University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Midterm Thursday, March 17

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed. (non cell phone)
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 65.1 , Stdev: 15.5

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

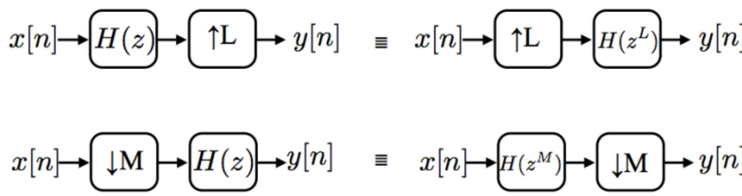
Z-Transform Equations:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$
Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

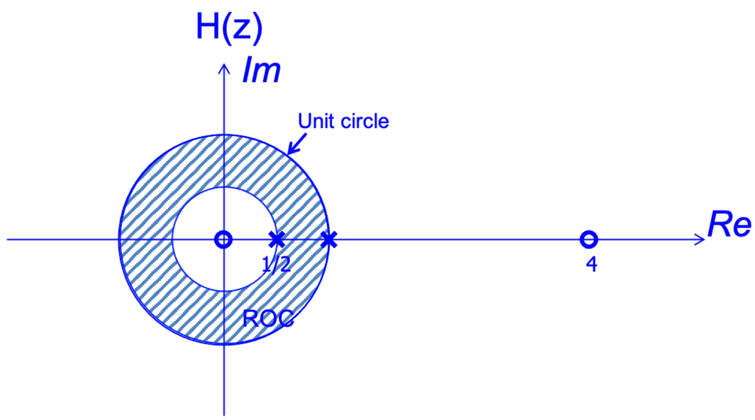
Interchange Identities:



1. (25 points) Consider the discrete-time system

$$H(z) = \frac{1 - 4.25z^{-1} + z^{-2}}{(1 - z^{-1})(1 - 0.5z^{-1})(1 - 0.25z^{-1})} \qquad \text{ROC: } .5 < |z| < 1 \qquad (1)$$

(a) Sketch the pole-zero diagram for the system.



(b) Is the system stable? Is the system causal? Explain your reasoning for both.
Not stable - ROC does not include unit circle. Not causal - ROC doesn't extend from outermost pole to infinity. Indicates a two-sided impulse response.

(c) What is the impulse response, $h[n]$, of the system?

$$H(z) = \frac{1 - 4.25z^{-1} + z^{-2}}{(1 - z^{-1})(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$
$$H(z) = \frac{(1 - 4z^{-1})(1 - \frac{1}{4}z^{-1})}{(1 - z^{-1})(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$
$$H(z) = \frac{1 - 4z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Using partial fractions

$$H(z) = \frac{-6}{1 - z^{-1}} + \frac{7}{1 - 0.5z^{-1}}$$

Using the transforms pairs

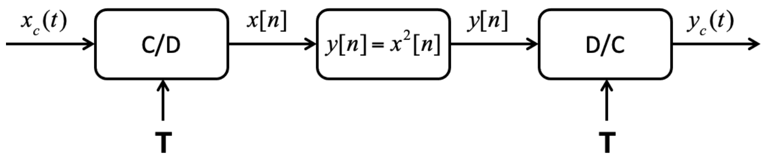
$$h[n] = 6u[-n-1] + 7\left(\frac{1}{2}\right)^n u[n]$$

(d) If the input to the system is $x[n] = \left(\frac{2}{3}\right)^n e^{j\pi n}$, what is the output $y[n]$?

$$x[n] = \left(\frac{2}{3}\right)^n e^{j\pi n} = \left(\frac{2}{3}e^{j\pi}\right)^n = \left(-\frac{2}{3}\right)^n$$

Since $-\frac{2}{3}$ is in the region of convergence, the output is the input modulated by the eigenvalue $H(-\frac{2}{3}) = 1.6$. Therefore, $y[n] = 1.6\left(\frac{2}{3}\right)^n e^{j\pi n}$

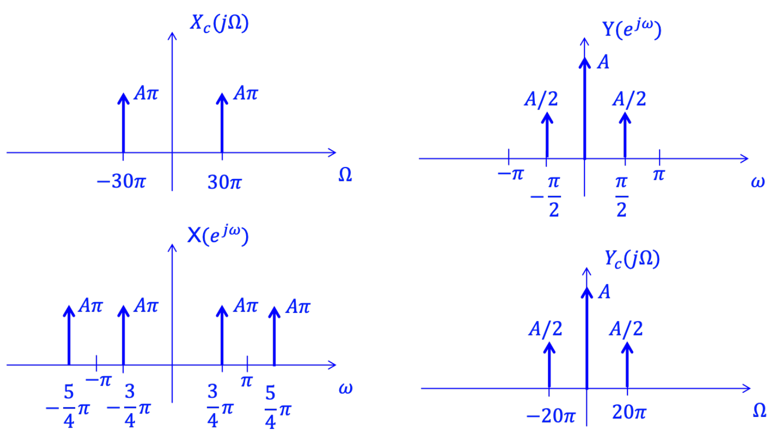
2. (25 points) Consider the signal processing system below, where the C/D and D/C converters are ideal. Let the input $x_c(t) = Acos(30\pi t)$ and the sampling rate be $\frac{1}{T} = 40Hz$.



(a) Sketch $X(e^{j\omega})$, $Y(e^{j\omega})$, and $Y_c(j\Omega)$.

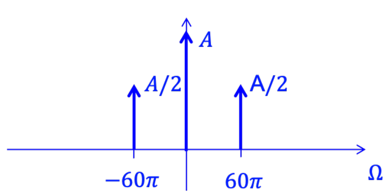
(b) Is $y_c(t) = x_c^2(t)$? Explain why or why not.

a)

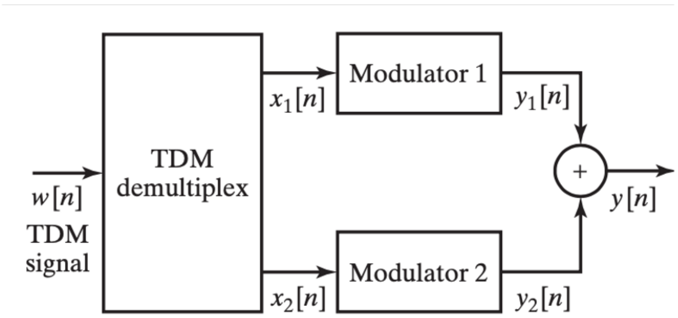


If the frequency locations and relative heights were correct, full credit was given.

b) No. Because of aliasing. The spectrum of $x_c^2(t)$ is as below which is not equal to $Y_c(j\Omega)$.



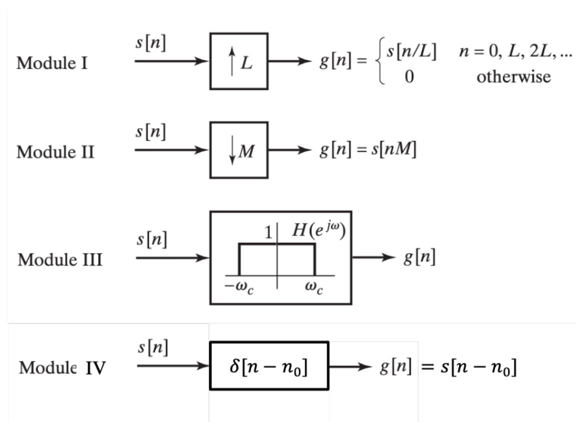
3. (30 points) Communication systems often require conversion from time-division multiplexing (TDM) to frequency-division multiplexing (FDM). In this problem, we examine a simple example of such a system. The full block diagram of the system is given below.

