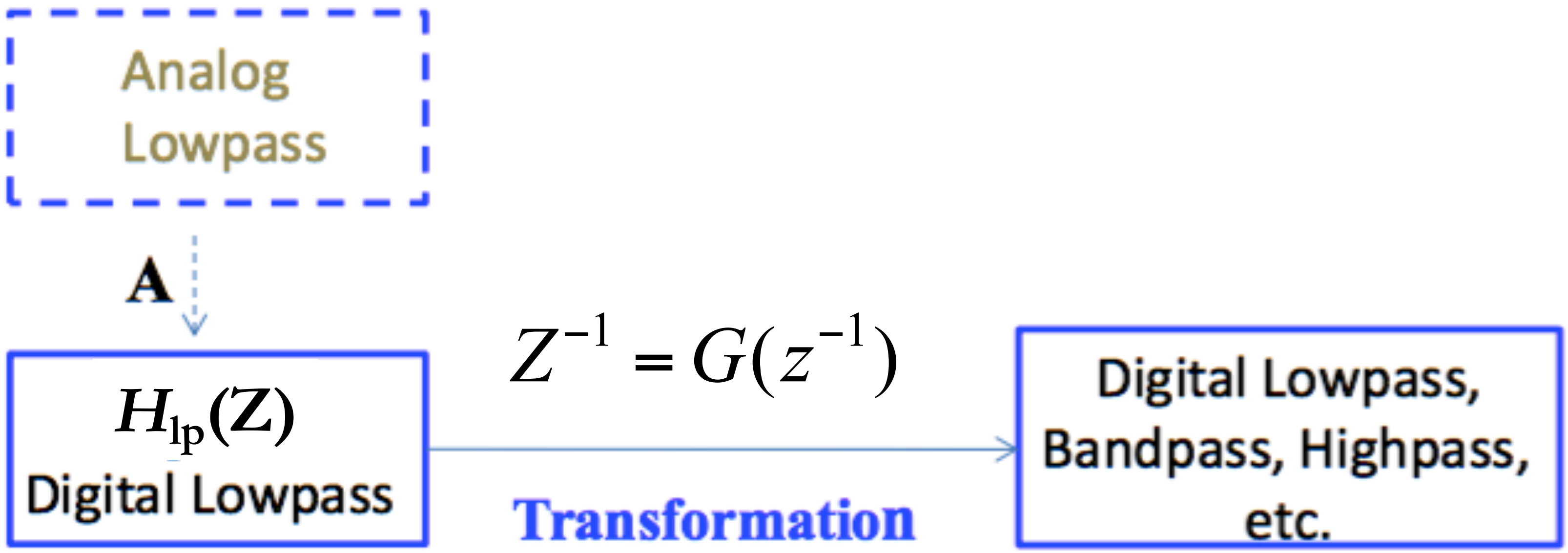
Transformation of DT Filters



- Z – complex variable for the LP filter
- z – complex variable for the transformed filter

- Map Z-plane→z-plane with transformation G

Transformation of DT Filters



- Map Z-plane→z-plane with transformation G

$$H(z) = H_{\mathrm{lp}}(Z)\Big|_{Z^{-1}=G(z^{-1})}$$

Example 1:

- Lowpass→highpass
  - Shift frequency by π

so $\omega \rightarrow \omega - \pi$ (Lowpass to highpass)

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$$G(z^{-1}) = -z^{-1} \quad \text{or} \quad e^{-j\omega} \rightarrow e^{-j(\omega-\pi)}$$

Example 1:

- Lowpass→highpass
  - Shift frequency by π

$$G(z^{-1}) = -z^{-1}$$

$\boldsymbol{\omega}$	$\mathbf{z}$	$\lvert H_{lp}(z)\rvert = \left\lvert\frac{0.1}{1-0.9z^{-1}}\right\rvert$	$\lvert H_{hp}(z)\rvert = \left\lvert\frac{0.1}{1+0.9z^{-1}}\right\rvert$
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			

Example 2:

