

ESE 531: Digital Signal Processing

Lecture 3: January 20, 2022

Discrete Time Signals and Systems, Pt 2



Lecture Outline

- ❑ Discrete Time Systems
- ❑ System Properties
- ❑ LTI Systems
- ❑ Difference Equations

Discrete-Time Systems

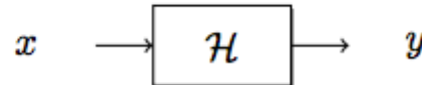


Discrete Time Systems

DEFINITION

A discrete-time **system** \mathcal{H} is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- ❑ Systems manipulate the information in signals
- ❑ Examples
 - Speech recognition system that converts acoustic waves into text
 - Radar system transforms radar pulse into position and velocity
 - fMRI system transform frequency into images of brain activity
 - Moving average system smooths out the day-to-day variability in a stock price



System Properties

❑ Causality

- $y[n]$ only depends on $x[m]$ for $m \leq n$

❑ Linearity

- Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
 - $Ax_1[n] + Bx_2[n] \rightarrow Ay_1[n] + By_2[n]$

❑ Memoryless

- $y[n]$ depends only on $x[n]$

❑ Time Invariance

- Shifted input results in shifted output
 - $x[n-q] \rightarrow y[n-q]$

❑ BIBO Stability

- A bounded input results in a bounded output (ie. max signal value exists for output if max)



Examples

□ Causal? Linear? Time-invariant? Memoryless?
BIBO Stable?

□ Time Shift:

- $y[n] = x[n - m]$

□ Accumulator:

- $$y[n] = \sum_{k=-\infty}^n x[k]$$

□ Compressor ($M > 1$):

$$y[n] = x[Mn]$$



Non-Linear System Example

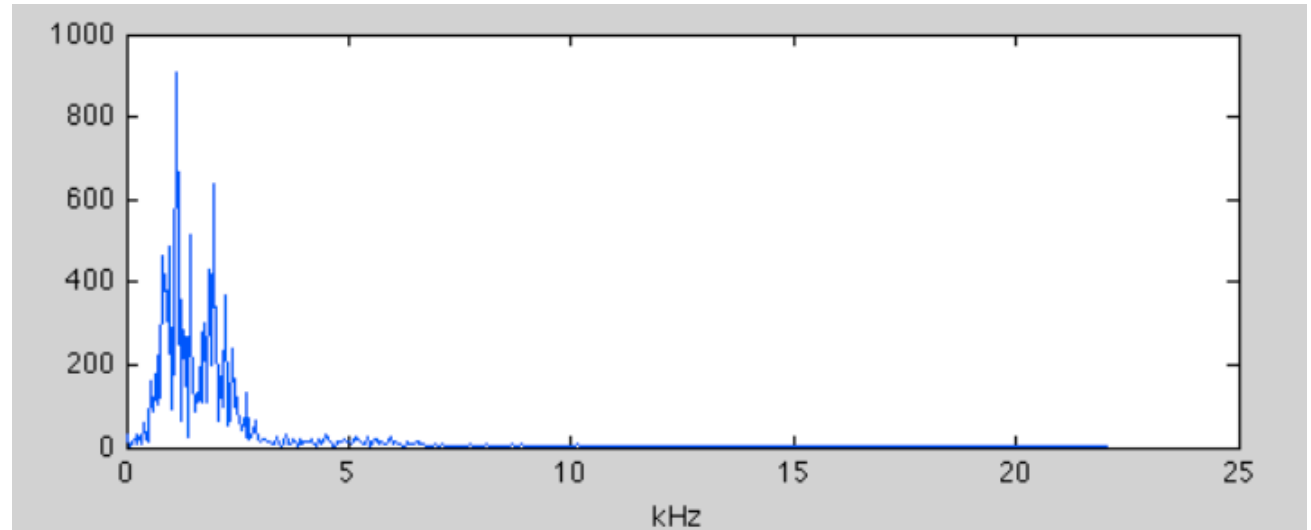
□ Median Filter

- $y[n] = \text{MED} \{x[n-k], \dots, x[n+k]\}$
- Let $k=1$
- $y[n] = \text{MED} \{x[n-1], x[n], x[n+1]\}$



