ESE 5310: Digital Signal Processing

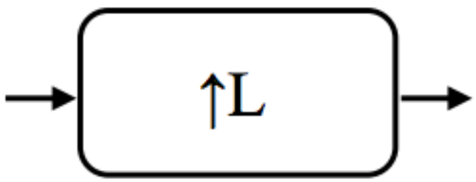
Lecture 11: February 16, 2023

Polyphase Decomposition and Multi-rate Filter Banks

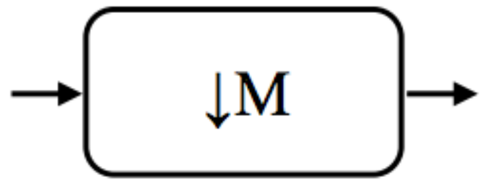
Lecture Outline

- Review: Interchanging Operations
- Polyphase Decomposition
- Multi-Rate Filter Banks
- Haar Filter Example

Expander and Compressor



"expander"



"compressor"

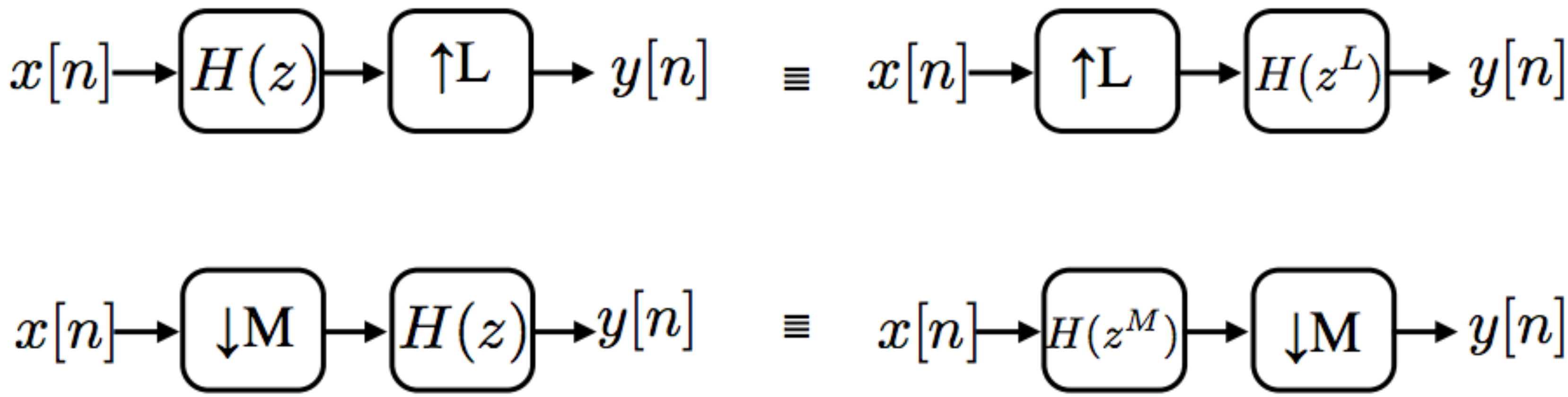
<u>not LTI!</u>

Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Summary

Filter and expander — Expander and expanded filter*



Compressor and filter — Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

Motivation

- Multirate DSP finds application in communications, speech processing, image compression, antenna systems, analog voice privacy systems, and in the digital audio industry to enable efficient processing
  - subband coding of waveforms
  - voice privacy systems
  - integral and fractional sampling rate conversion (such as in digital audio)
  - digital crossover networks
  - multirate coding of narrow-band filter coefficients.

P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial," in Proceedings of the IEEE, vol. 78, no. 1, pp. 56-93, Jan. 1990, doi: 10.1109/5.52200.

Polyphase Decomposition

- The polyphase decomposition of a sequence is obtained by representing it as a superposition of M subsequences, each consisting of every M^{th} value of successively delayed versions of the sequence
- When this decomposition is applied to a filter impulse response, it can lead to efficient implementation structures for linear filters in several contexts

Polyphase Decomposition

- We can decompose an impulse response (of our filter) to:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

Polyphase Decomposition

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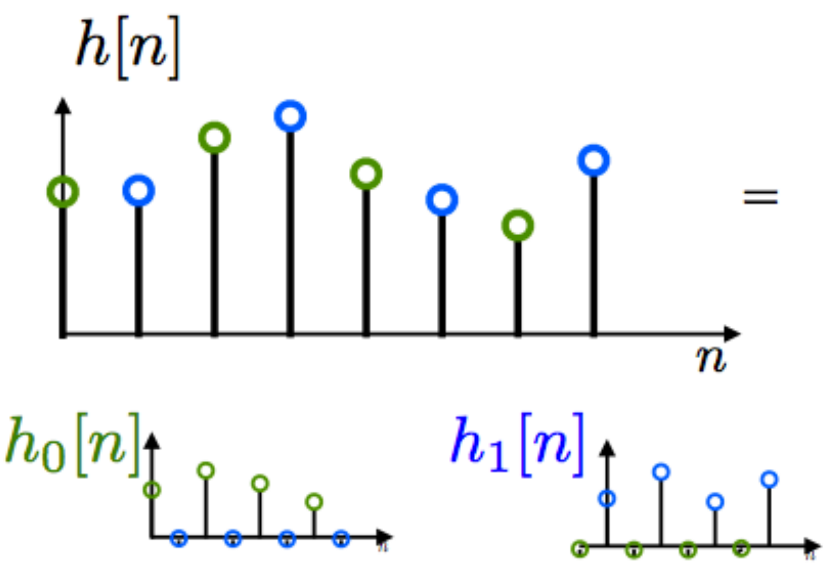
M=2



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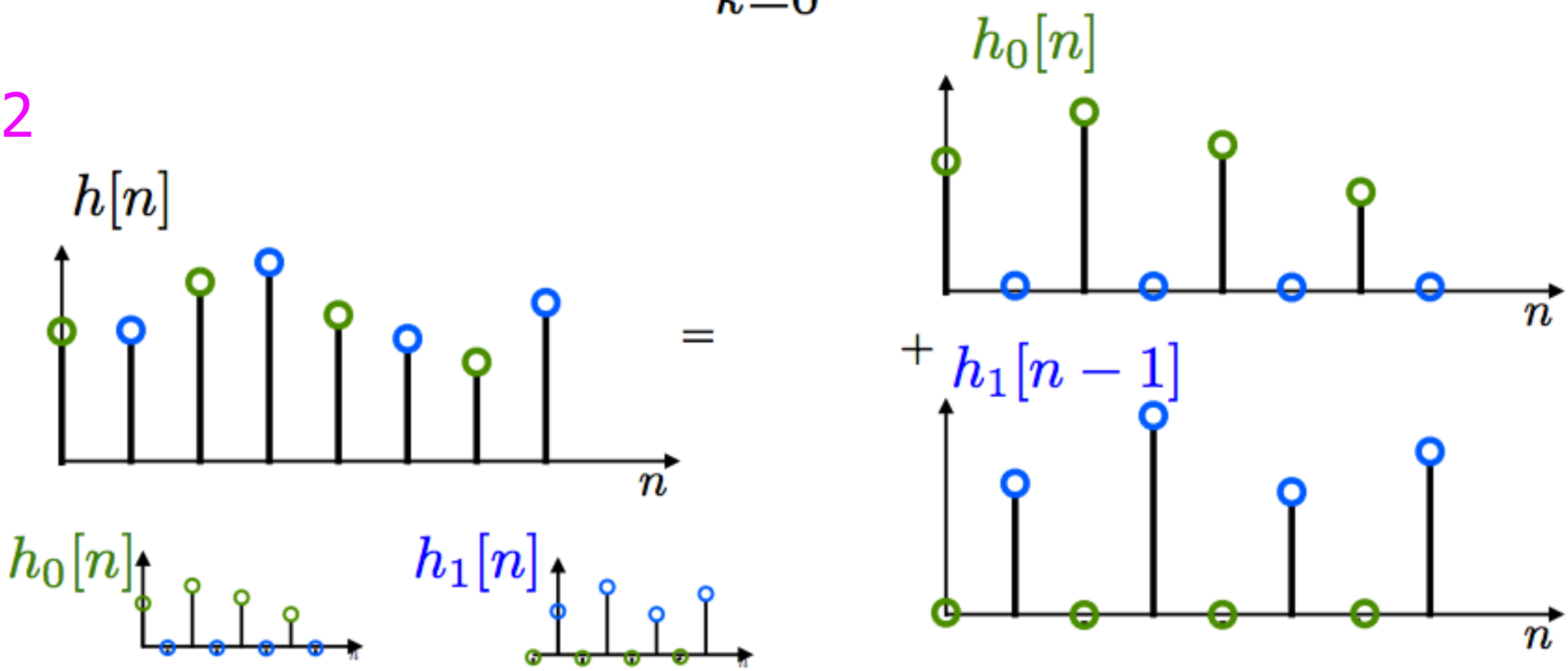
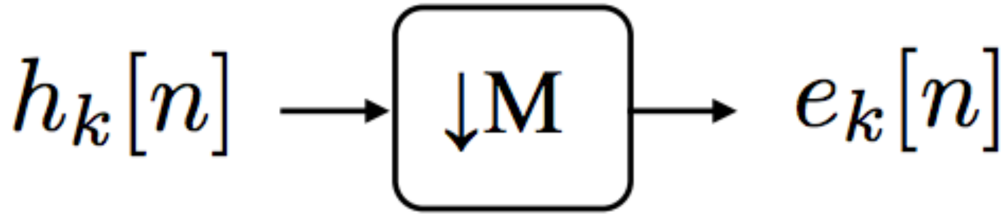
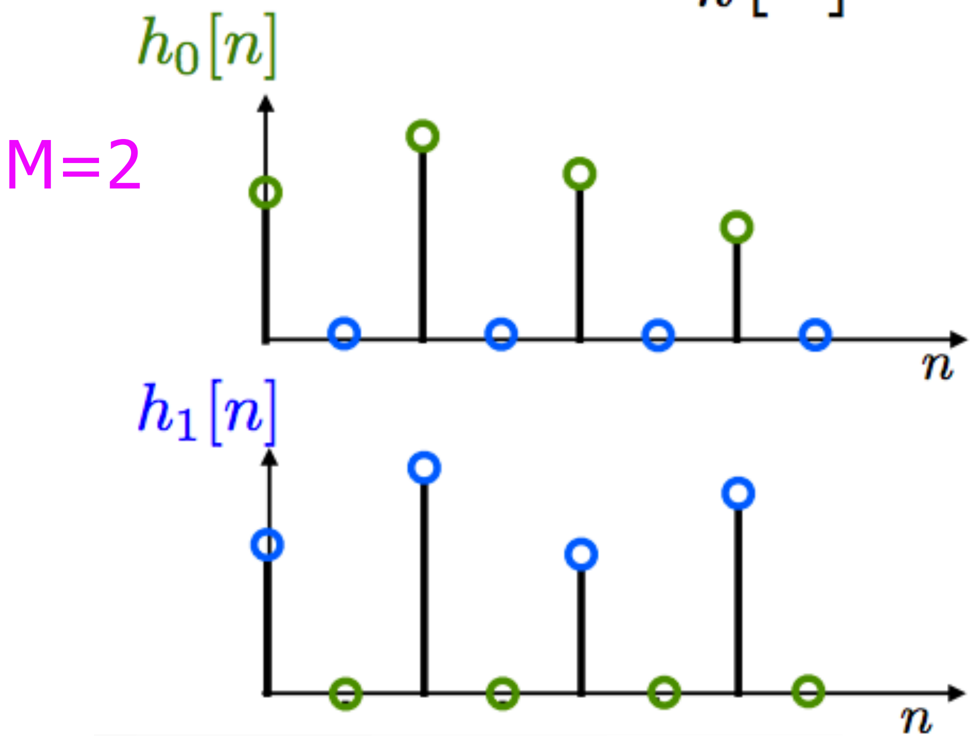


Polyphase Decomposition



$$e_k[n] = h_k[nM]$$



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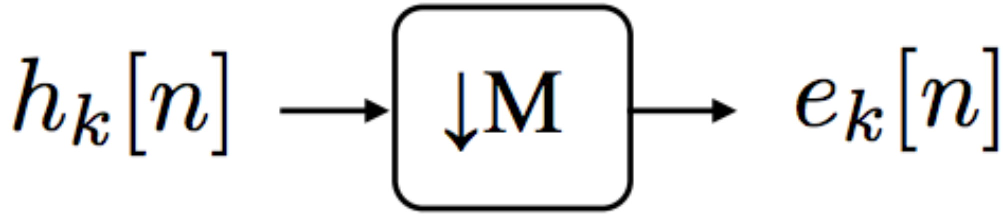

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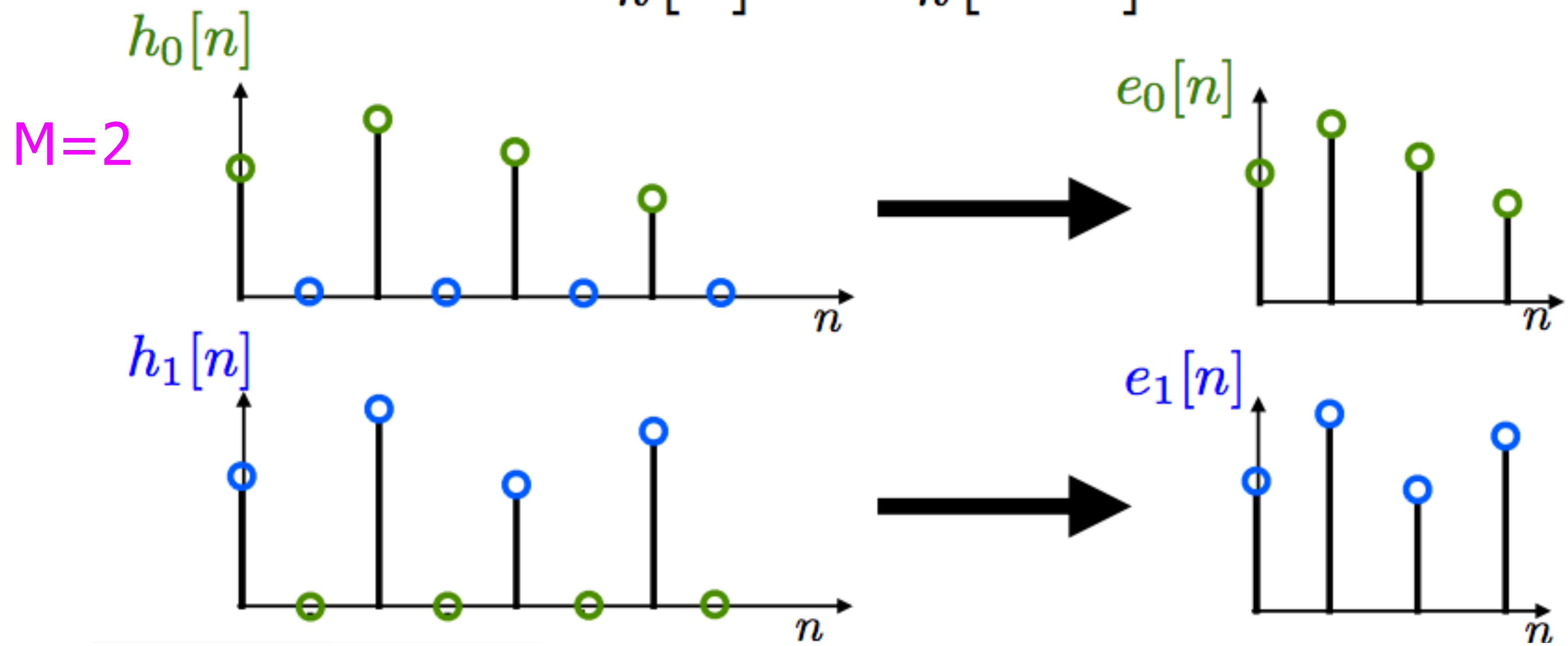

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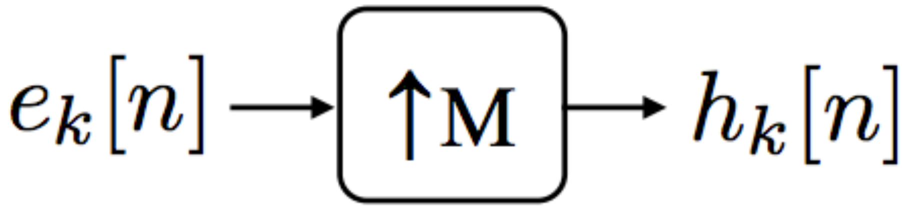

Polyphase Decomposition

$$e_k[n] \longrightarrow \boxed{\uparrow M} \longrightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Also, recall:

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

So,

$$\boxed{H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}}$$

Polyphase Decomposition

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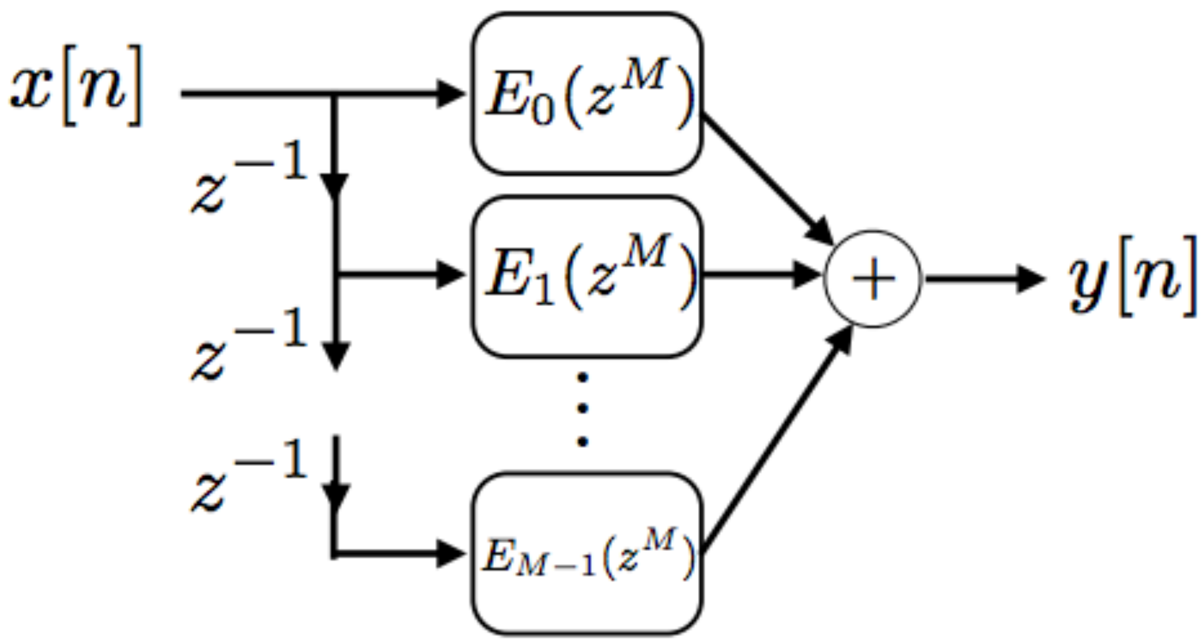
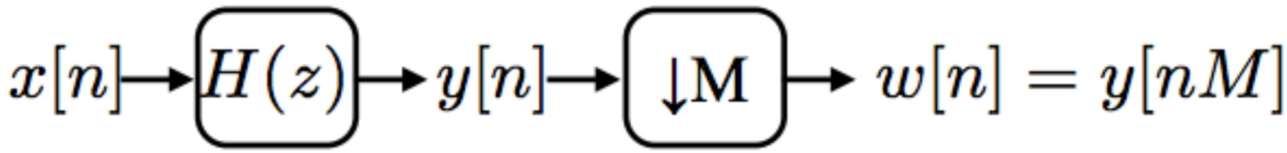


Polyphase Implementation of Decimation



- Problem:
  - Compute all y[n] and then throw away -- wasted computation!

