

ESE 531: Digital Signal Processing

Lecture 5: January 27, 2022
z-Transform

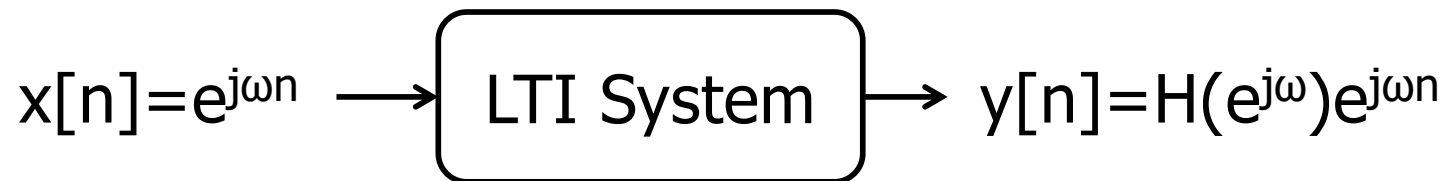


Lecture Outline

- ❑ Frequency Response wrap-up
- ❑ LP filter example
- ❑ z-Transform
 - Regions of convergence (ROC) & properties
 - z-Transform properties

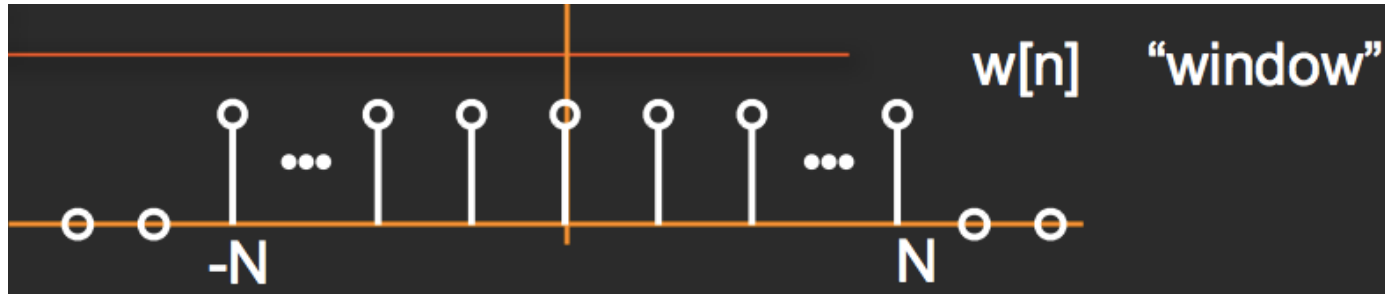
LTI System Frequency Response

- (DT)Fourier Transform of impulse response

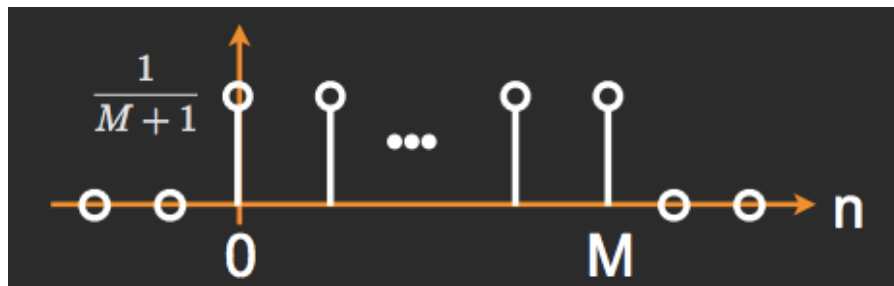


$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

Example: Moving Average

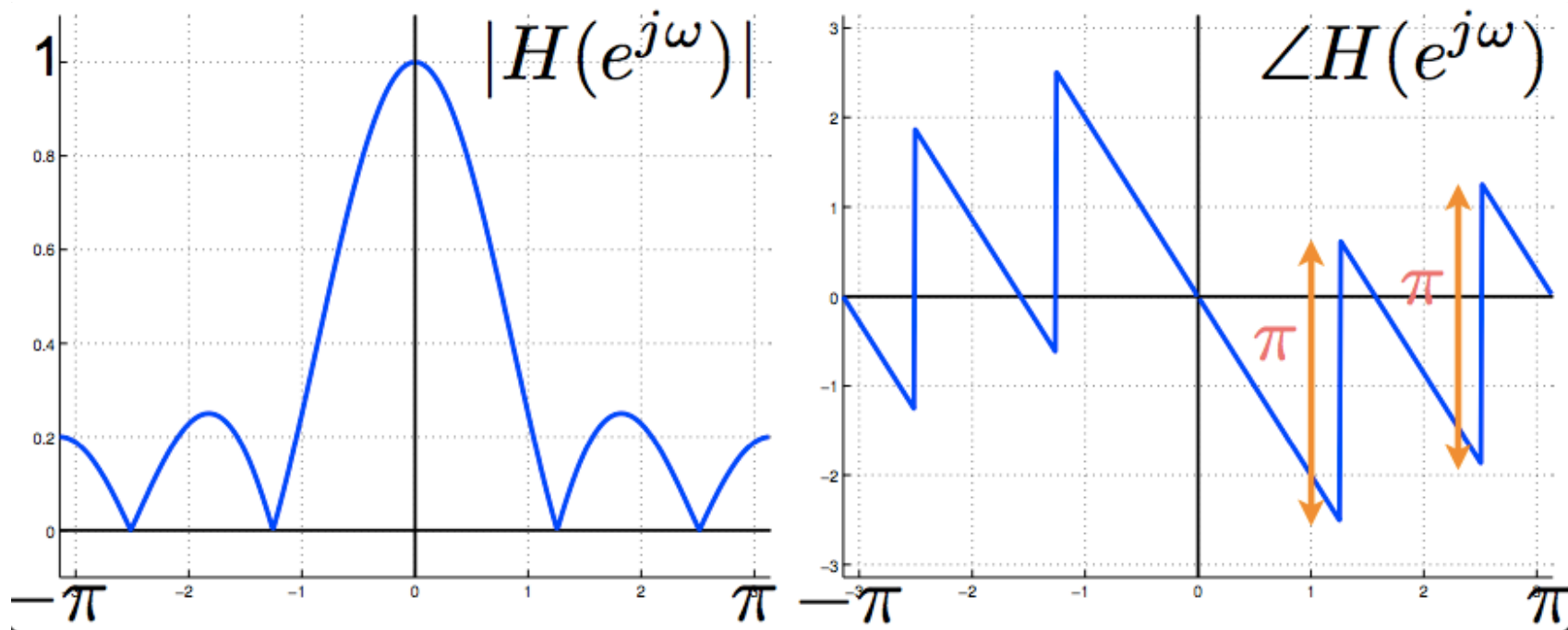


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average



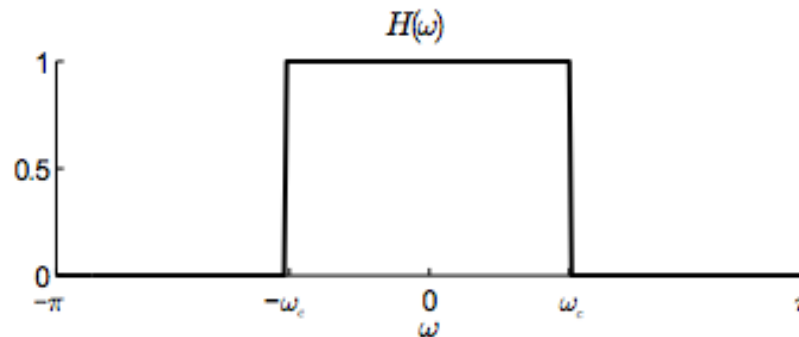
$M=4$
($N=2$)

$$H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

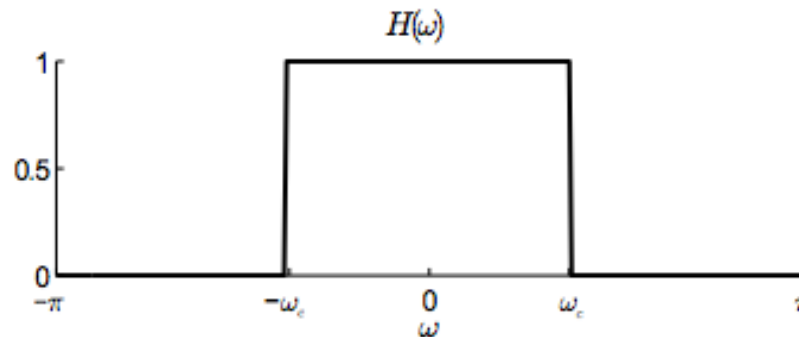
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



