ESE 5310: Digital Signal Processing

Lecture 14: February 28, 2023

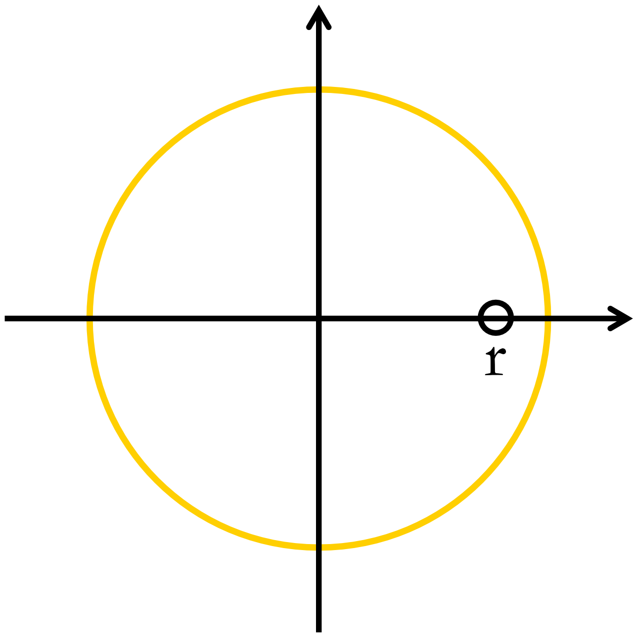
All-Pass Systems, Min Phase Decomposition

Penn ESE 5310 Spring 2023 – Khanna
Adapted from M. Lustig, EECS Berkeley

Example: Zero on Real Axis

- Geometric Interpretation for (θ=0)

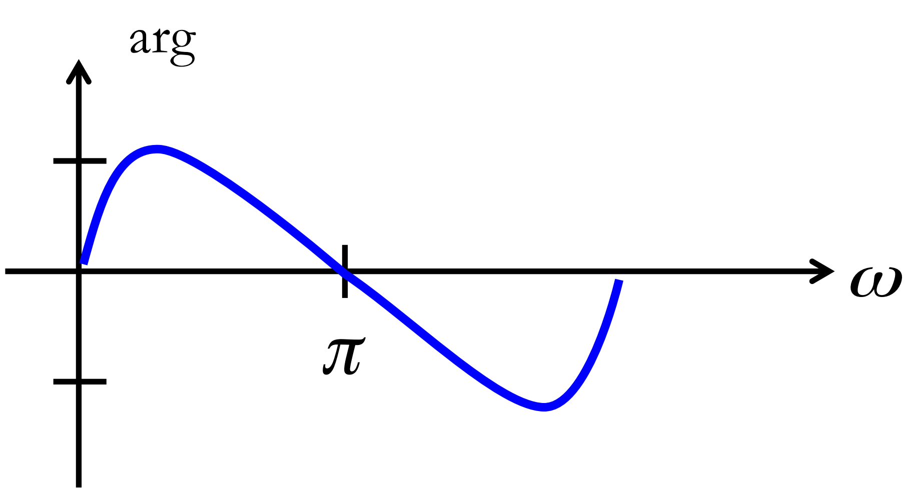
$$\arg[1 - re^{-j\omega}]$$

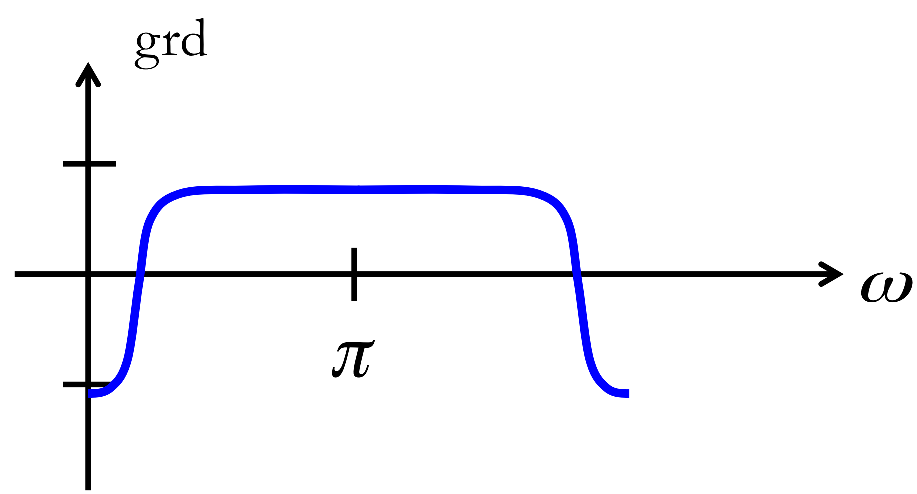


Example: Zero on Real Axis

- Geometric Interpretation for (θ=0)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$





Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \text{grd}[1 - d_k e^{-j\omega}]$$

- Look at each factor:

θ≠0?