Polyphase Implementation of Decimation

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  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N → N multiplications/unit time

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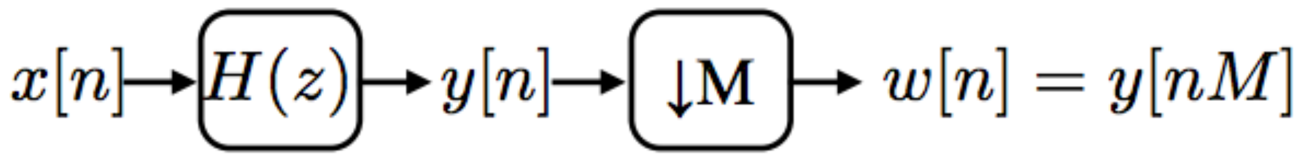

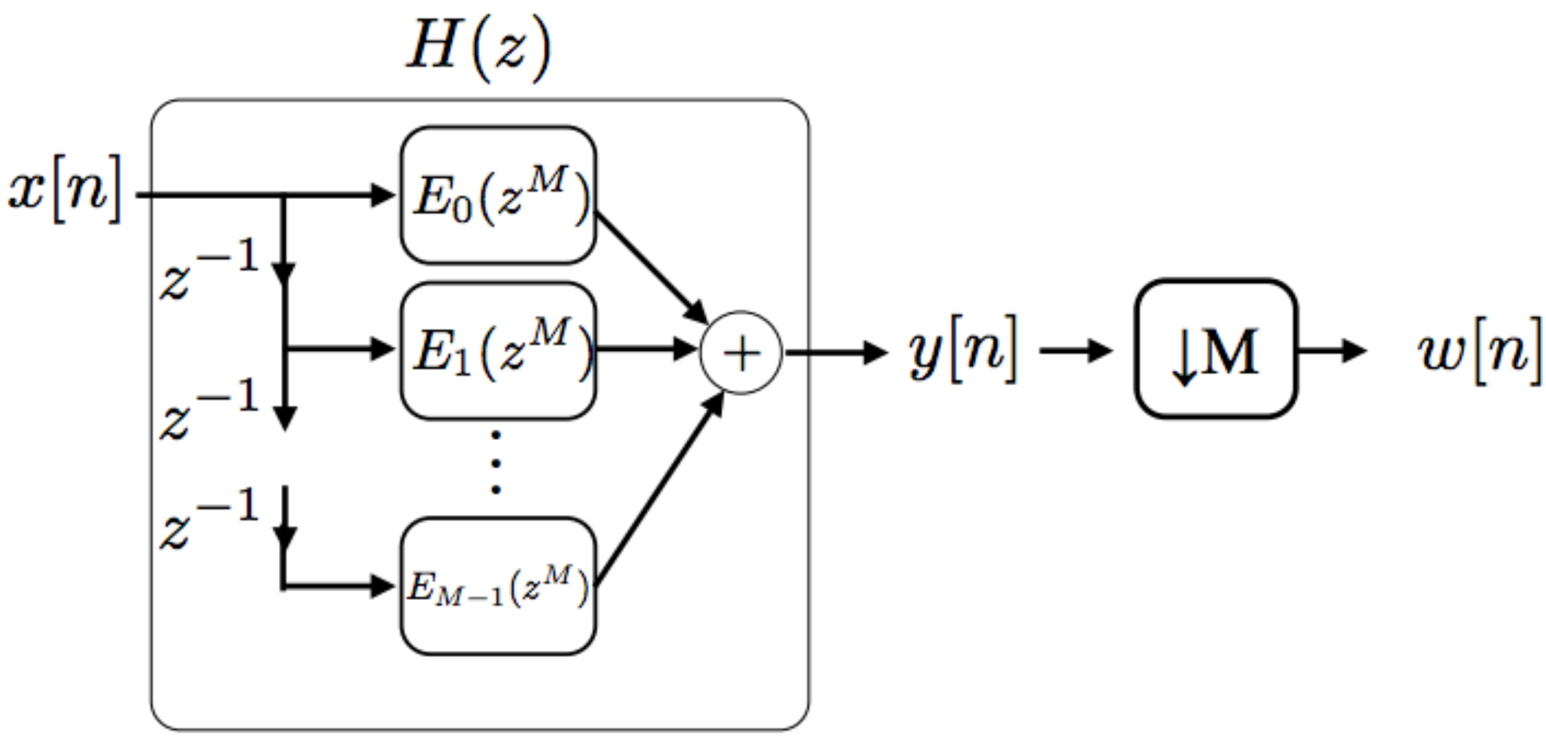

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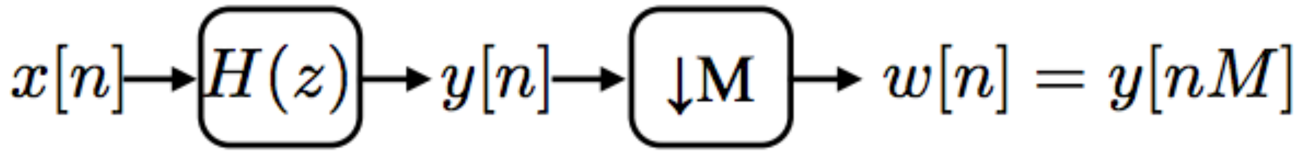

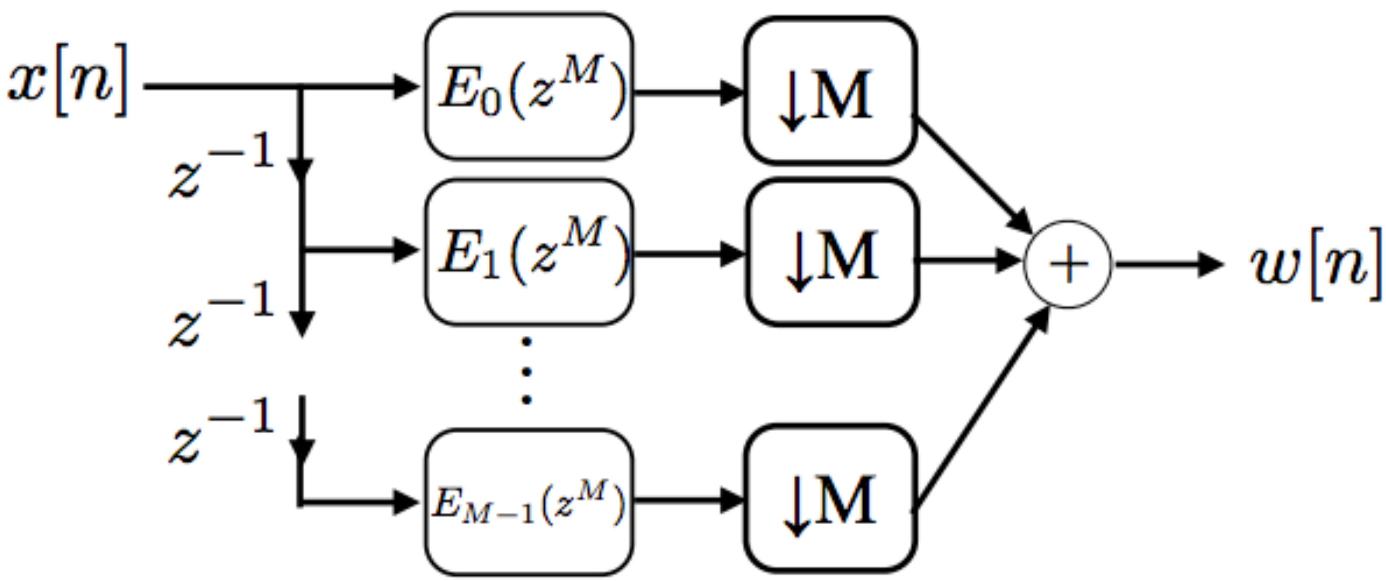

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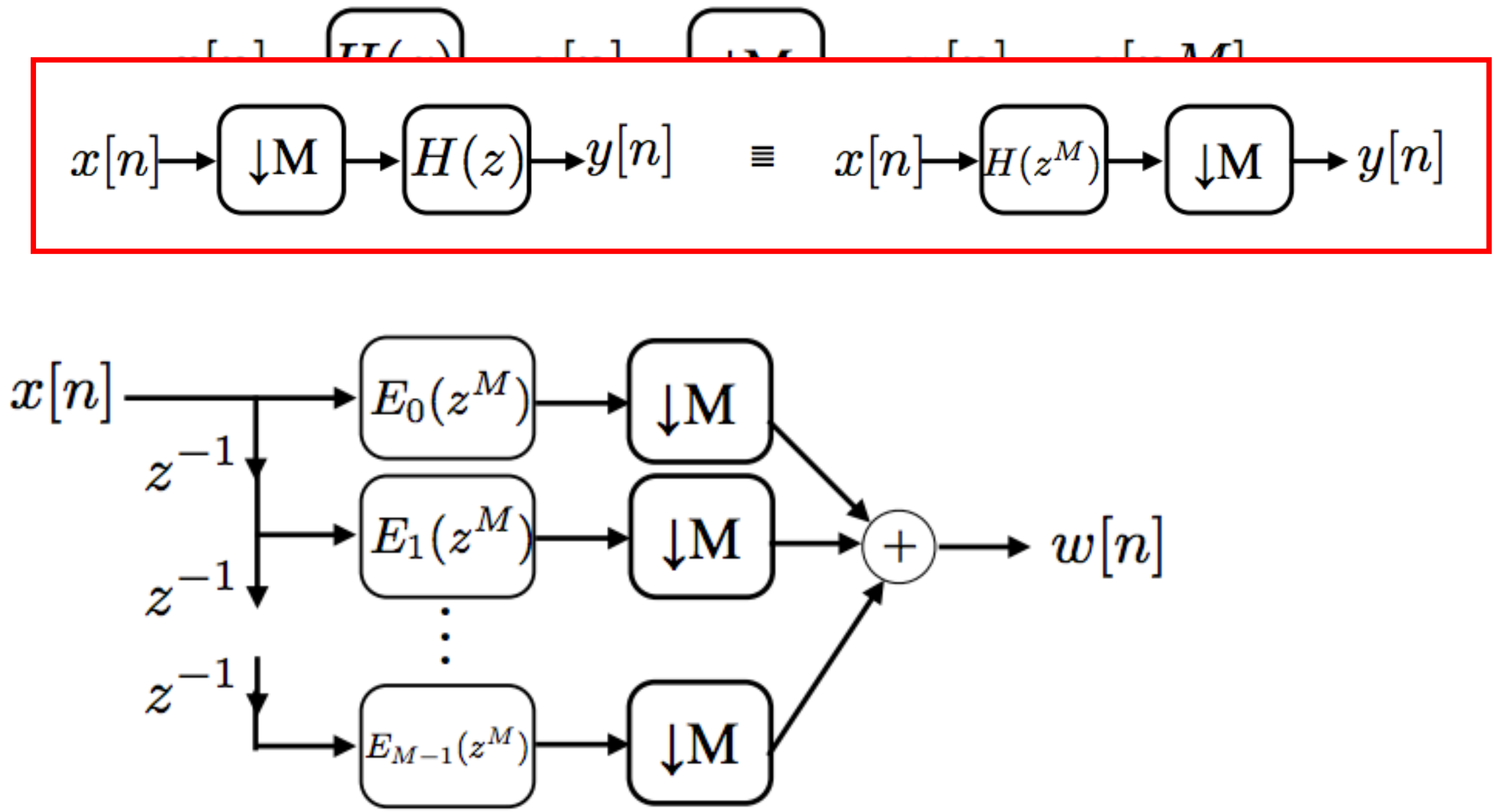

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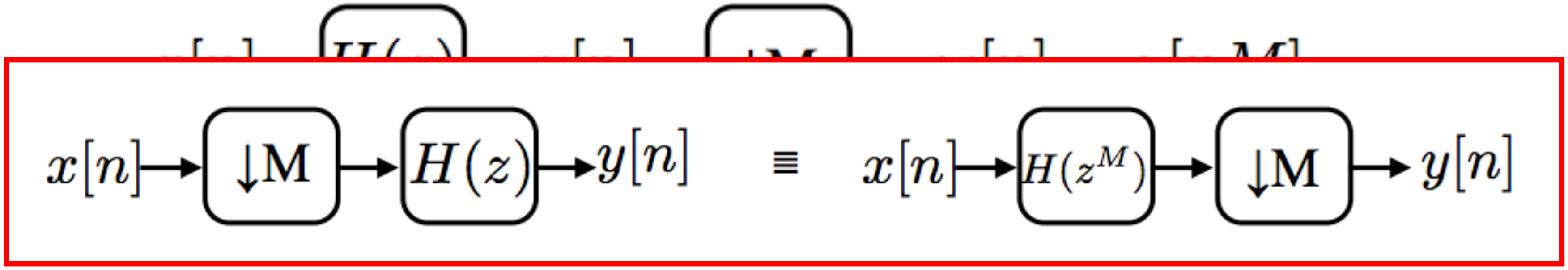

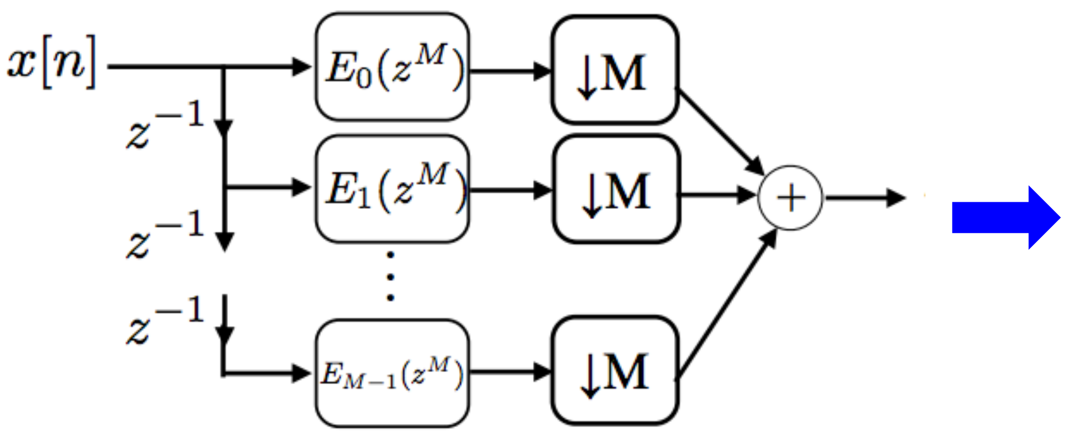

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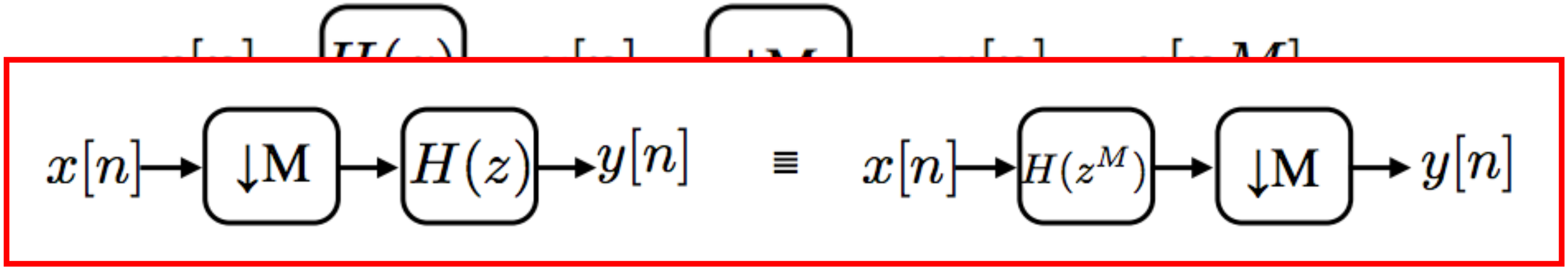