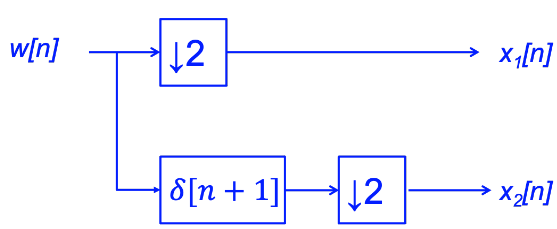


(a) The TDM input, $w[n]$, is assumed to be the sequence of interleaved samples of two signals, $x_1[n]$ and $x_2[n]$, such that

$$w[n] = \begin{cases} x_1[n/2] & \text{for } n \text{ an even integer,} \\ x_2[(n-1)/2] & \text{for } n \text{ an odd integer,} \end{cases} \tag{2}$$

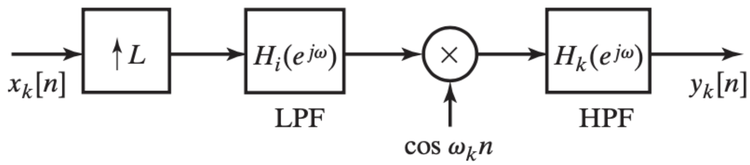
Design and draw a block diagram of a TDM demultiplex system that produces $x_1[n]$ and $x_2[n]$ as outputs from an input $w[n]$. Use only the simple modules given below and specify all parameters for each module you use. State whether your system is linear, time invariant, causal, and stable.





System is stable, linear, not causal (delay system is not causal), and not time invariant (due to downsampling).

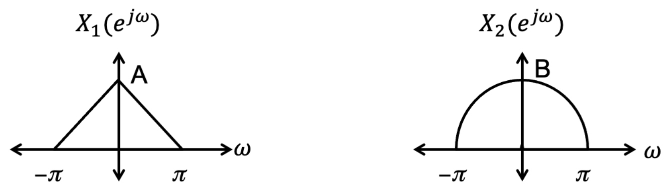
(b) The k^{th} modulator system ($k = 1$ or 2) is defined by the block diagram below.