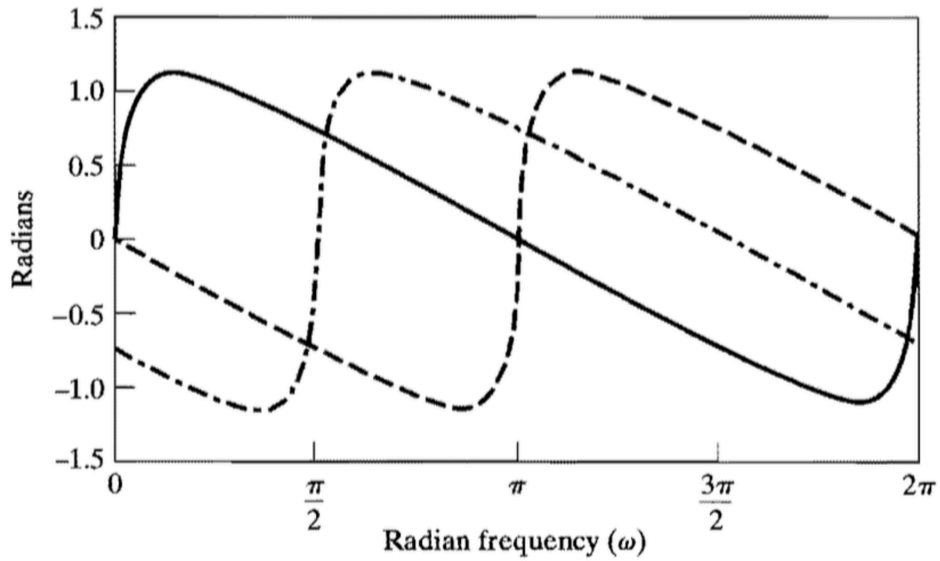
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega-\theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