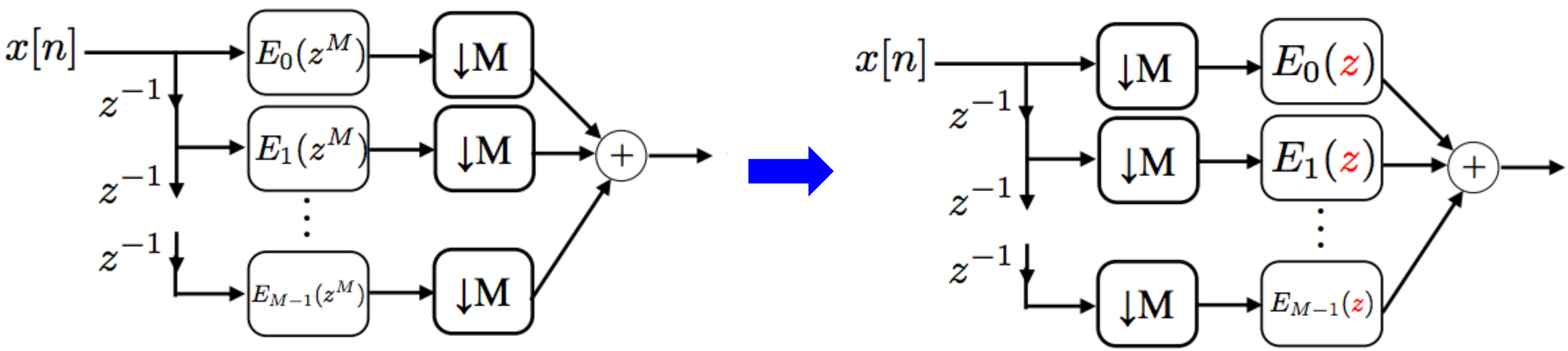

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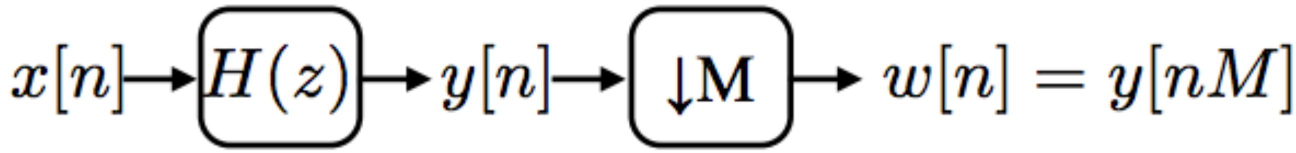

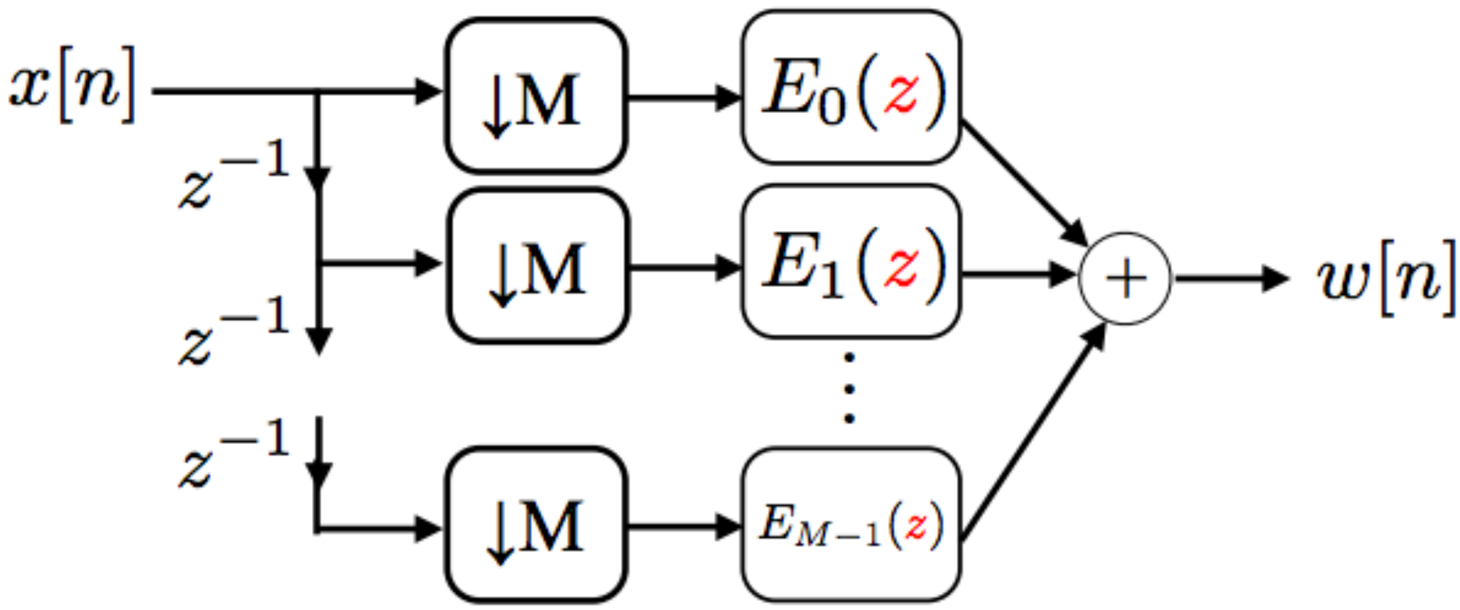

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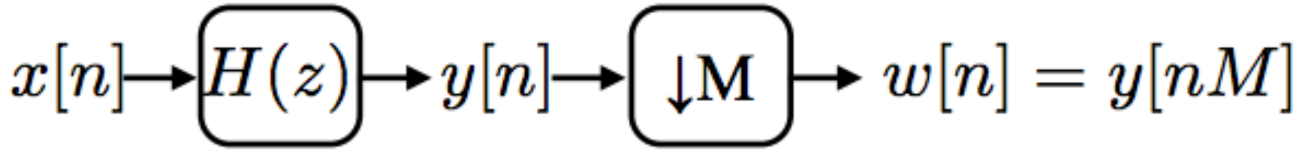
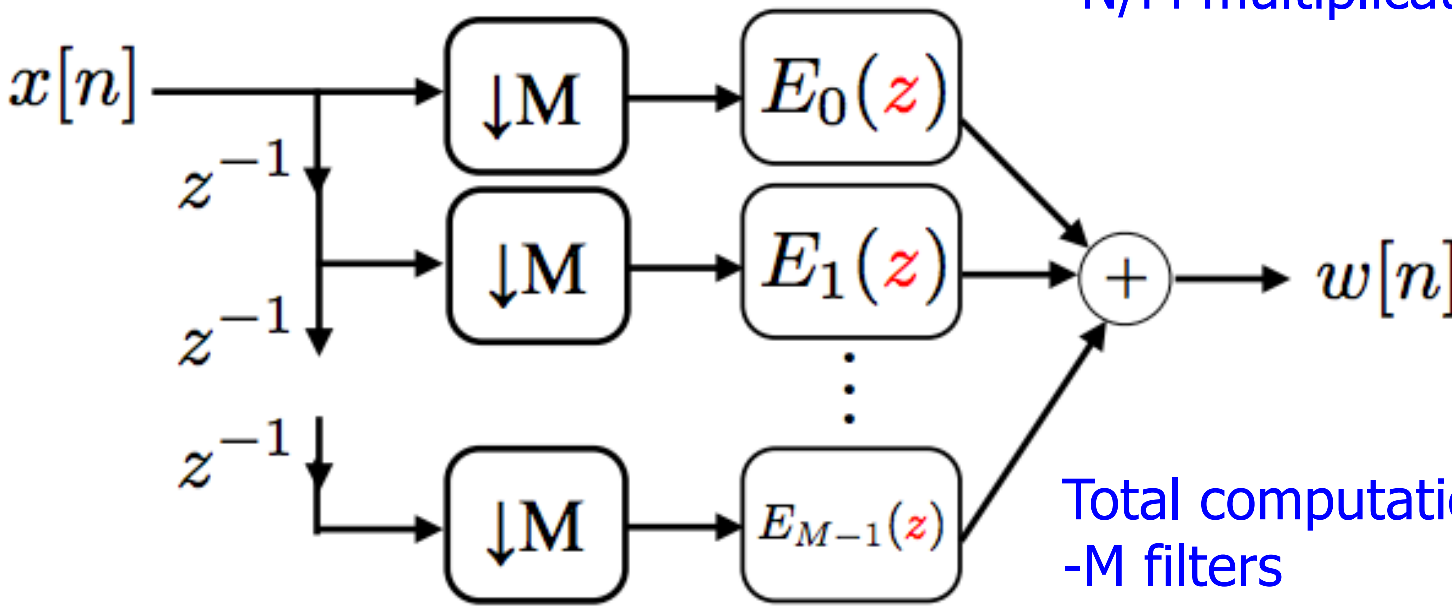
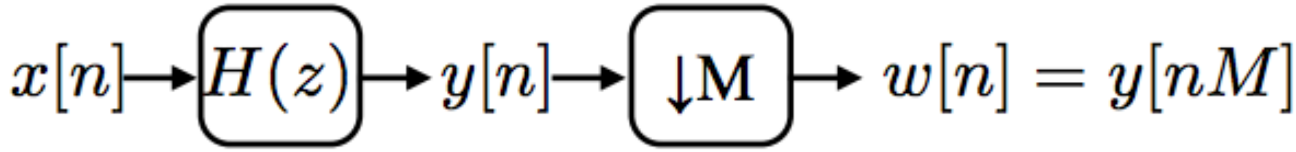


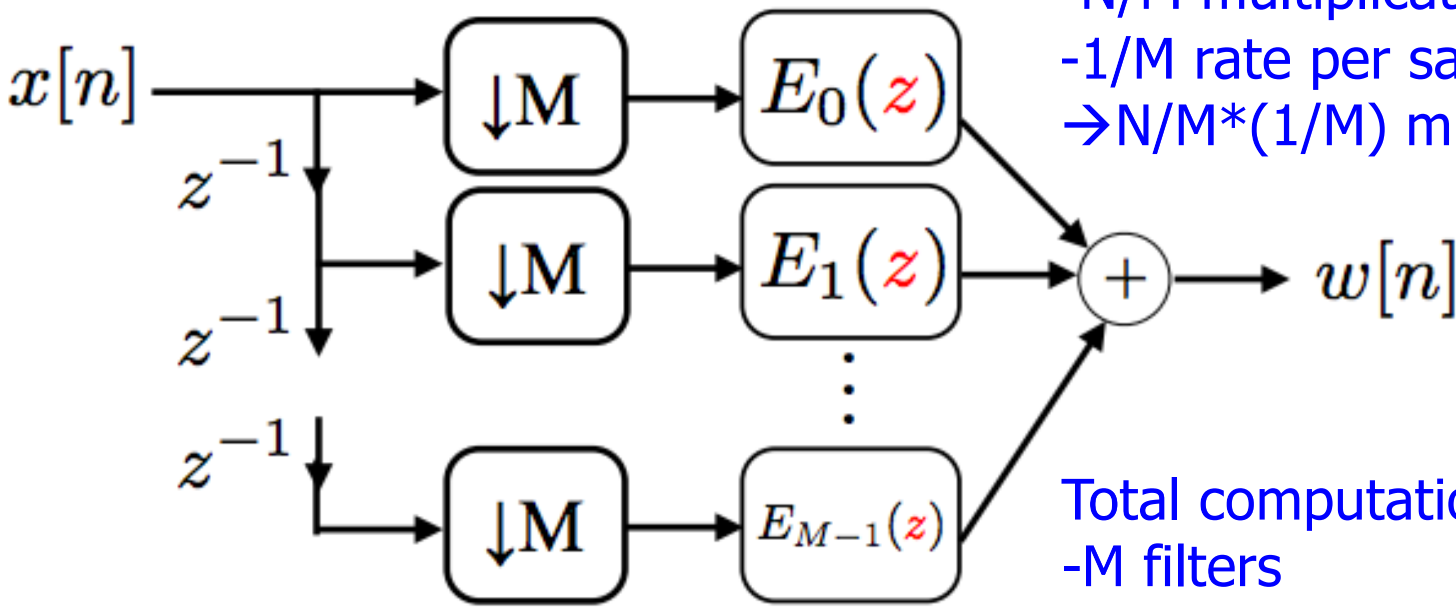
Each filter computation:
-N/M multiplications



Total computation:
-M filters
→N mults??

Polyphase Implementation of Decimation

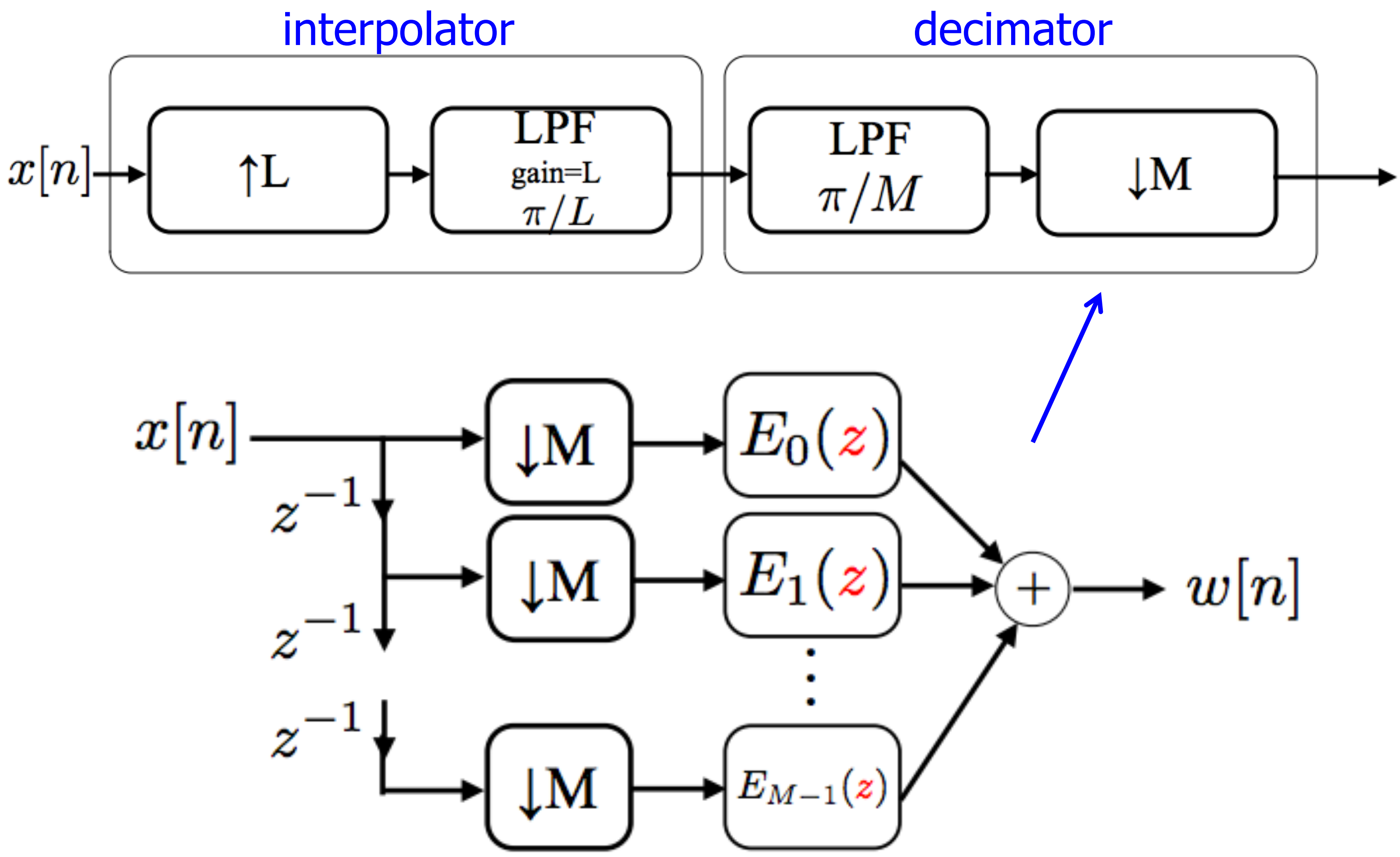




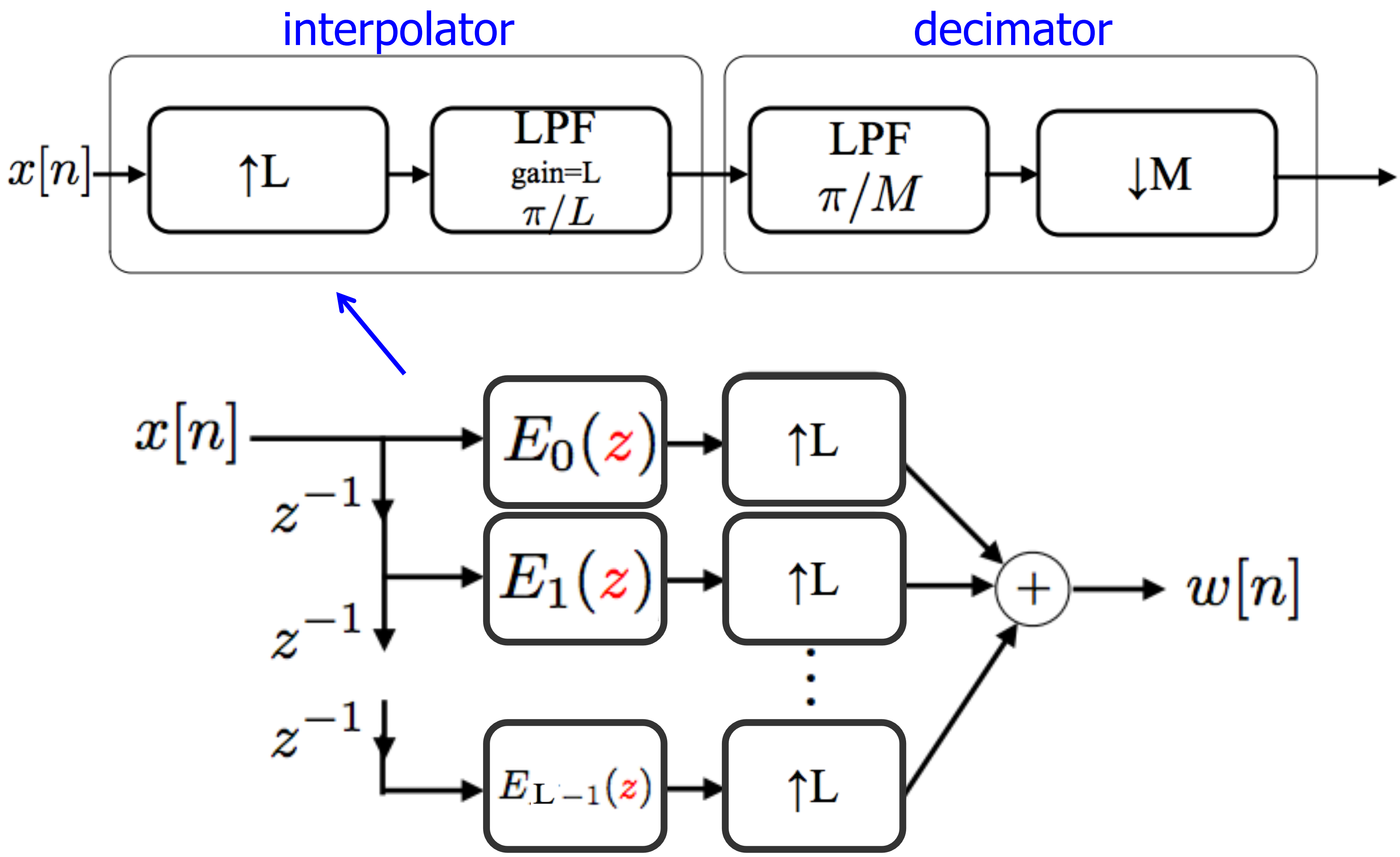
Each filter computation:
-N/M multiplications
-1/M rate per sample
→N/M*(1/M) mults/unit time

