ESE5310 Spring 2023

University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Final

Monday, May 1

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators (non-cellphone) allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Final answers here.
- Additional workspace in exam book. Note where to find work in exam book if relevant.

Name:

Grade:

Q1	
Q2	
Q3	
Q4	
Total	

DTFT Equations:	**Z-Transform Equations:**
$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad (n_d$ an integer$)$	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$

DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

Trigonometric Identities:	**Geometric Series:**
$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$ $cos(\Theta) = \frac{1}{2}(e^{j\Theta} + e^{-j\Theta})$ $sin(\Theta) = \frac{1}{2j}(e^{j\Theta} - e^{-j\Theta})$	$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$ $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \|r\| < 1$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

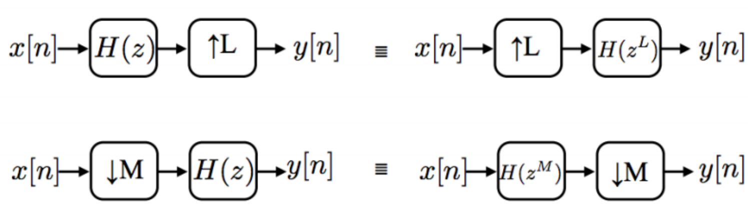
Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$

Generalized Linear Phase Systems:

	Type I	Type II
Symmetry	Even, $h[n] = h[M-n]$	Even, $h[n] = h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$A(e^{j\omega})e^{-j\omega M/2}$	$A(e^{j\omega})e^{-j\omega M/2}$
$A(e^{j\omega})$	$\sum_{k=0}^{M/2} a[k]cos(\omega k)$ $a[0] = h[M/2]$ $a[k] = 2h[(M/2) - k]$ for $k = 1, 2, ..., M/2$	$\sum_{k=1}^{(M+1)/2} b[k]cos(\omega(k - \frac{1}{2}))$ $b[k] = 2h[(M+1)/2 - k]$ for $k = 1, 2, ..., (M+1)/2$

	Type III	Type IV
Symmetry	Odd, $h[n] = -h[M-n]$	Odd, $h[n] = -h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$A(e^{j\omega})je^{-j\omega M/2}$	$A(e^{j\omega})je^{-j\omega M/2}$
$A(e^{j\omega})$	$\sum_{k=1}^{M/2} c[k]sin(\omega k)$ $c[k] = 2h[(M/2) - k]$ for $k = 1, 2, ..., M/2$	$\sum_{k=1}^{(M+1)/2} d[k]sin(\omega(k - \frac{1}{2}))$ $d[k] = 2h[(M+1)/2 - k]$ for $k = 1, 2, ..., (M+1)/2$

Interchange Identities:



1. (25 pts) Determine the the impulse response $h_{eq}[n]$ in Figure 2(b) so that the I/O relationship of the system in Figure 2(b) is exactly the same as the I/O relationship of the system in Figure 2(a). Hint: Analyze the system of Figure 2(a) in the time domain with the interchange identities.

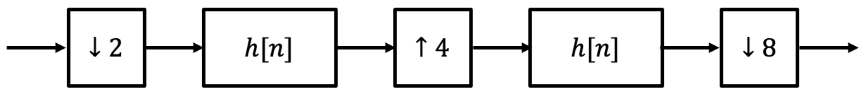


Figure 2(a)

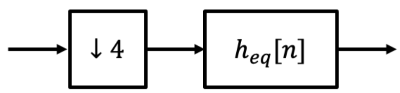


Figure 2(b)

where $h[n] = \delta[n] + \delta[n-8]$.

$h_{eq}[n]$:

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2. (30 pts) Consider a discrete time system $H(e^{j\omega})$ defined as

$$H(e^{j\omega}) = \begin{cases} j, & \text{if } -\pi < \omega < 0 \\ 0, & \text{if } \omega = 0 \\ -j, & \text{if } 0 < \omega < \pi \end{cases}$$

This is a very useful system, called a Hilbert-filter, and is often used in communication. The system diagram is given below.

$$x[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n]$$

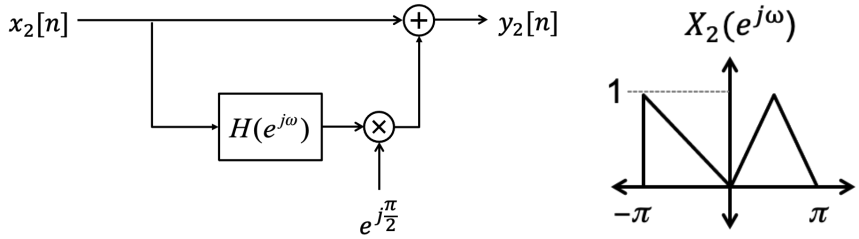
(a) Plot the magnitude and phase response of $H(e^{j\omega})$.

(b) We apply an input to this system $x[n] = cos(\frac{\pi}{8}n)$. Find the output $y[n]$.