- Lowpass→bandpass

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$$H_{lp}(z) = \frac{1}{1-az^{-1}} \longrightarrow H_{bp}(z) = \frac{1}{1+az^{-2}}$$

Pole at z=a

Pole at z=±j√a

Example 2:

- Lowpass→bandpass

$$G(z^{-1}) = -z^{-2}$$

$\boldsymbol{\omega}$	**z**	$\lvert H_{lp}(z)\rvert = \left\lvert\frac{0.1}{1-0.9z^{-1}}\right\rvert$	$\lvert H_{hp}(z)\rvert = \left\lvert\frac{0.1}{1+0.9z^{-2}}\right\rvert$
0	1	1	
$\frac{\pi}{2}$	j	0.074	
π	-1	0.05	
$\frac{3\pi}{2}$	-j	0.074	
2π	1	1	

Example 3:

- Lowpass→bandstop

$$Z^{-1} = G(z^{-1}) = z^{-2}$$

$$H_{lp}(z) = \frac{1}{1-az^{-1}} \longrightarrow H_{bs}(z) = \frac{1}{1-az^{-2}}$$

Pole at z=±√a

Transformation Constraints on $G(z^{-1})$

- If $H_{lp}(Z)$ is the rational system function of a causal and stable system, we naturally require that the transformed system function *H(z)* be a rational function and that the system also be causal and stable.
  - $G(Z^{-1})$ must be a rational function of z^{-1}
  - The inside of the unit circle of the Z-plane must map to the inside of the unit circle of the z-plane
  - The unit circle of the Z-plane must map onto the unit circle of the z-plane.

Transformation Constraints on $G(z^{-1})$

- Respective unit circles in both planes

$$Z = e^{j\theta} \text{ and } z = e^{j\omega}$$

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- Respective unit circles in both planes

$$Z = e^{j\theta} \text{ and } z = e^{j\omega}$$

$$Z^{-1} = G(z^{-1})$$

$$e^{-j\theta} = G(e^{-j\omega})$$

$$e^{-j\theta} = \left|G(e^{-j\omega})\right| e^{j\angle G(e^{-j\omega})}$$

Transformation Constraints on $G(z^{-1})$

- Respective unit circles in both planes

$$Z = e^{j\theta} \text{ and } z = e^{j\omega}$$

$$Z^{-1} = G(z^{-1})$$

$$e^{-j\theta} = G(e^{-j\omega})$$

$$e^{-j\theta} = \left|G(e^{-j\omega})\right| e^{j\angle G(e^{-j\omega})}$$

$$\boxed{1 = \left|G(e^{-j\omega})\right| \qquad -\theta = \angle G(e^{-j\omega})}$$

Transformation Constraints on $G(z^{-1})$

- General form that meets all constraints:
  - a_k real and $|a_k|<1$

$$G(z^{-1}) = \pm\prod_{k=1}^{N}\frac{z^{-1}-\alpha_k}{1-\alpha_k z^{-1}}$$

General Transformation

- Lowpass→lowpass

$$G(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

- Changes passband/stopband edge frequencies

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$$\text{From } e^{-j\theta} = \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}}, \text{ get}$$

$$\omega(\theta) = \tan^{-1}\left(\frac{(1-\alpha^2)\sin(\theta)}{2\alpha + (1+\alpha^2)\cos(\theta)}\right)$$

General Transformation

- Lowpass→lowpass

$$G(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

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From $e^{-j\theta} = \dfrac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}}$, get