Total computation:
-M filters
→N/M mults/unit time

Polyphase Implementation of Decimator



Polyphase Implementation of Interpolation



Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

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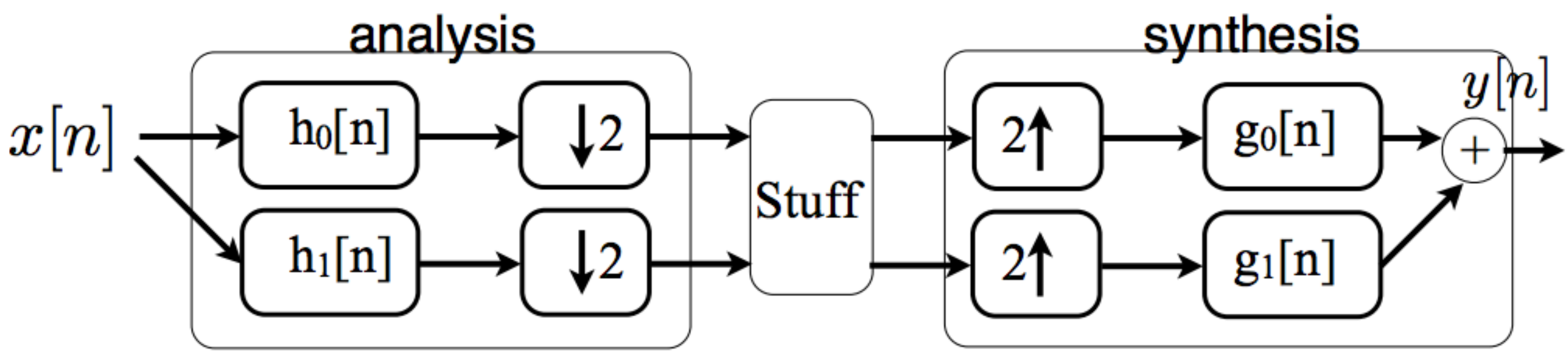
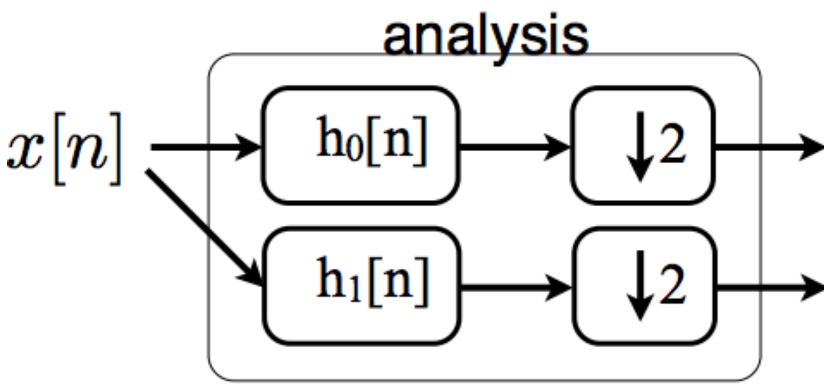
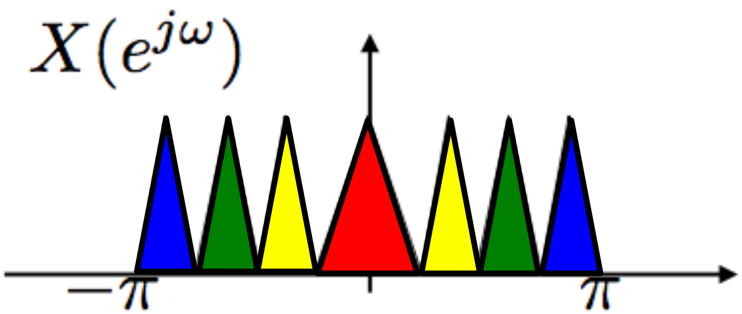
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Multi-Rate Filter Banks: Analysis

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$





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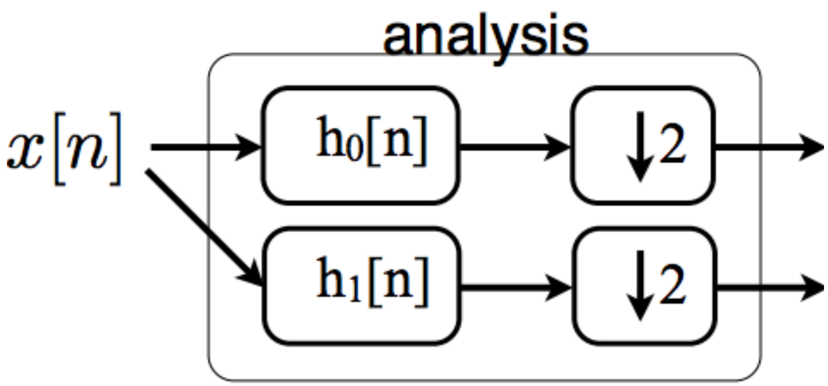

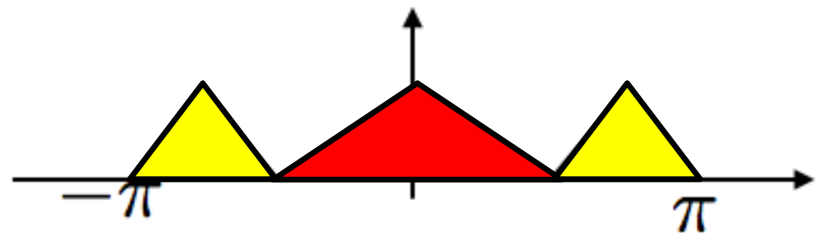

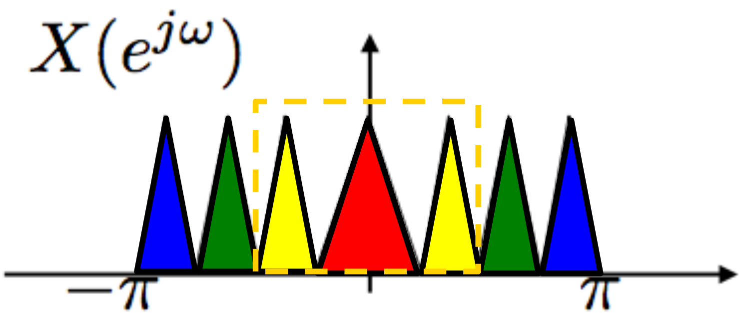

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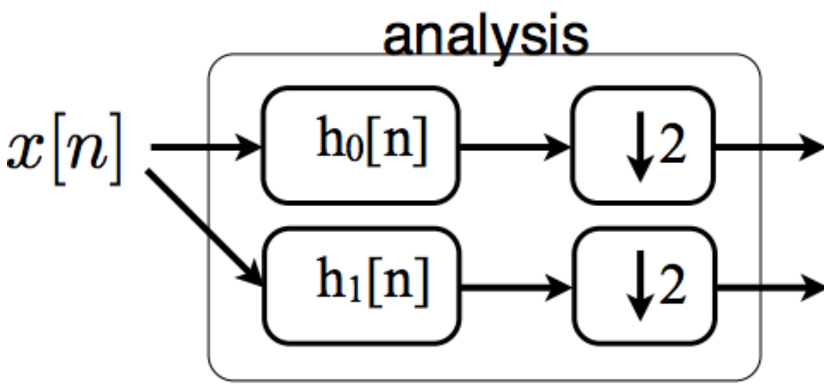

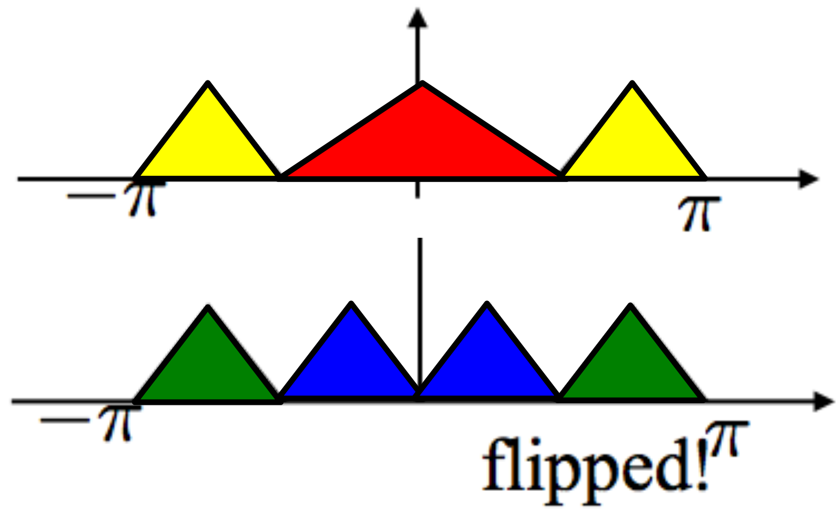

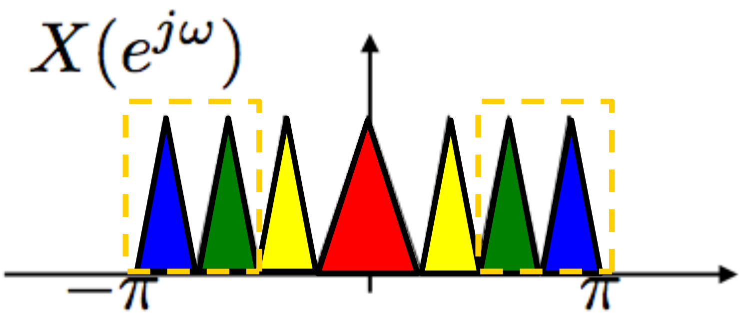

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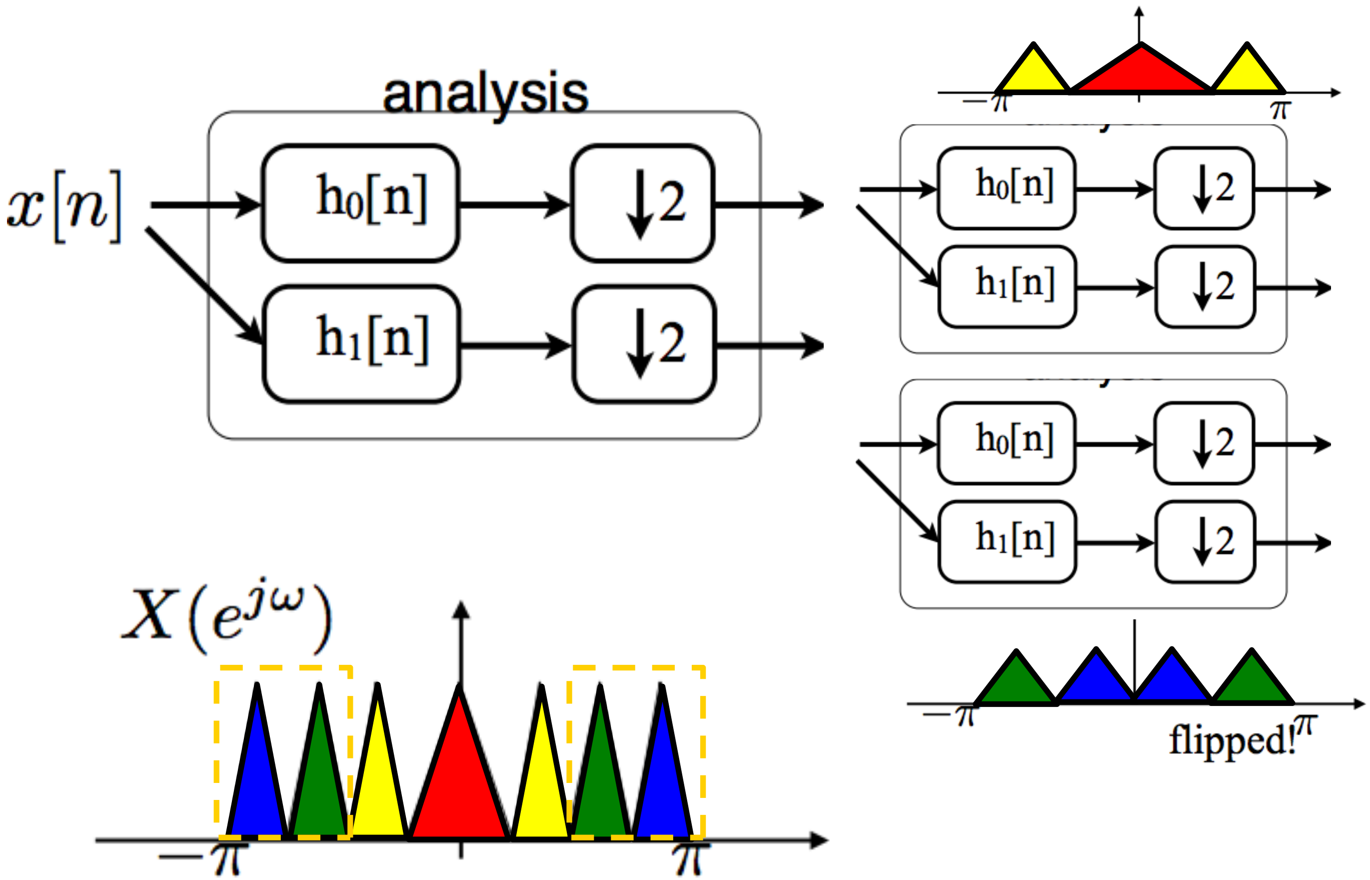

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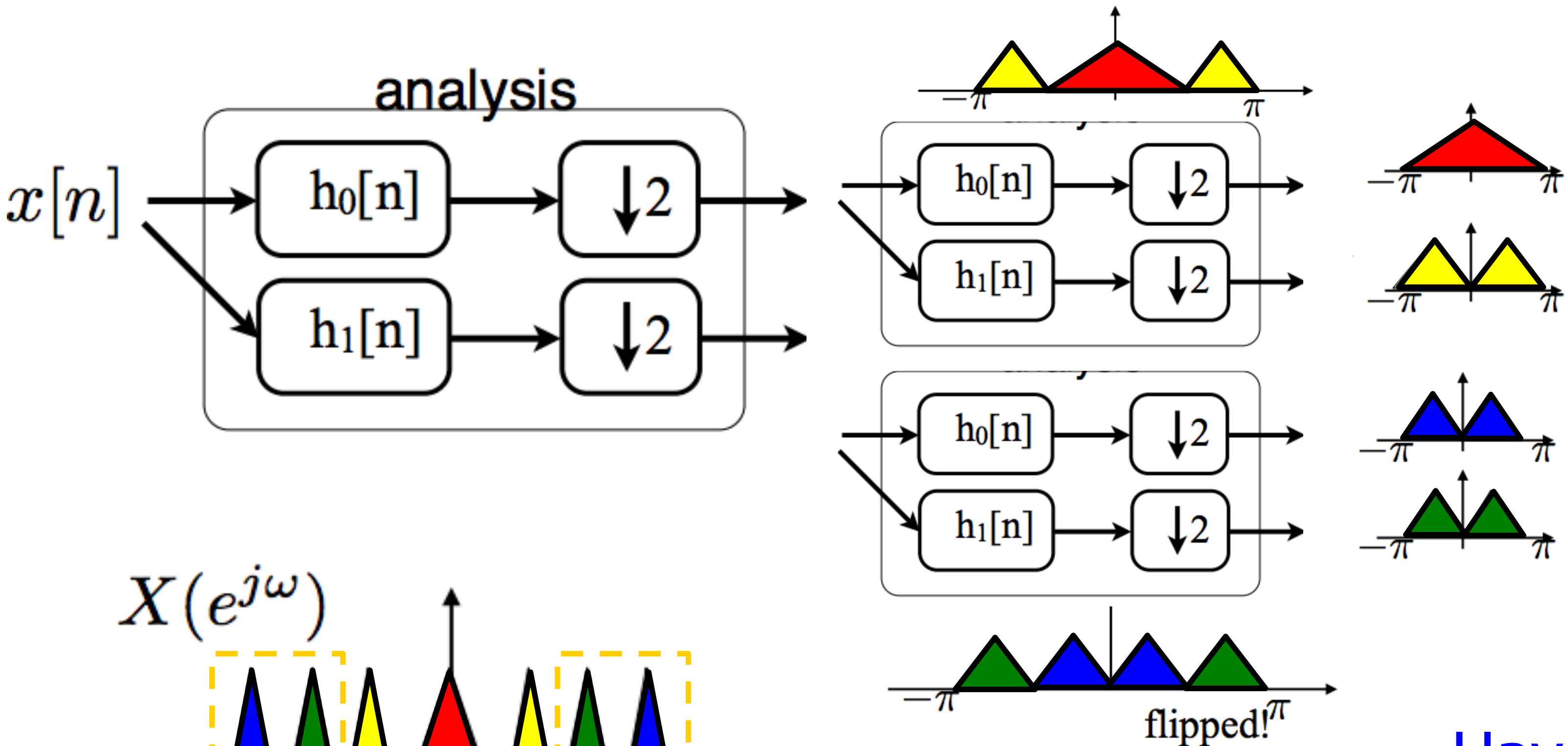

Have to be careful with order!

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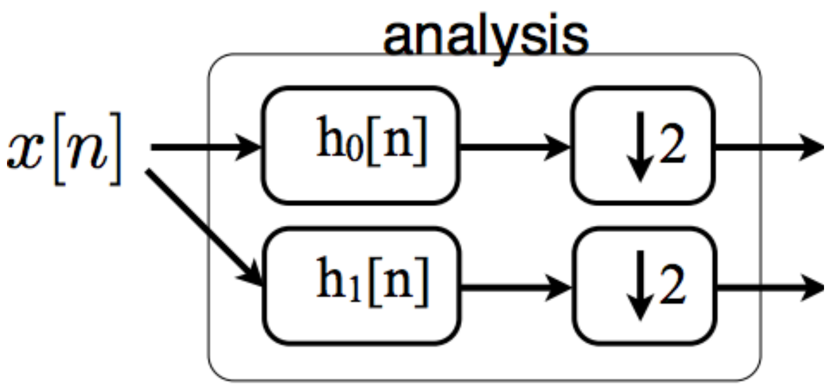

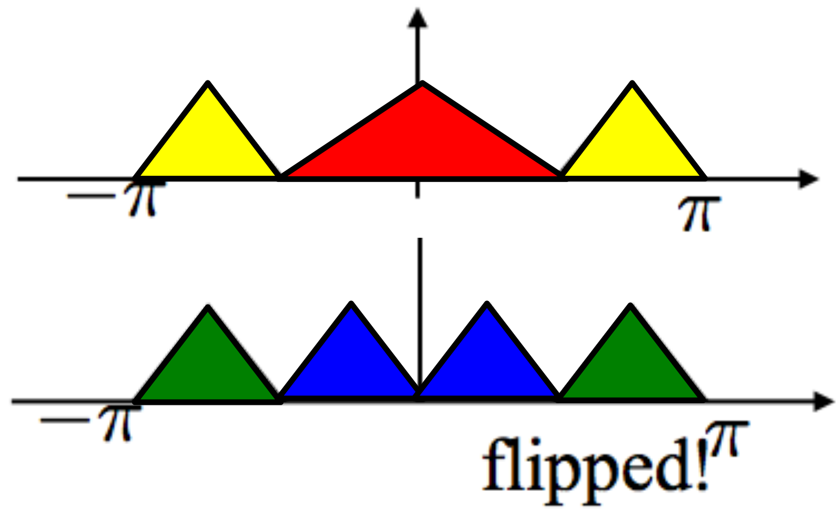

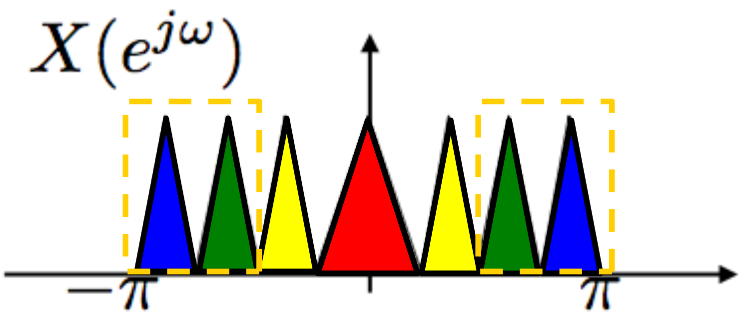

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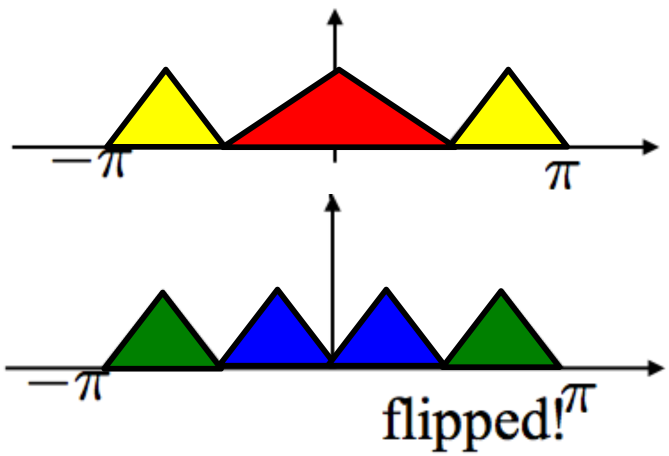

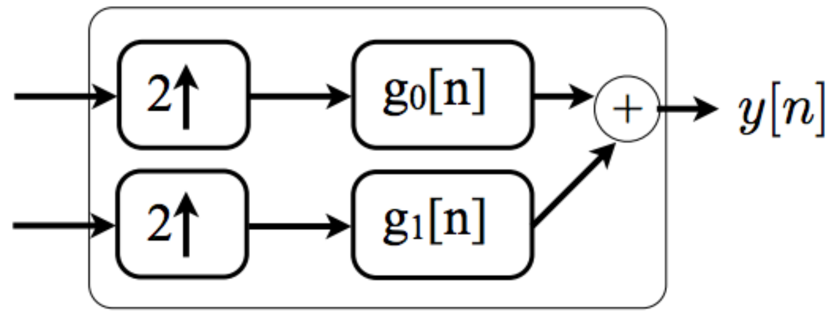

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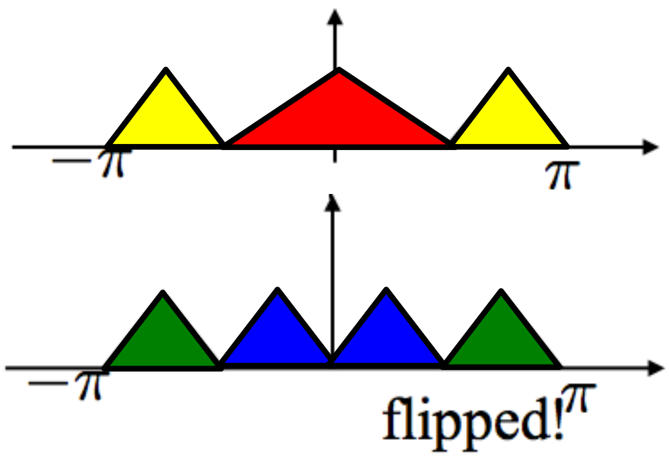

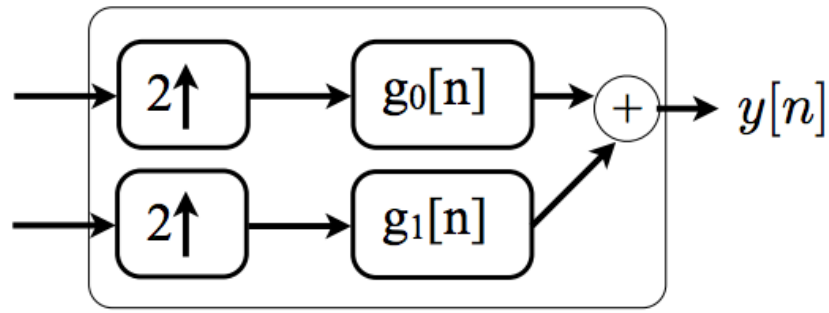

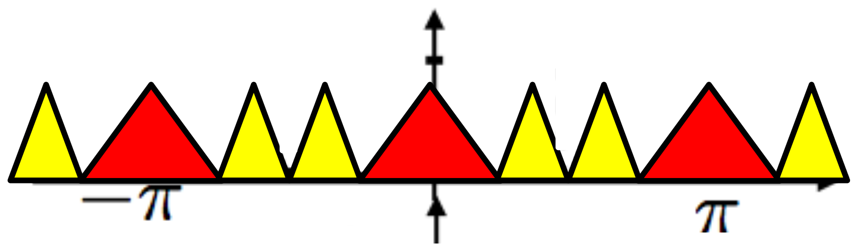

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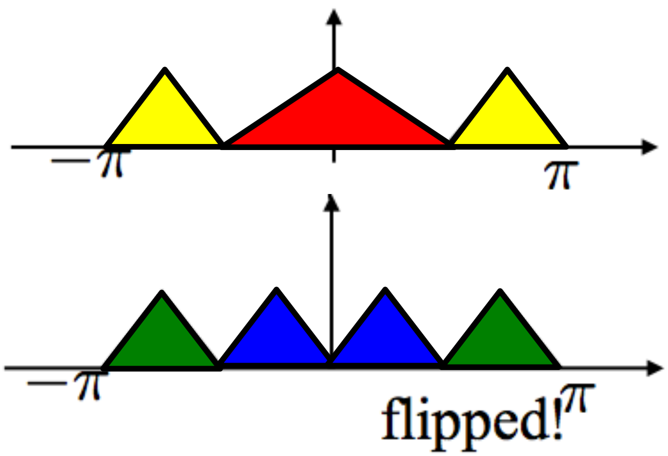

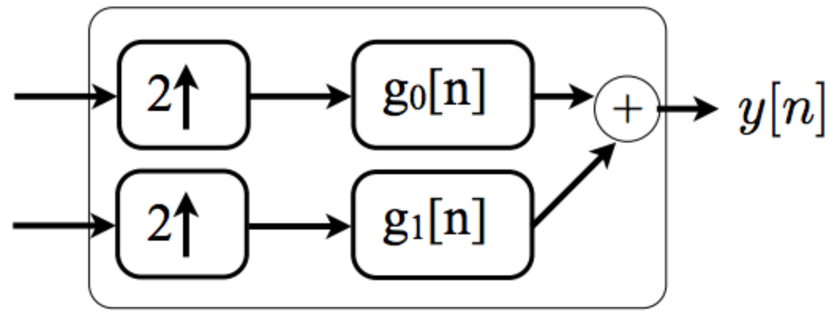

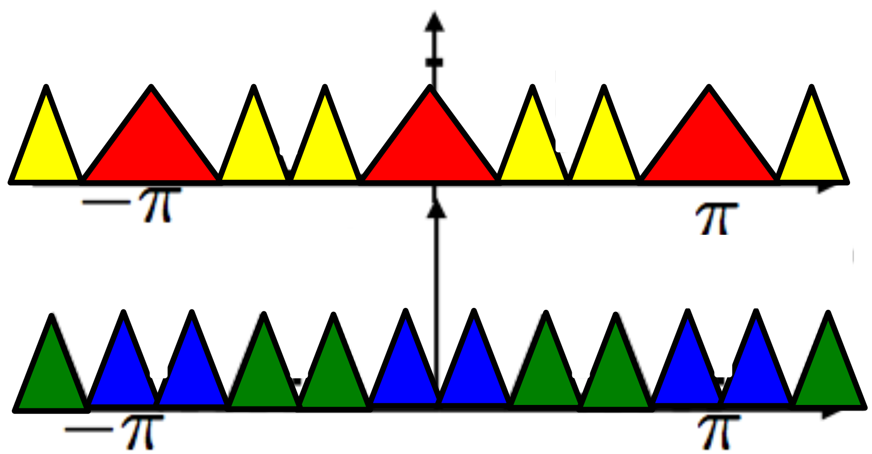

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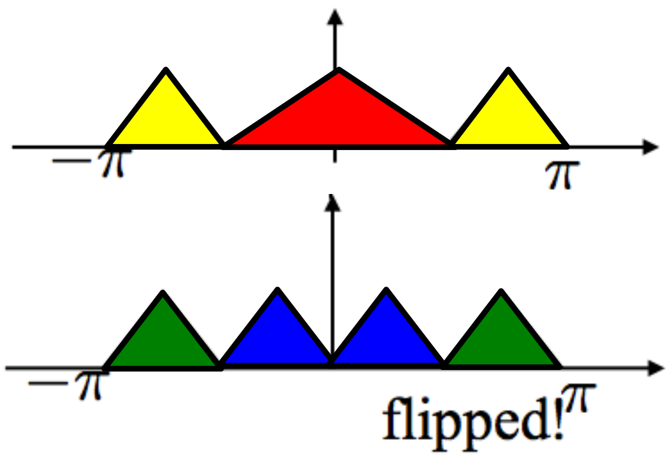

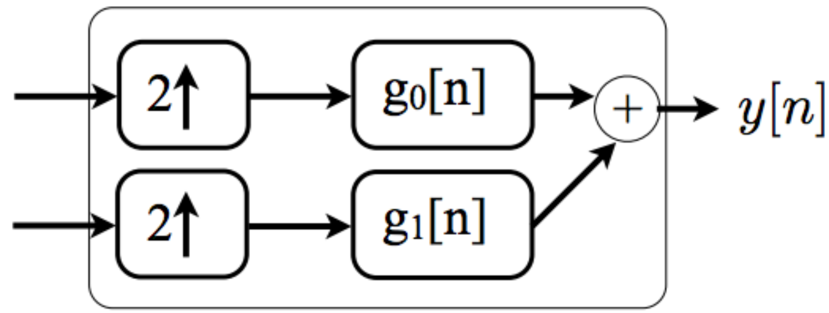

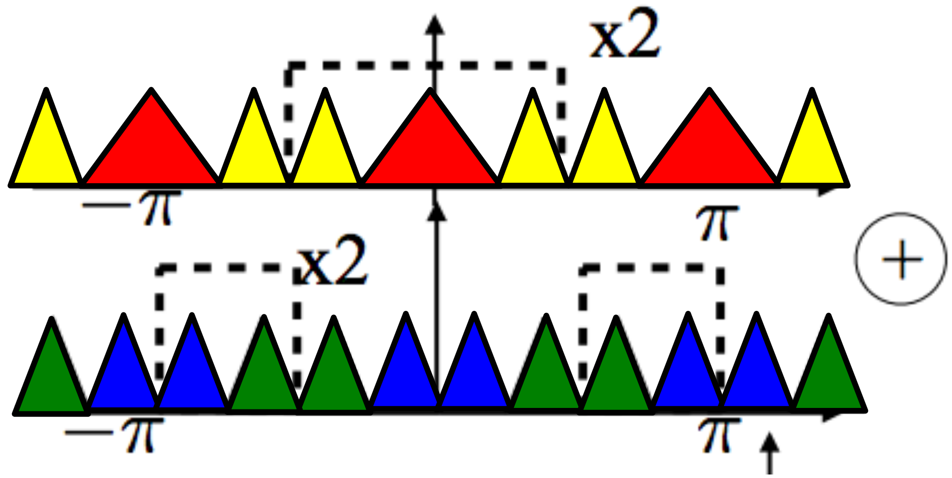

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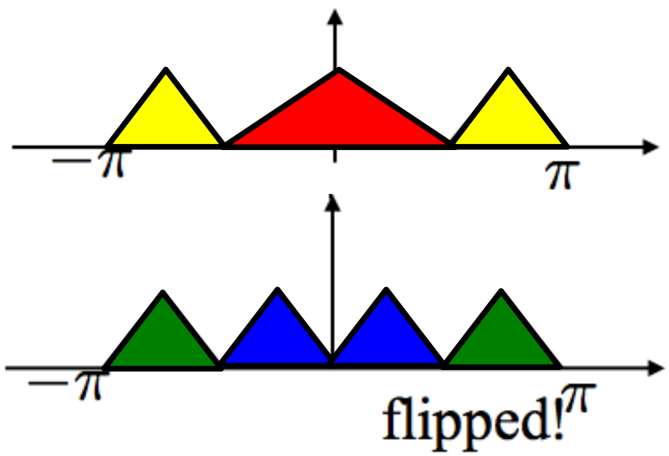

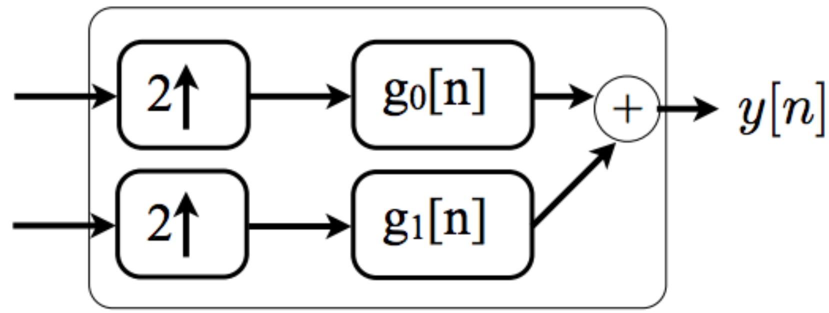

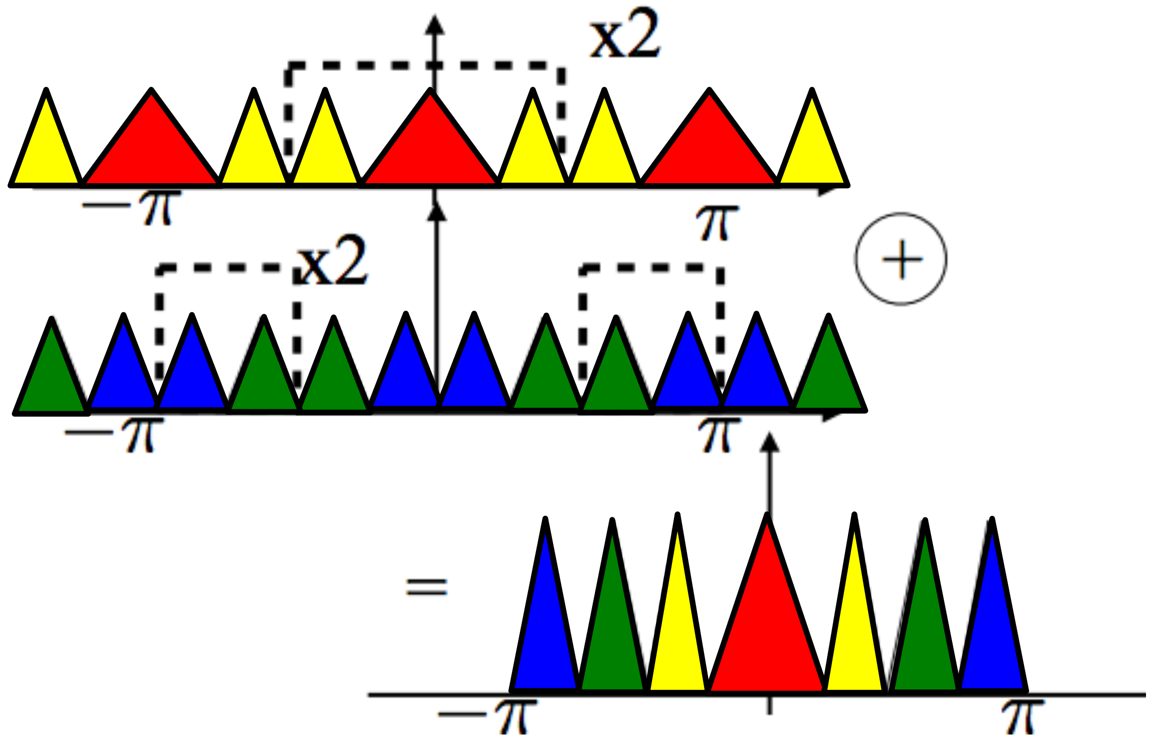

Multi-Rate Filter Banks

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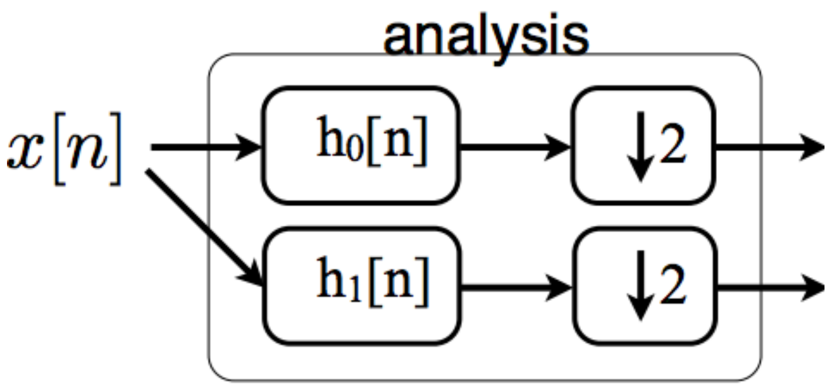