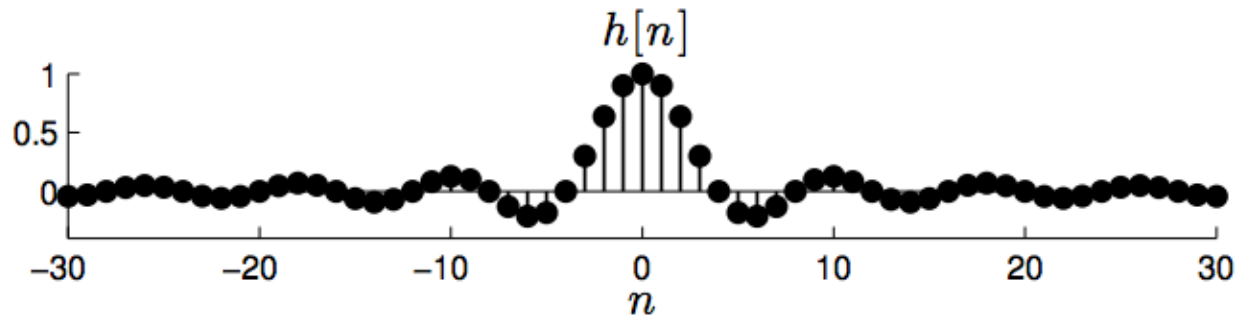
- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

Example: Ideal Low-Pass Filter

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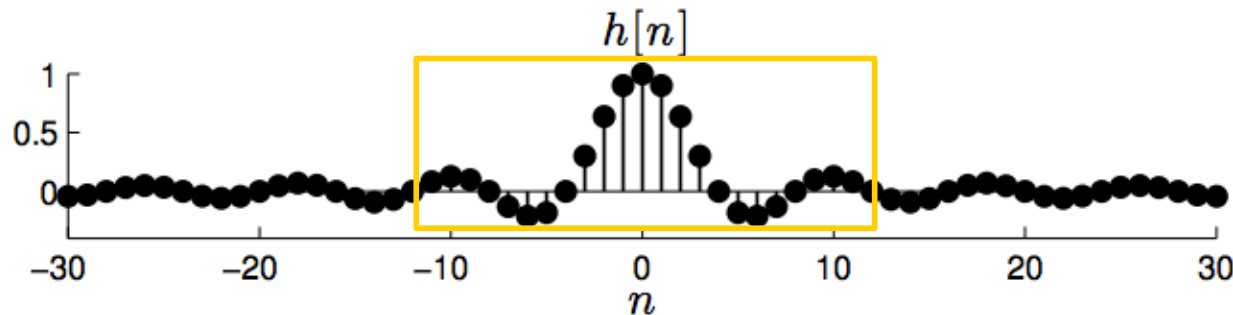
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

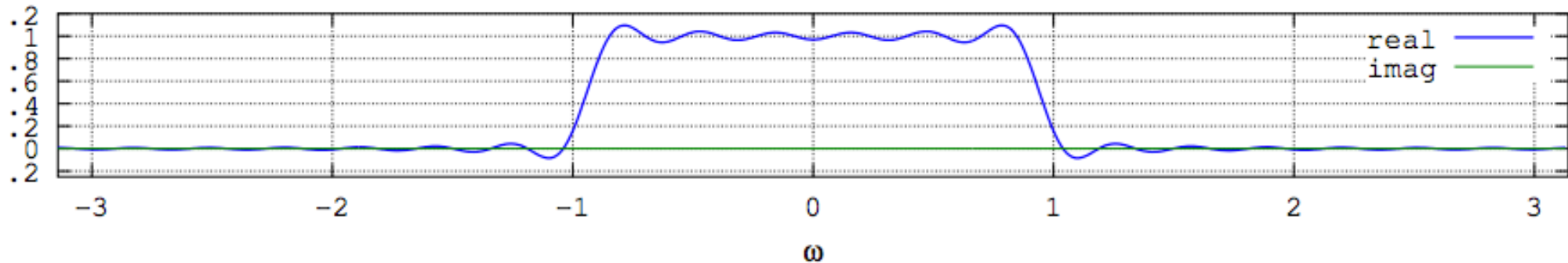
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



Truncate
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$

Example: Practical LP Filter



- ❑ Pass band smeared and rippled
 - Smearing determined by width of main lobe
 - Rippling determined by size of side lobes

z-Transform



z-Transform

- ❑ The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- ❑ Very useful for designing and analyzing signal processing systems
- ❑ Properties are very similar to the DTFT with a few caveats



Reminder: DTFT Definition

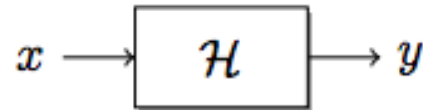
- ❑ The core “basis functions” (I.e eigenfunctions) of the DTFT are the complex sinusoids $e^{j\omega n}$ with arbitrary frequencies ω
- ❑ The sinusoids $e^{j\omega n}$ are eigenvectors of LTI systems for infinite-length signals

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

Reminder: Frequency Response of LTI System

- We can use the DTFT to characterize an LTI system



$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

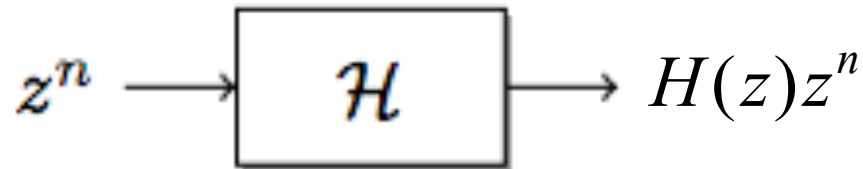
- and relate the DTFTs of the input and output

$$X(\omega) = \sum_{m=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad H(\omega) = \sum_{m=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$Y(\omega) = X(\omega)H(\omega)$$

Complex Exponentials as Eigenfunctions

- Fact: A more general set of eigenfunctions of an LTI system are the complex exponentials z^n , $z \in \mathbb{C}$



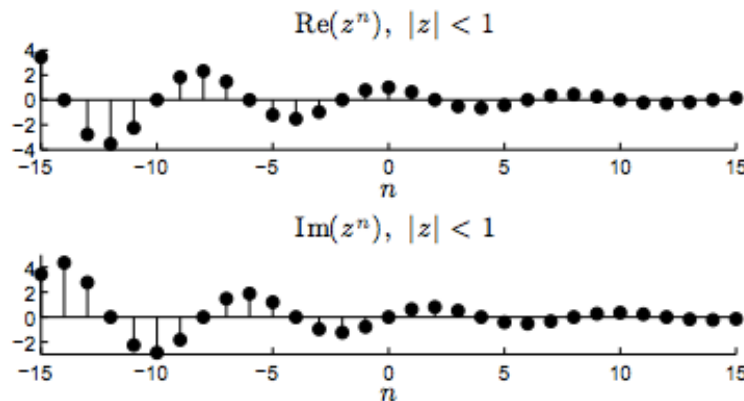
Reminder: Complex Exponentials

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

$|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)

