University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Final

Monday, May 1

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators (non-cellphone) allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- Final answers here.
- Additional workspace in exam book. Note where to find work in exam book if relevant.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 64.4, Stdev: 22.8

DTFT Equations:	Z-Transform Equations:
$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega}) \quad$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

DFT Equations:

N-point DFT of $\{x[n], n = 0, 1, ..., N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0, 1, ..., N-1$

N-point IDFT of $\{X[k], k = 0, 1, ..., N-1\}$ is $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0, 1, ..., N-1$

Trigonometric Identities:	**Geometric Series:**
$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$ $cos(\Theta) = \frac{1}{2}(e^{j\Theta} + e^{-j\Theta})$ $sin(\Theta) = \frac{1}{2j}(e^{j\Theta} - e^{-j\Theta})$	$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$ $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

Generalized Linear Phase Systems:

	Type I	Type II
Symmetry	Even, $h[n] = h[M-n]$	Even, $h[n] = h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$A(e^{j\omega})e^{-j\omega M/2}$	$A(e^{j\omega})e^{-j\omega M/2}$
$A(e^{j\omega})$	$\sum_{k=0}^{M/2} a[k]cos(\omega k)$ $a[0] = h[M/2]$ $a[k] = 2h[(M/2) - k]$ for $k = 1, 2, ..., M/2$	$\sum_{k=1}^{(M+1)/2} b[k]cos(\omega(k - \frac{1}{2}))$ $b[k] = 2h[(M+1)/2 - k]$ for $k = 1, 2, ..., (M+1)/2$

	Type III	Type IV
Symmetry	Odd, $h[n] = -h[M-n]$	Odd, $h[n] = -h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$A(e^{j\omega})je^{-j\omega M/2}$	$A(e^{j\omega})je^{-j\omega M/2}$
$A(e^{j\omega})$	$\sum_{k=1}^{M/2} c[k]sin(\omega k)$ $c[k] = 2h[(M/2) - k]$ for $k = 1, 2, ..., M/2$	$\sum_{k=1}^{(M+1)/2} d[k]sin(\omega(k - \frac{1}{2}))$ $d[k] = 2h[(M+1)/2 - k]$ for $k = 1, 2, ..., (M+1)/2$

Interchange Identities:

$$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y[n] \equiv x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y[n] \equiv x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

1. (25 pts) Determine the the impulse response $h_{eq}[n]$ in Figure 2(b) so that the I/O relationship of the system in Figure 2(b) is exactly the same as the I/O relationship of the system in Figure 2(a). Hint: Analyze the system of Figure 2(a) in the time domain with the interchange identities.

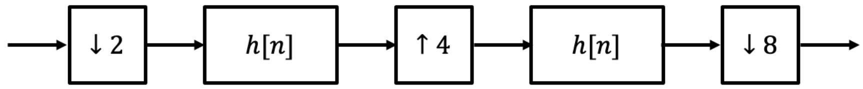


Figure 2(a)

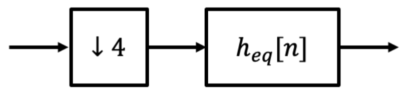
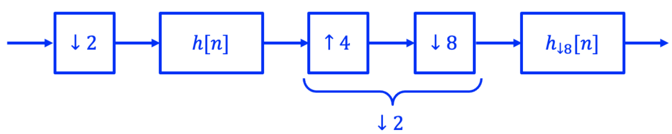


Figure 2(b)

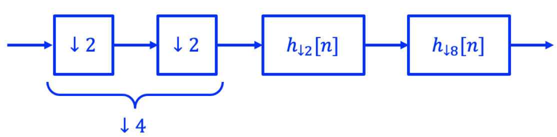
where $h[n] = \delta[n] + \delta[n-8]$.