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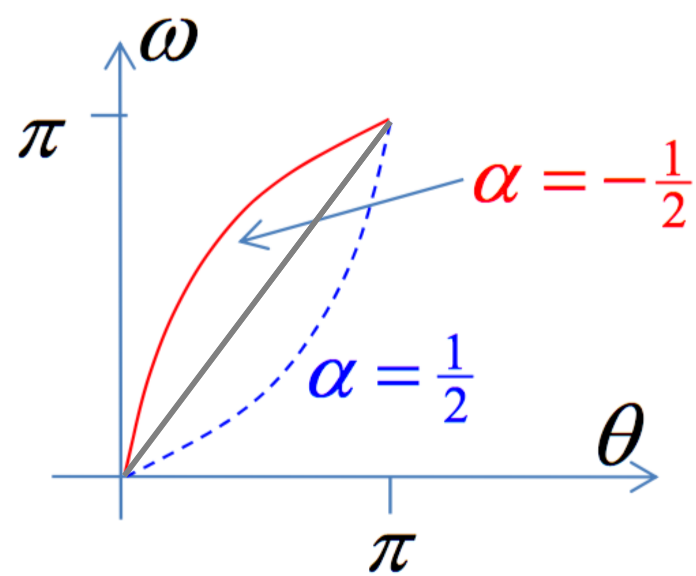


General Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

CT Filters

- Butterworth
  - Monotonic in pass and stop bands
- Chebyshev, Type I
  - Equiripple in pass band and monotonic in stop band
- Chebyshev, Type II
  - Monotonic in pass band and equiripple in stop band
- Elliptic
  - Equiripple in pass and stop bands

- Appendix B in textbook

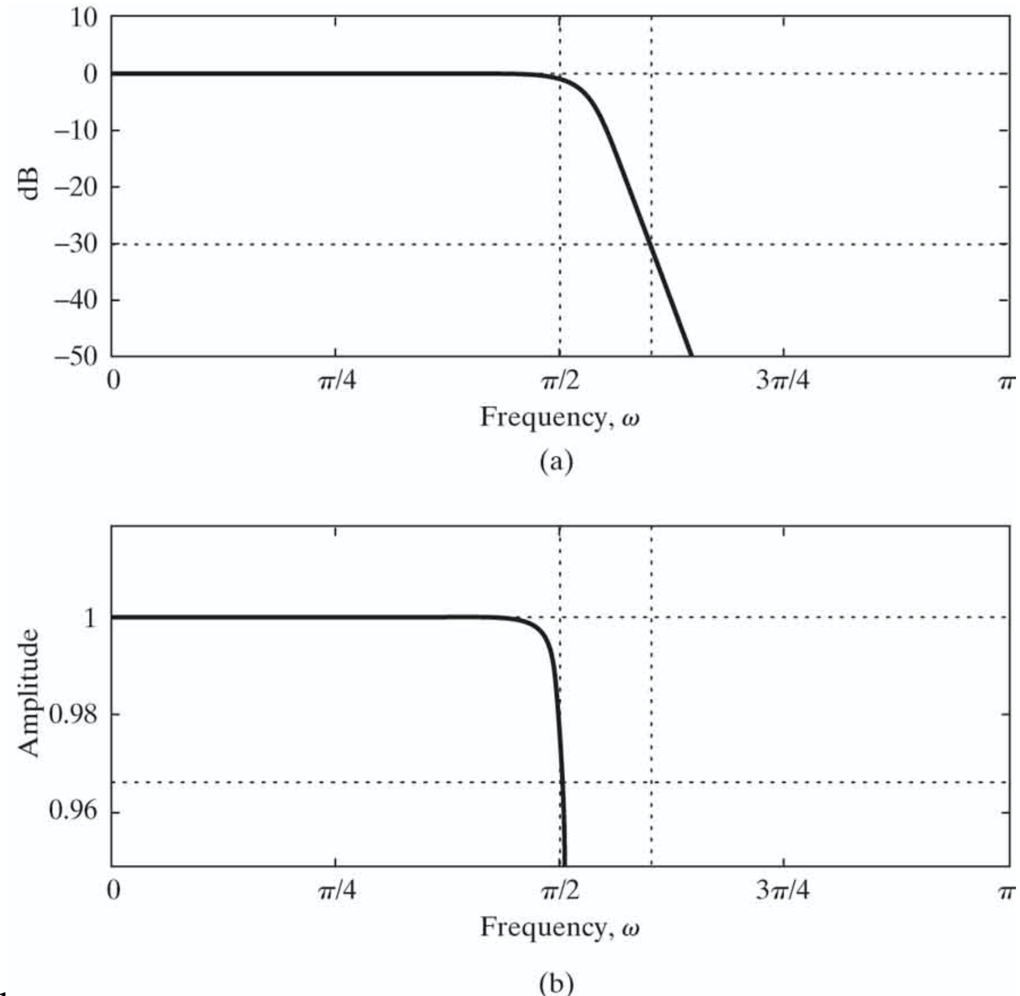
Design Comparison

- Design specifications
  - passband edge frequency $\omega_p = 0.5\pi$
  - stopband edge frequency $\omega_s = 0.6\pi$
  - maximum passband gain = 0 dB
  - minimum passband gain = -0.3dB
  - maximum stopband gain =-30dB

- Use bilinear transformation to design DT low pass filter for each type

Butterworth

- Butterworth
  - Monotonic in pass and stop bands

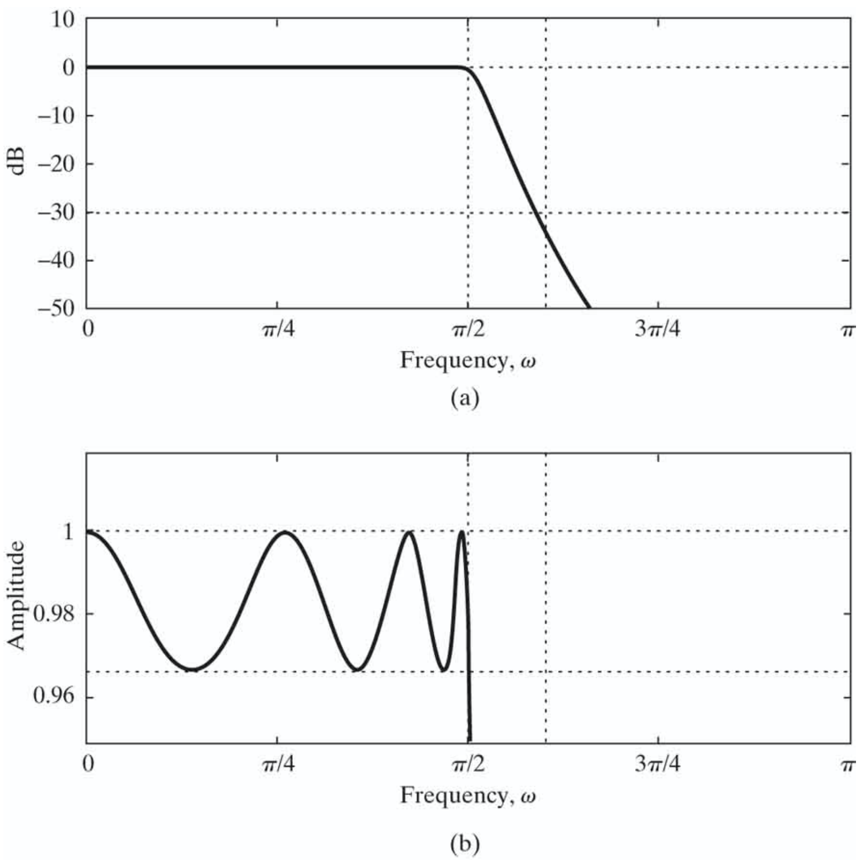


(a)

(b)

Chebyshev

- Type I
  - Equiripple in pass band and monotonic in stop band



(a)