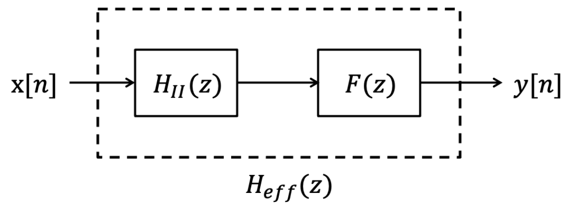
(c) We now apply an arbitrary input $x_1[n]$ to two Hilbert-filters cascaded together in series, as shown below. Derive an expression for the output $y_1[n]$ in terms of the input $x_1[n]$.

$$x_1[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y_1[n]$$

(d) The Hilbert-filter is used in the system configuration given below with an input $x_2[n]$ with the spectrum $X_2(e^{j\omega})$. Draw the output spectrum $Y_2(e^{j\omega})$.



3. (25 pts) The system function $H_{II}(z)$ represents a Type II FIR generalized linear-phase (GLP) system with impulse response $h_{II}[n]$. This system is cascaded with an LTI system, $F(z) = 1 - z^{-1}$, to produce a third system, $H_{eff}(z)$ and impulse response $h_{eff}[n]$. The corresponding diagram is below:

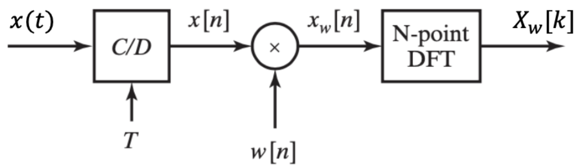


(a) For the LTI system $F(z)$, write an expression for the magnitude response, $|F(e^{j\omega})|$, and the phase response, $\angle F(e^{j\omega})$.

(b) Prove that the cascaded system $H_{eff}(z)$ is also a linear-phase system and indicate what type GLP system it is.

4. (20 pts) Consider the system in the figure below with input $x(t) = e^{j(\frac{3\pi}{8})10^4 t}$, sampling period $T = 10^{-4}$s, and

$$w[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq N-1 \\ 0, & \text{else} \end{cases}$$



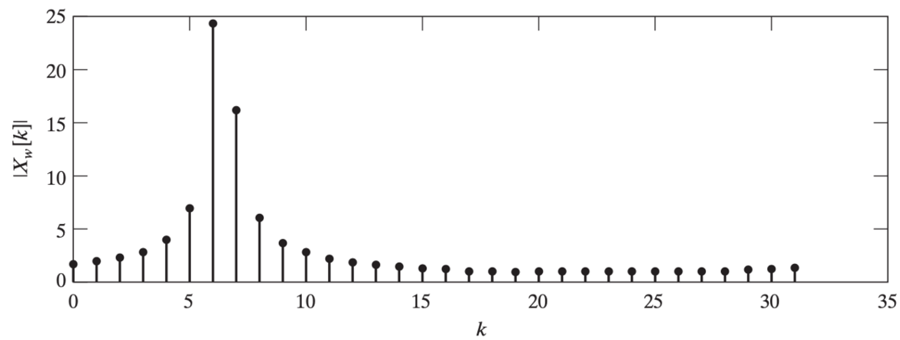
(a) What is the smallest nonzero value of N such that $X_w[k]$ is nonzero at exactly one value of k?

(b) Suppose now that $N = 32$, the input signal is $x(t) = e^{j\Omega_0 t}$, and the sampling period T is chosen such that no aliasing occurs during the sampling process. The figures below show the magnitude of the sequence $X_w[k]$ for $k = 0, ..., 31$ for the following two different choices of $w[n]$:

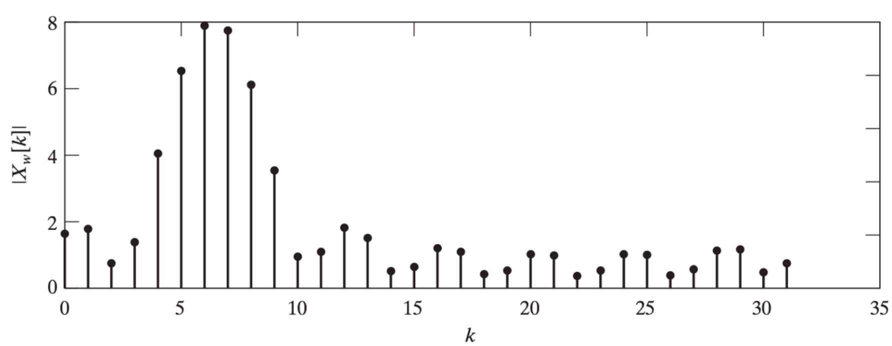
$$w_1[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq 31 \\ 0, & \text{else} \end{cases} \qquad w_2[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq 7 \\ 0, & \text{else} \end{cases}$$

Indicate which figure corresponds to which choice of $w[n]$. State your reasoning clearly.

i. Circle one: $w_1[n]$ or $w_2[n]$



ii. Circle one: $w_1[n]$ or $w_2[n]$



iii. Reasoning: