# ESE 531: Digital Signal Processing

Lecture 7: February 3, 2022

Sampling and Reconstruction

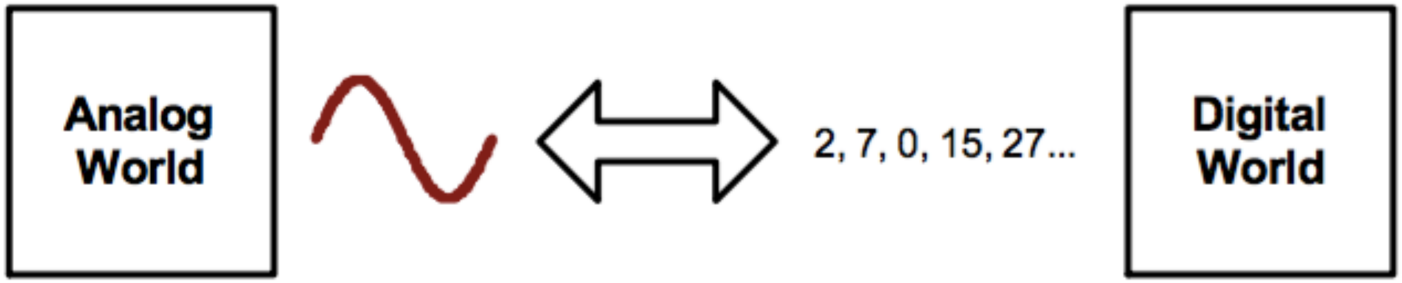
# Lecture Outline

- Sampling
  - Frequency Response of Sampled Signal
- Reconstruction
- Anti-aliasing Filter

# Video Example

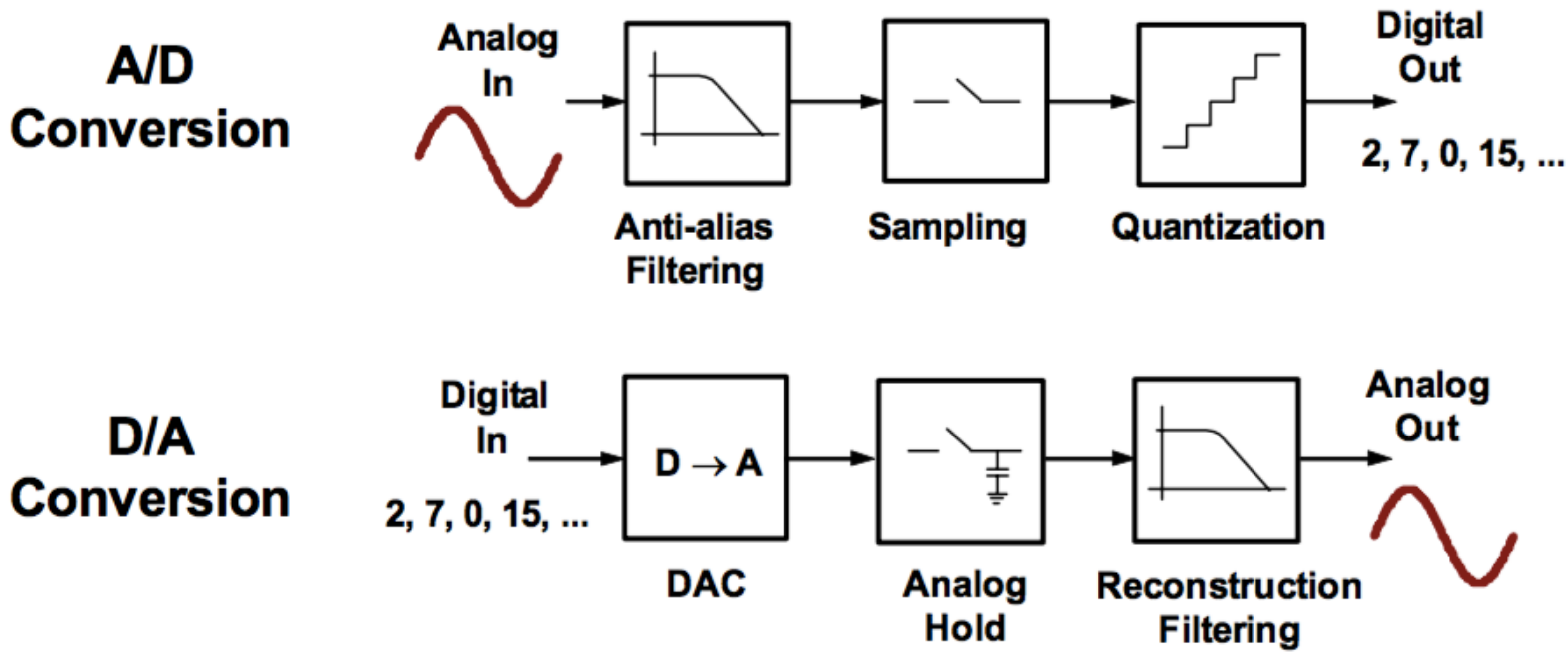

- https://www.youtube.com/watch?v=ByTsISFXUoY

# The Data Conversion Problem



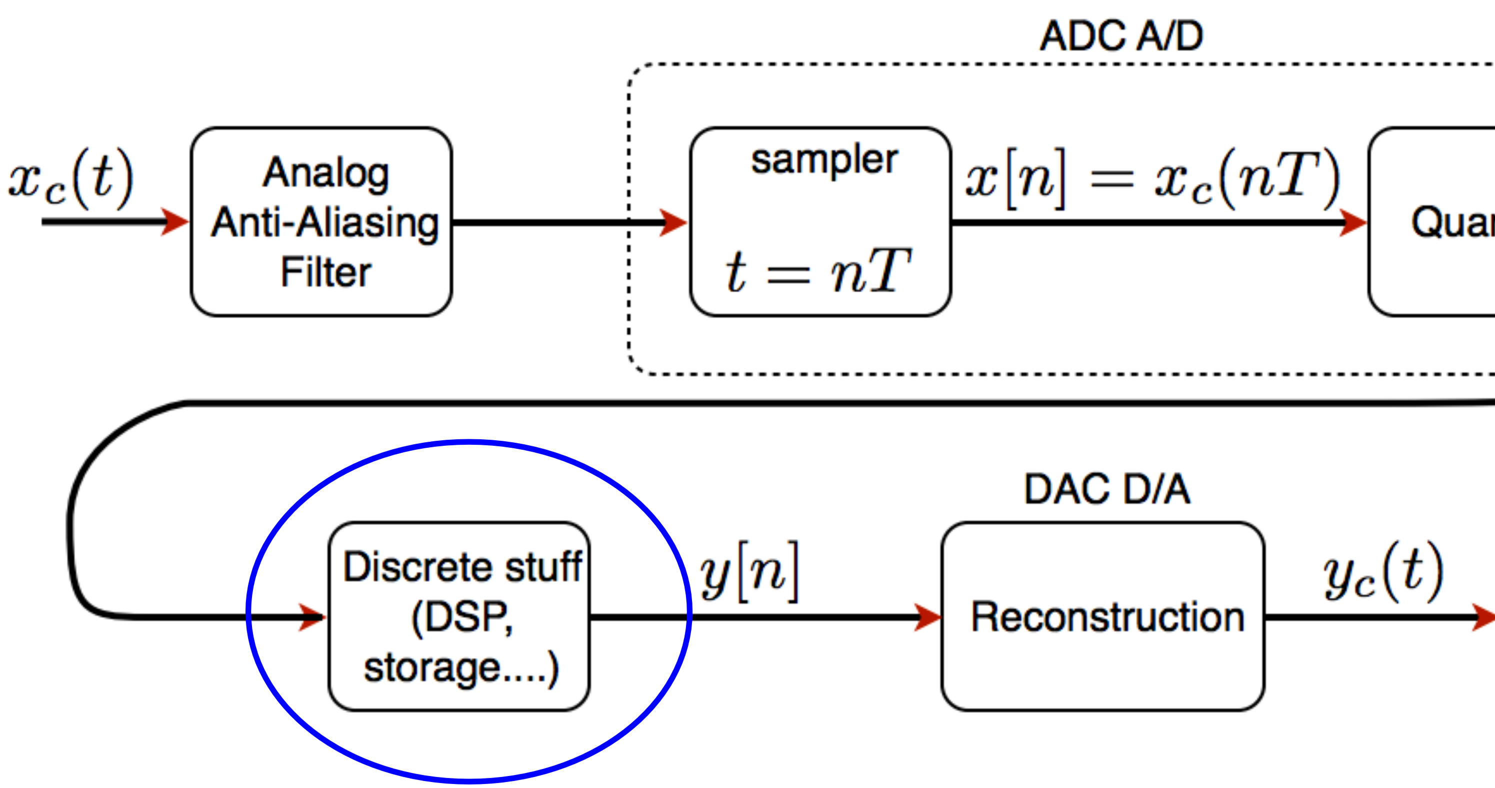

- Real world signals
  - Continuous time, continuous amplitude
- Digital abstraction
  - Discrete time, discrete amplitude
- Two problems
  - How to go discretize in time and amplitude
    - A/D conversion
  - How to "undescretize" in time and amplitude
    - D/A conversion

# Overview



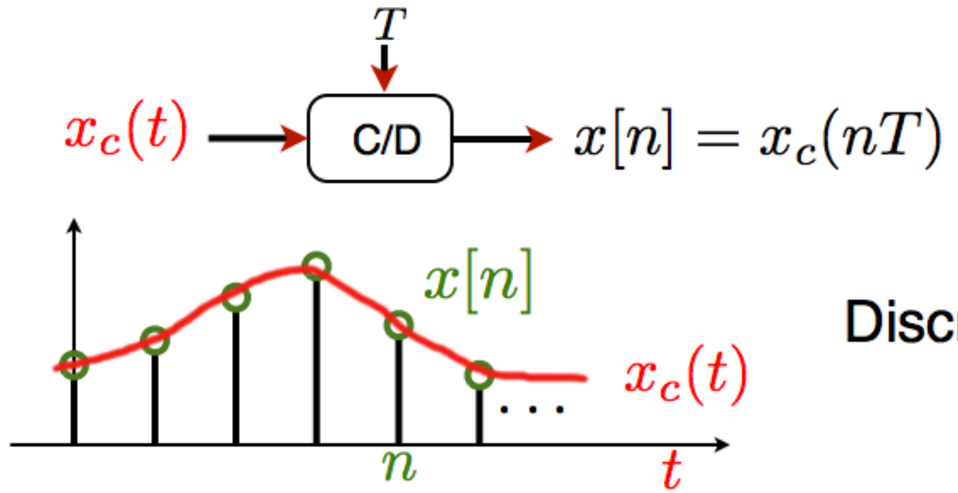

- We'll first look at these building blocks from a functional, "black box" perspective

# DSP System



ADC A/D
$x_c(t)$
Analog
Anti-Aliasing
Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff
(DSP,
storage....)
$y[n]$
Reconstruction
$y_c(t)$

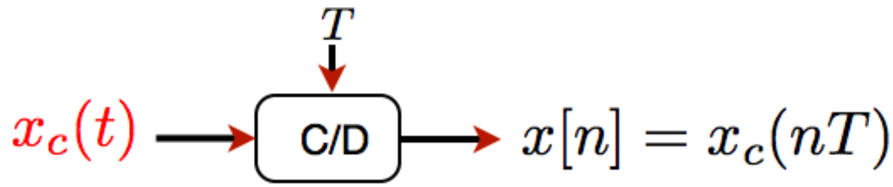
# Ideal Sampling Model

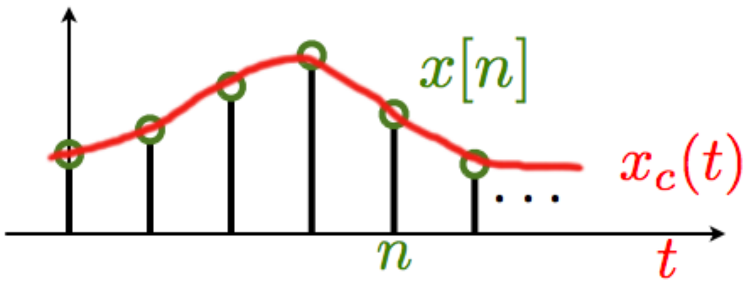



Discrete and Continuous

- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - $f_s$=1/T is the sampling frequency
  - $\Omega_s$=2π/T

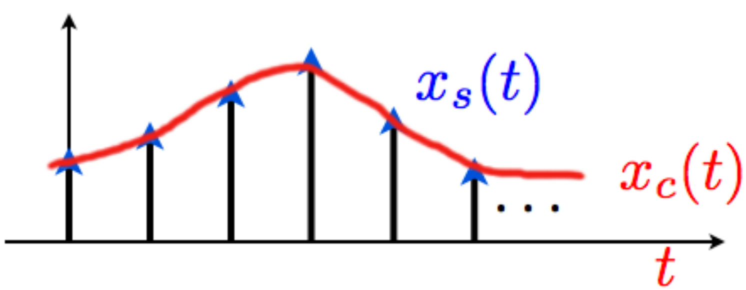
# Ideal Sampling Model




Discrete and Continuous

define impulsive sampling:




Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

# Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Dirac delta function, δ(t)
  - Infinitely high and thin, area of 1
  - Not physical—for modeling

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- Three signals. How are they related? In time? In frequency?