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Chebyshev

- Type I
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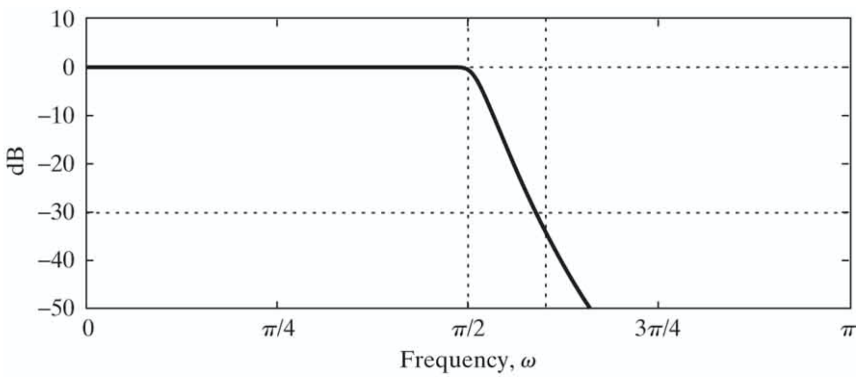
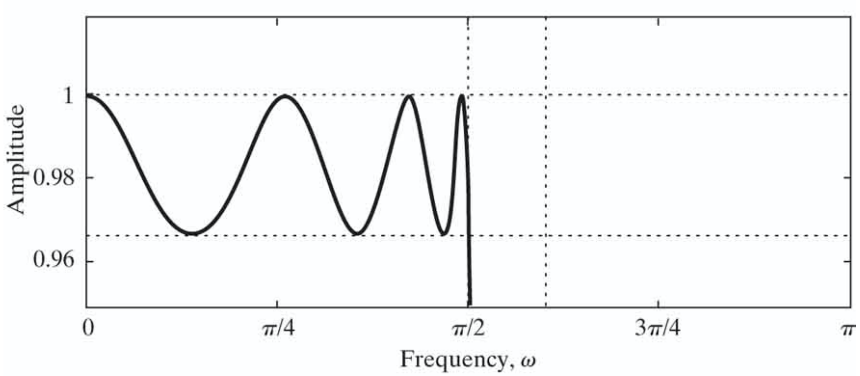


(a)



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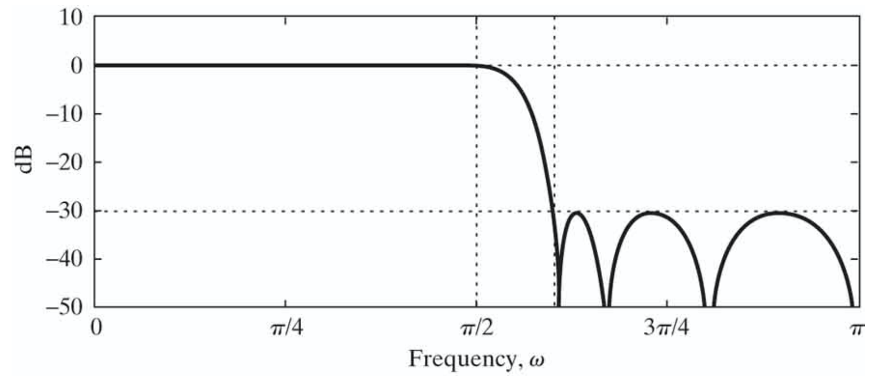

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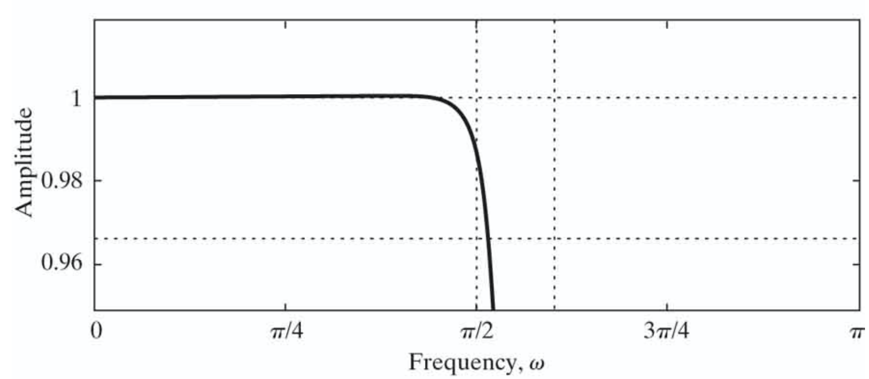