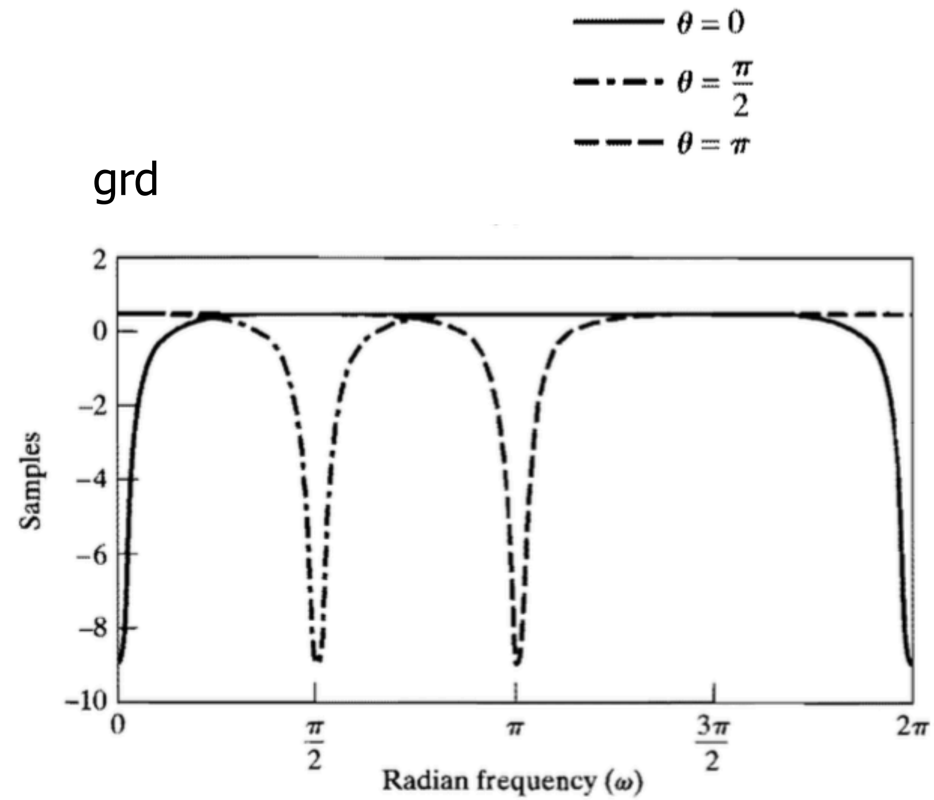
Example: Zero on Real Axis

- For θ≠0

arg



grd



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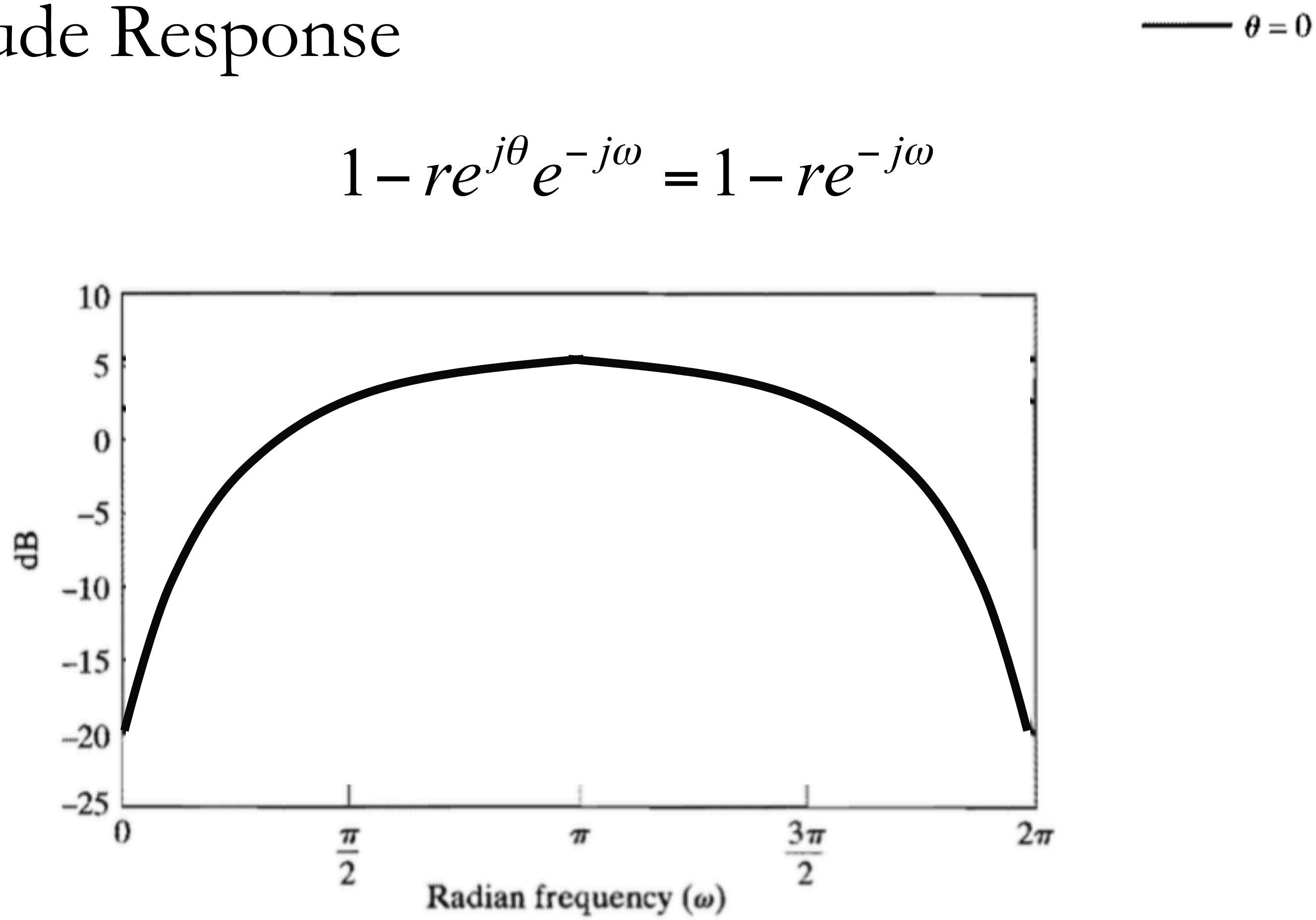
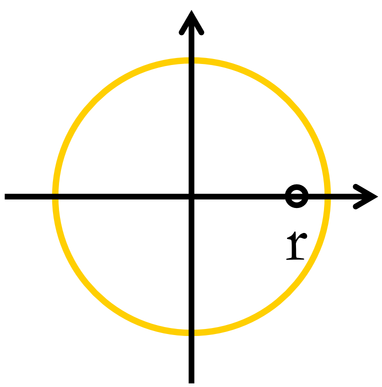
Example: Zero on Real Axis

- Magnitude Response

$$1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j\omega}$$





Example: Zero on Real Axis

- Magnitude Response

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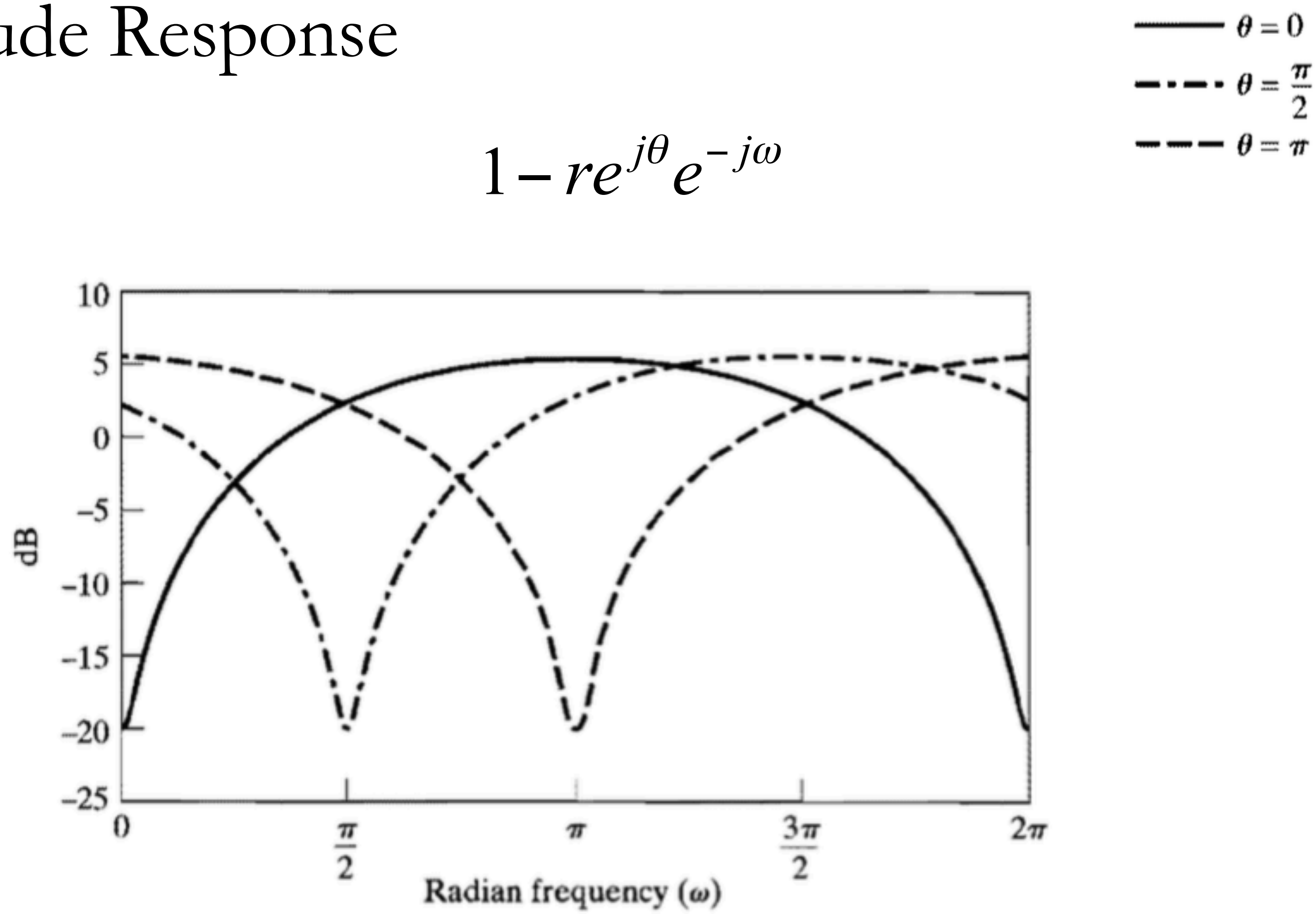