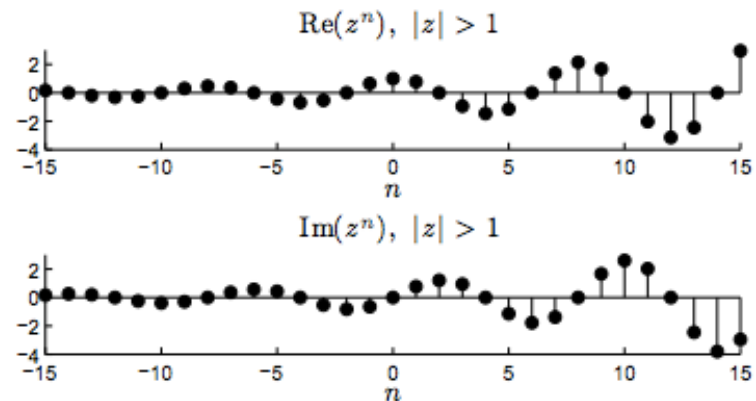
$e^{j\omega n}$ is a **complex sinusoid**

$$|z| < 1$$



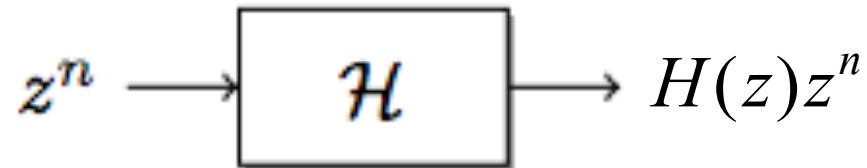
Bounded

$$|z| > 1$$



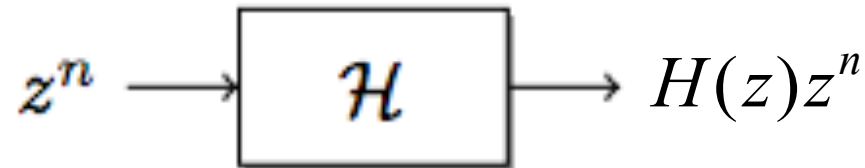
Unbounded

Proof: Complex Exponentials as Eigenfunctions



- Prove by computing the convolution with input $x[n] = z^n$

Proof: Complex Exponentials as Eigenfunctions



- Prove by computing the convolution with input $x[n] = z^n$

$$\begin{aligned} z^n * h[n] &= \sum_{m=-\infty}^{\infty} z^{n-m} h[m] = \sum_{m=-\infty}^{\infty} z^n z^{-m} h[m] \\ &= \left(\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right) z^n \\ &= H(z) z^n \checkmark \end{aligned}$$



z-Transform

- Define the **forward z-transform** of $x[n]$ as

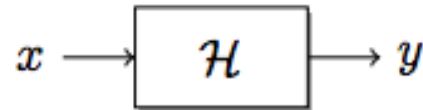
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The core “basis functions” of the z-transform are the complex exponentials z^n with arbitrary $z \in \mathbb{C}$; these are the eigenfunctions of LTI systems for infinite-length signals
- **Notation abuse alert:** We use $X(\bullet)$ to represent both the DTFT $X(e^{j\omega})$ and the z-transform $X(z)$; they are, in fact, intimately related

$$X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Transfer Function of LTI System

- We can use the z-Transform to characterize an LTI system



$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

- and relate the z-transforms of the input and output

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$Y(z) = X(z) H(z)$$



Z-transform

What are we missing?

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



Z-Transform

$$z^n = (|z| e^{j\omega n})^n = |z|^n e^{j\omega n}$$

$|z|^n$ is a **real exponential** envelope (a^n with $a = |z|$)

$e^{j\omega n}$ is a **complex sinusoid**

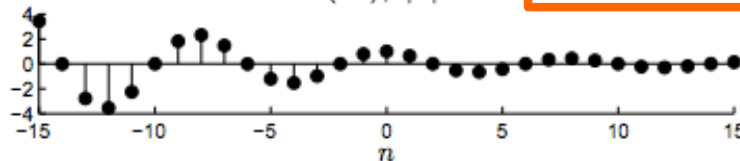
What are we missing?

$$|z| < 1$$

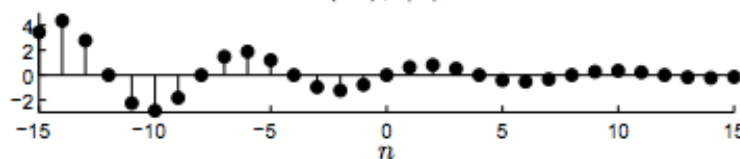
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$|z| > 1$$

$\text{Re}(z^n), |z| < 1$

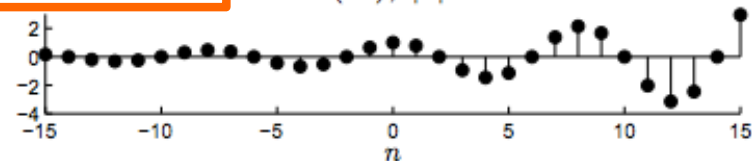


$\text{Im}(z^n), |z| < 1$

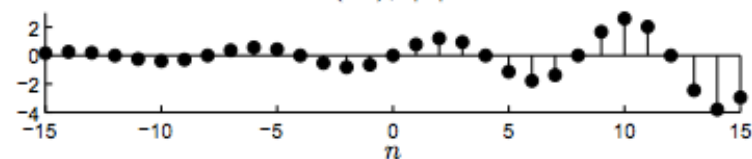


Bounded

$\text{Re}(z^n), |z| > 1$



$\text{Im}(z^n), |z| > 1$



Unbounded

Region of Convergence (ROC)



Region of Convergence (ROC)

DEFINITION

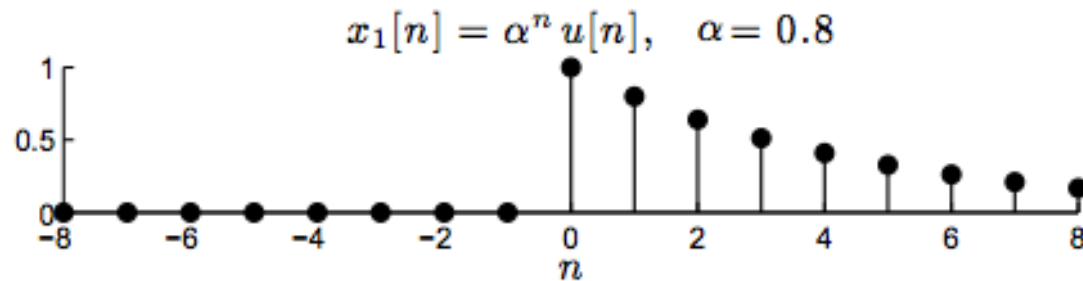
Given a time signal $x[n]$, the **region of convergence** (ROC) of its z -transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n] z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| < \infty$$

ROC Example 1 (preclass 1)