$h_{eq}[n]$: $\boxed{h_{eq}[n] = \delta[n] + \delta[n-1] + \delta[n-4] + \delta[n-5]}$

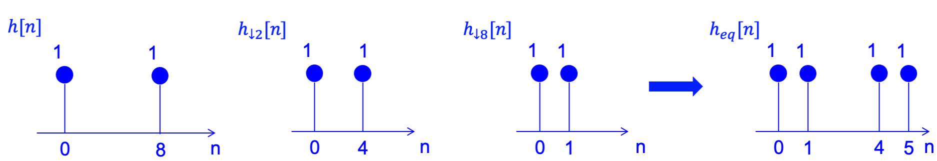
We can use the interchange relations and swap the $\downarrow 8$ block and the right filter with the compressed filter. The $\uparrow 4$ and $\downarrow 8$ can be combined into $\downarrow 2$:



The new $\downarrow 2$ and left filter can be interchanged with the compressed filter again and downsampling blocks combined resulting in:



This means that $h_{eq}[n] = h_{\downarrow 2} * h_{\downarrow 8}$:



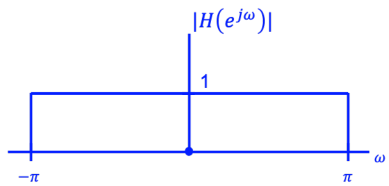
2. (30 pts) Consider a discrete time system $H(e^{j\omega})$ defined as

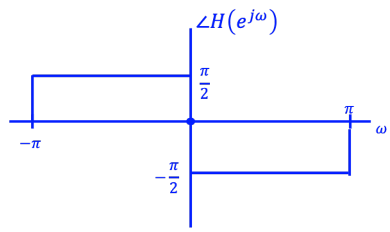
$$H(e^{j\omega}) = \begin{cases} j, & \text{if } -\pi < \omega < 0 \\ 0, & \text{if } \omega = 0 \\ -j, & \text{if } 0 < \omega < \pi \end{cases}$$

This is a very useful system, called a Hilbert-filter, and is often used in communication. The system diagram is given below.

$$x[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y[n]$$

(a) Plot the magnitude and phase response of $H(e^{j\omega})$.





(b) We apply an input to this system $x[n] = cos(\frac{\pi}{8}n)$. Find the output $y[n]$.

$x[n] = cos(\frac{\pi}{8}n) = \frac{1}{2}(e^{j\frac{\pi}{8}n} + e^{-j\frac{\pi}{8}n})$, therefore

$$\begin{aligned} y[n] &= \frac{1}{2}(-j \cdot e^{j\frac{\pi}{8}n} + j \cdot e^{-j\frac{\pi}{8}n}) \\ &= \frac{1}{2j}(e^{j\frac{\pi}{8}n} - e^{-j\frac{\pi}{8}n}) = sin(\frac{\pi}{8}n) \end{aligned}$$

(c) We now apply an arbitrary input $x_1[n]$ to two Hilbert-filters cascaded together in series, as shown below. Derive an expression for the output $y_1[n]$ in terms of the input $x_1[n]$.

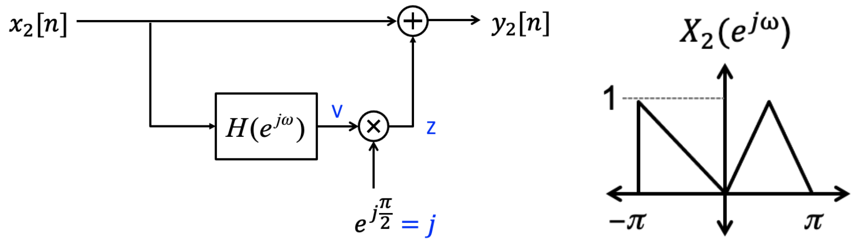
$$x_1[n] \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y_1[n]$$