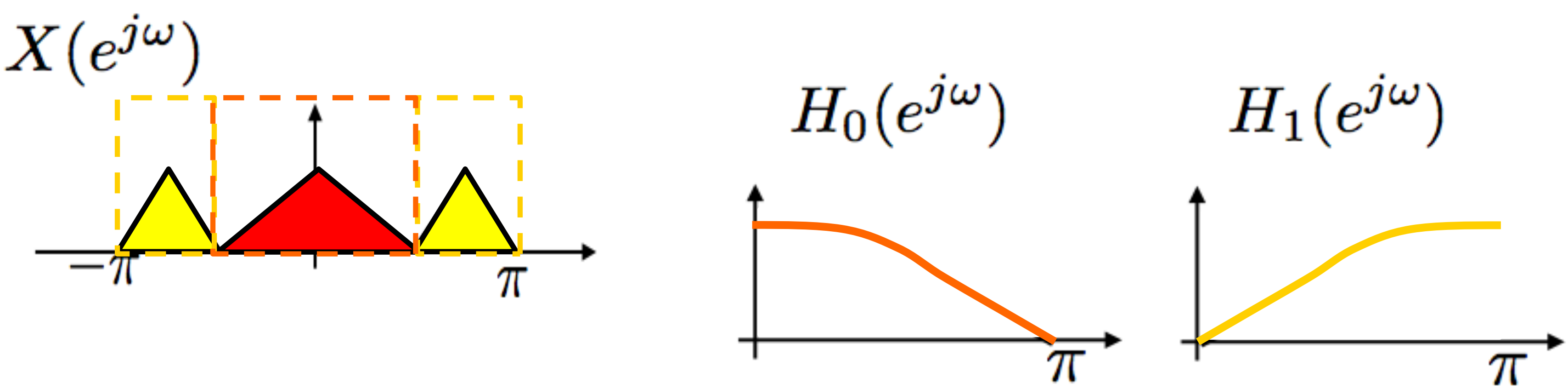

Non Ideal Filters

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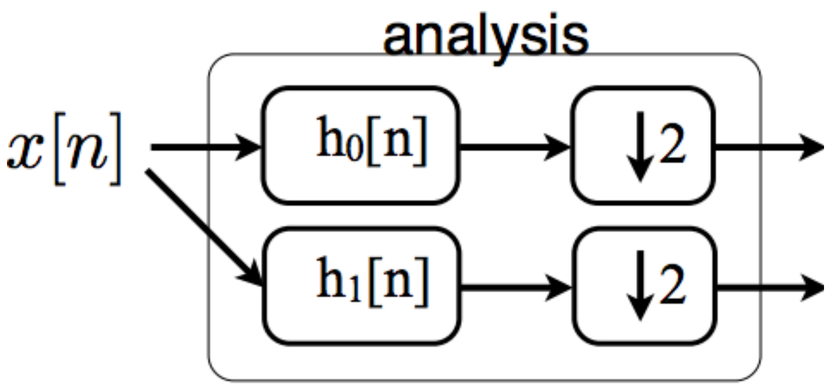

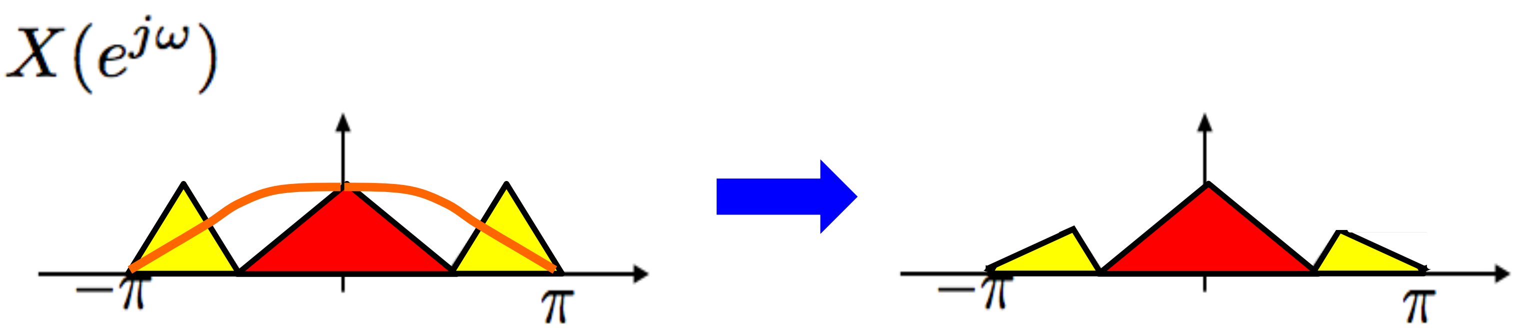

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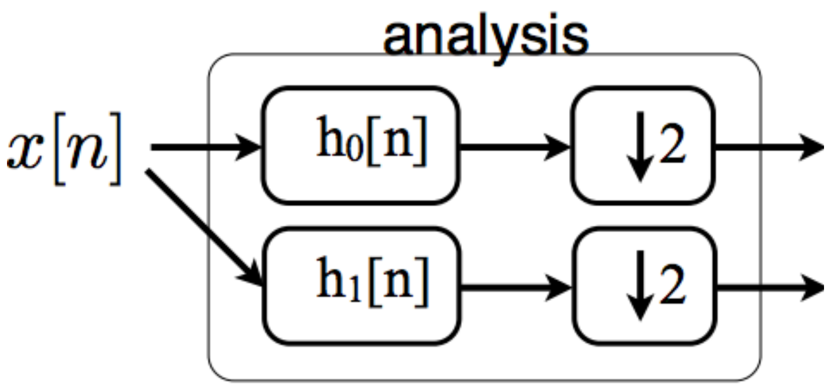
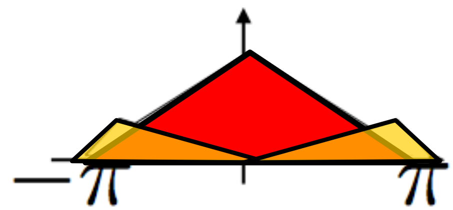
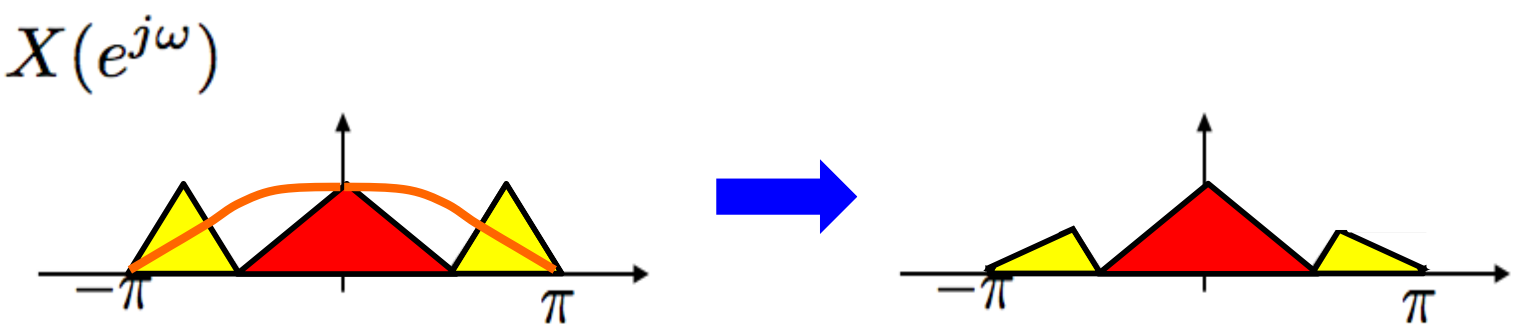
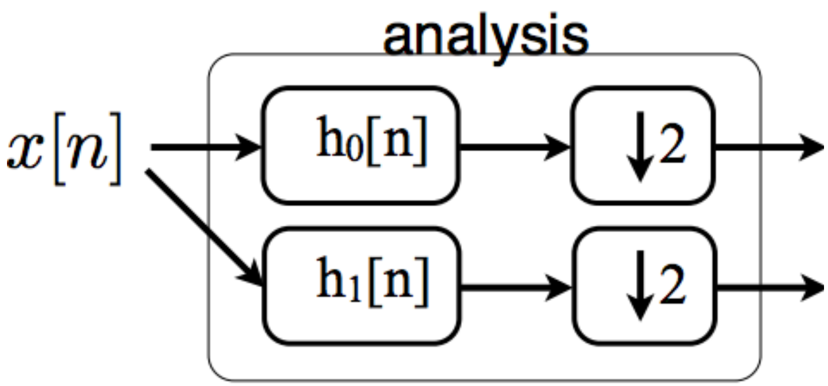