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence

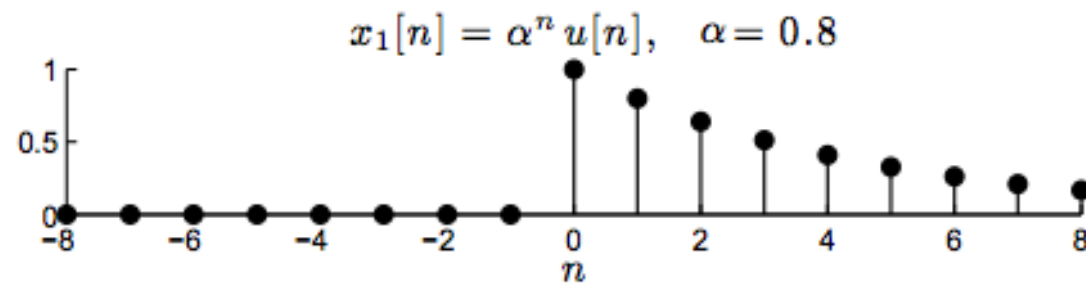
- Example for $\alpha = 0.8$



ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal) Right-sided sequence

- Example for $\alpha = 0.8$



- The **forward** z -transform of $x_1[n]$

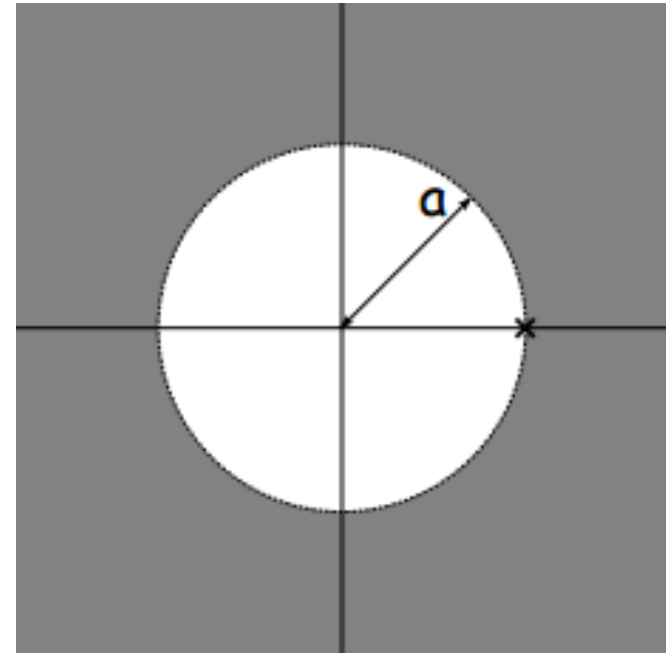
$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$

ROC Example 1

- Signal $x_1[n] = \alpha^n u[n]$, $\alpha \in \mathbb{C}$ (causal signal)

$$ROC = \{z : |z| > |\alpha|\}$$



- The **forward** z -transform of $x_1[n]$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

- **Important:** We can apply the geometric sum formula only when $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$

Poles and zeros?



ROC Example 1



□ What is the DTFT of $x_1[n] = a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

ROC Example 1

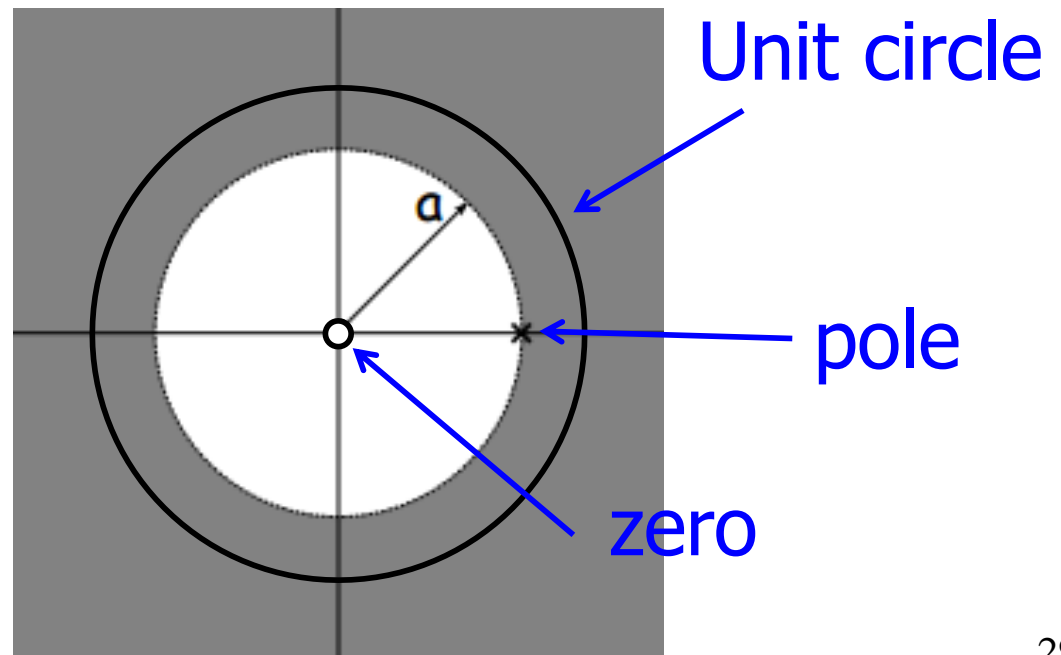
□ What is the DTFT of $x_1[n] = a^n u[n]$?

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}, \quad -\pi \leq \omega < \pi$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad -\infty \leq n < \infty$$

$$X_1(z) = \frac{z}{z - a}$$

$$ROC = \{z : |z| > |a|\}$$





ROC Example 2

□ What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$



ROC Example 2

□ What is the z-transform of $x_2[n]$? ROC?

$$x_2[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

□ Hint: $x_1[n] = a^n u[n] \xleftrightarrow{Z} \frac{1}{1 - az^{-1}}$

ROC Example 3

□ What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

Left-sided sequence

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$



ROC Example 3

- What is the z-transform of $x_3[n]$? ROC?

$$x_3[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The z-transform without ROC does not uniquely define a sequence!



ROC Example 4

□ What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

ROC Example 4

□ What is the z-transform of $x_4[n]$? ROC?

$$x_4[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

two-sided sequence

□ Hint:

$$x_1[n] = a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| < |a|\}$$

ROC Example 5

□ What is the z-transform of $x_5[n]$? ROC?

$$x_5[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

two-sided sequence

$$x_1[n] = a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| > |a|\}$$

$$x_3[n] = -a^n u[-n-1] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad ROC = \{z : |z| < |a|\}$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$



ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

finite length sequence

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?

$$x_6[n] = a^n u[n] u[-n + M - 1]$$

$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

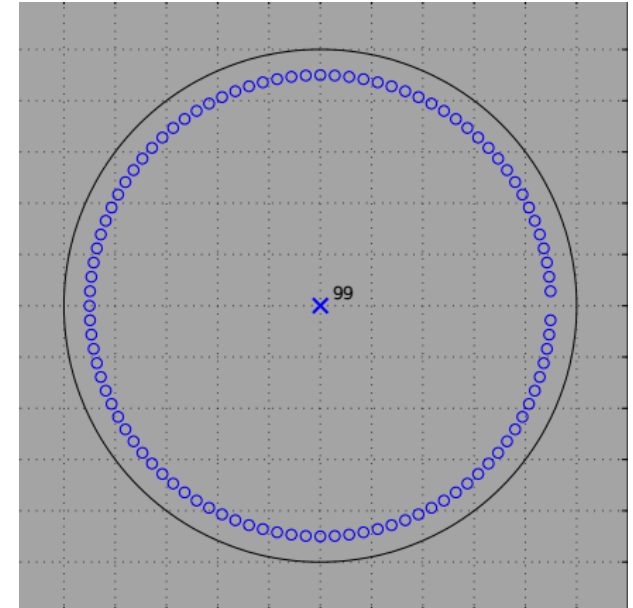
Zero cancels pole

$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$

ROC Example 6

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

□ What is the z-transform of $x_6[n]$? ROC?



$$X_6(z) = \frac{1 - a^M z^{-M}}{1 - az^{-1}}$$

$M=100$

Zero cancels pole

$$= \prod_{k=1}^{M-1} (1 - ae^{j2\pi k/M} z^{-1})$$



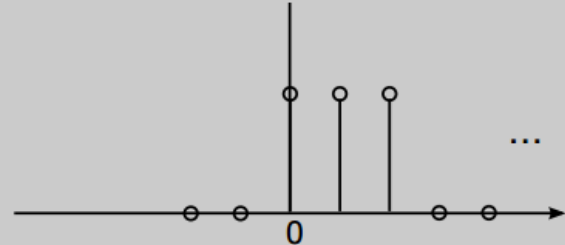
Properties of ROC

- ❑ For right-sided sequences: ROC extends outward from the outermost pole to infinity
 - Examples 1,2
- ❑ For left-sided: inwards from inner most pole to zero
 - Example 3
- ❑ For two-sided, ROC is a ring - or do not exist
 - Examples 4,5