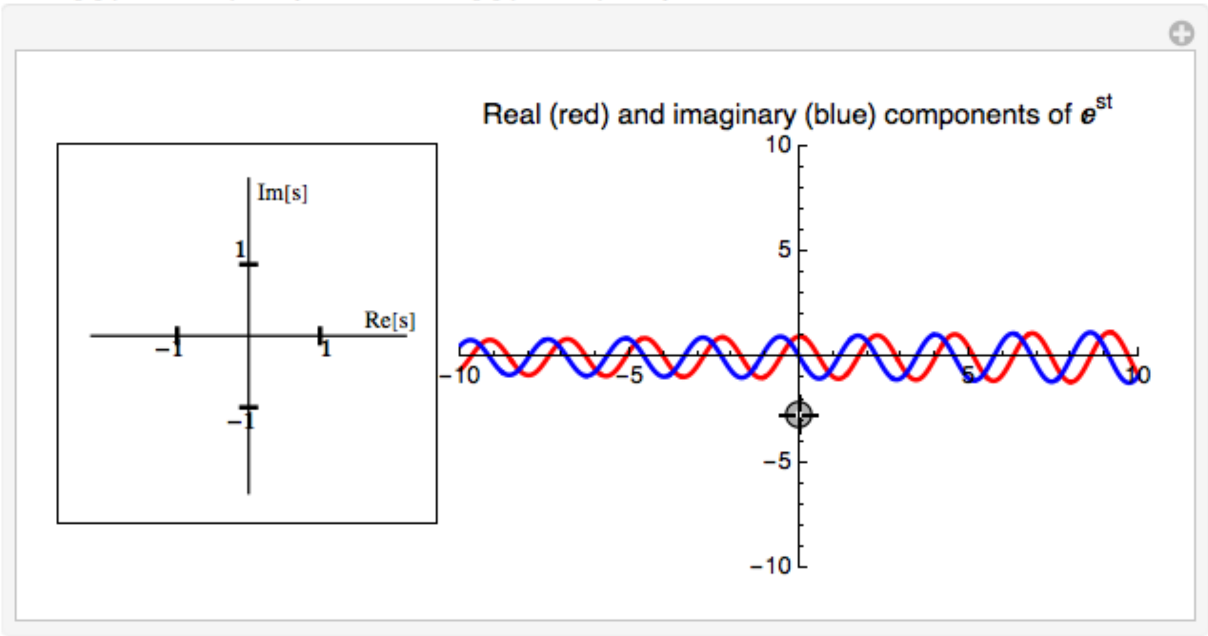
# Reminder: Laplace Transform

- The Laplace transform takes a function of time, t, and transforms it to a function of a complex variable, s.
- Because the transform is invertible, no information is lost, and it is reasonable to think of a function f(t) and its Laplace transform F(s) as two views of the same phenomenon.
- Each view has its uses, and some features of the phenomenon are easier to understand in one view or the other.

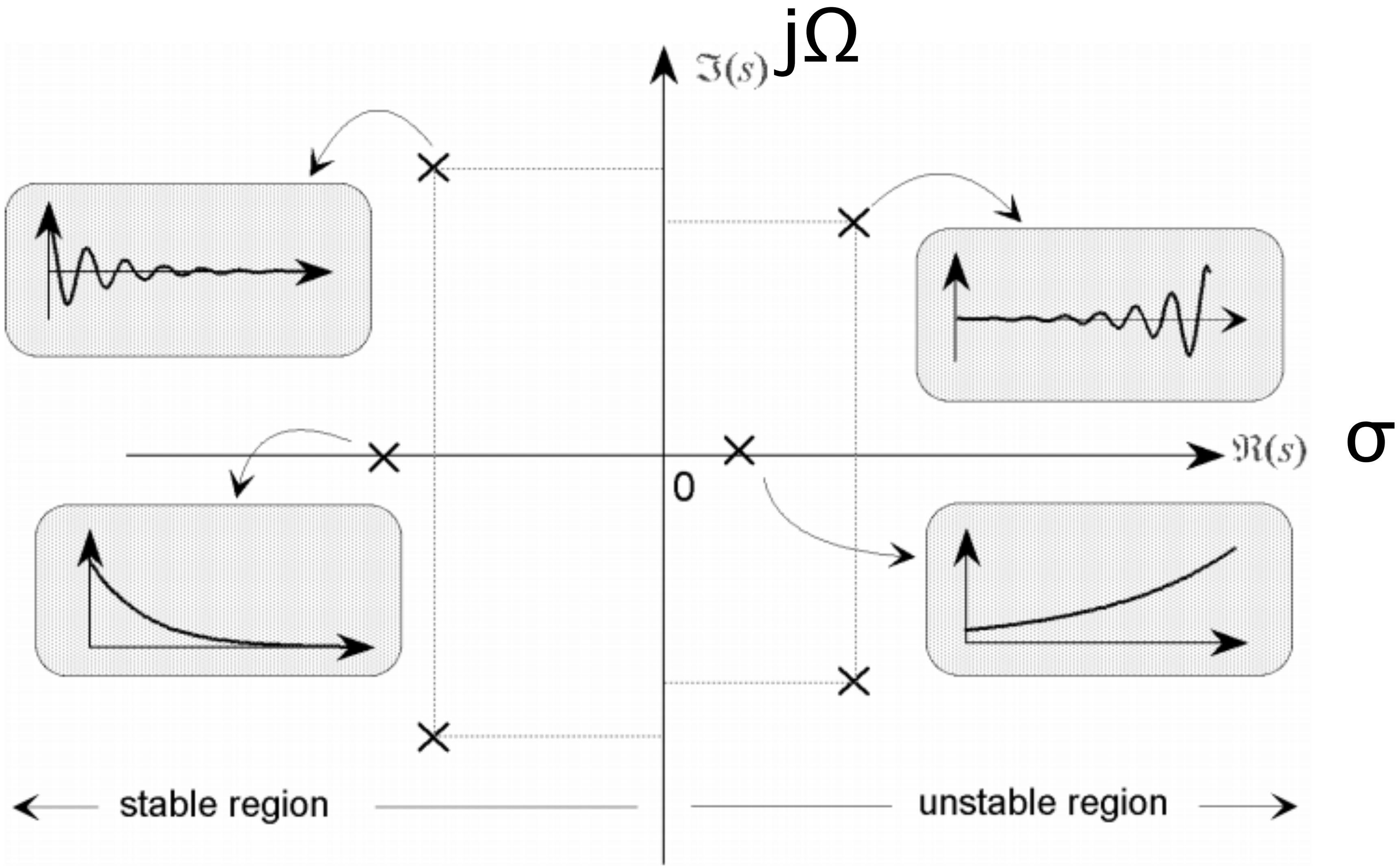
# S-Plane

- $s=\sigma+j\Omega$

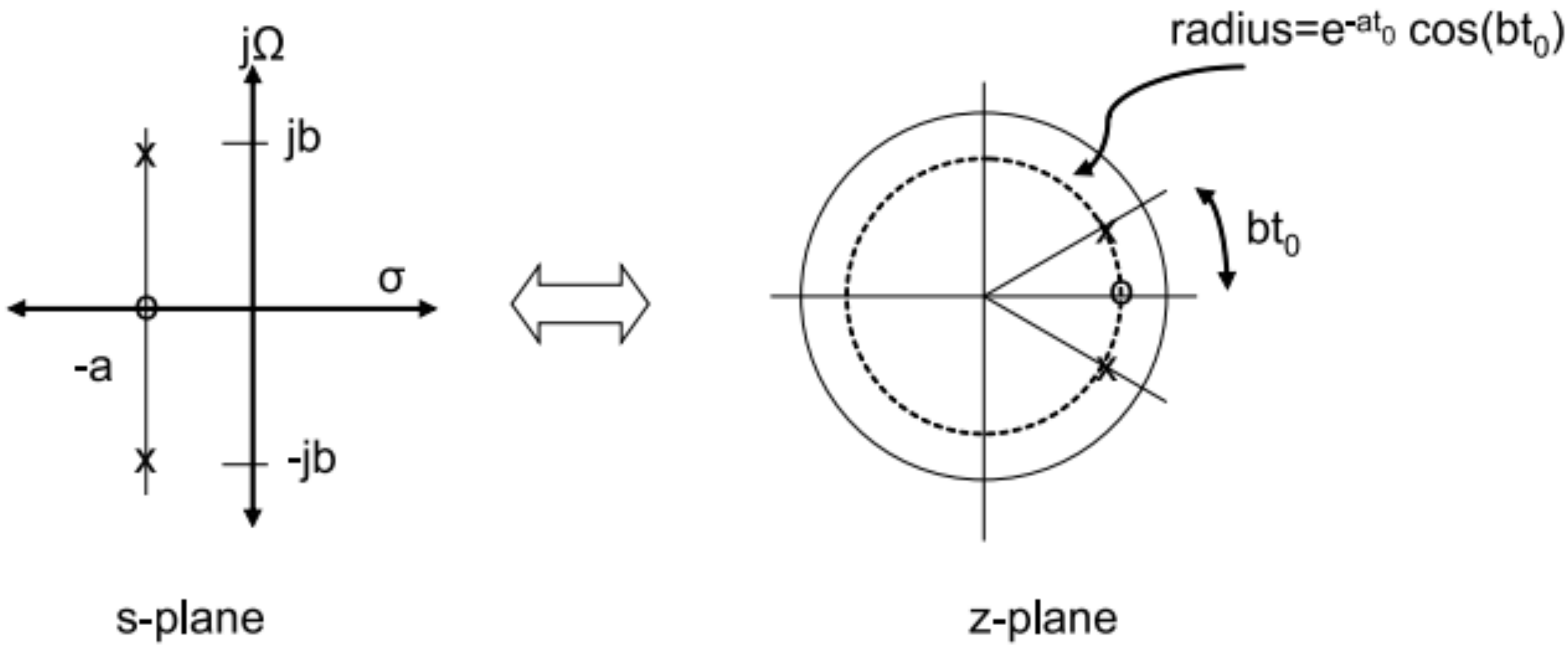
- Wolfram Demo



# S-plane and stability



# S-Plane Mapping to Z-Plane




$t_0$=sampling period

# Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

$$x[n] = x_c(nT) \qquad\qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

# Frequency Domain Analysis

- How is x[n] related to $x_s(t)$ in frequency domain?