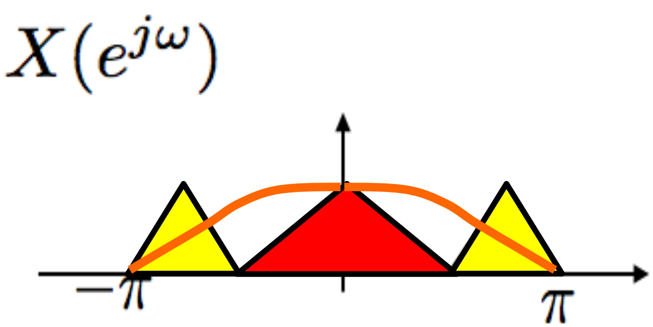


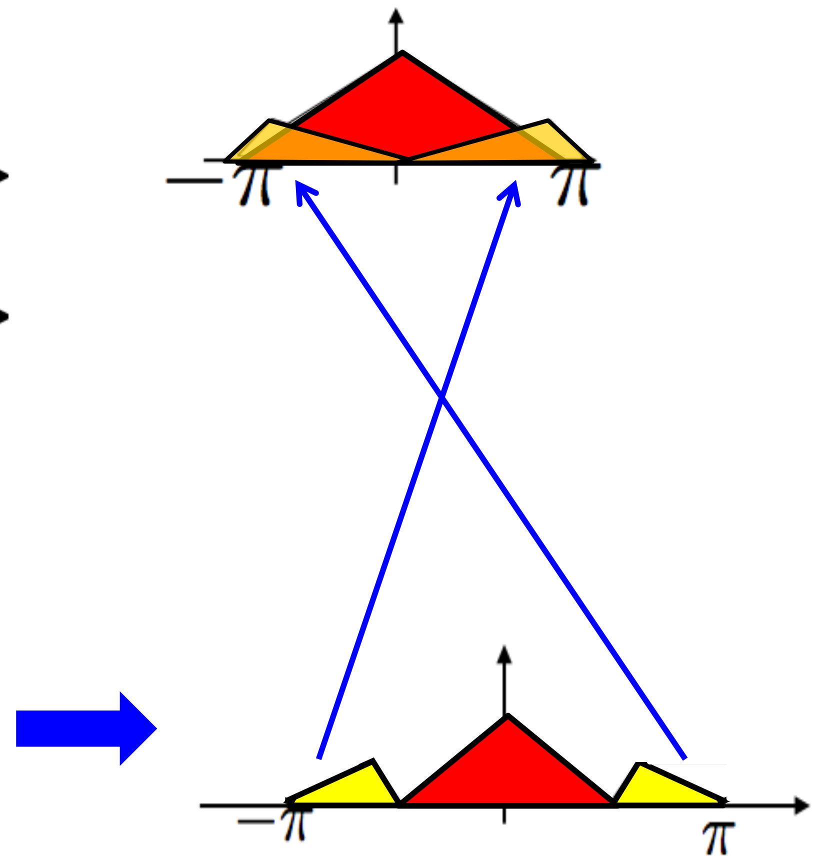


Non Ideal Filters

- h_0, h_1 are NOT ideal low/high pass

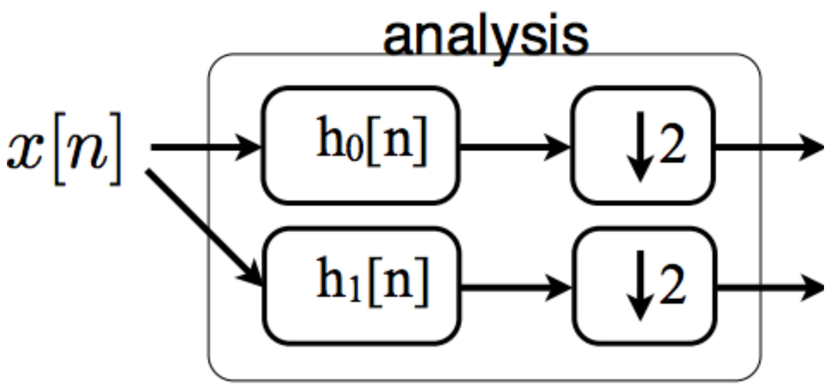


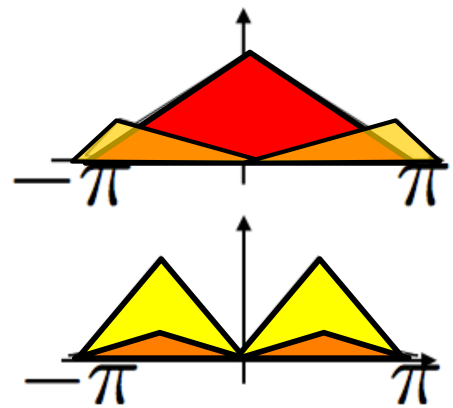


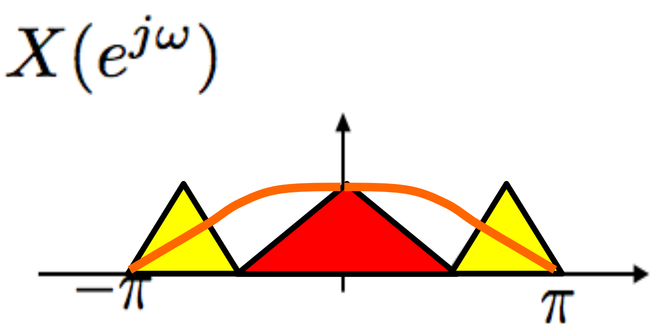


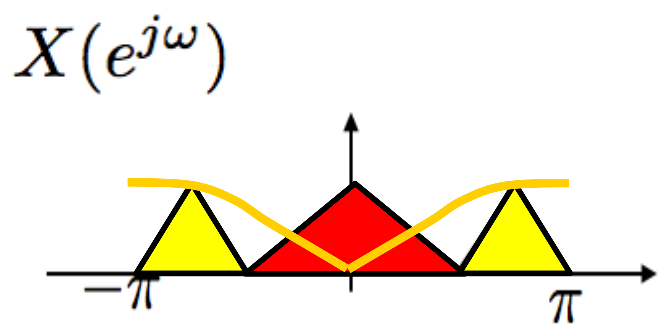
Non Ideal Filters

- h_0, h_1 are NOT ideal low/high pass



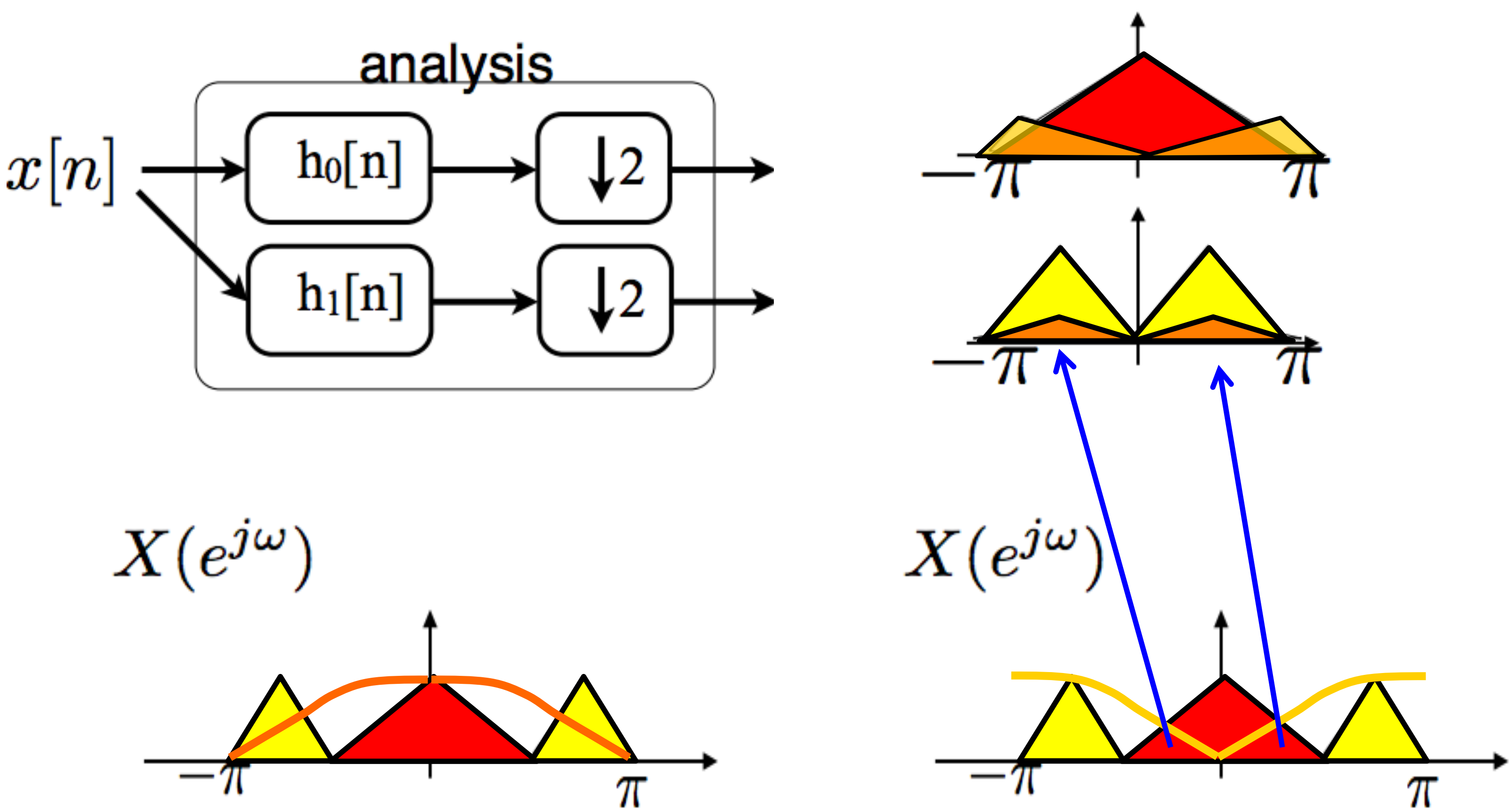




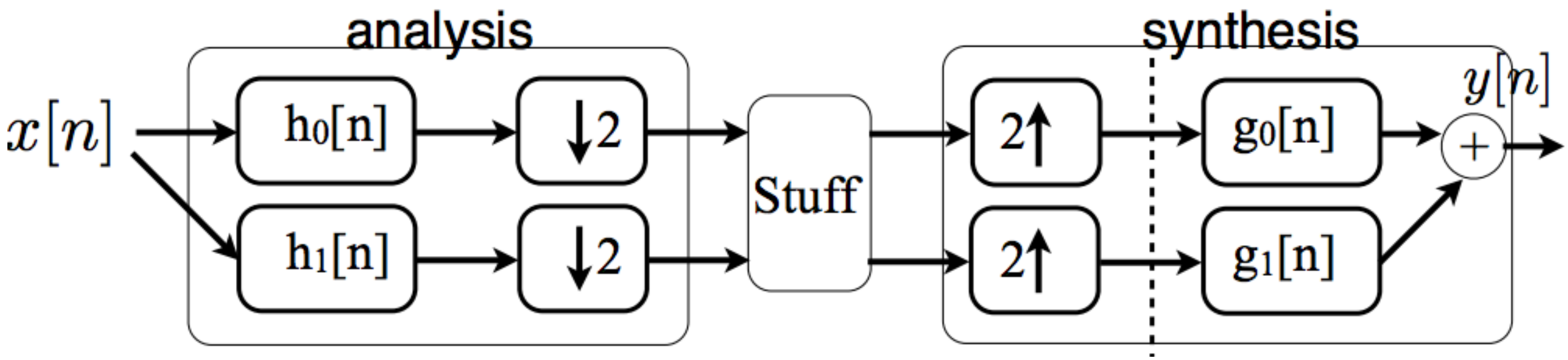


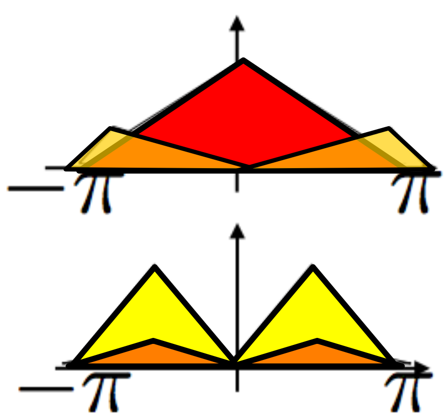
Non Ideal Filters

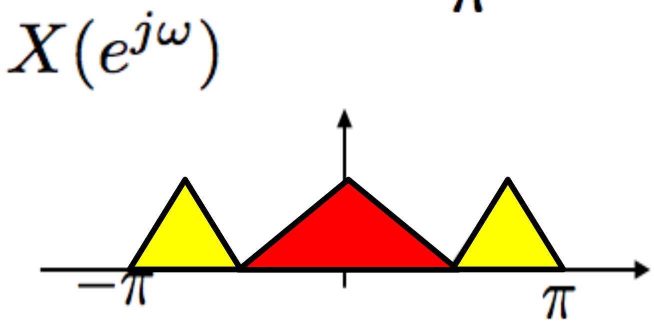
- h_0, h_1 are NOT ideal low/high pass



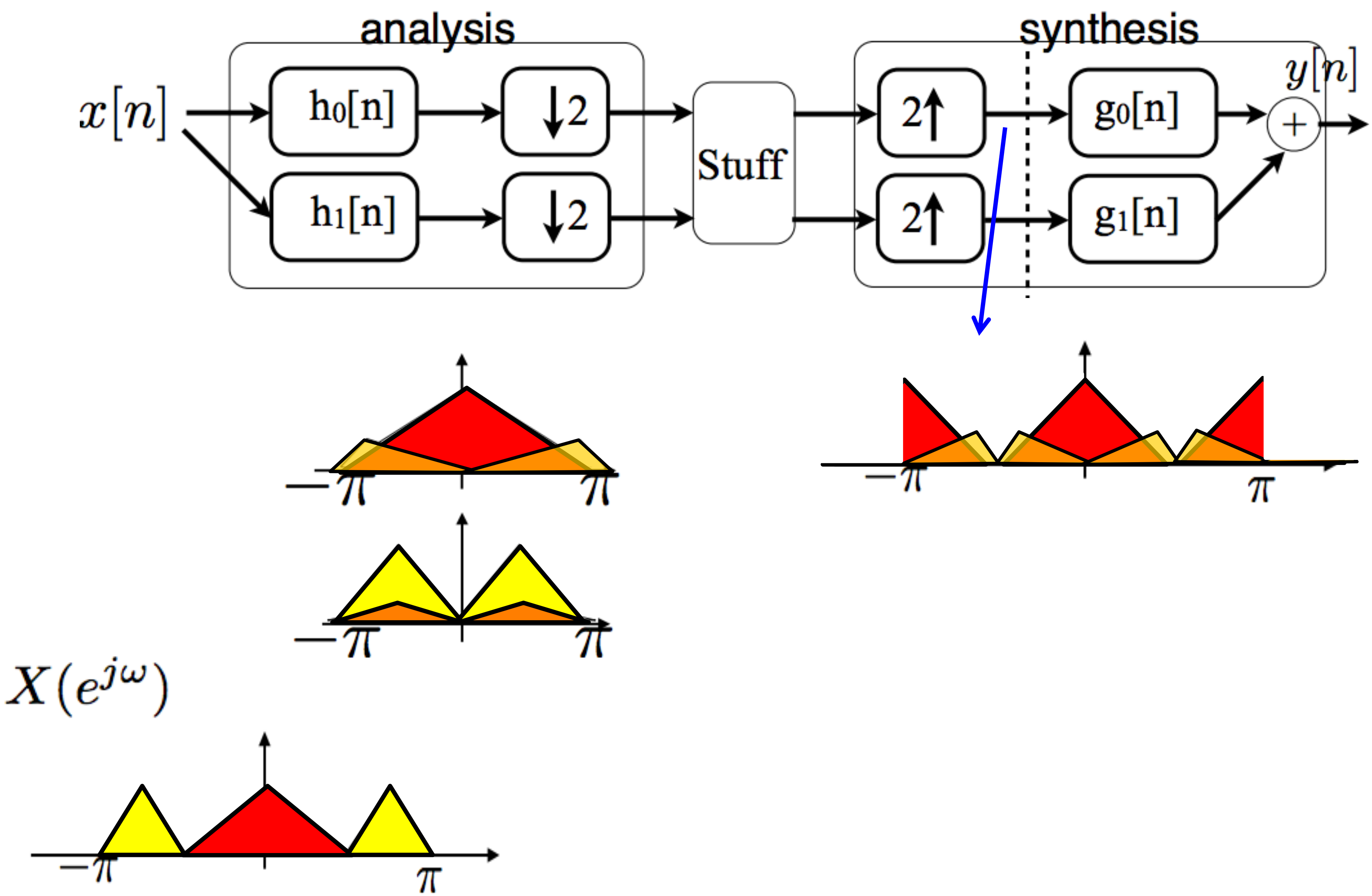
Non Ideal Filters



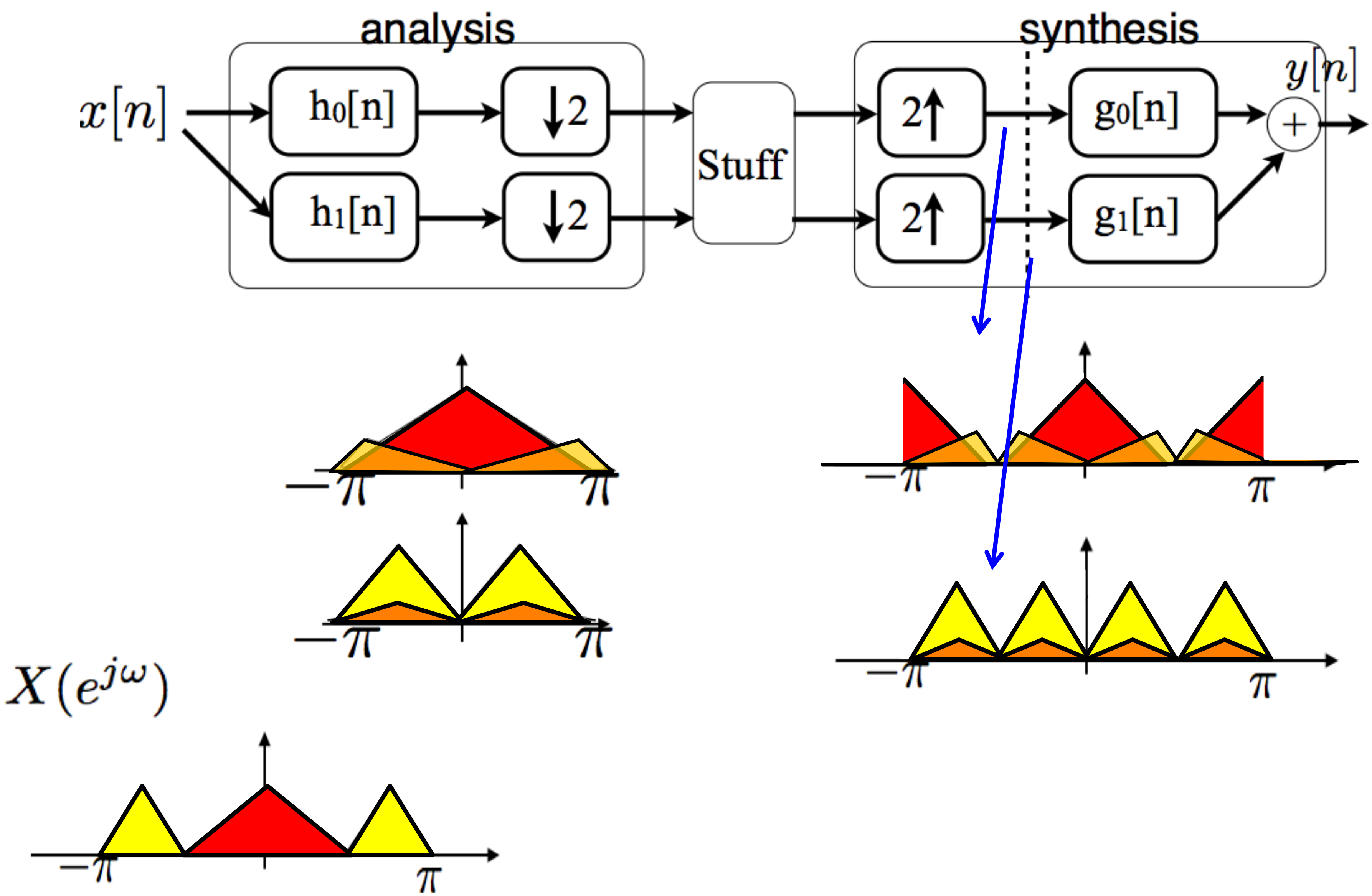




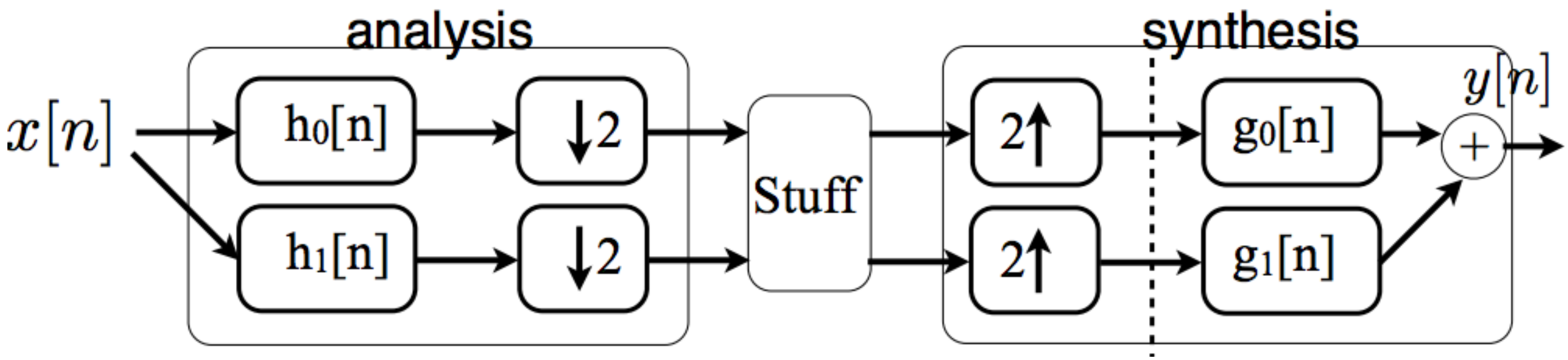
Non Ideal Filters



Non Ideal Filters



Perfect Reconstruction non-Ideal Filters

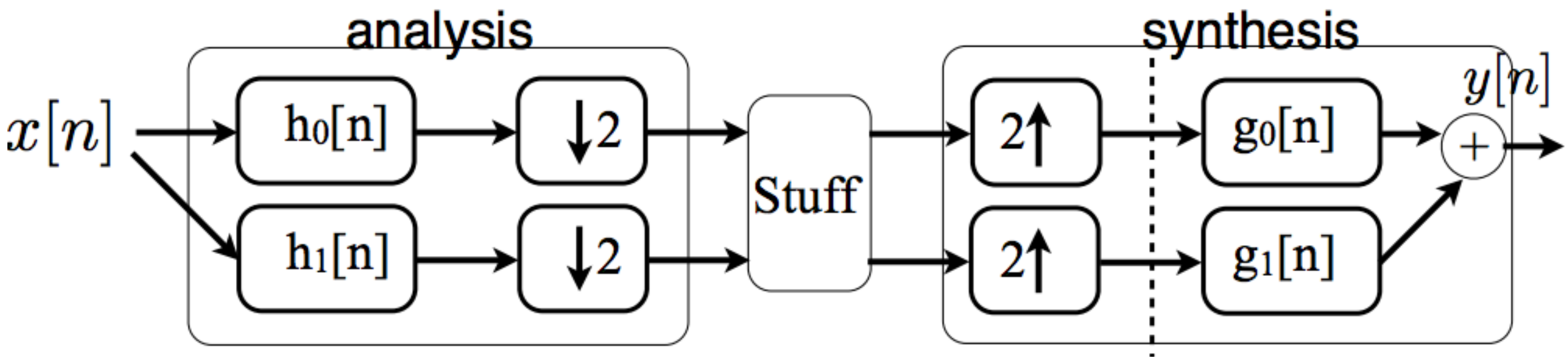


$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)})$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

Perfect Reconstruction non-Ideal Filters

$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

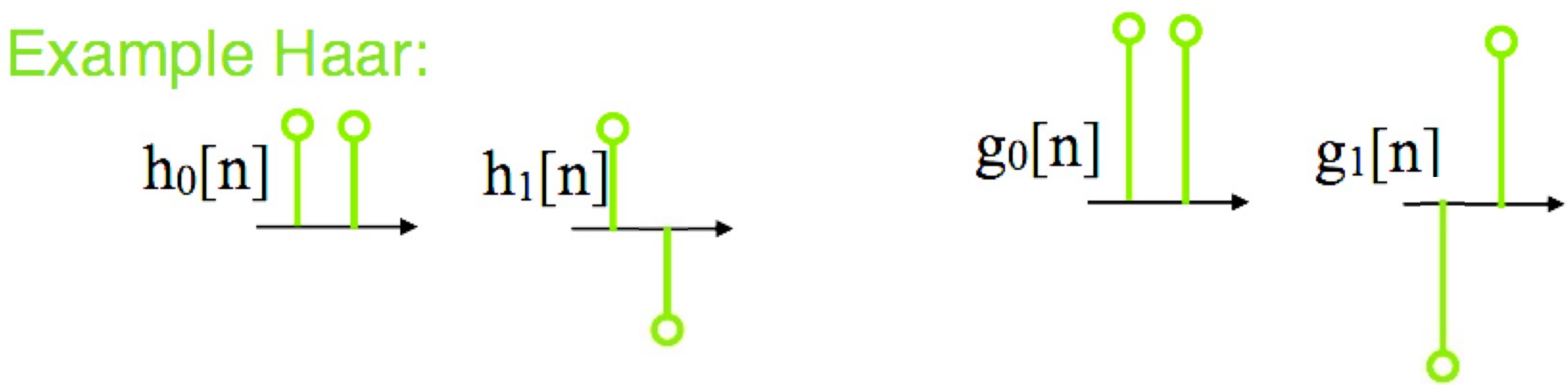
need to cancel!