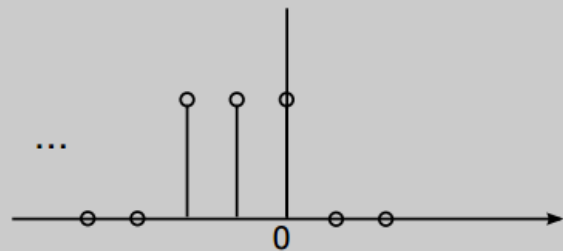


Properties of ROC

- For finite duration sequences, ROC is the entire z -plane, except possibly $z=0$, $z=\infty$ (Example 6)



$$X(z) = 1 + z^{-1} + z^{-2} \quad \text{ROC excludes } z = 0$$



$$X(z) = 1 + z^1 + z^2 \quad \text{ROC excludes } z = \infty$$



Formal Properties of the ROC

□ PROPERTY 1:

- The ROC will either be of the form $0 < r_R < |z|$, or $|z| < r_L < \infty$, or, in general the annulus, i.e., $0 < r_R < |z| < r_L < \infty$.

□ PROPERTY 2:

- The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z -transform of $x[n]$ includes the unit circle.

□ PROPERTY 3:

- The ROC cannot contain any poles.

□ PROPERTY 4:

- If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 < n < N_2 < \infty$, then the ROC is the entire z -plane, except possibly $z = 0$ or $z = \infty$.



Formal Properties of the ROC

❑ PROPERTY 5:

- If $x[n]$ is a *right-sided sequence*, the ROC extends outward from the *outermost* finite pole in $X(z)$ to (and possibly including) $z = \infty$.

❑ PROPERTY 6:

- If $x[n]$ is a *left-sided sequence*, the ROC extends inward from the *innermost* nonzero pole in $X(z)$ to (and possibly including) $z=0$.

❑ PROPERTY 7:

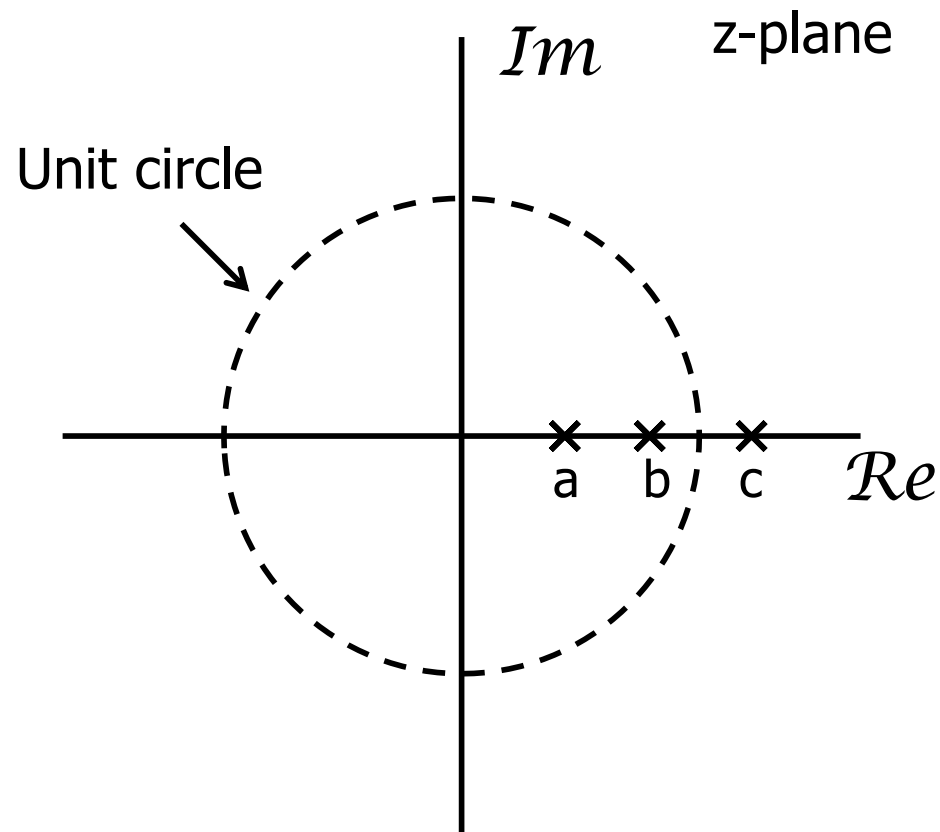
- A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z -plane, bounded on the interior and exterior by a pole and, consistent with Property 3, not containing any poles.

❑ PROPERTY 8:

- The ROC must be a connected region.

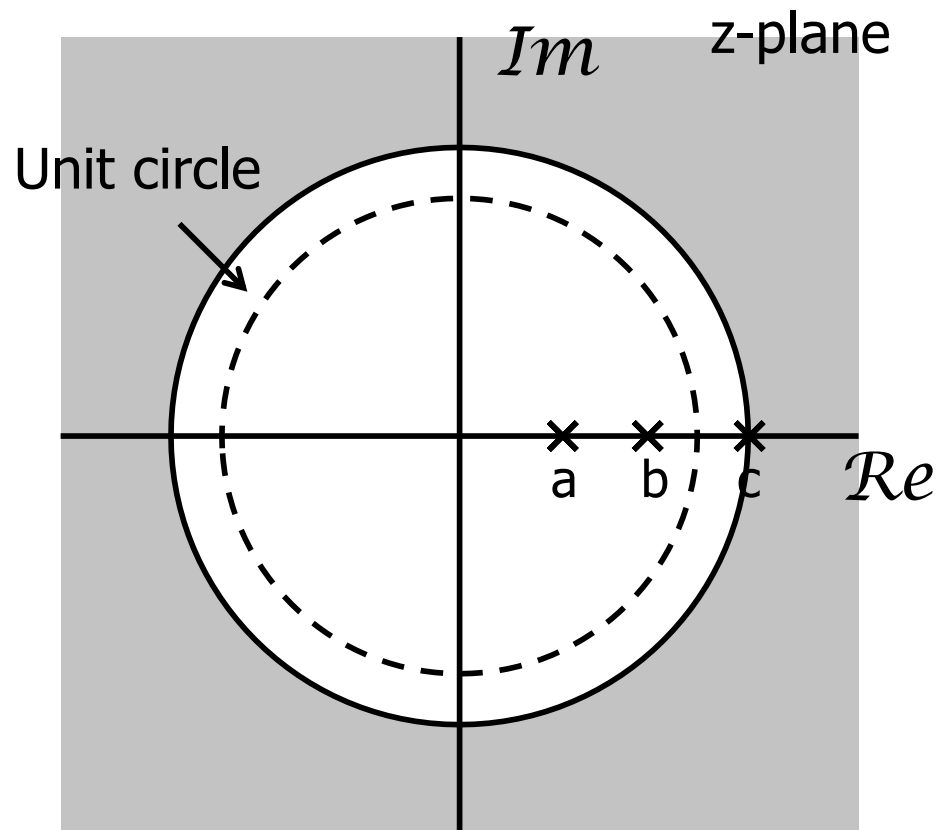
Example: ROC from Pole-Zero Plot

□ How many possible ROCs?