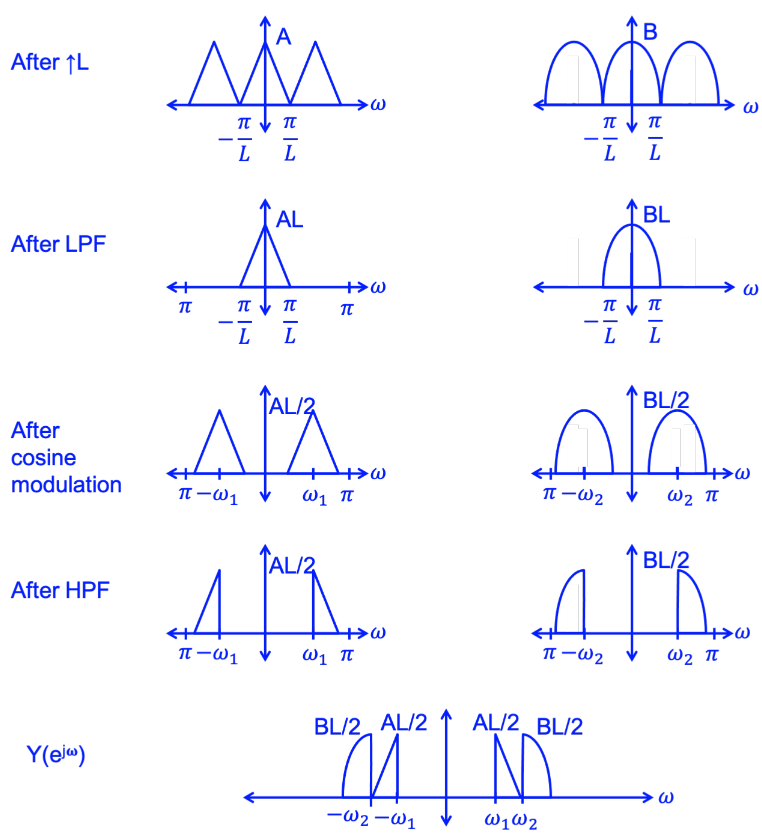


All filters are ideal. The lowpass filter $H_i(e^{j\omega})$, which is the same for both channels, has gain L and cutoff frequency π/L, and the highpass filters $H_k(e^{j\omega})$ have unity gain and cutoff frequency ω_k. The modulator frequencies are such that

$$\omega_2 = \omega_1 + \pi/L \qquad \text{and} \qquad \omega_2 + \pi/L \leq \pi \qquad (\text{assume } \omega_1 > \pi/2) \tag{3}$$

Assume the DTFT of $x_1[n]$ and $x_2[n]$ are as given below, sketch the DTFT of the full system output $y[n]$. Sketch the Fourier transforms at each point in the modulator system for partial credit.

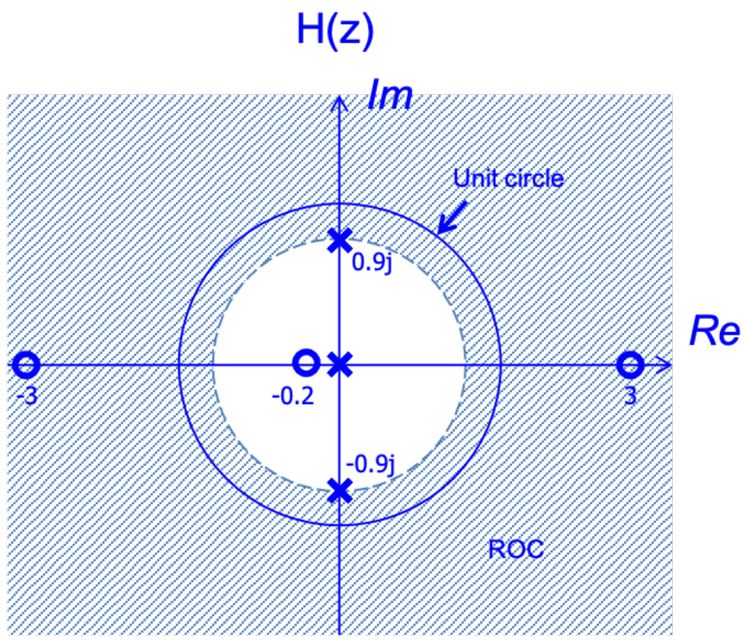




4. (20 points) A discrete-time causal LTI system has the transfer function

$$H(z) = \frac{(1 + 0.2z^{-1})(1 - 9z^{-2})}{(1 + 0.81z^{-2})} \tag{4}$$

(a) Draw the pole-zero diagram for the system and indicate the ROC. Is the system stable?



The ROC includes the unit circle, so it is stable.

(b) Determine expressions for a minimum-phase system, $H_{min}(z)$ and an all-pass system $H_{ap}(z)$ such that $H(z) = H_{min}(z)H_{ap}(z)$.

First factor $H(z)$ into two parts. The first is the minimum phase system and should have all its poles and zeros inside the unit circle. The second part should contain the remaining poles and zeros.

$$H(z) = \frac{1 + 0.2z^{-1}}{1 + 0.81z^{-2}} \cdot \frac{1 - 9z^{-2}}{1} \tag{5}$$

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1 - \frac{1}{9}z^{-2})$ in both parts of the equation above, the first part will remain min phase and the second will become allpass.

$$H(z) = \frac{(1 + 0.2z^{-1})(1 - \frac{1}{9}z^{-2})}{1 + 0.81z^{-2}} \cdot \frac{1 - 9z^{-2}}{(1 - \frac{1}{9}z^{-2})} \tag{6}$$