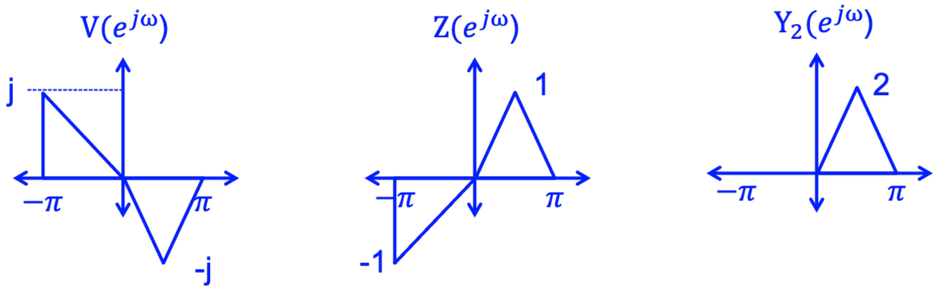
The two systems in cascade has a frequency response, $H_{eff}(e^{j\omega})$,

$$H_{eff}(e^{j\omega}) = \begin{cases} -1, & \text{if } -\pi < \omega < 0 \\ 0, & \text{if } \omega = 0 \\ -1, & \text{if } 0 < \omega < \pi \end{cases}$$

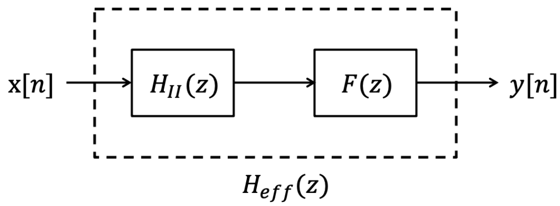
Therefore, $y_1[n] = -x_1[n]$.

(d) The Hilbert-filter is used in the system configuration given below with an input $x_2[n]$ with the spectrum $X_2(e^{j\omega})$. Draw the output spectrum $Y_2(e^{j\omega})$.





3. (25 pts) The system function $H_{II}(z)$ represents a Type II FIR generalized linear-phase (GLP) system with impulse response $h_{II}[n]$. This system is cascaded with an LTI system, $F(z) = 1 - z^{-1}$, to produce a third system, $H_{eff}(z)$ and impulse response $h_{eff}[n]$. The corresponding diagram is below:



(a) For the LTI system $F(z)$, write an expression for the magnitude response, $|F(e^{j\omega})|$, and the phase response, $\angle F(e^{j\omega})$.

$$\begin{aligned} F(e^{j\omega}) &= 1 - e^{-j\omega} \\ &= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \\ &= e^{-j\omega/2}(2jsin(\omega/2)) = 2sin(\omega/2)e^{-j\omega/2+j\pi/2} \end{aligned}$$

Therefore $|F(e^{j\omega})| = 2sin(\omega/2)$ and $\angle F(e^{j\omega}) = -\frac{\omega}{2} + \frac{\pi}{2}$. It is also noted that F is a type IV GLP system.

(b) Prove that the cascaded system $H_{eff}(z)$ is also a linear-phase system and indicate what type GLP system it is.

Since H_{II} is a type two filter we can write:

$$H_{II}(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

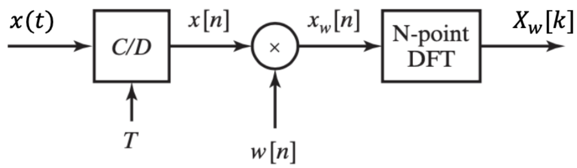
where $A_e(e^{j\omega})$ is real and even, and M is an odd integer. It follows

$$\begin{aligned} H_{eff}(e^{j\omega}) &= A_e(e^{j\omega}) \cdot 2sin(\omega/2) \cdot e^{-j\omega M/2} \cdot e^{-j\omega/2+j\pi/2} \\ &= A_0(e^{j\omega})e^{-j\omega M_0/2+j\pi/2} \end{aligned}$$

Where $A_0(e^{j\omega})$ is real and odd, and M_0 is an even integer. Thus this clearly has linear phase and is type III FIR GLP system.

4. (20 pts) Consider the system in the figure below with input $x(t) = e^{j(\frac{3\pi}{8})10^4 t}$, sampling period $T = 10^{-4}$s, and

$$w[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq N-1 \\ 0, & \text{else} \end{cases}$$



(a) What is the smallest nonzero value of N such that $X_w[k]$ is nonzero at exactly one value of k?