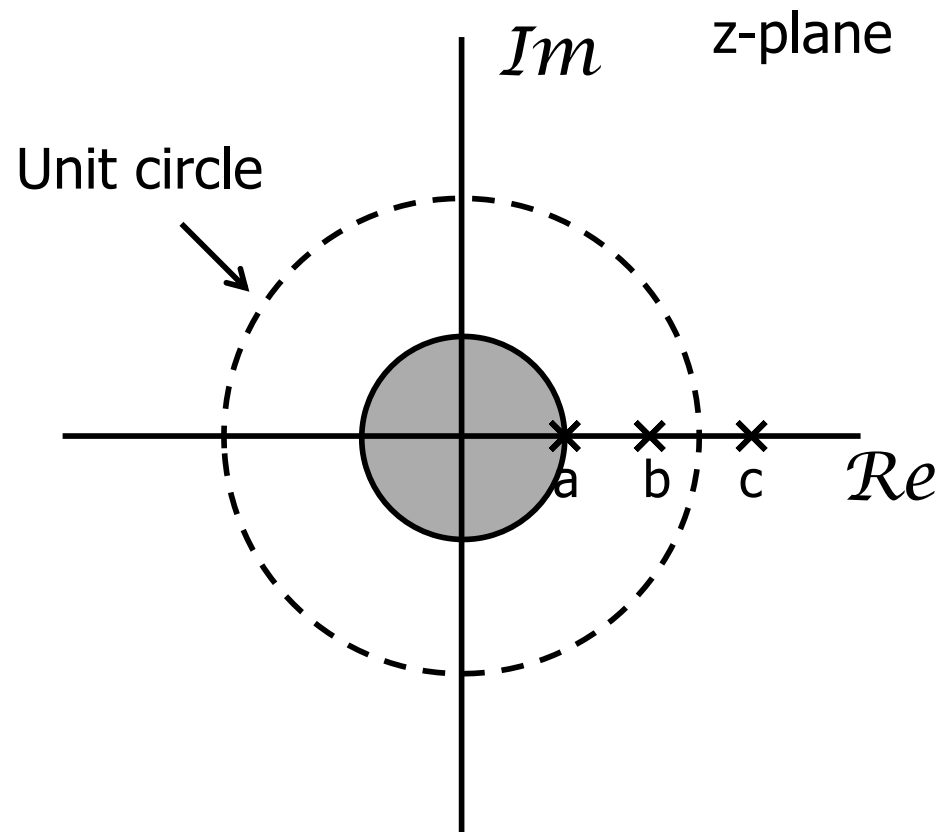
Example: ROC from Pole-Zero Plot

ROC 1: right-sided



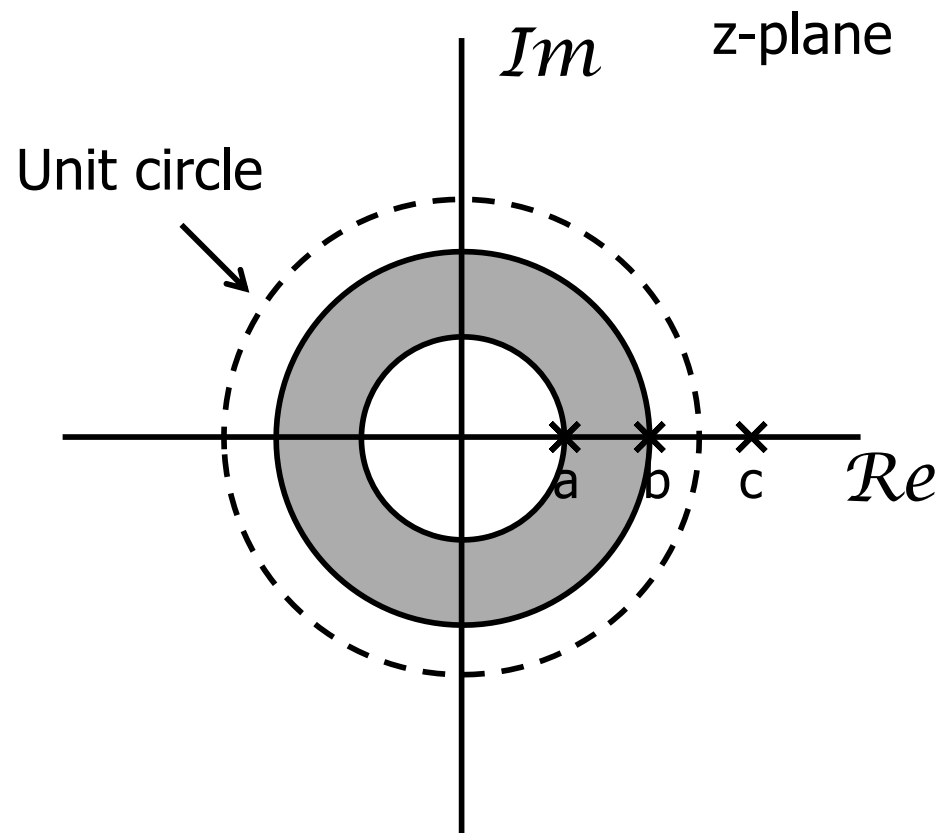
Example: ROC from Pole-Zero Plot

ROC 2: left-sided



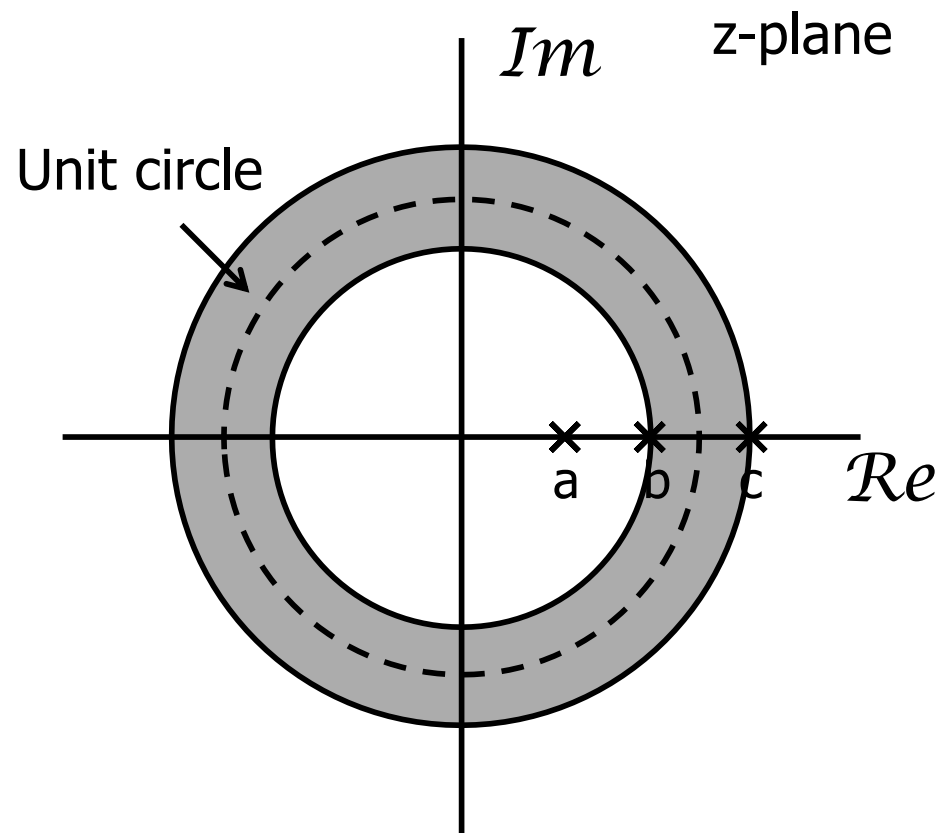
Example: ROC from Pole-Zero Plot

ROC 3: two-sided