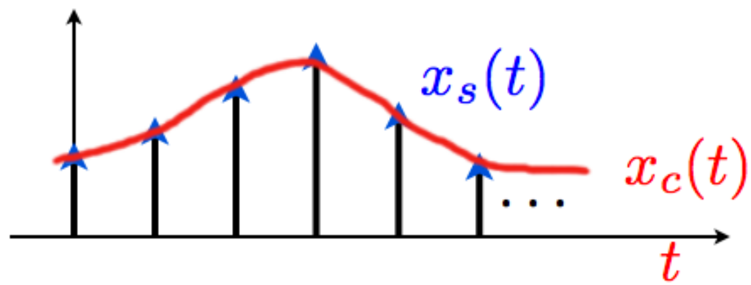
$$x[n] = x_c(nT) \qquad x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$x_s(t)$ :C.T
$$X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$$

$x[n]$ :D.T
$$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad \omega = \Omega T$$

$$\boxed{X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T}} \qquad \boxed{X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T}}$$

# Frequency Domain Analysis



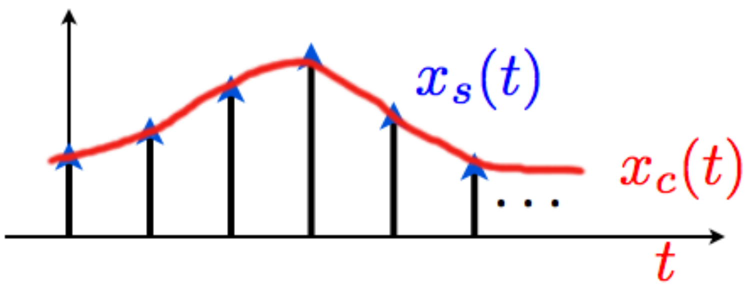

Continuous

$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-nT)}_{\triangleq s(t)}$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

# Frequency Domain Analysis

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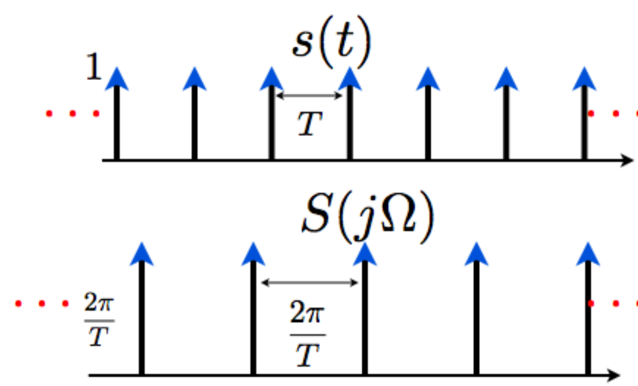
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$s(t) \quad \leftrightarrow \quad S(j\Omega)$$

$$S(j\Omega) \quad = \quad \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi}{T}k)$$

$= \Omega_s$



# Frequency Domain Analysis

$$x_s(t) = s(t) \cdot x_c(t)$$

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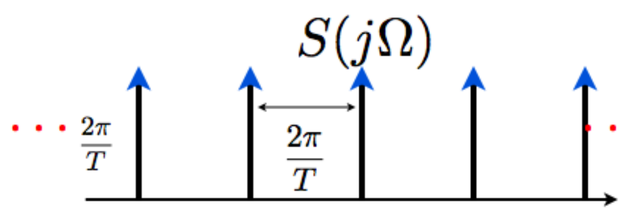

# Frequency Domain Analysis

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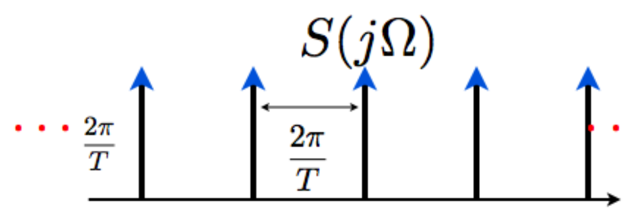

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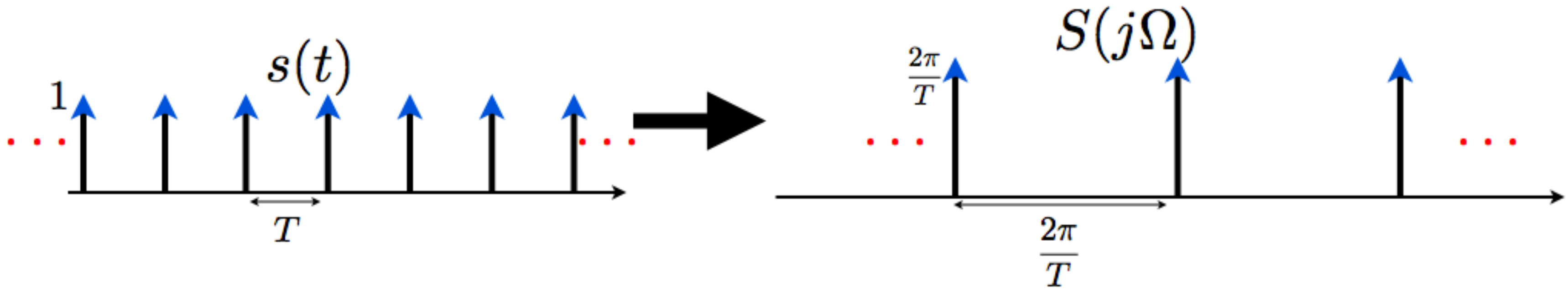
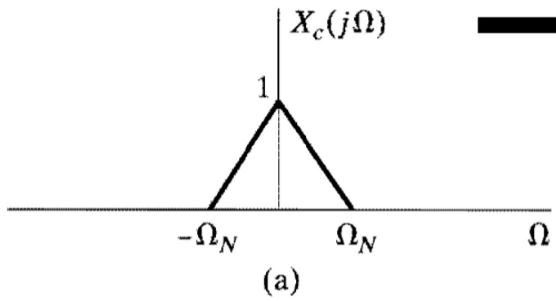



and $\Omega_s > 2\Omega_N$



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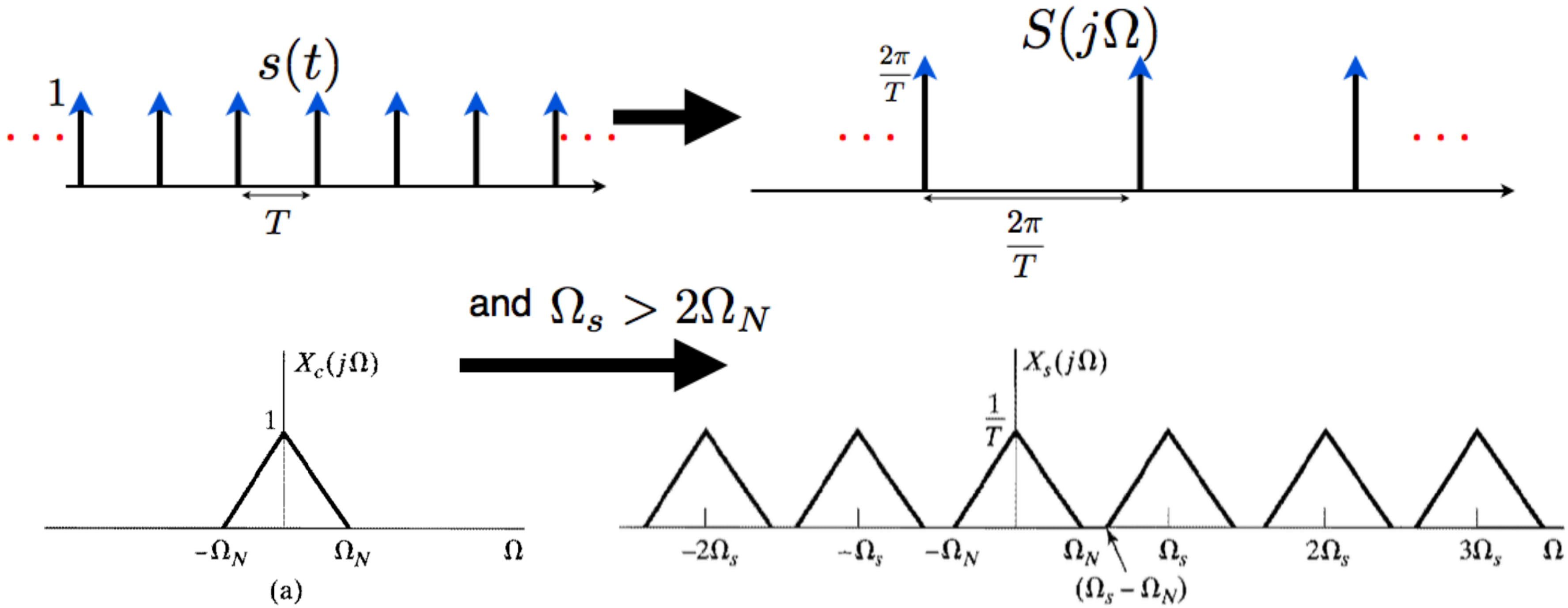

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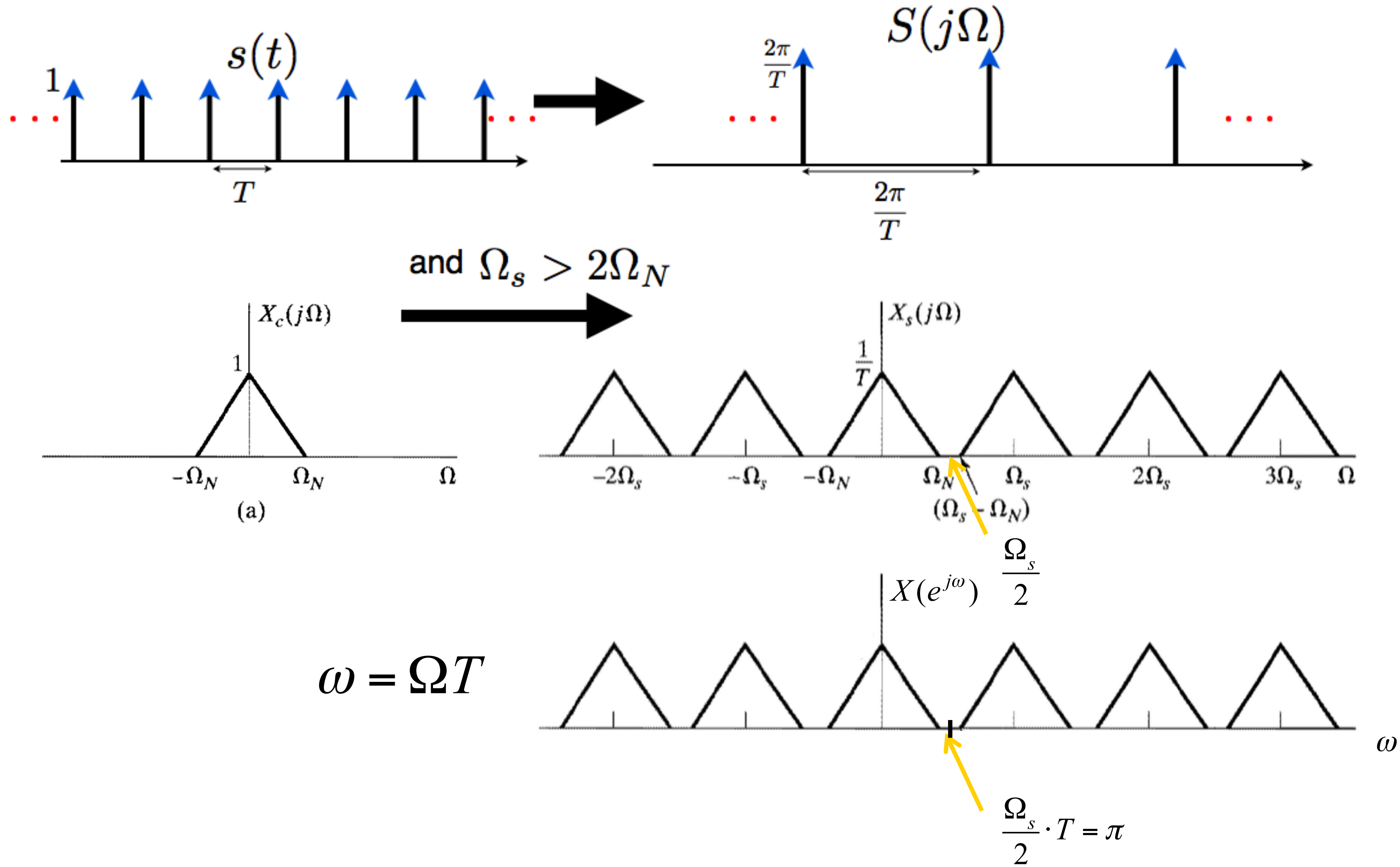
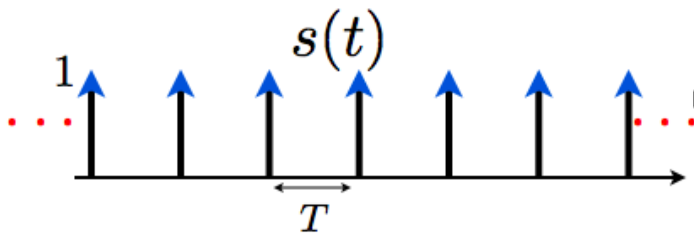
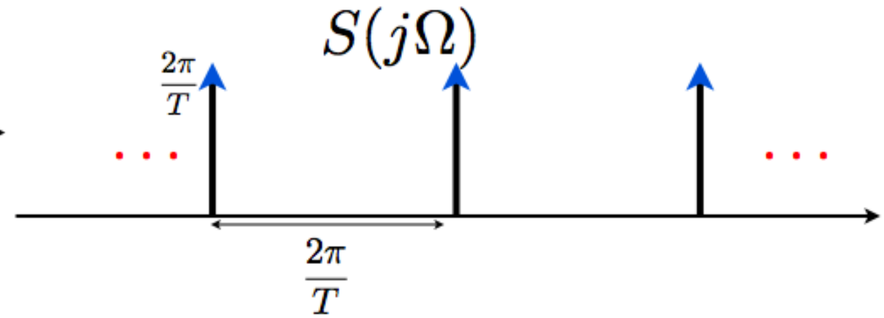


# Frequency Domain Analysis w/ Aliasing

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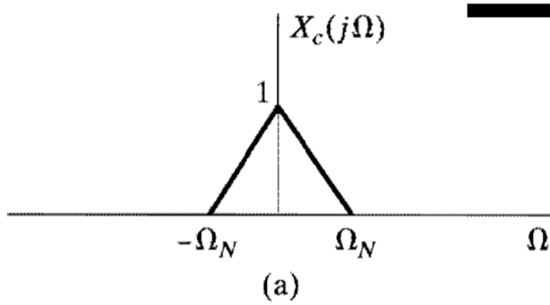

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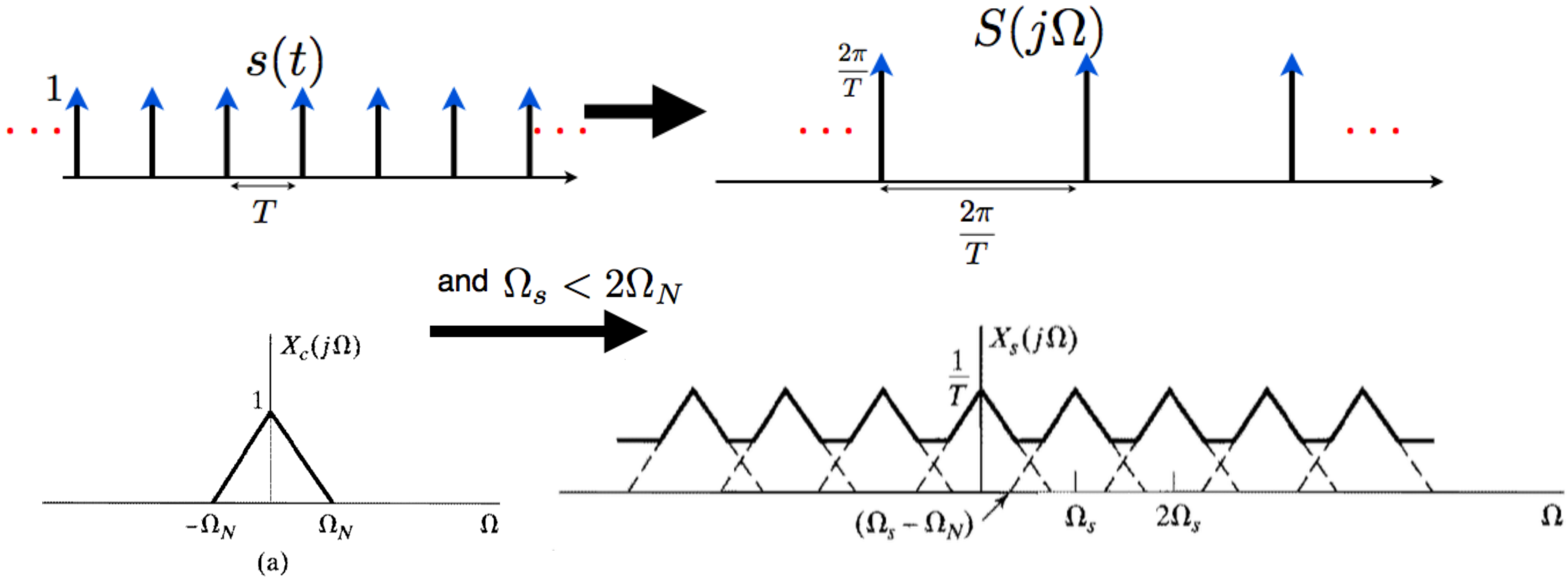

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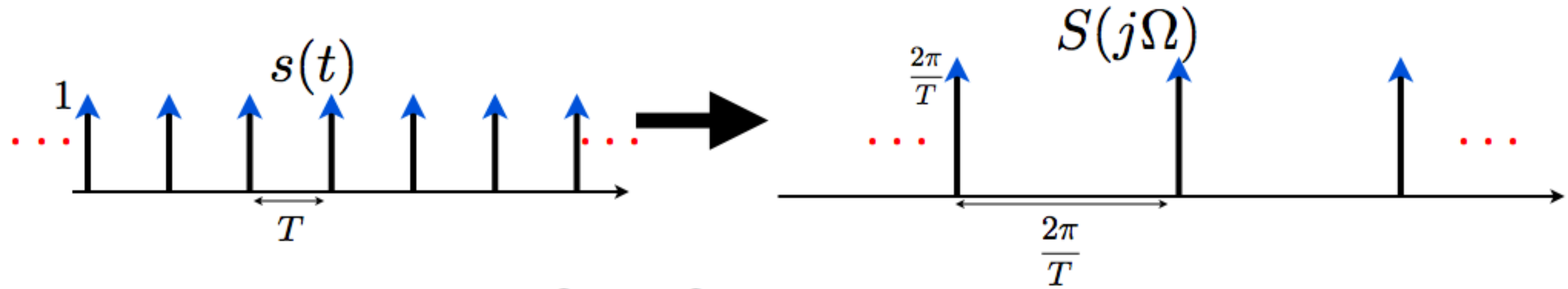
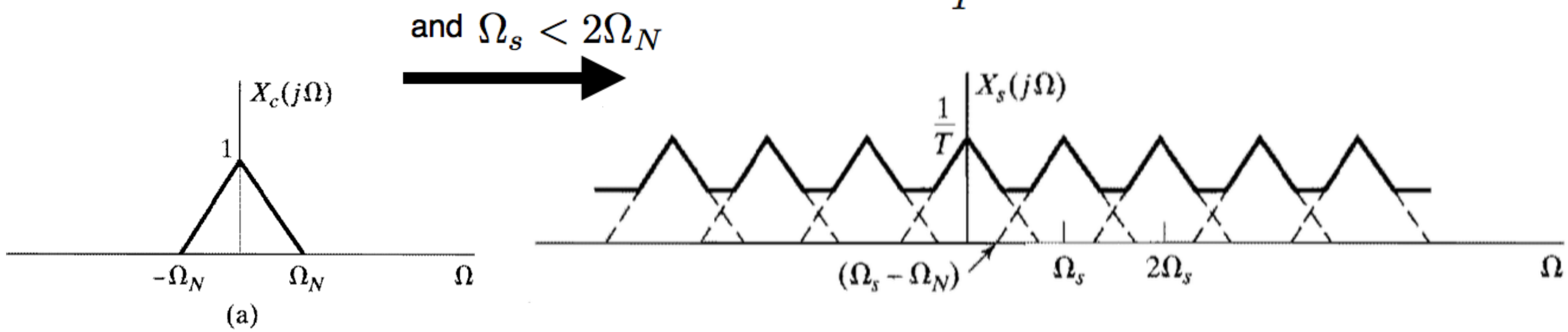
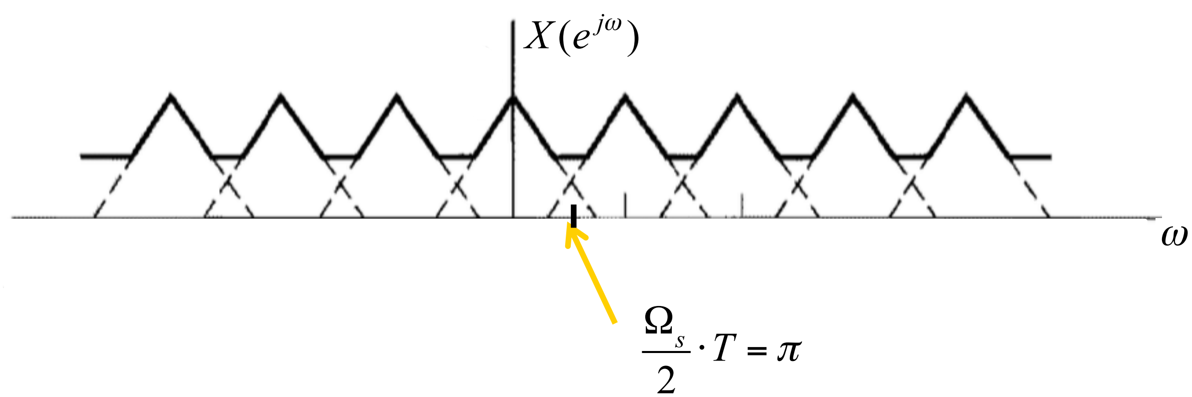



$$\omega = \Omega T$$



# Example: Cosine Input

- Sample the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period T=1/6000 s ($f_s$=6kHz)

$$\boxed{\begin{aligned} X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T} \end{aligned}}$$