After sampling, $x[n] = x(nT) = e^{j\frac{3\pi}{8}n}$. For $X_w[k]$ to only have one nonzero value, the frequency of the complex exponential must fall into exactly one DFT bin, thus,

$$\omega_k = \frac{3\pi}{8} = \frac{2\pi k}{N}$$
$$N = \frac{16k}{3}$$

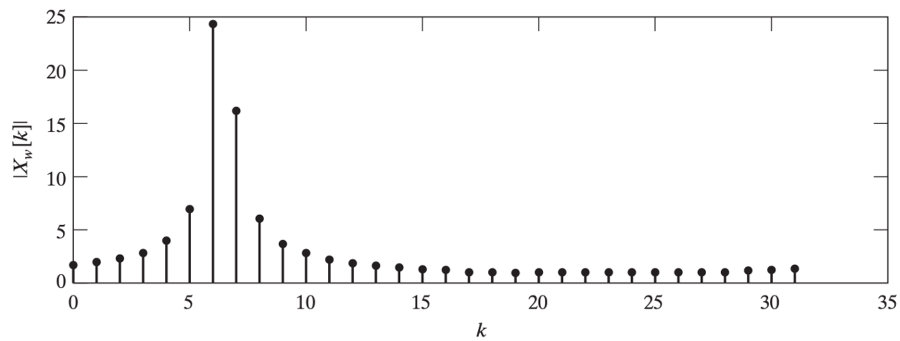
The smallest k such that N is an integer is $k = 3$, thus the smallest $N = 16$.

(b) Suppose now that $N = 32$, the input signal is $x(t) = e^{j\Omega_0 t}$, and the sampling period T is chosen such that no aliasing occurs during the sampling process. The figures below show the magnitude of the sequence $X_w[k]$ for $k = 0, ..., 31$ for the following two different choices of $w[n]$:

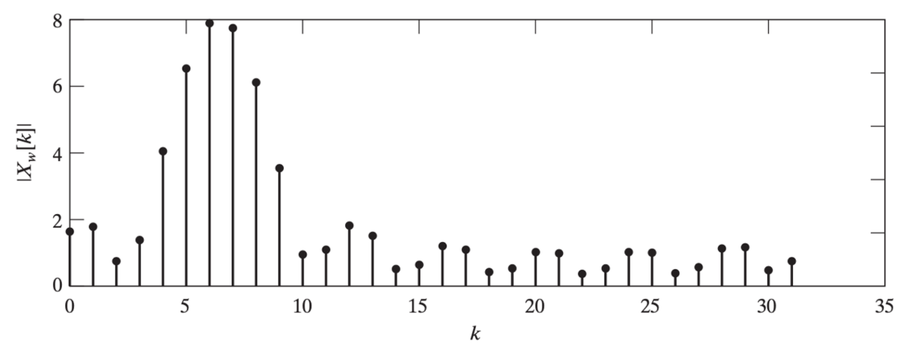
$$w_1[n] = \begin{cases} 1, & \text{if } 0 \le n \le 31 \\ 0, & \text{else} \end{cases} \qquad w_2[n] = \begin{cases} 1, & \text{if } 0 \le n \le 7 \\ 0, & \text{else} \end{cases}$$

Indicate which figure corresponds to which choice of $w[n]$. State your reasoning clearly.

i. Circle one: $w_1[n]$ or $w_2[n]$



ii. Circle one: $w_1[n]$ or $w_2[n]$



iii. Reasoning:

Our sampled signal still has a single frequency, but now we don't know what it is. The DTFT of our sampled signal would results in a delta spike at the corresponding frequency, however because of our window we have smearing and leakage in the spectrum.

The two rectangular windows differ only in length. Recall that compared to a longer window length, the Fourier transform of a shorter window has a larger mainlobe width and higher sidelobes. We notice that the second plot above has a wider mainlobe and higher sidelobes, therefore it corresponds to the smaller window length.