Spectrum of Speech

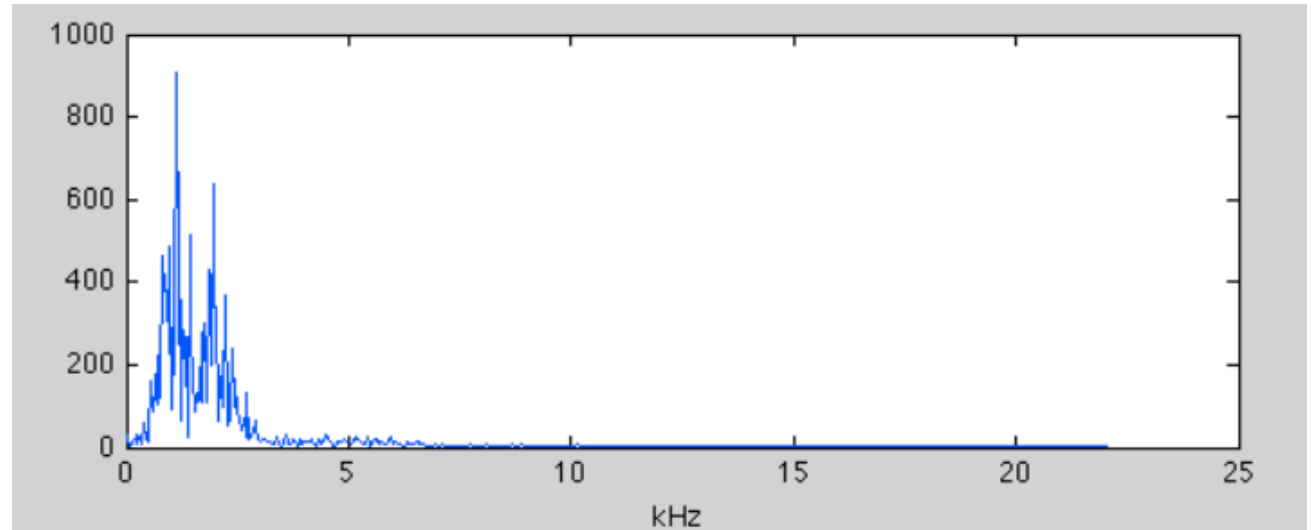
Speech



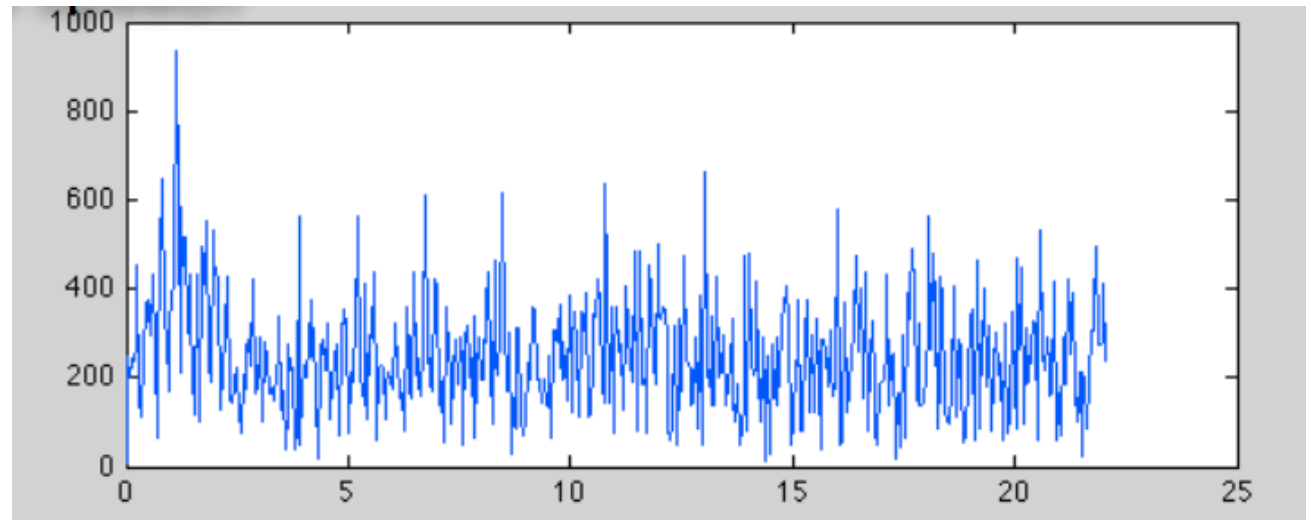


Spectrum of Speech

Speech

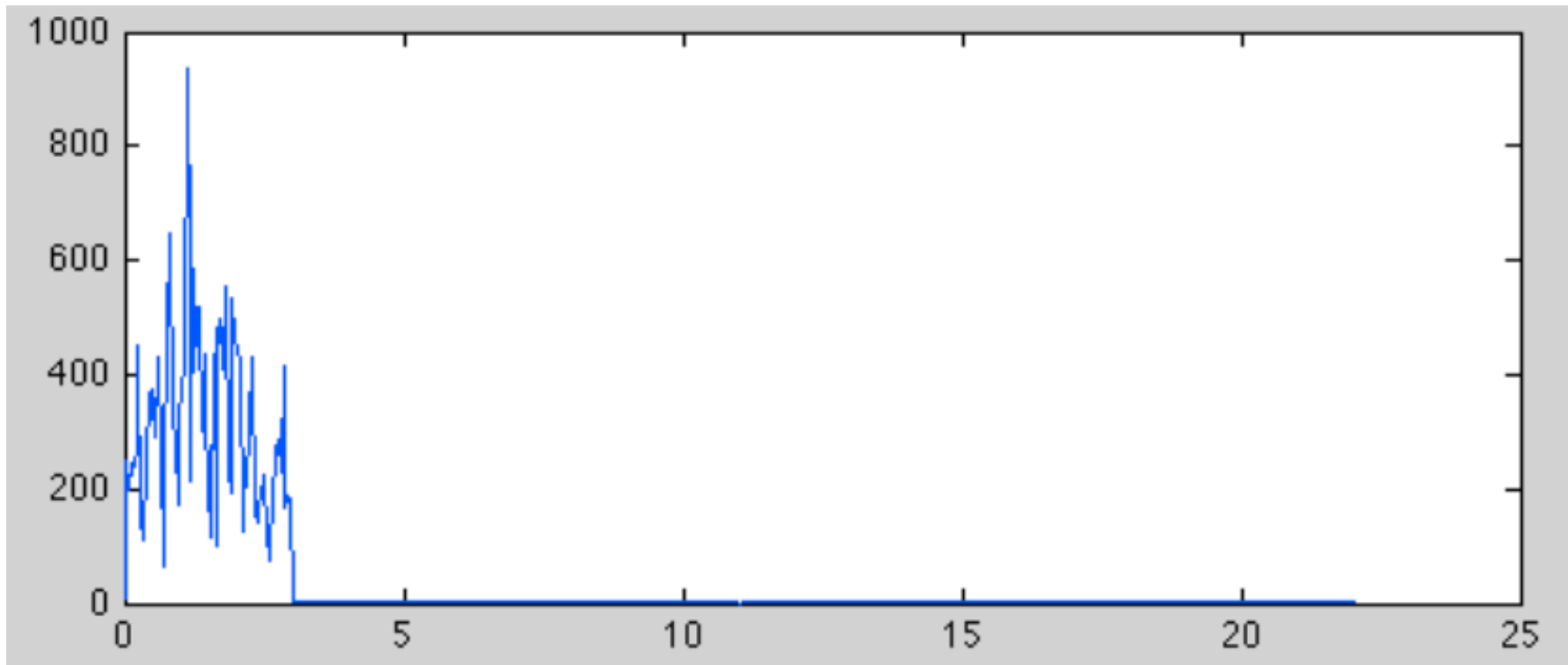


Corrupted
Speech





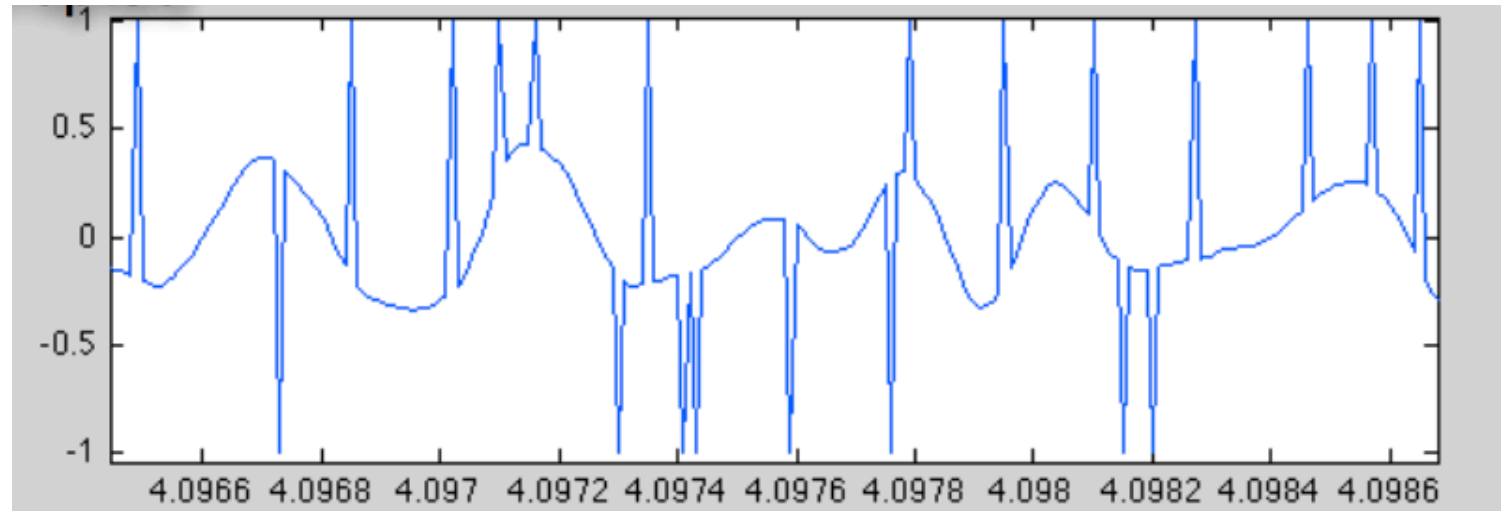
Low Pass Filtering





Speech in Time

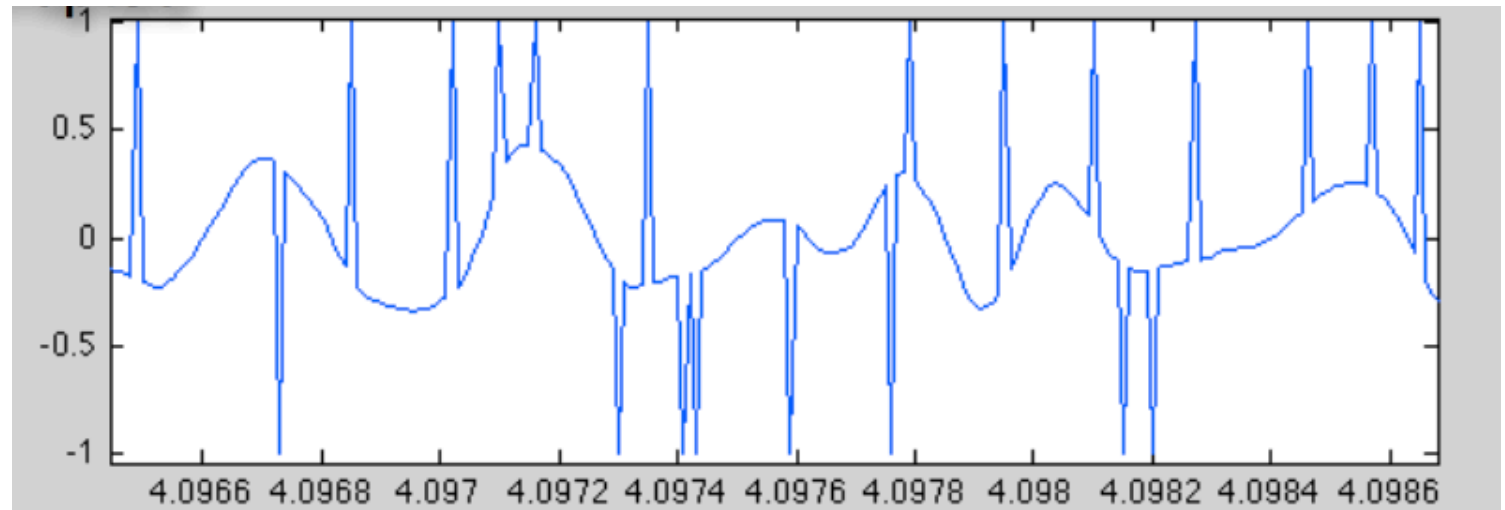
Corrupted
Speech



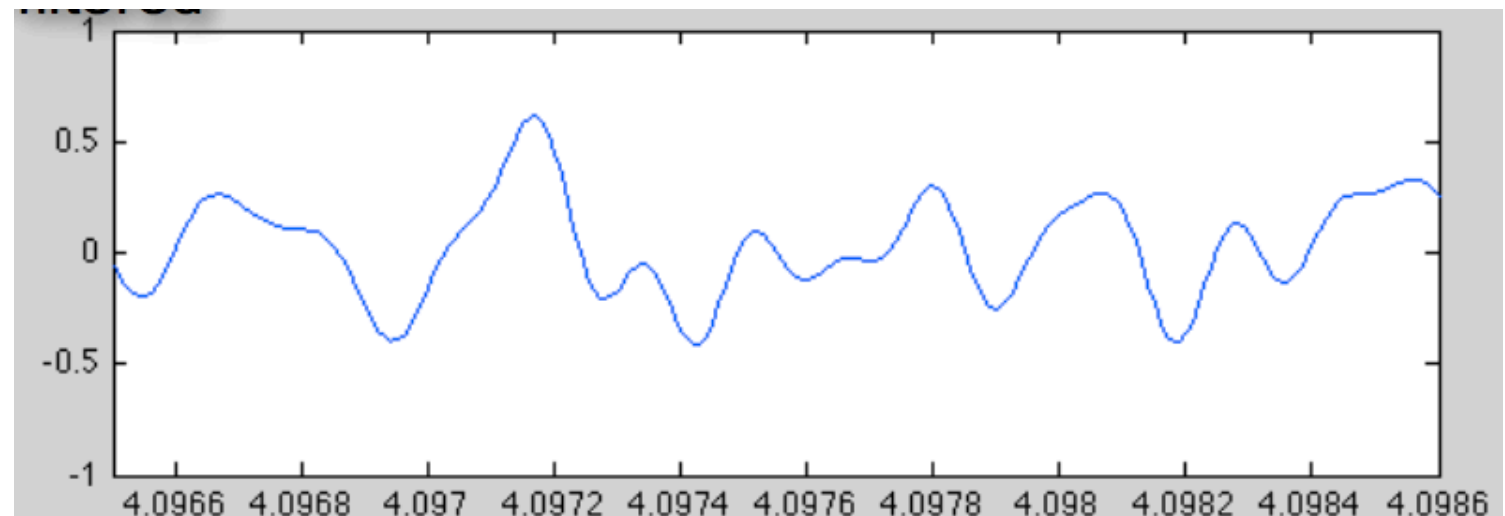