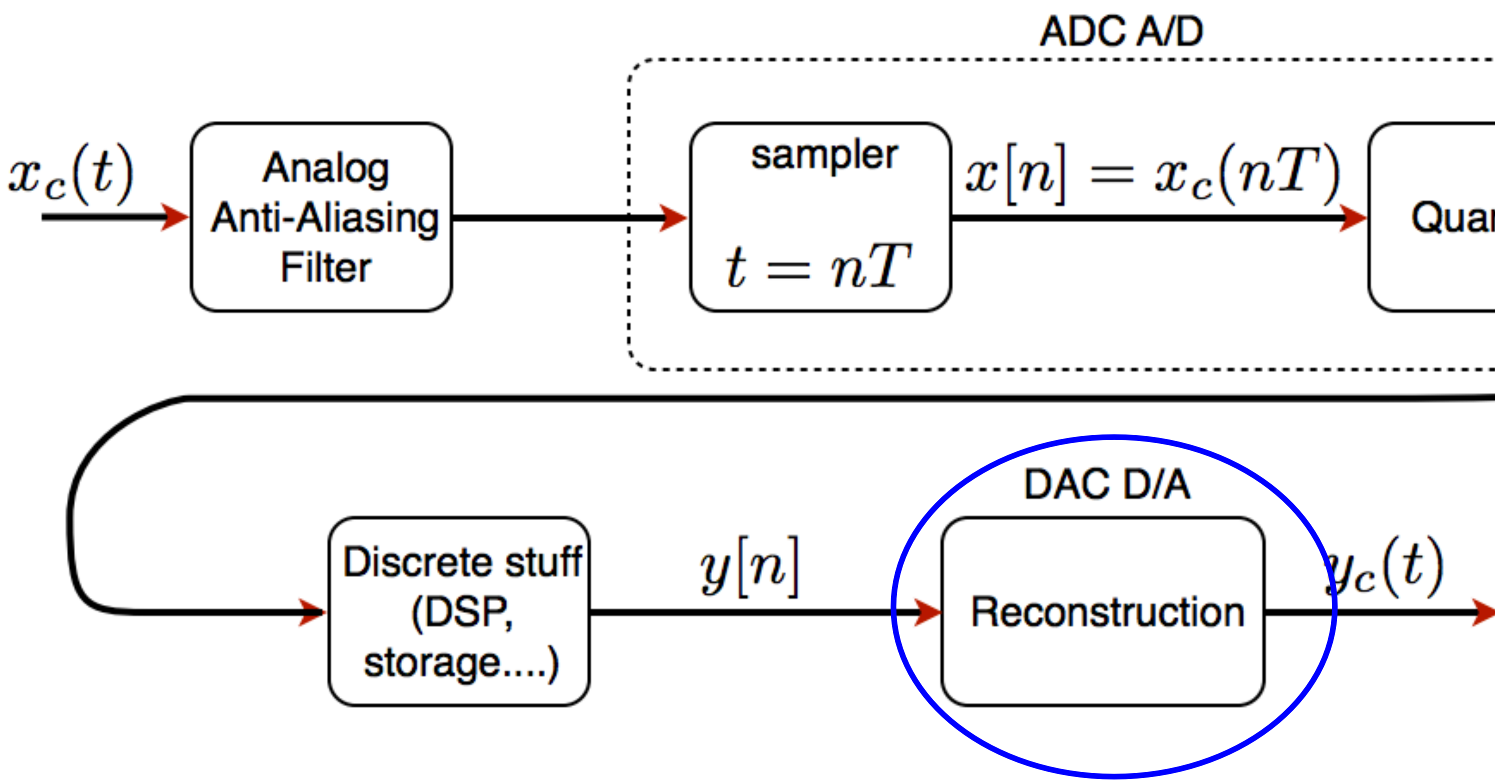
# Example: Cosine Input

- $x_c(t)=\cos(4000\pi t)$, $T=1/6000$ s ($f_s$=6kHz)

# Example: Cosine Input

- Sample the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period $T=1/6000$ ($f_s$=6kHz)

# DSP System



ADC A/D
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Analog
Anti-Aliasing
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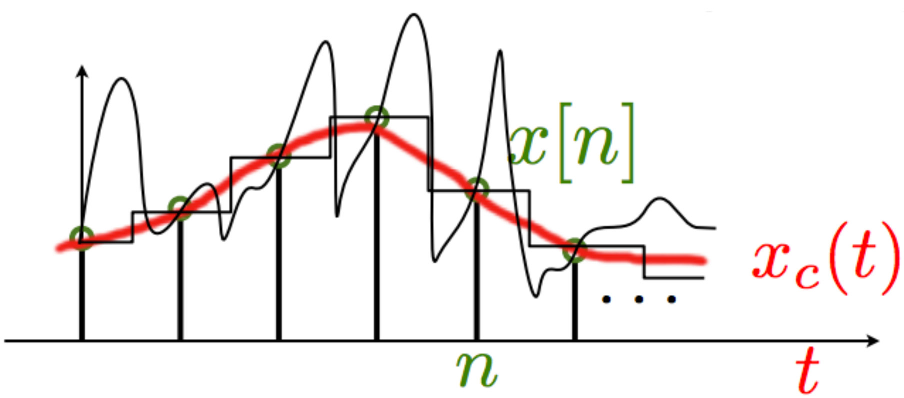
$t = nT$
$x[n] = x_c(nT)$
Quantizer
Discrete stuff
(DSP,
storage....)
$y[n]$
DAC D/A
Reconstruction
$y_c(t)$

# Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

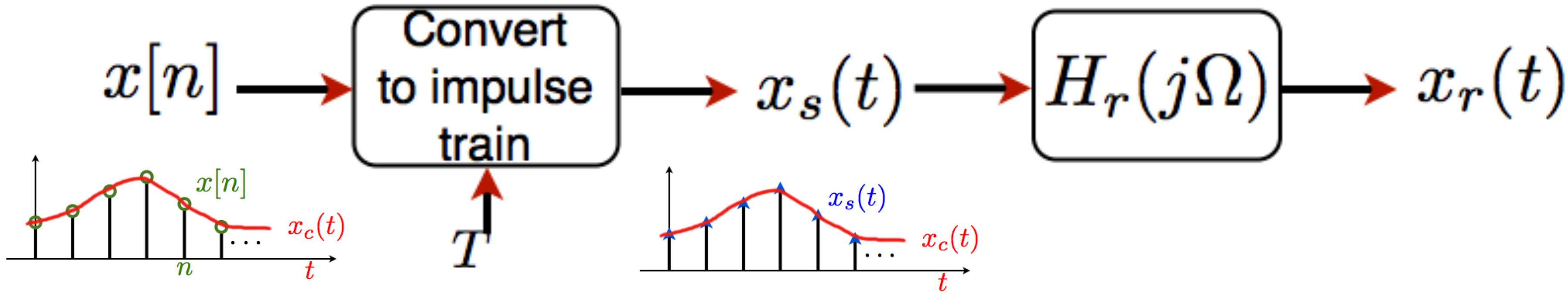
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

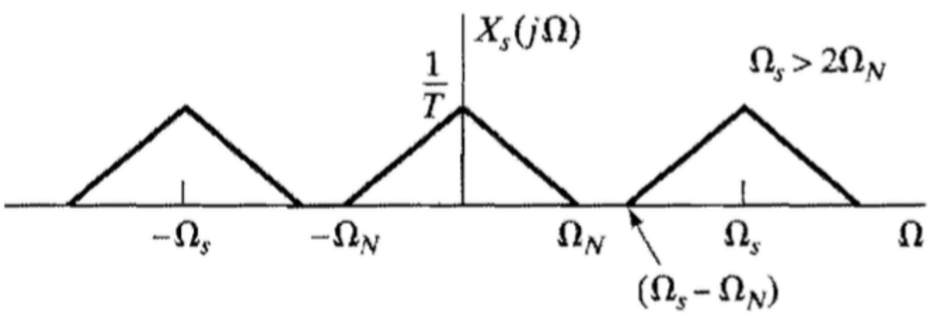
- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness




Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

# Reconstruction in Frequency Domain



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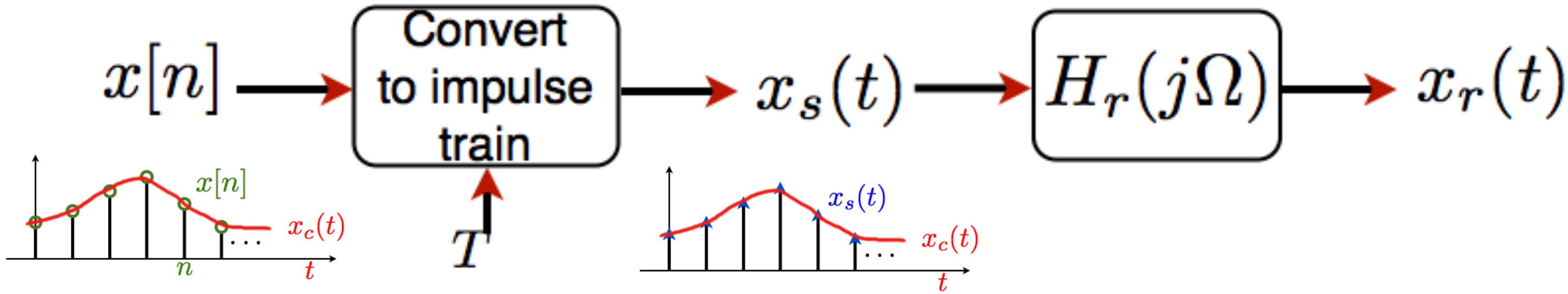


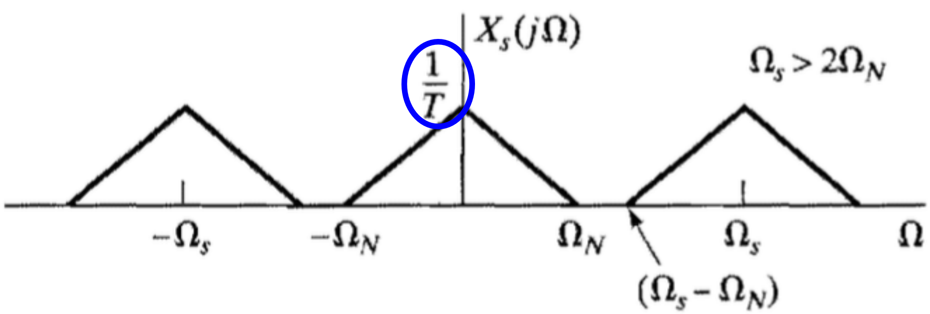


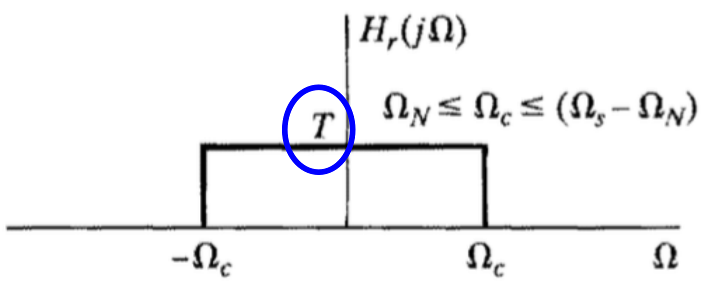

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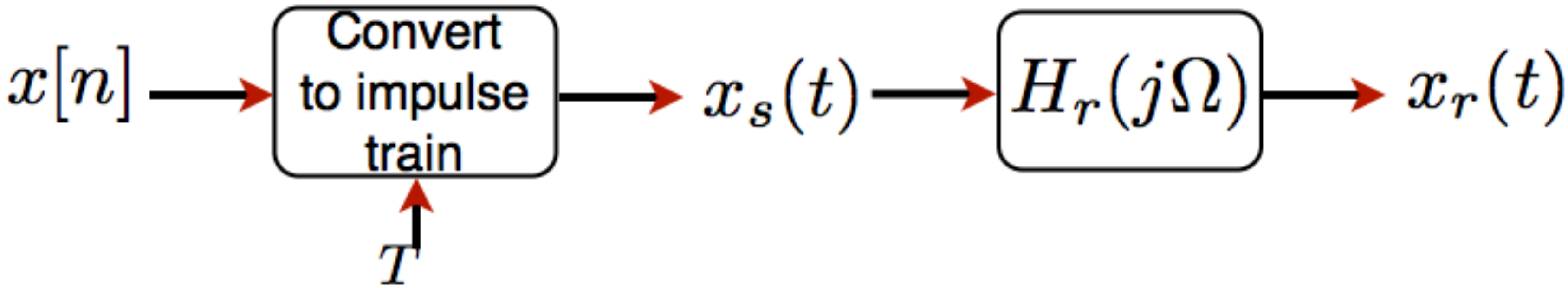


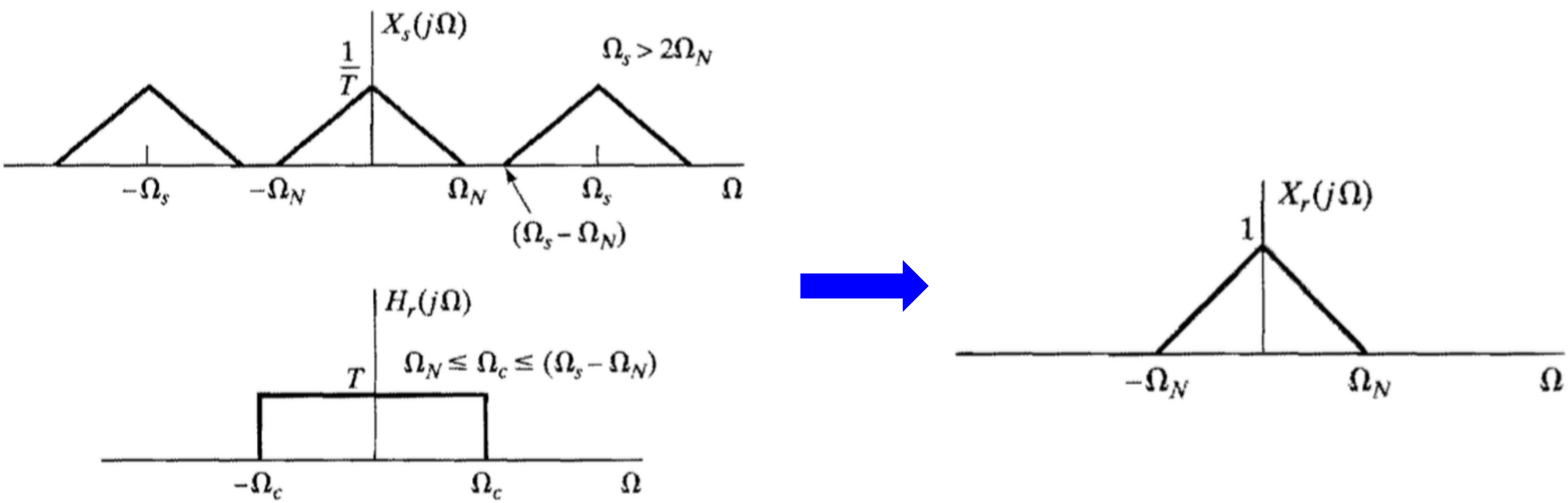

# Example: Cosine Input

- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(4000\pi t)$ with sampling period $T=1/6000$ s ($f_s$=6kHz)

$$
\begin{aligned}
X_s(j\Omega) &= \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}
\end{aligned}
$$

- Sample and reconstruct the continuous-time signal $x_c(t)=\cos(16000\pi t)$ with sampling period $T=1/6000$ ($f_s$=6kHz)

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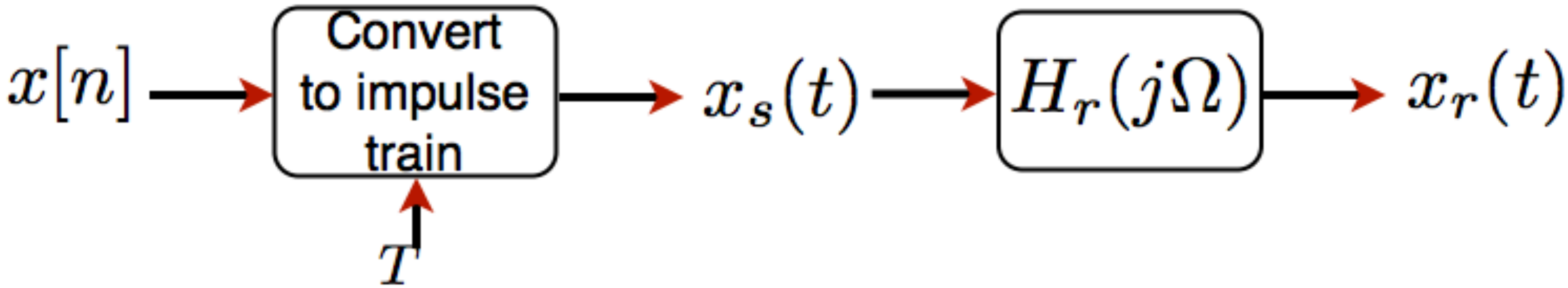
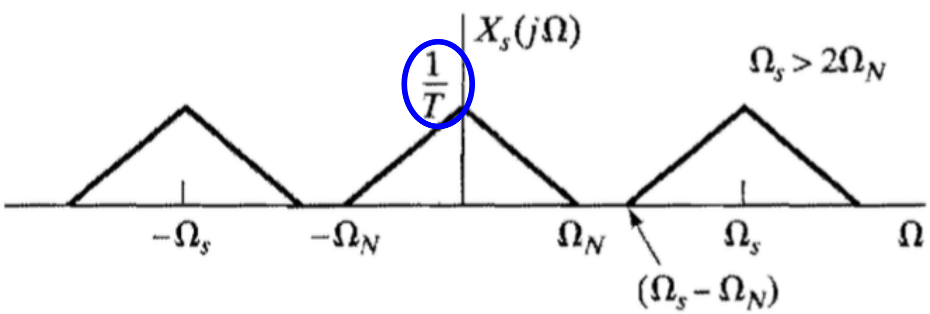
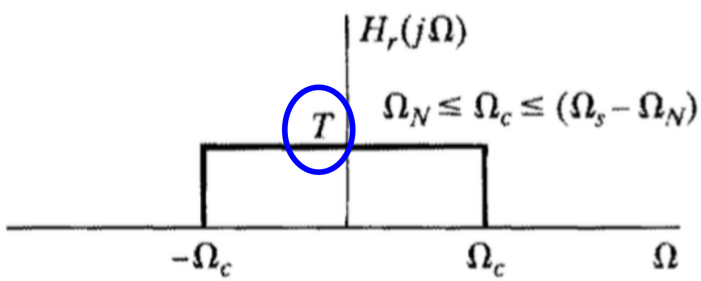
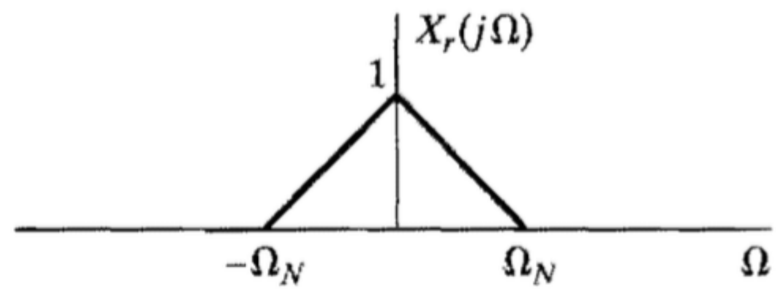


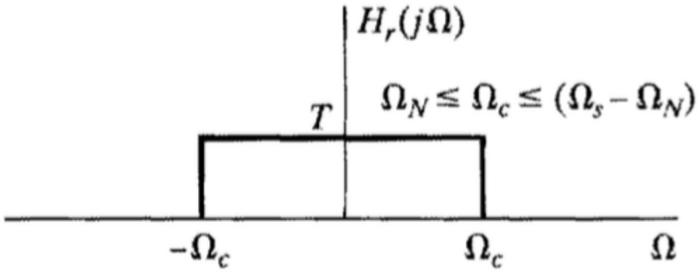

$2\Omega_N = \Omega_s$

$\Omega_N = \Omega_C = \Omega_s/2$

Sample at Nyquist and filter at signal badnwidth

# Reconstruction in Time Domain

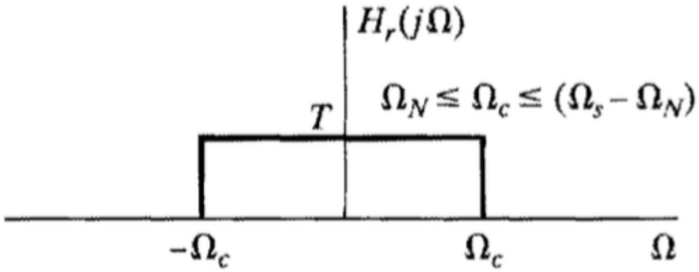
$\Omega_N = \Omega_C = \Omega_s/2$



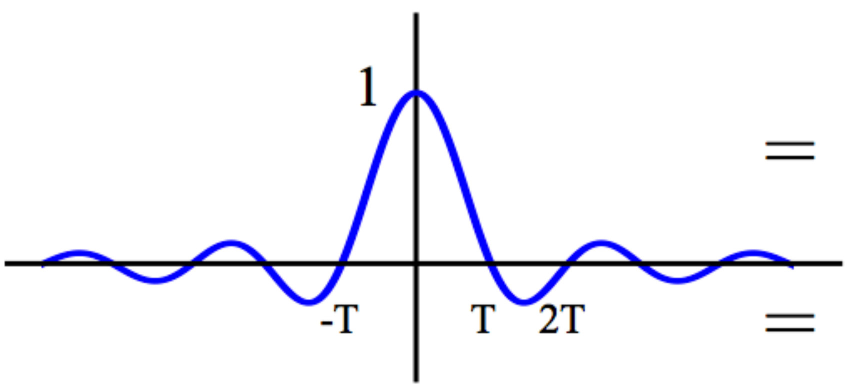

$$h_r(t) = \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t}\, d\Omega$$

# Reconstruction in Time Domain

$\Omega_N = \Omega_C = \Omega_s/2$



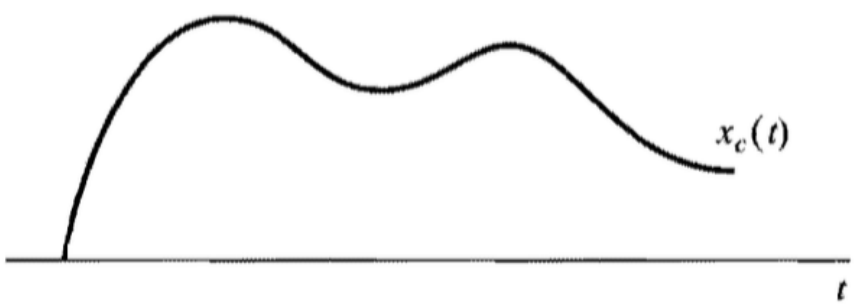

$$
\begin{aligned}
h_r(t) &= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} T e^{j\Omega t}\, d\Omega \\
&= \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} \\
&= \frac{T}{\pi t}\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} \\
&= \frac{T}{\pi t}\sin(\frac{\Omega_s}{2}t) = \frac{T}{\pi t}\sin(\frac{\pi}{T}t) \\
&= \operatorname{sinc}(\frac{t}{T})
\end{aligned}
$$