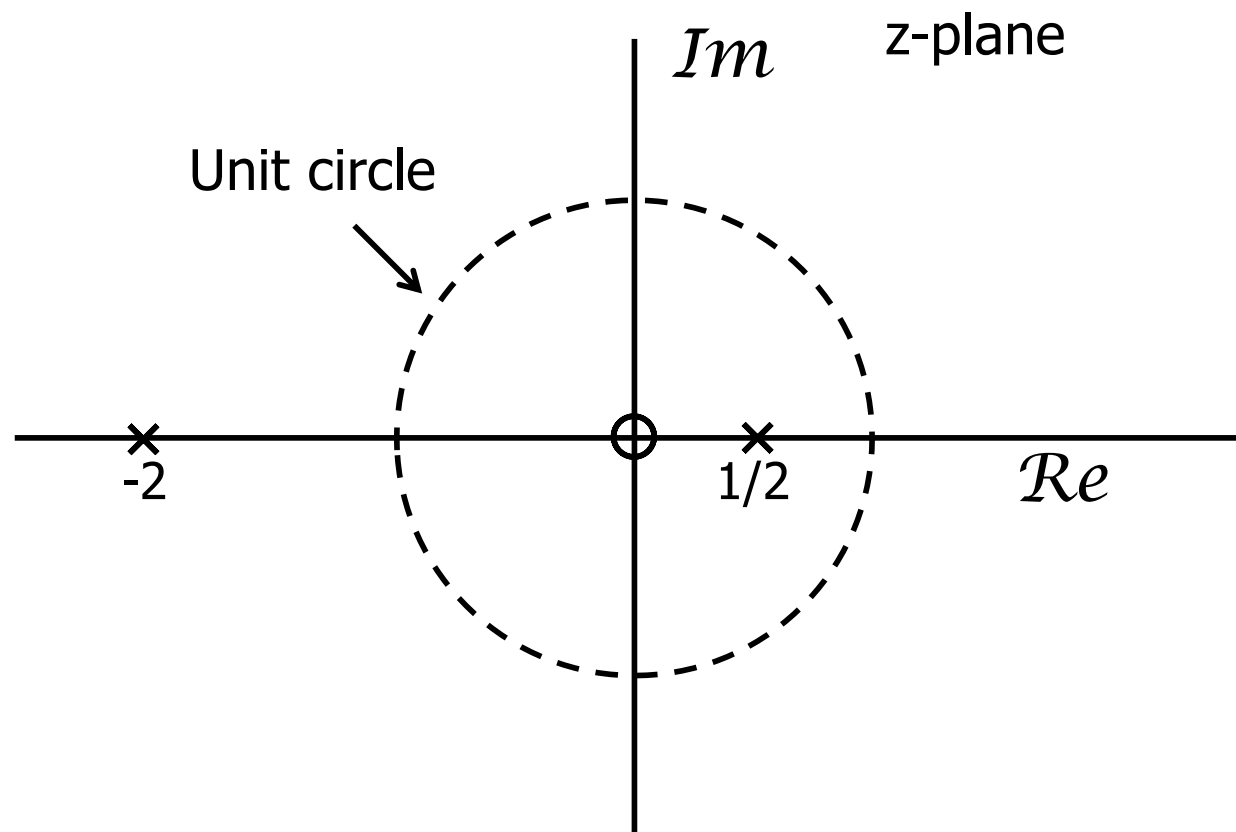
Example: ROC from Pole-Zero Plot

ROC 4: two-sided



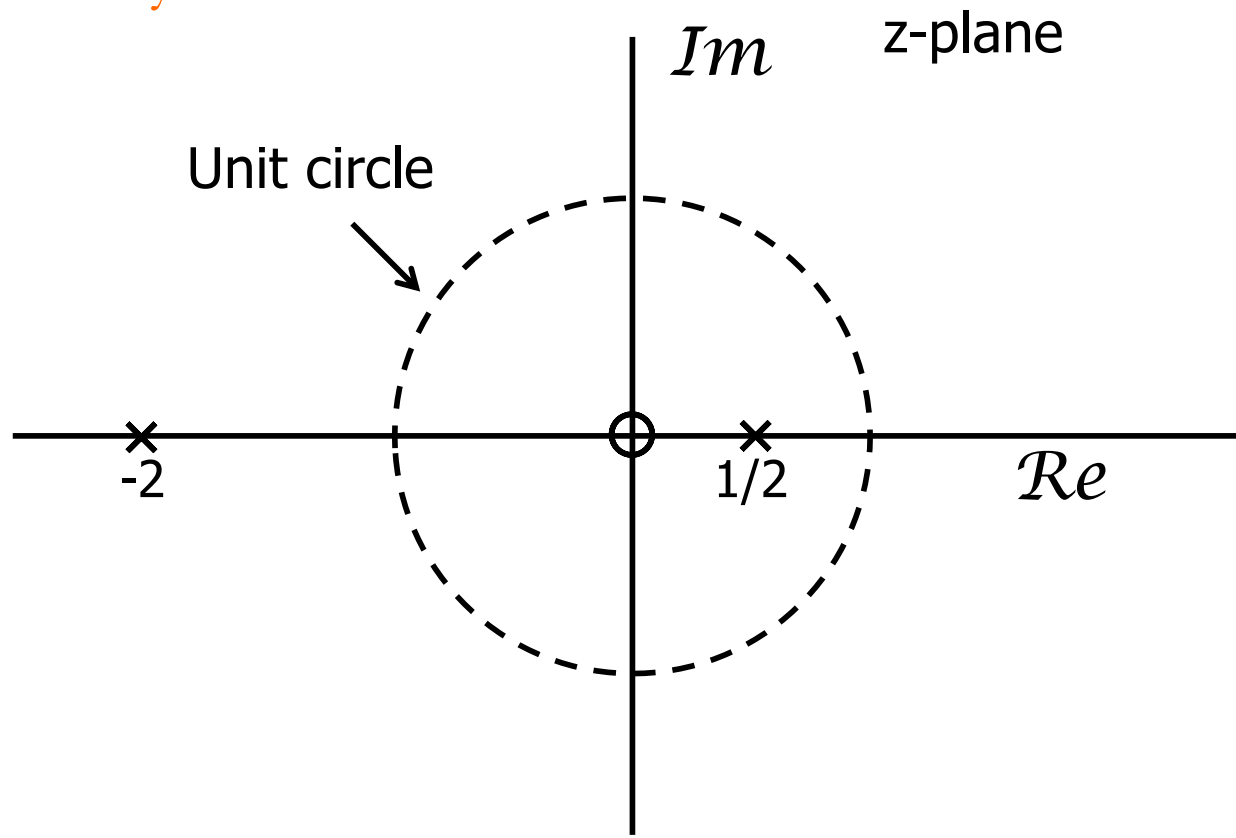
Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?



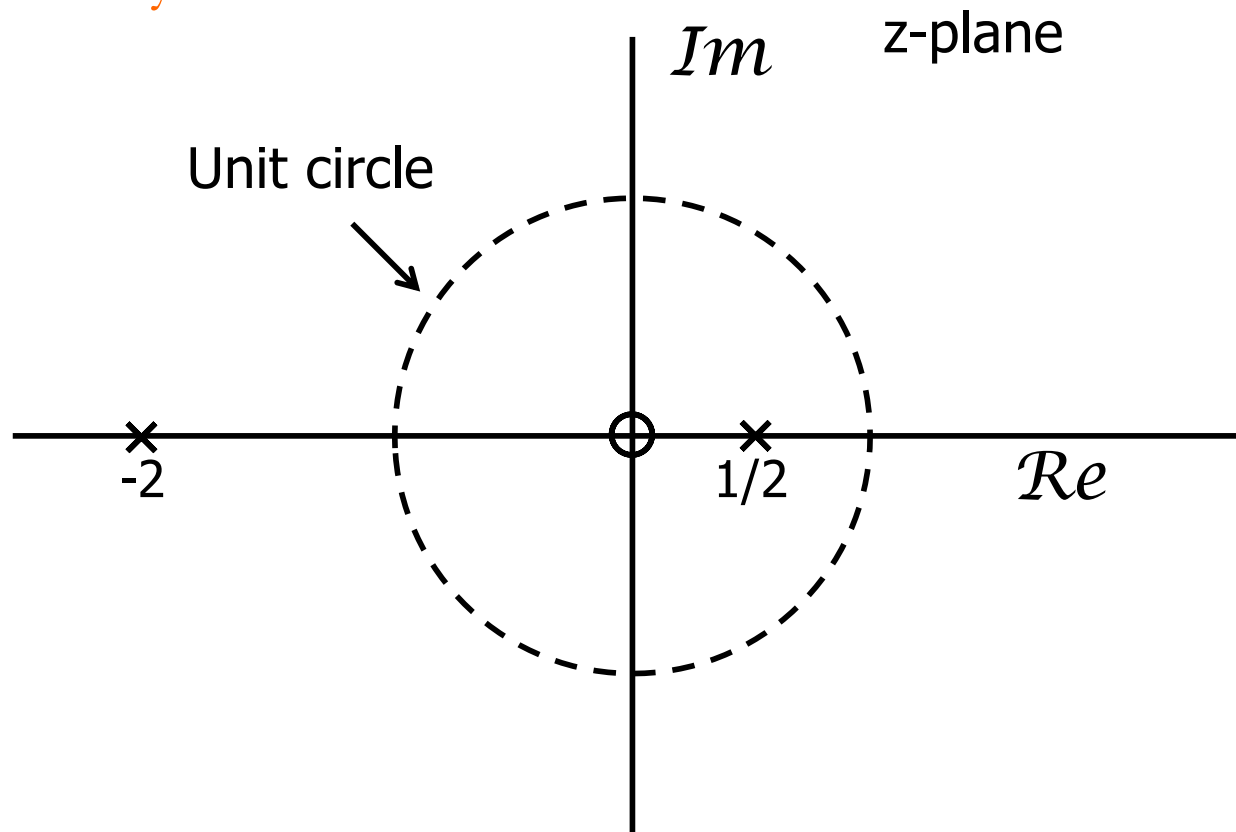
Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - What if system is causal?



Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - What if system is stable?

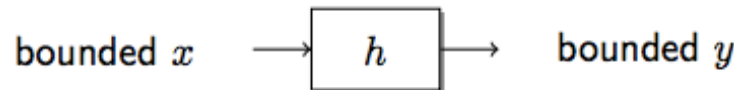


BIBO Stability Revisited

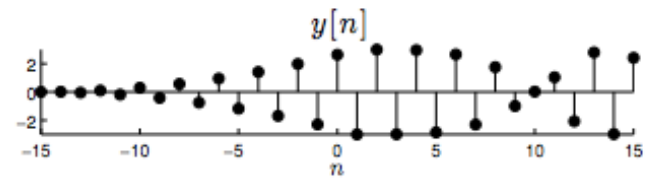
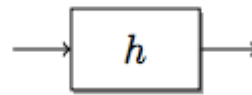
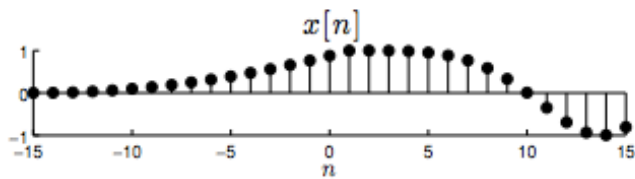
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output** (BIBO) stable if every bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_{\infty} < \infty$ and $\|y\|_{\infty} < \infty$



- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

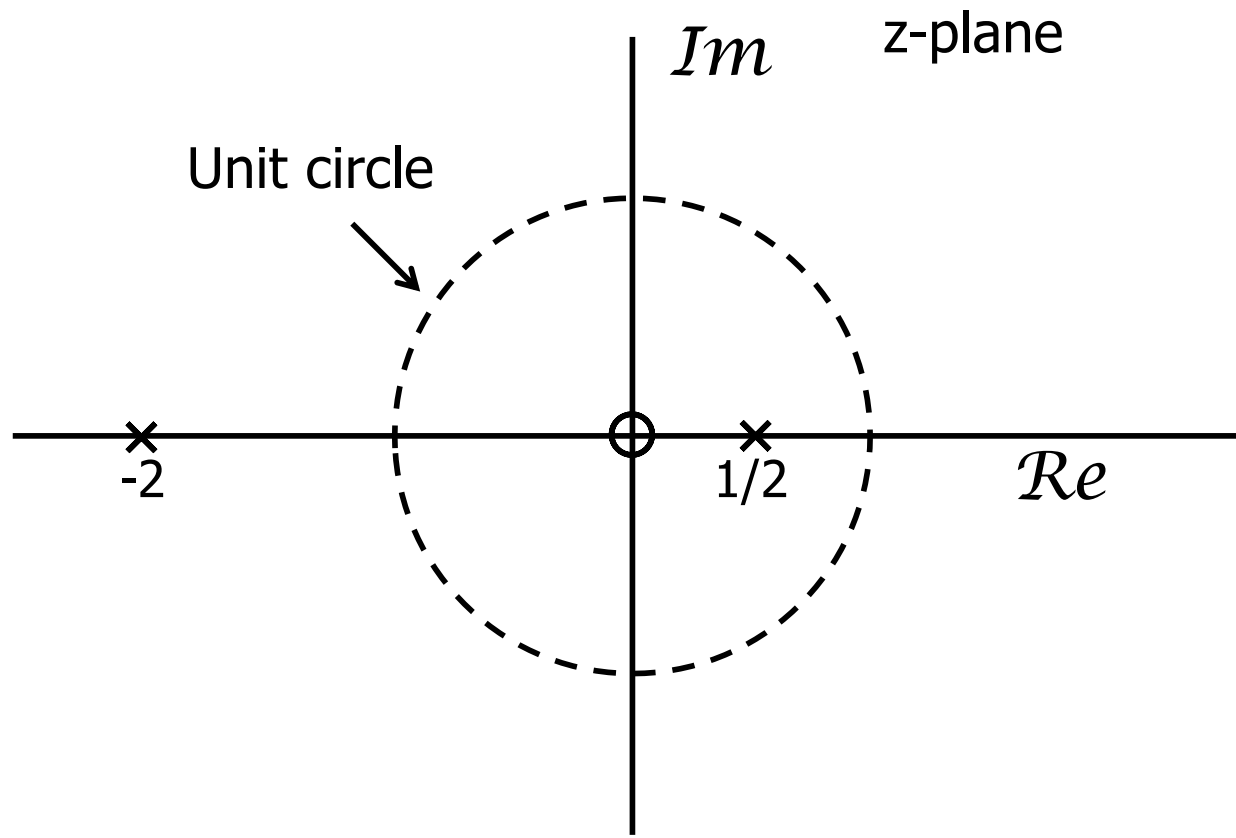
$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$



Stability

Example: Pole-Zero Plot

- $H(z)$ for an LTI System
 - How many possible ROCs?
 - Stable and causal?



z-transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z > a $
6. $-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z < a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z > a $
8. $-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z < a $
9. $\cos(\omega_0 n) u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
10. $\sin(\omega_0 n) u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
11. $r^n \cos(\omega_0 n) u[n]$	$\frac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z > r$
12. $r^n \sin(\omega_0 n) u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2z^{-2}}$	$ z > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z > 0$



Properties of z-Transform

- Linearity:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z)$$

- Time shifting:

$$x[n] \leftrightarrow X(z)$$

$$x[n - n_d] \leftrightarrow z^{-n_d} X(z)$$

- Multiplication by exponential sequence

$$x[n] \leftrightarrow X(z)$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right)$$



Properties of z-Transform

□ Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

□ Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z \frac{dX(z)}{dz}$$

□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

ROC_Y at least $\text{ROC}_X \wedge \text{ROC}_H$

z-transform Properties

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x , except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$ z_0 R_x$
4	3.4.4	$nx[n]$	$-z \frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$



Big Ideas

□ z-Transform

- Uses complex exponential eigenfunctions to represent discrete time sequence
 - DTFT is z-Transform where $z=e^{j\omega}$, $|z|=1$
- Draw pole-zero plots
- Must specify region of convergence (ROC)

□ z-Transform properties

- Similar to DTFT



Admin

- ❑ HW 1 out now
 - Due 1/30 at midnight
 - Submit in Canvas
- ❑ HW 2 posted on Sunday