aliasing

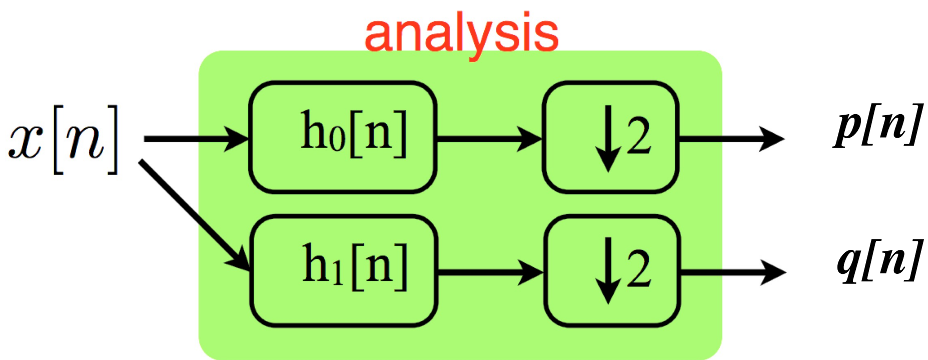
$$H_1(e^{j\omega}) = H_0(e^{j(\omega-\pi)})$$
$$G_0(e^{j\omega}) = 2H_0(e^{j\omega})$$
$$G_1(e^{j\omega}) = -2H_1(e^{j\omega})$$

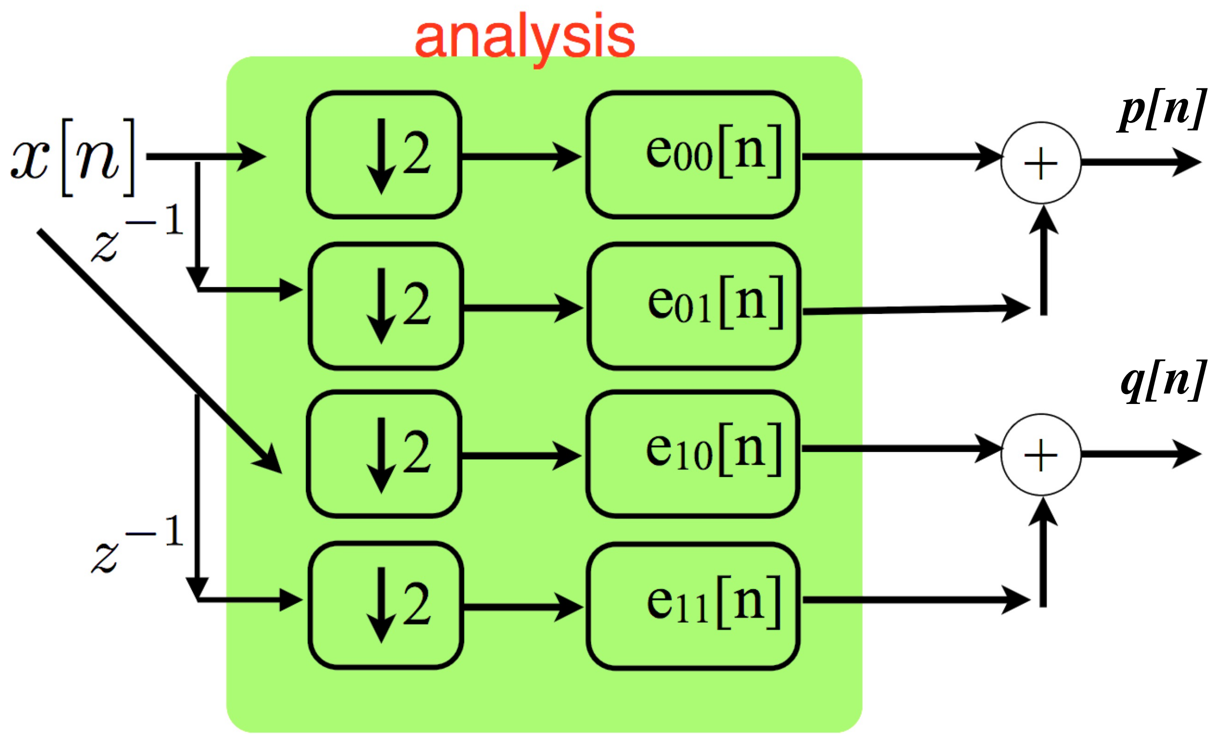
Haar Filter Example

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

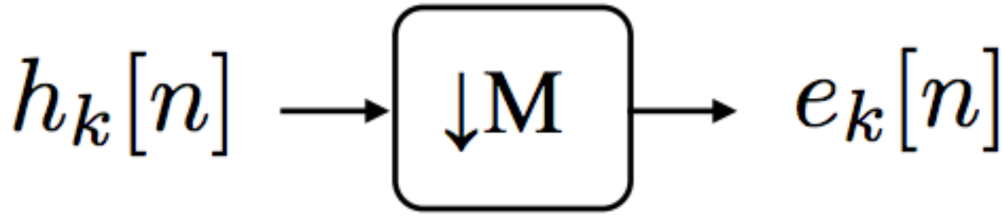


Polyphase Filter Bank

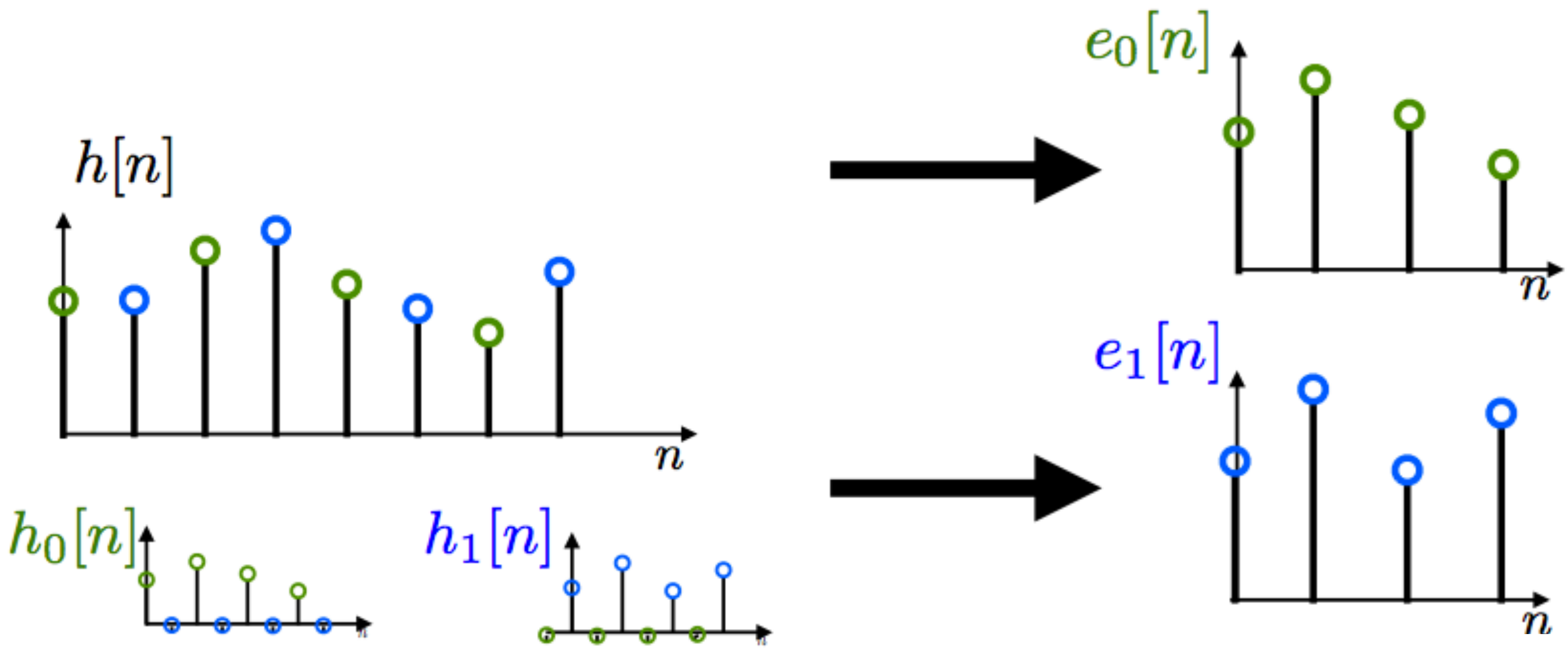




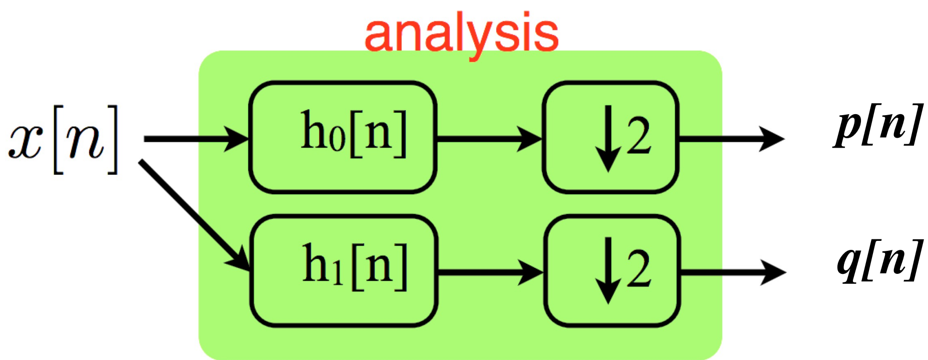
Polyphase Decomposition



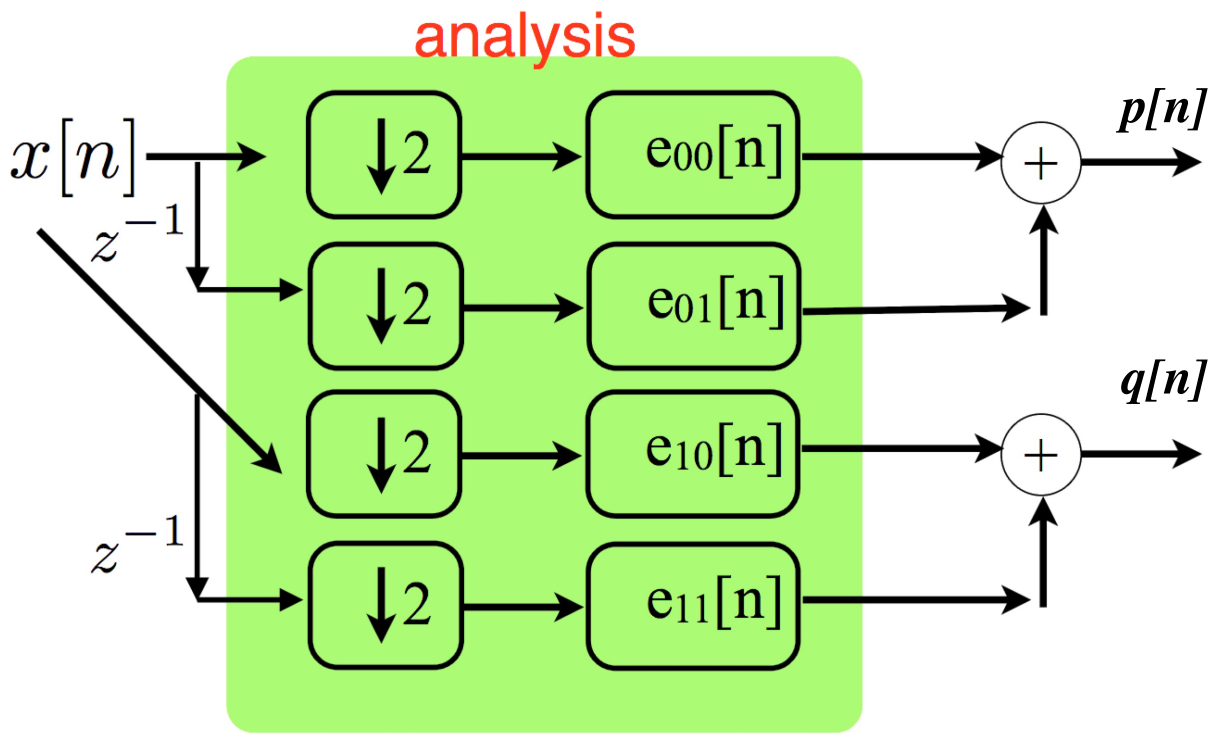
$$e_k[n] = h_k[nM]$$



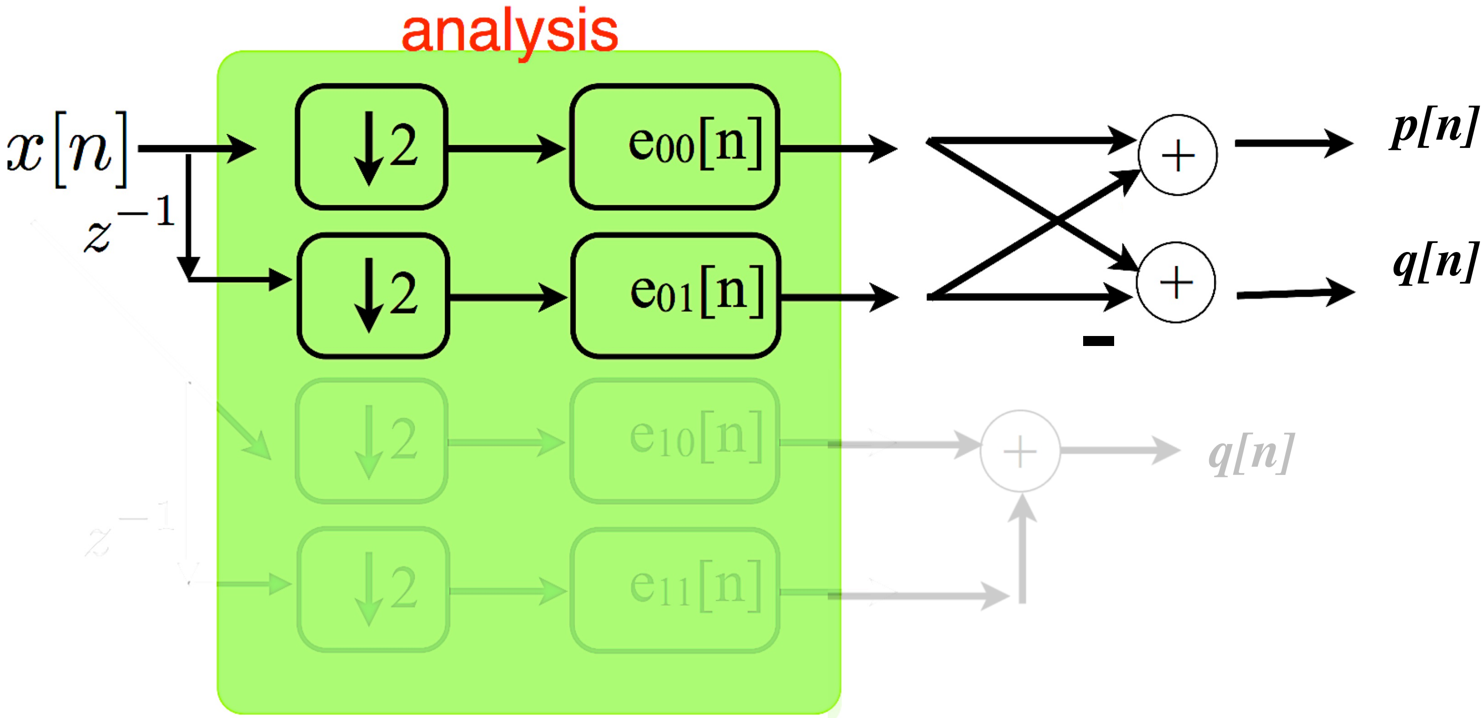
Polyphase Filter Bank



$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{aligned}
$$



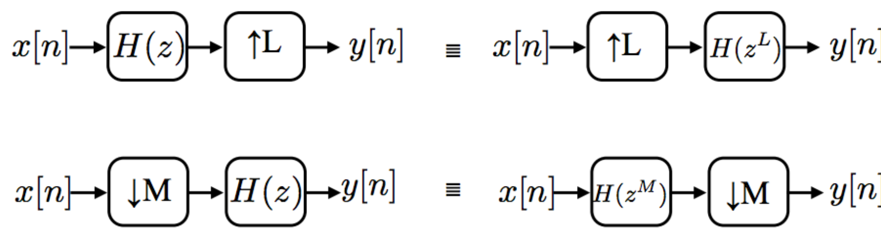
Polyphase Filter Bank

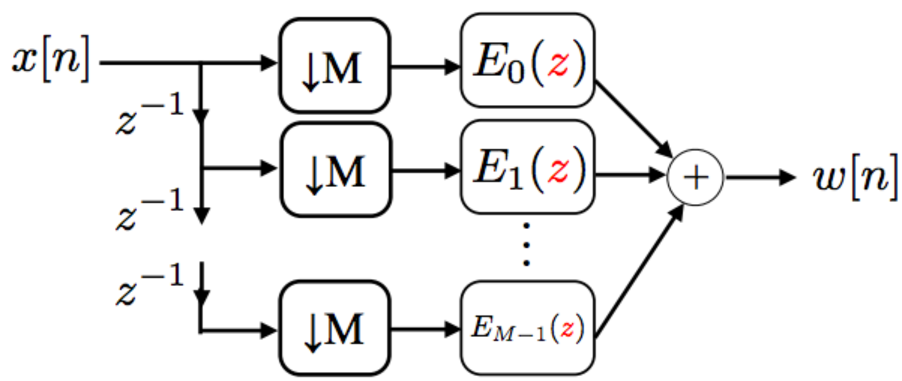


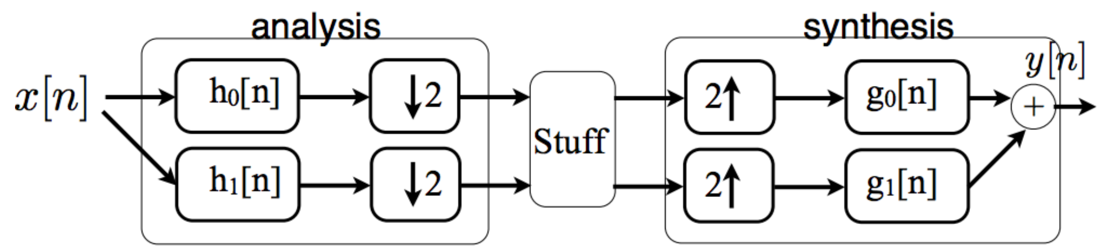
$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{aligned}
$$

Big Ideas

- Multi-rate systems enable more efficient processing
  - Interchanging Operations
  - Polyphase Decomposition
  - Multi-Rate Filter Banks







Admin

- HW 4 due Sunday