Low Pass Filtering

Corrupted
Speech



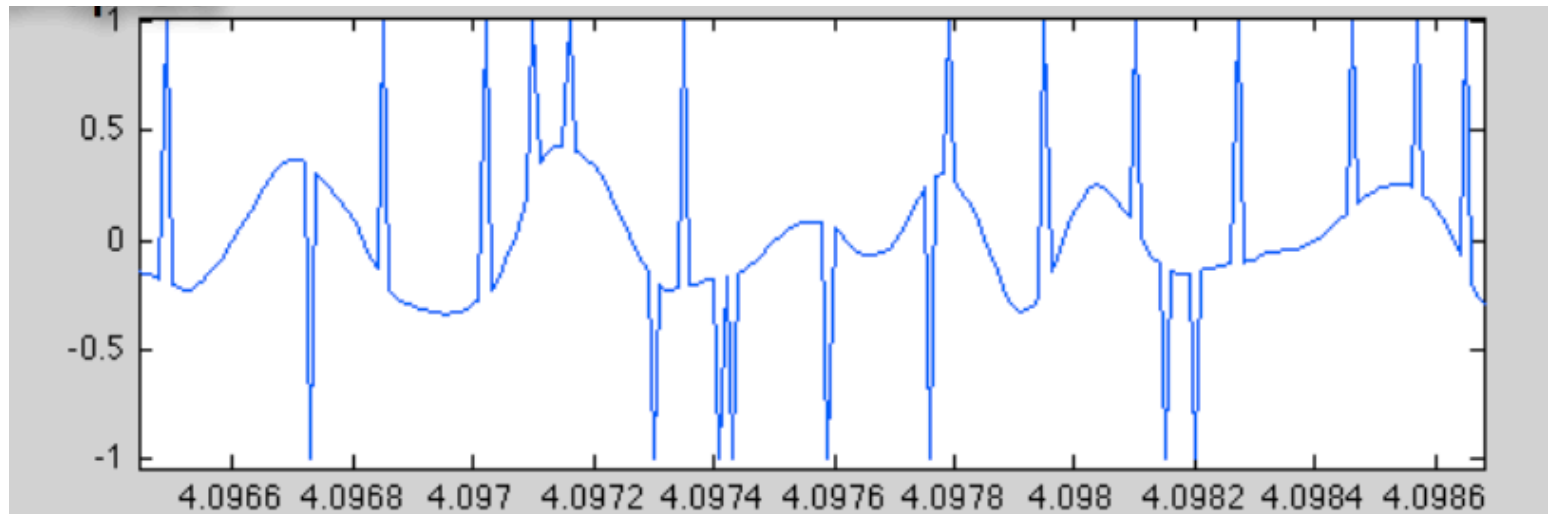
LP-Filtered
Speech



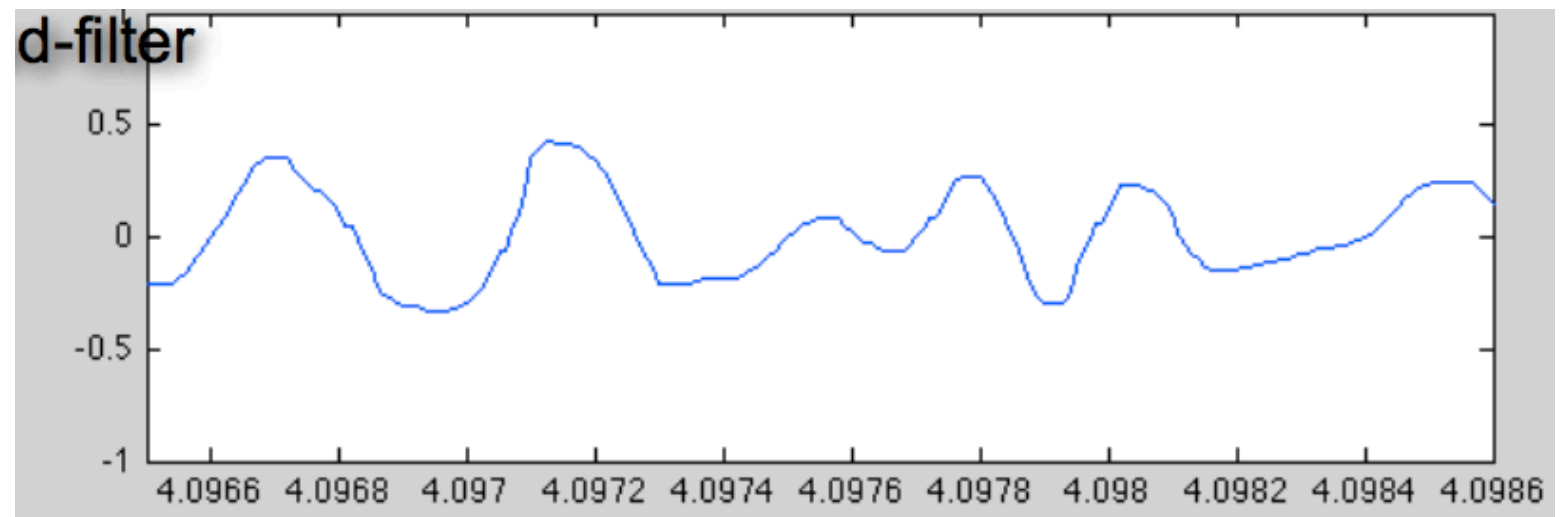


Median Filtering

Corrupted
Speech



Med-Filter
Speech



LTI Systems

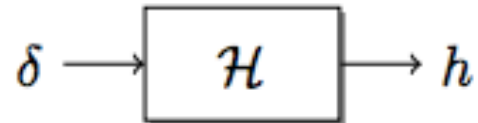


LTI Systems

DEFINITION

A system \mathcal{H} is **linear time-invariant** (LTI) if it is both linear and time-invariant

- ❑ LTI system can be completely characterized by its impulse response



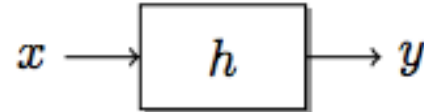
- ❑ Then the output for an arbitrary input is a sum of weighted, delay impulse responses

A block diagram showing an arbitrary input signal x entering a rectangular block labeled h . The output of the block is the signal y .