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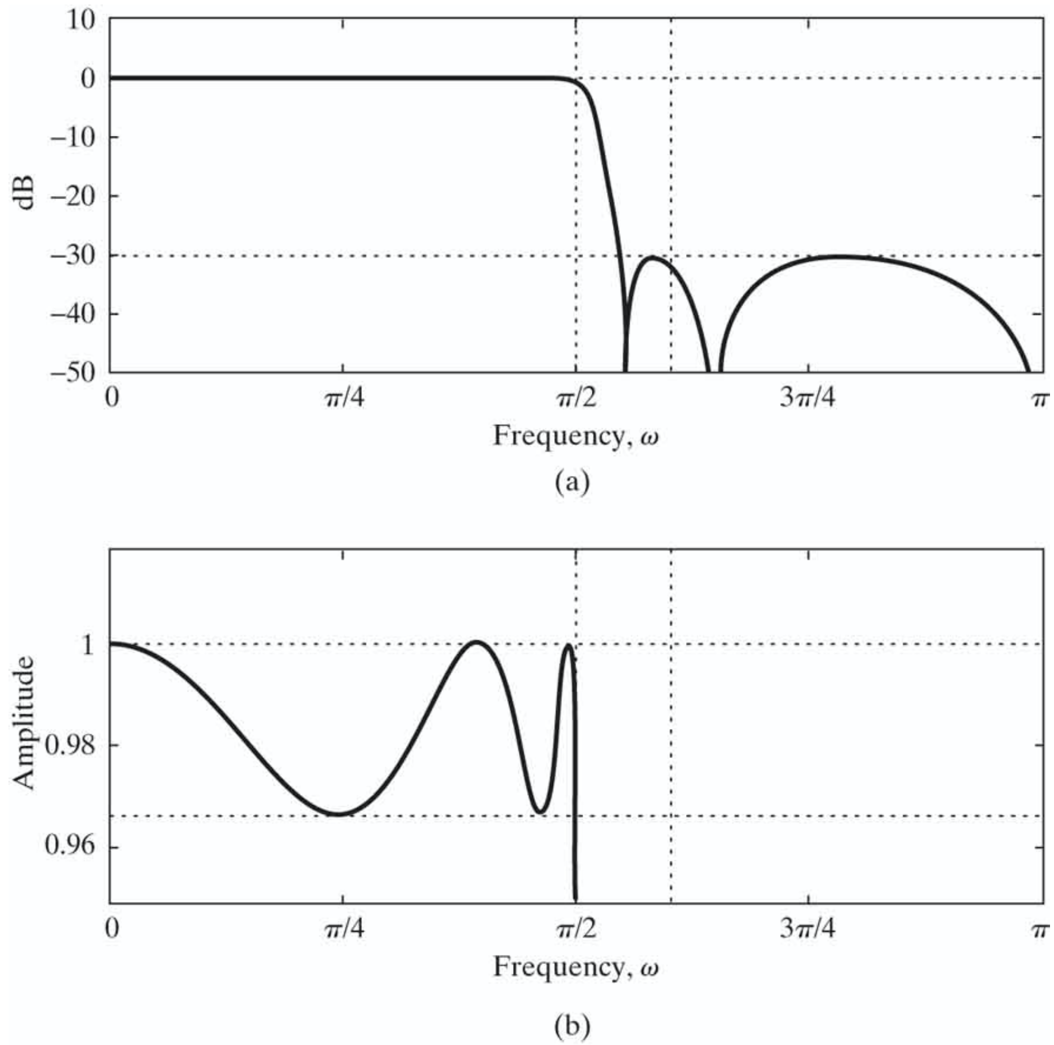
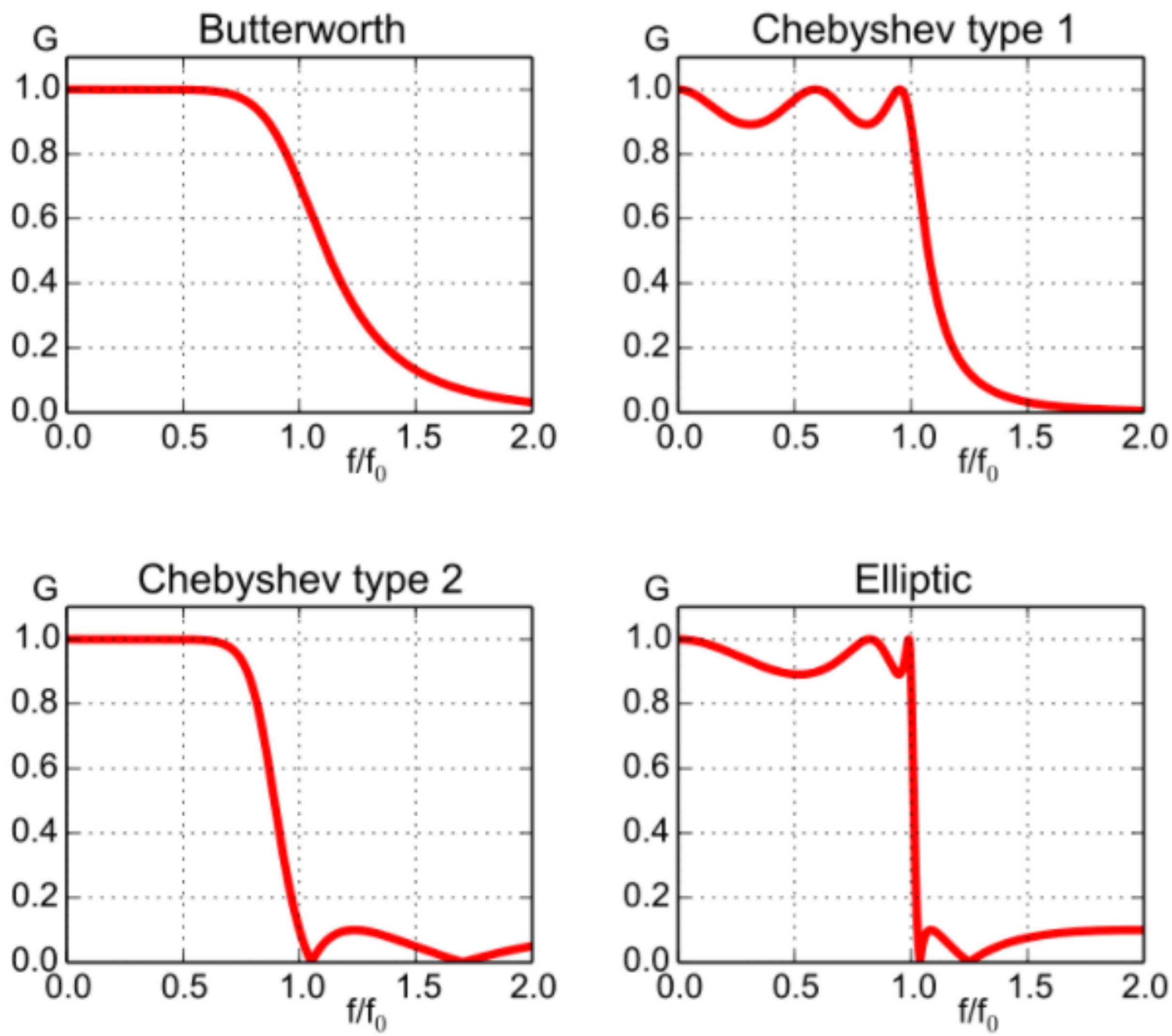
Elliptic

- Elliptic
  - Equiripple in pass and stop bands



Comparisons



Big Idea

- IIR
  - Design from continuous time filters with mapping of s-plane onto z-plane
    - Linear mapping – impulse invariance
    - Non-linear mapping – bilinear transformation
- DT filter transformations
  - Transform z-plane with rational function $G(z^{-1})$
    - Constraints on G for causal/stable systems

Admin

- Midterm
  - Thursday 3/17 in person during class in DRLB A1
  - Covers lectures 1-14
    - Doesn’t cover data converters and noise shaping
  - Old exams online
    - Disclaimer: 2020/2021 had different exam coverage
  - Closed book/notes
  - Can bring 1 double-sided 8”x11” cheat sheet and non-cell phone calculator