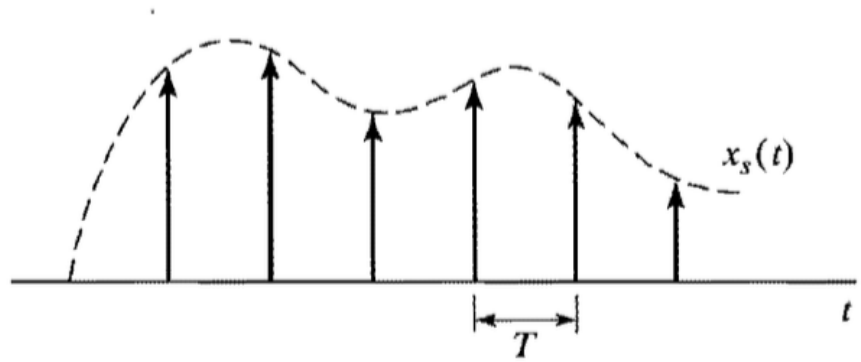


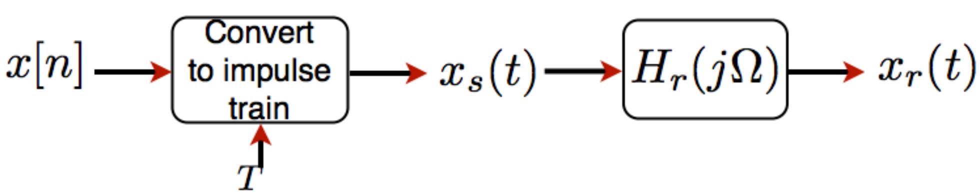
# Reconstruction in Time Domain

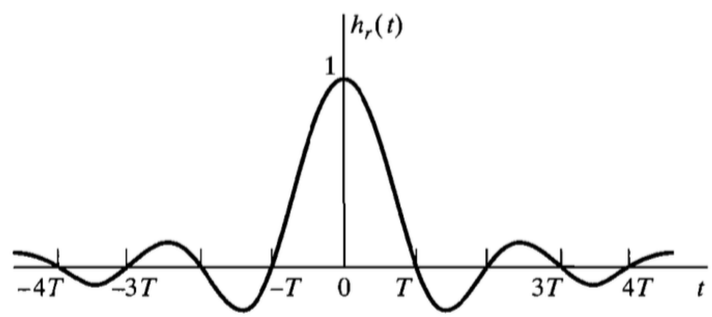
$$x_r(t) = x_s(t) * h_r(t) = \left( \sum_n x[n]\delta(t - nT) \right) * h_r(t)$$

$$= \sum_n x[n]h_r(t - nT)$$



# Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t)$$
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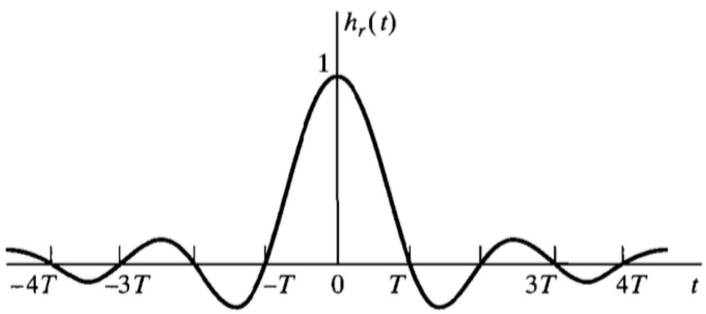
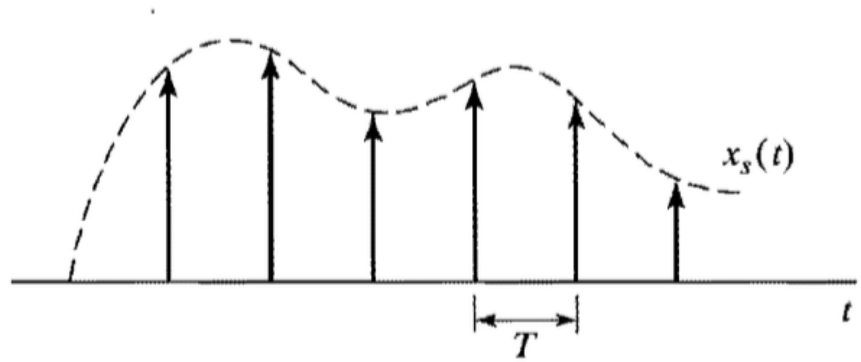



$*$

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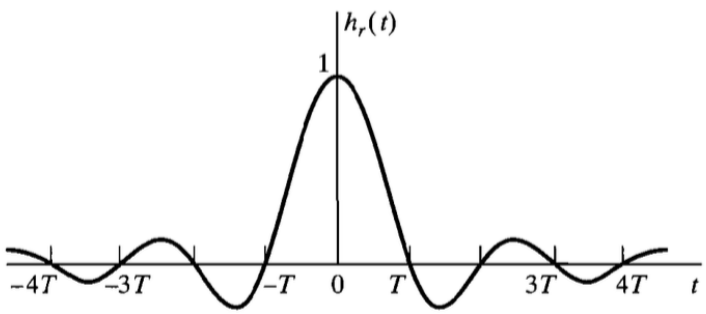
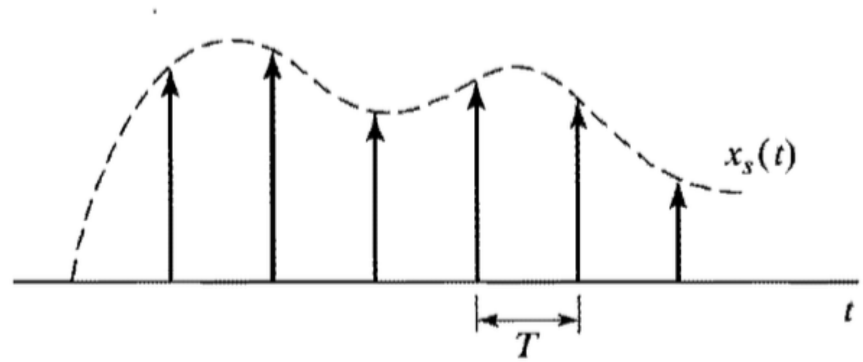
# Reconstruction in Time Domain

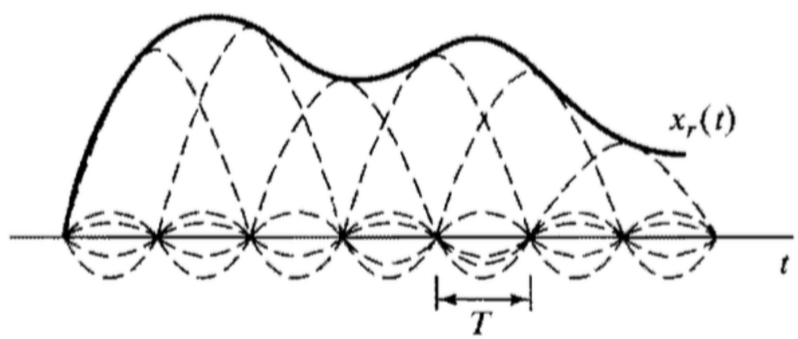
$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t)$$
$$= \sum_n x[n]h_r(t-nT)$$




$*$



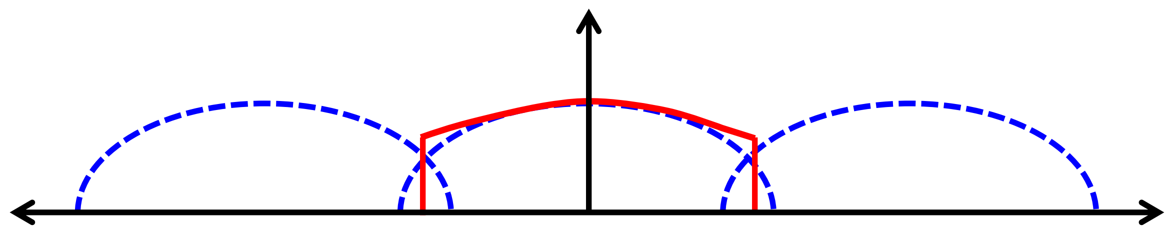

$=$




The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

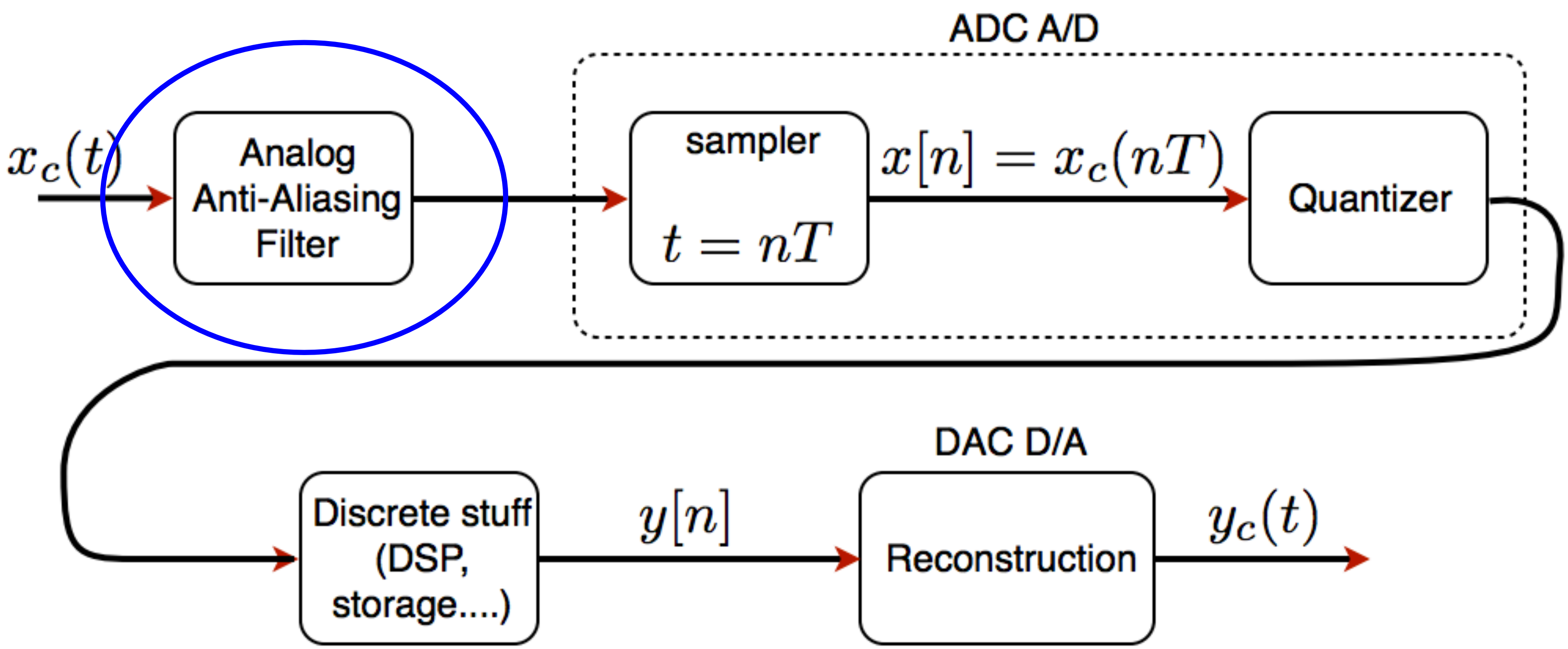
# Aliasing

- If $\Omega_N > \Omega_s/2$, $x_r(t)$ is an aliased version of $x_c(t)$

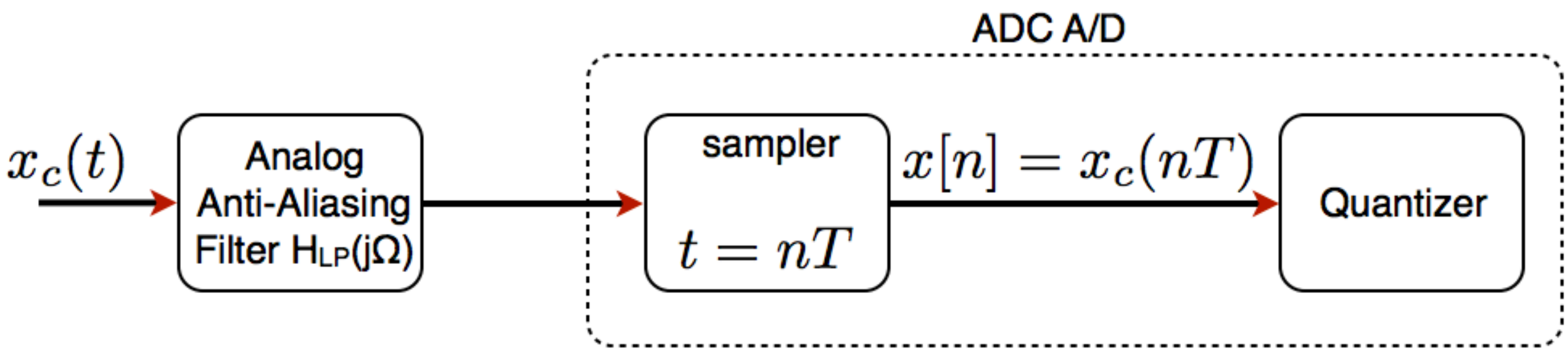


$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & \text{if} \quad |\Omega| \leq \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