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution



- Convolution formula:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- Convolution method:

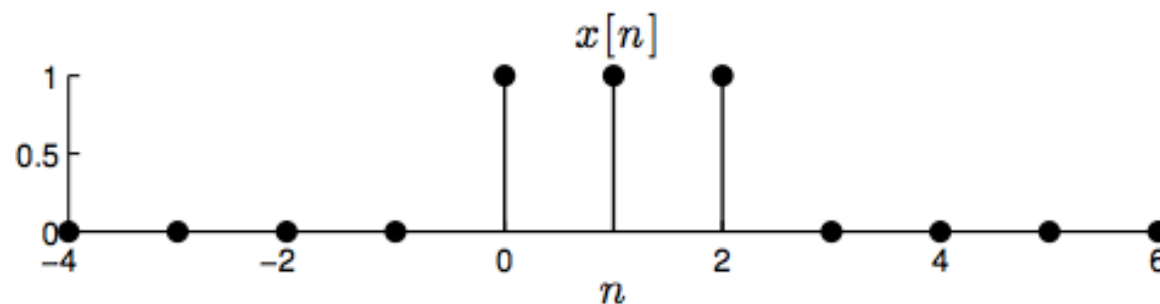
- 1) Time reverse the impulse response and shift it n time steps to the right
- 2) Compute the inner product between the shifted impulse response and the input vector
- Repeat for every n



Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- Convolve a unit pulse with itself



Convolution is Commutative

- Convolution is commutative:

$$x * h = h * x$$

- These block diagrams are equivalent

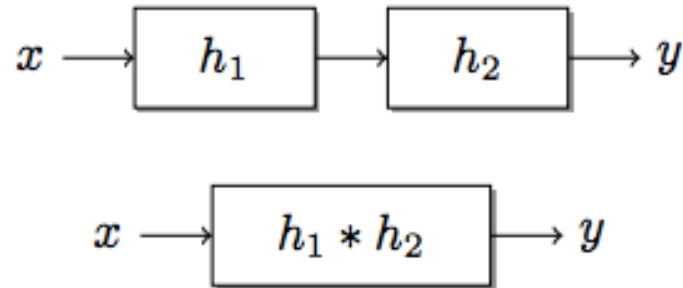


- Implication: pick either h or x to flip and shift when convolving



LTI Systems in Series

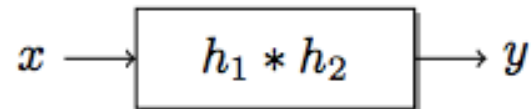
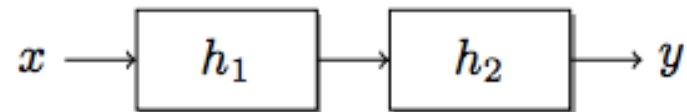
- Impulse response of the cascade of two LTI systems:



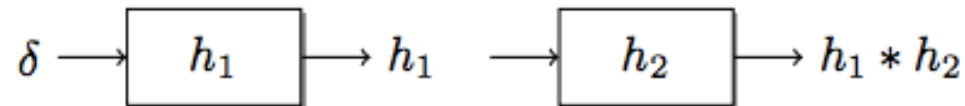


LTI Systems in Series

- Impulse response of the cascade of two LTI systems:



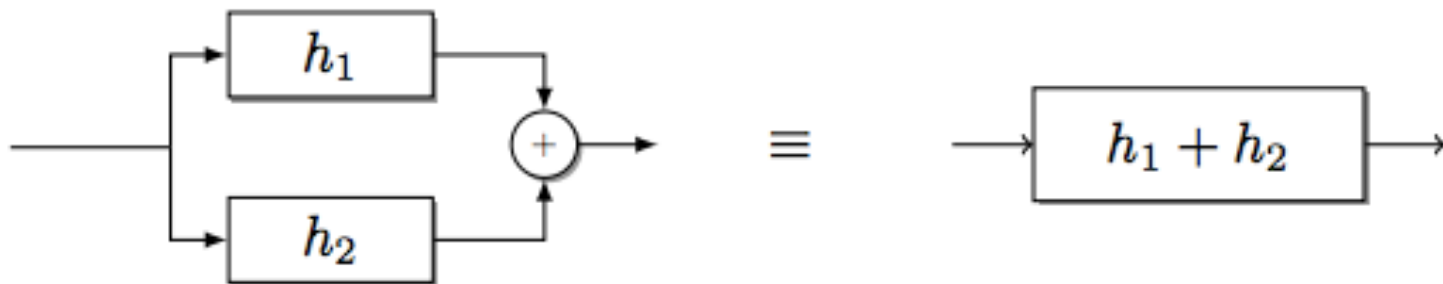
- Proof by picture





LTI Systems in Parallel

- Impulse response of the parallel connection of two LTI systems:



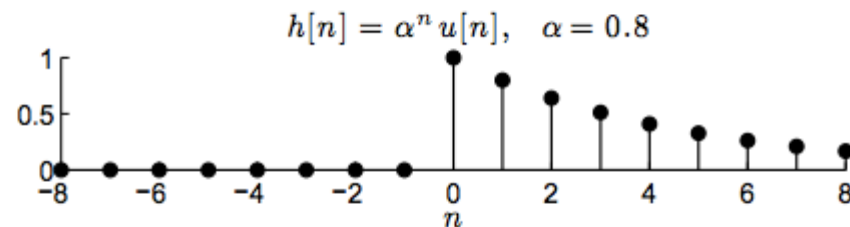
Causal System Revisited

DEFINITION

A system \mathcal{H} is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$



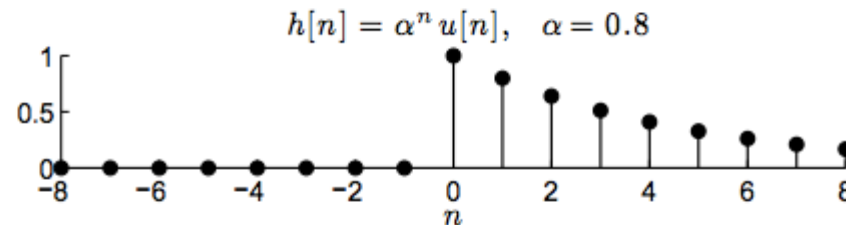
Causal System Revisited

DEFINITION

A system \mathcal{H} is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$



- To prove, note that the convolution does not look into the future if the impulse response is causal

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m] \qquad h[n-m] = 0 \text{ when } m > n;$$



Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite



Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average

$$y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2} (x[n] + x[n-1])$$



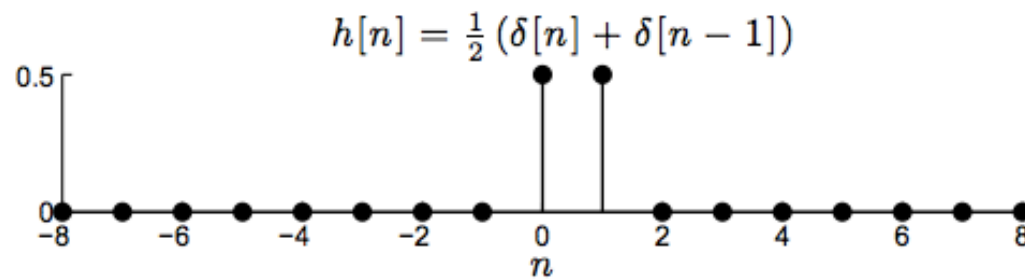
Duration of Impulse

DEFINITION

An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

- Example: Moving average

$$y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2} (x[n] + x[n-1])$$





Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite



Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

- ❑ Example: Recursive average

$$y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha y[n - 1]$$



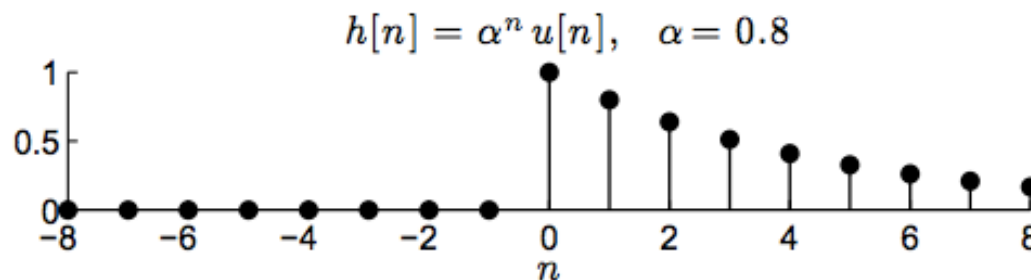
Duration of Impulse

DEFINITION

An LTI system has an **infinite impulse response** (IIR) if the duration of its impulse response h is infinite

- Example: Recursive average

$$y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha y[n - 1]$$





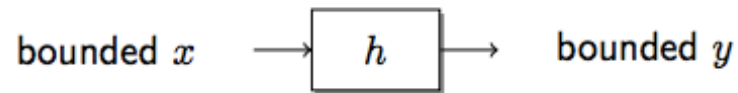
Ended here



BIBO Stability Revisited

DEFINITION

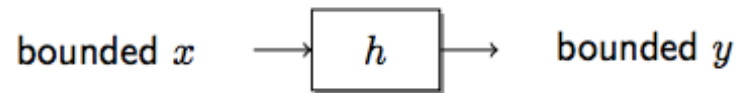
An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



BIBO Stability Revisited

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

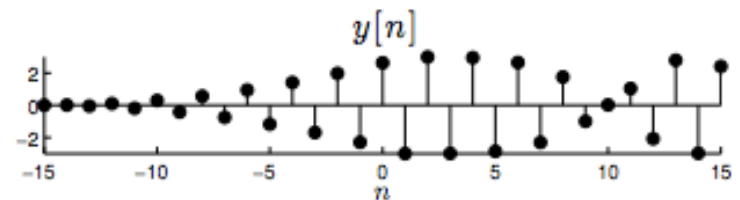
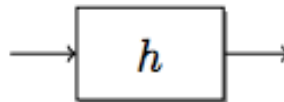
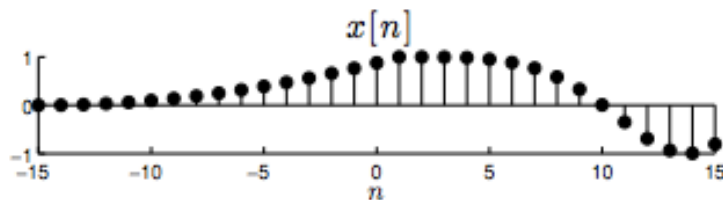


- Bounded input and output:

$$\|x\|_{\infty} < \infty \quad \text{and} \quad \|y\|_{\infty} < \infty$$

- Where

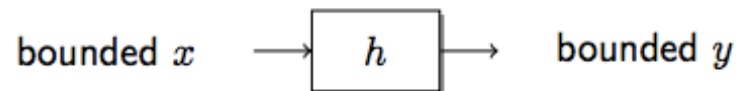
$$\|x\|_{\infty} = \max |x[n]|$$



BIBO Stability Revisited

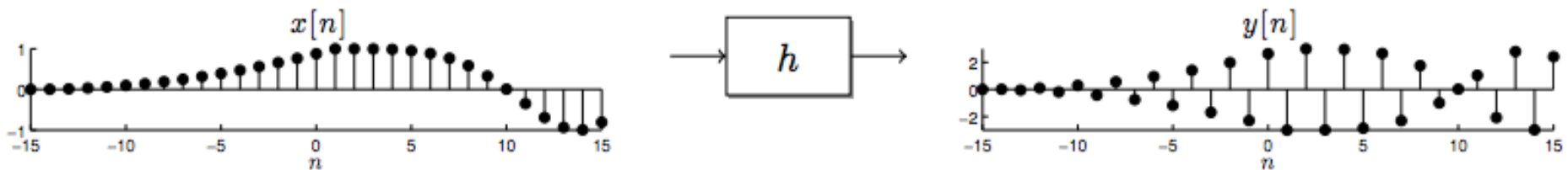
DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output:

$$\|x\|_{\infty} < \infty \quad \text{and} \quad \|y\|_{\infty} < \infty$$



- An LTI system is BIBO stable **if and only if**

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

BIBO Stability – Sufficient Condition



- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable, then for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$
- Recall that $\|x\|_\infty < \infty$ means there exist a constant A such that $|x[n]| < A < \infty$ for all n

BIBO Stability – Sufficient Condition

- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable, then for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$
- Recall that $\|x\|_\infty < \infty$ means there exist a constant A such that $|x[n]| < A < \infty$ for all n
- Let $\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| = B < \infty$
- Compute a bound on $|y[n]|$ using the convolution of x and h and the bounds A and B

$$\begin{aligned} |y[n]| &= \left| \sum_{m=-\infty}^{\infty} h[n-m] x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]| |x[m]| \\ &< \sum_{m=-\infty}^{\infty} |h[n-m]| A = A \sum_{k=-\infty}^{\infty} |h[k]| = AB = C < \infty \end{aligned}$$

- Since $|y[n]| < C < \infty$ for all n , $\|y\|_\infty < \infty$ ✓

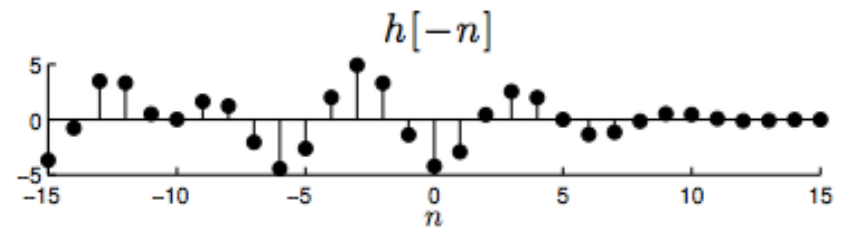
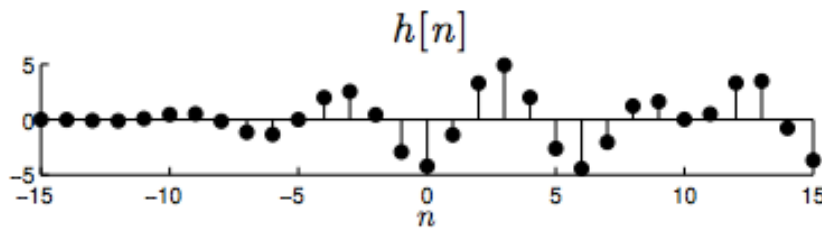


BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
 - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical

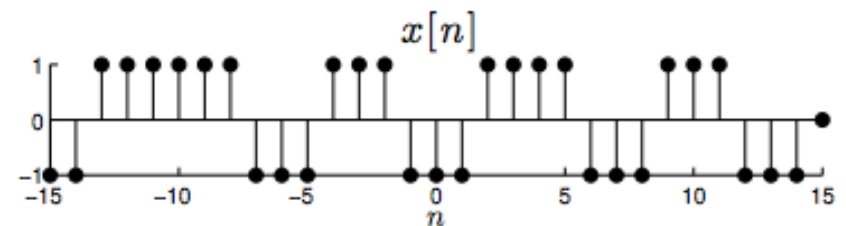
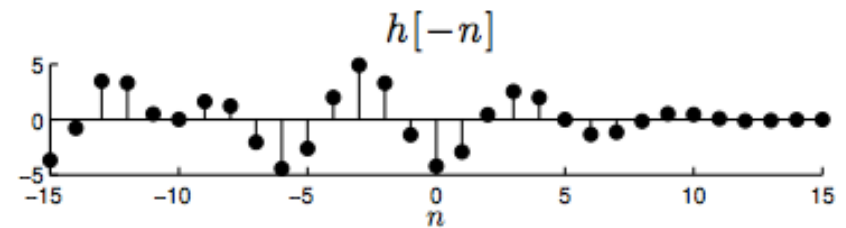
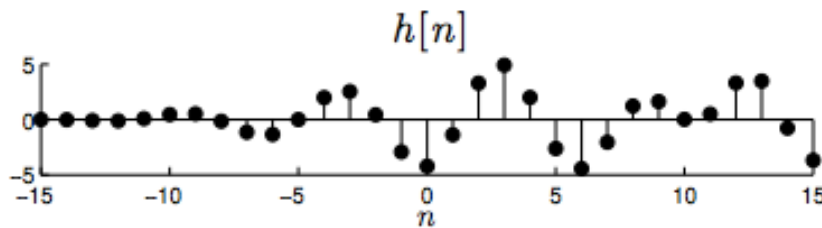
BIBO Stability – Necessary Condition

- ❑ Prove that if $\|h\|_1 = \infty$ the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
 - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical
- ❑ Given an impulse response h with $\|h\|_1 = \infty$, form the tricky special signal $x[n] = \text{sgn}(h[-n])$
 - $x[n]$ is the sign of the time-reversed impulse response $h[-n]$



BIBO Stability – Necessary Condition

- ❑ Prove that if $\|h\|_1 = \infty$ the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
 - Assume that x and h are real-valued; the proof for complex-valued signals is nearly identical
- ❑ Given an impulse response h with $\|h\|_1 = \infty$, form the tricky special signal $x[n] = \text{sgn}(h[-n])$
 - $x[n]$ is the sign of the time-reversed impulse response $h[-n]$
 - Note that x is bounded $|x[n]| \leq 1$ for all n





BIBO Stability – Necessary Condition



- We are proving that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
- Armed with the tricky signal x , compute the output $y[n]$ at $n=0$

BIBO Stability – Necessary Condition

- We are proving that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
- Armed with the tricky signal x , compute the output $y[n]$ at $n=0$

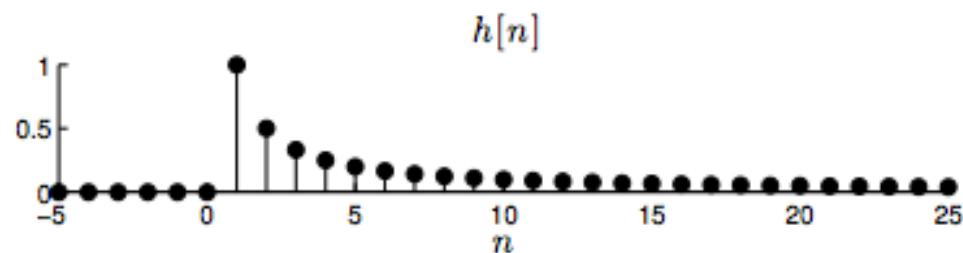
$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} h[0-m] x[m] = \sum_{m=-\infty}^{\infty} h[-m] \operatorname{sgn}(h[-m]) \\ &= \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty \end{aligned}$$

- Thus y is not bounded while x is bounded, so the system is not BIBO stable

Examples

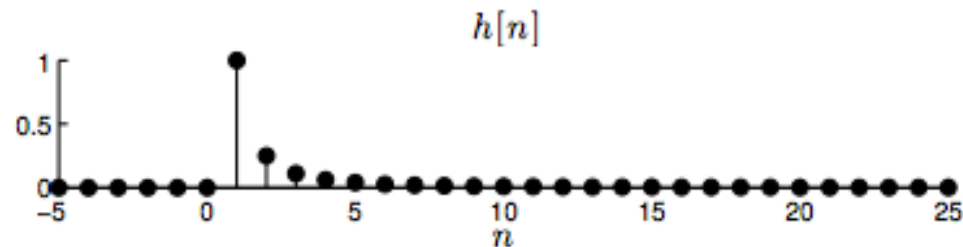
Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n} \right| = \infty \Rightarrow \text{not BIBO}$$



Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

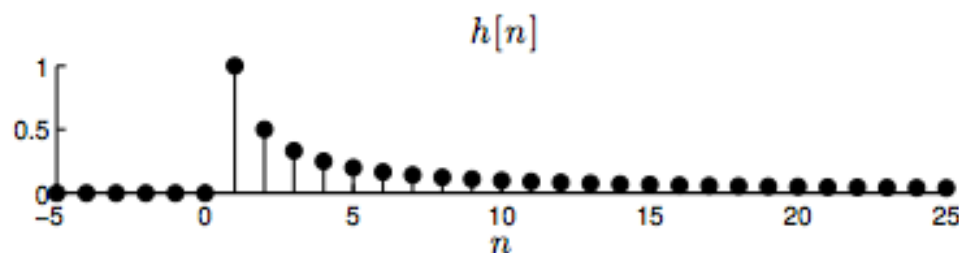
$$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| = \frac{\pi^2}{6} \Rightarrow \text{BIBO}$$



Examples

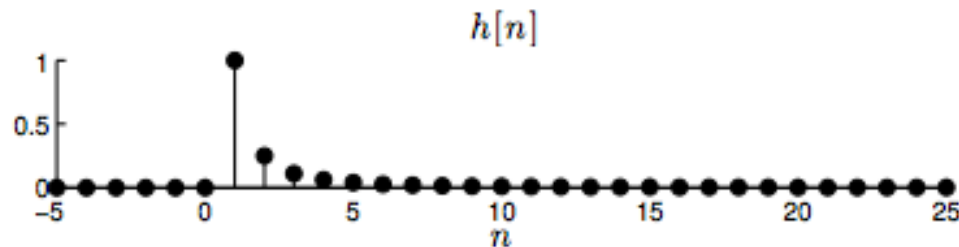
Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n} \right| = \infty \Rightarrow \text{not BIBO}$$

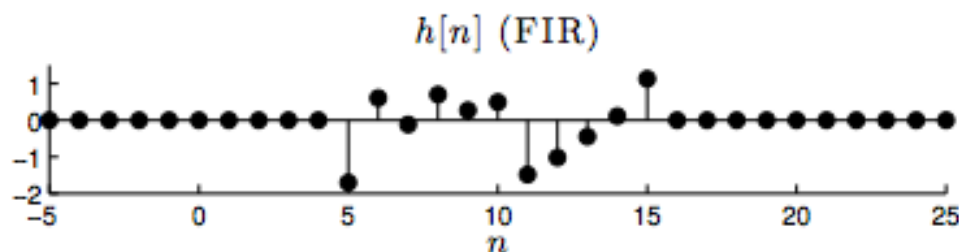


Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\|h\|_1 = \sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right| = \frac{\pi^2}{6} \Rightarrow \text{BIBO}$$



Example: h FIR \Rightarrow BIBO





Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha y[n-1]$
- Impulse response: $h[n] = \alpha^n u[n]$

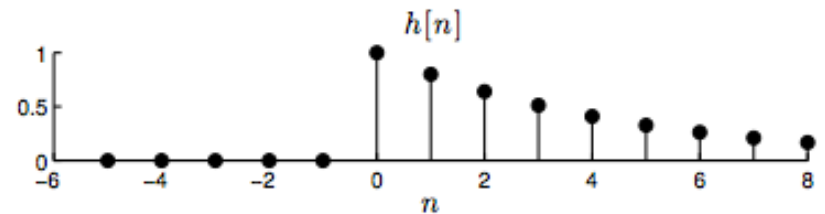


Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha y[n-1]$
- Impulse response: $h[n] = \alpha^n u[n]$

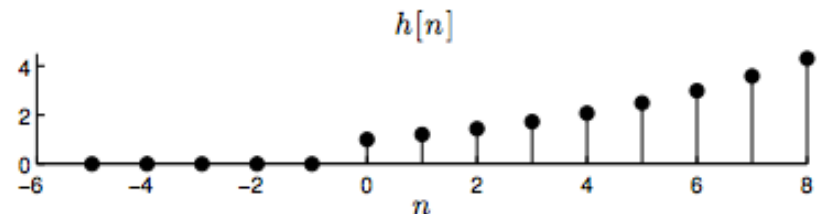
For $|\alpha| < 1$

$$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \Rightarrow \text{BIBO}$$



For $|\alpha| > 1$

$$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \Rightarrow \text{not BIBO}$$





Difference Equations

□ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



Difference Equations

□ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

Difference Equations

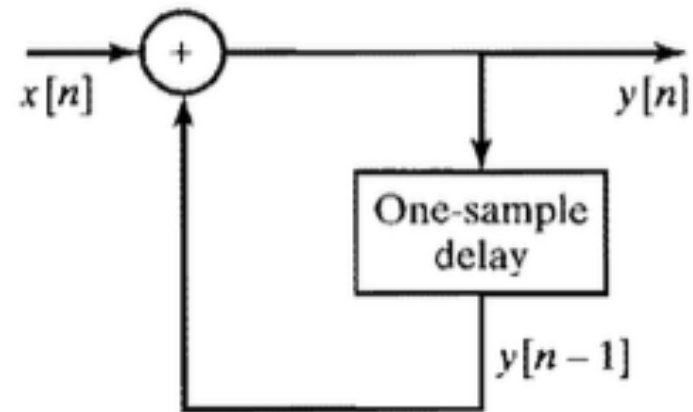
❑ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$



Example: Difference Equation

□ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

□ Causal?



Example: Difference Equation

□ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

□ Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$



Big Ideas

- ❑ LTI Systems are a special class of systems with significant signal processing applications
 - Can be characterized by the impulse response
- ❑ LTI System Properties
 - Causality and stability can be determined from impulse response
- ❑ Difference equations suggest implementation of systems
 - Give insight into complexity of system



Admin

- ❑ Enroll in Piazza site:
 - piazza.com/upenn/spring2022/ese531
- ❑ Complete Diagnostic Quiz by midnight
 - Solutions posted after due date
- ❑ HW 0: Brush up on background and Matlab tutorial
- ❑ HW 1 out now
- ❑ First recitation posted after lecture
 - Basic Matlab usage