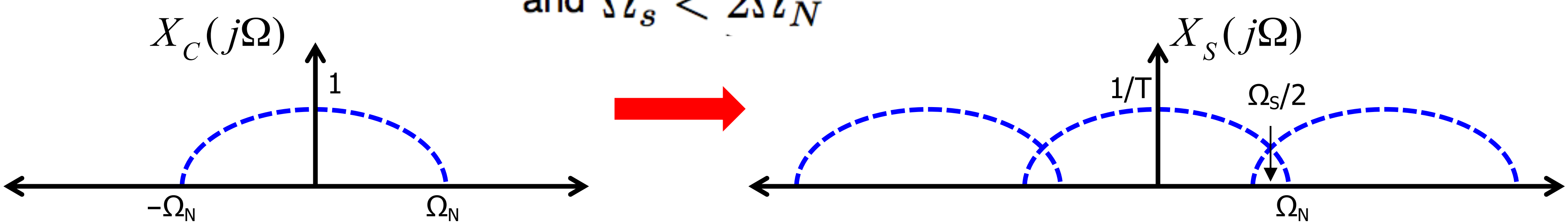
# DSP System


ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

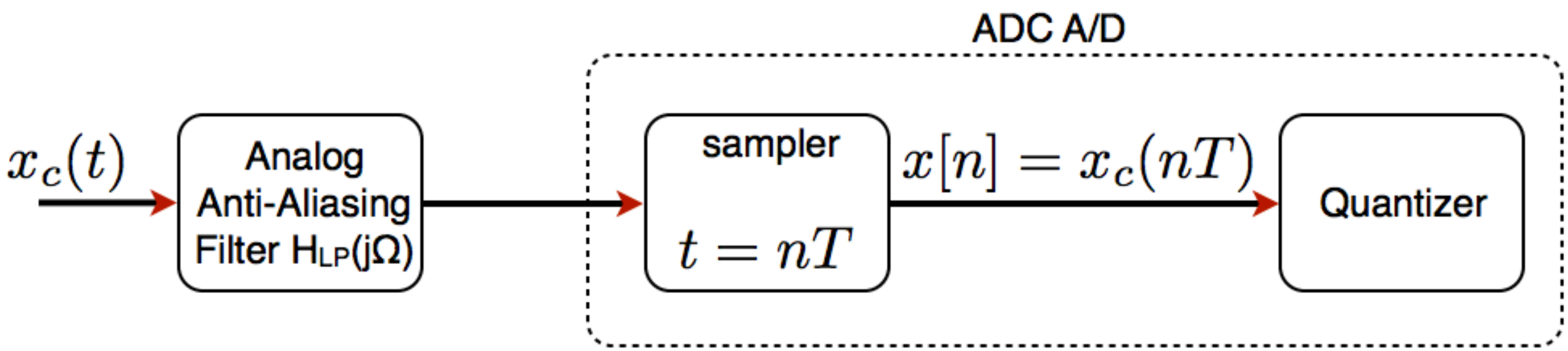
# Anti-Aliasing Filter



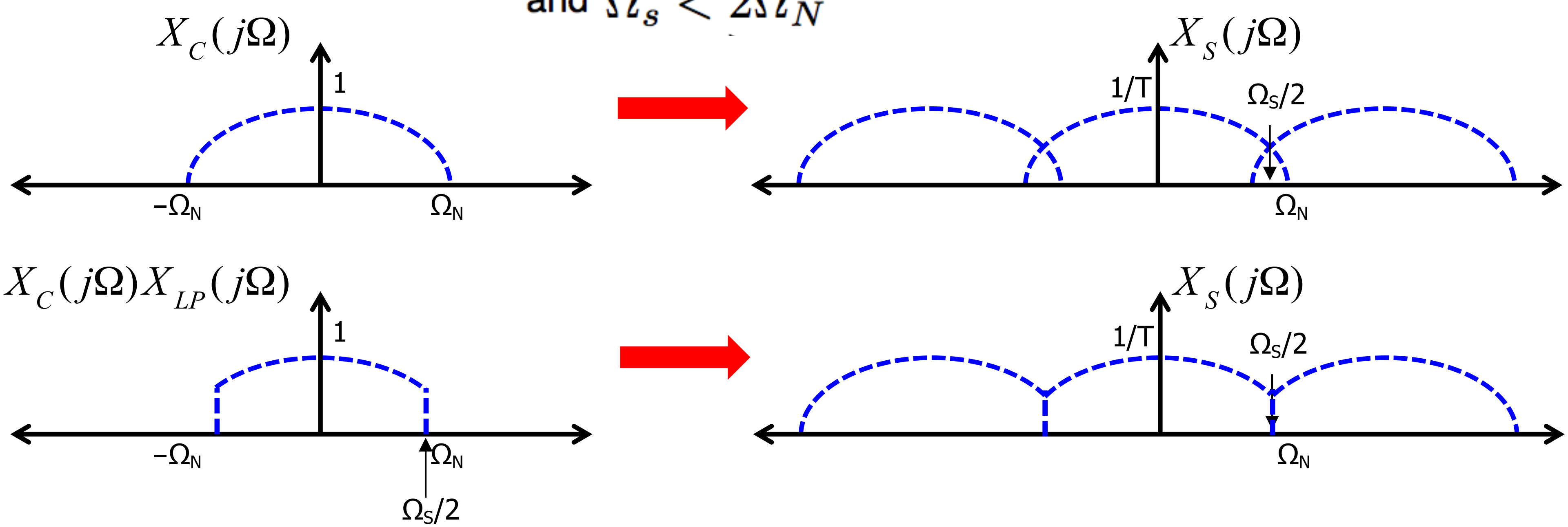

and $\Omega_s < 2\Omega_N$



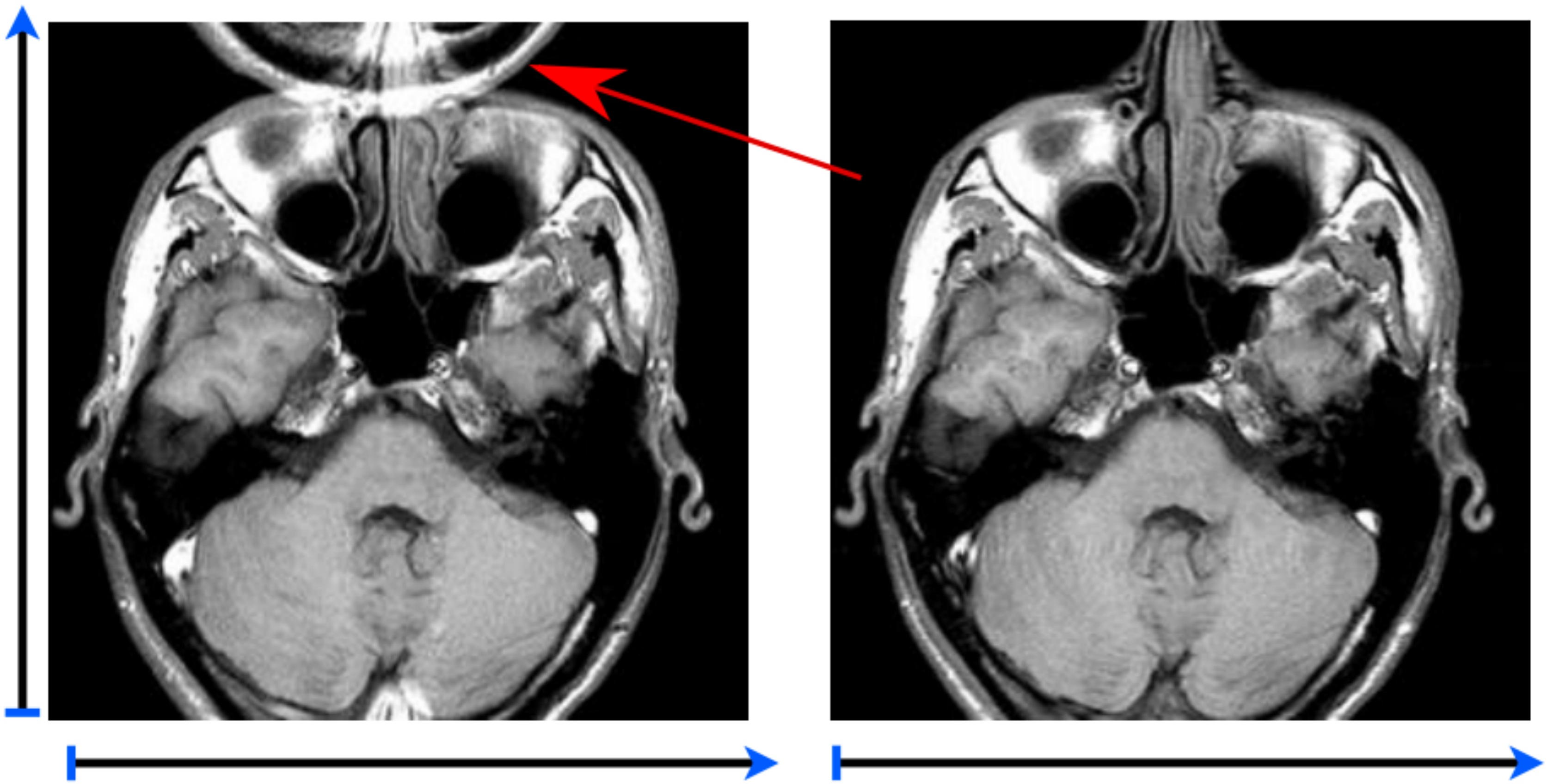
# Anti-Aliasing Filter




and $\Omega_s < 2\Omega_N$



# MRI aliasing example



# MRI anti-aliasing example

# MRI anti-aliasing example

# Big Ideas

- Sampling
  - Ideal sampling modeled as impulsive sampling
  - Sample at Nyquist rate for recovery of unique bandlimited signal (i.e. avoid aliasing)
- Frequency Response of Sampled Signal
  - Sampled signal is period replicated input CT signal
- Reconstruction
  - Low pass/reconstruction filter results in sum of sincs
- Anti-aliasing filtering
  - Force input signal to be bandlimited

# Admin

- HW 2 out due Sunday 2/6 at midnight
  - Double check that your submission by viewing it!