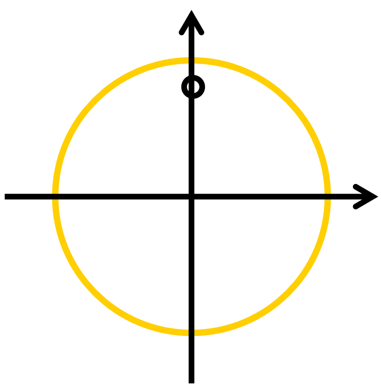
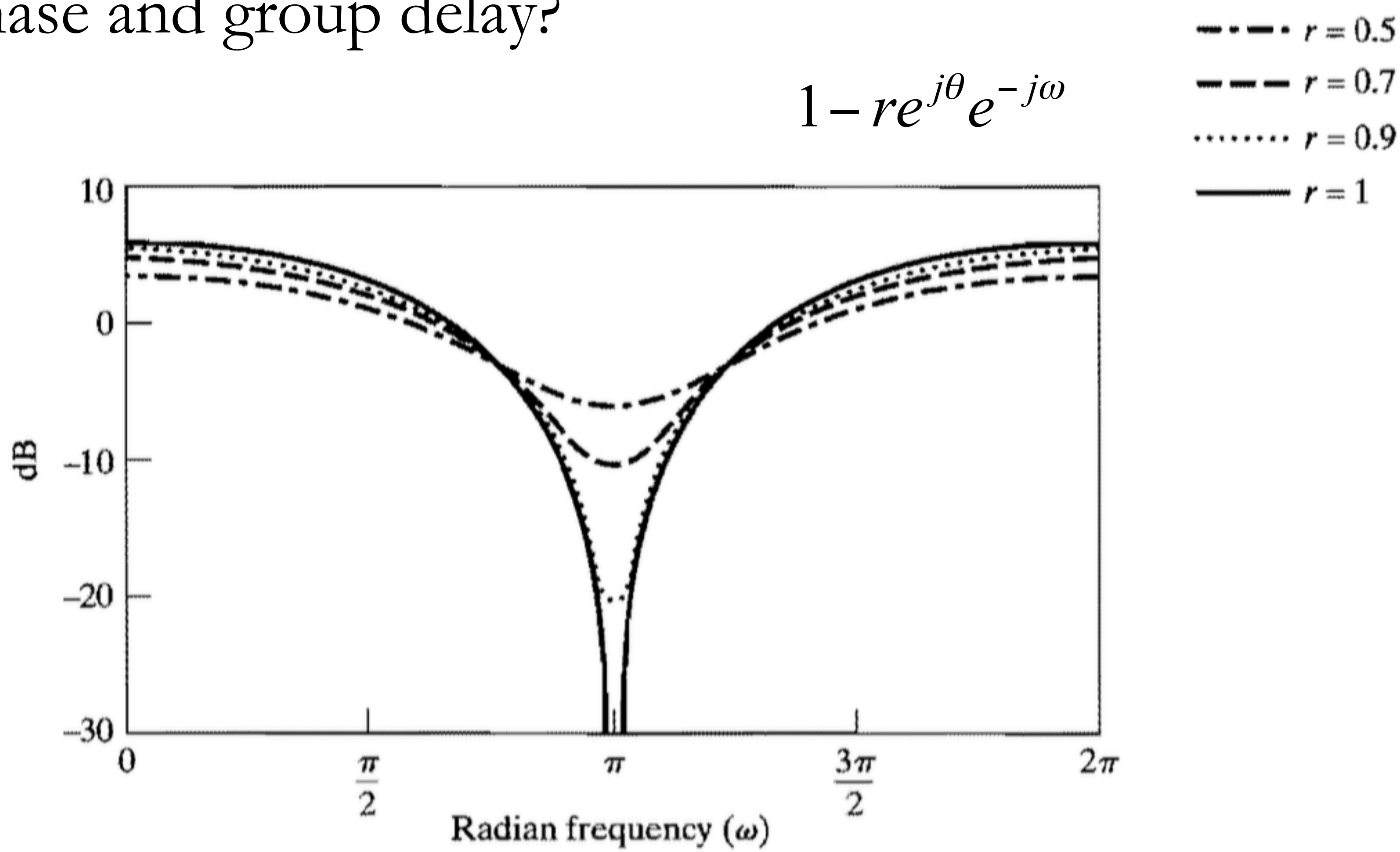




Example: Zero on Real Axis

- For θ=π, how does zero location effect magnitude, phase and group delay?

$$1 - re^{j\theta}e^{-j\omega}$$



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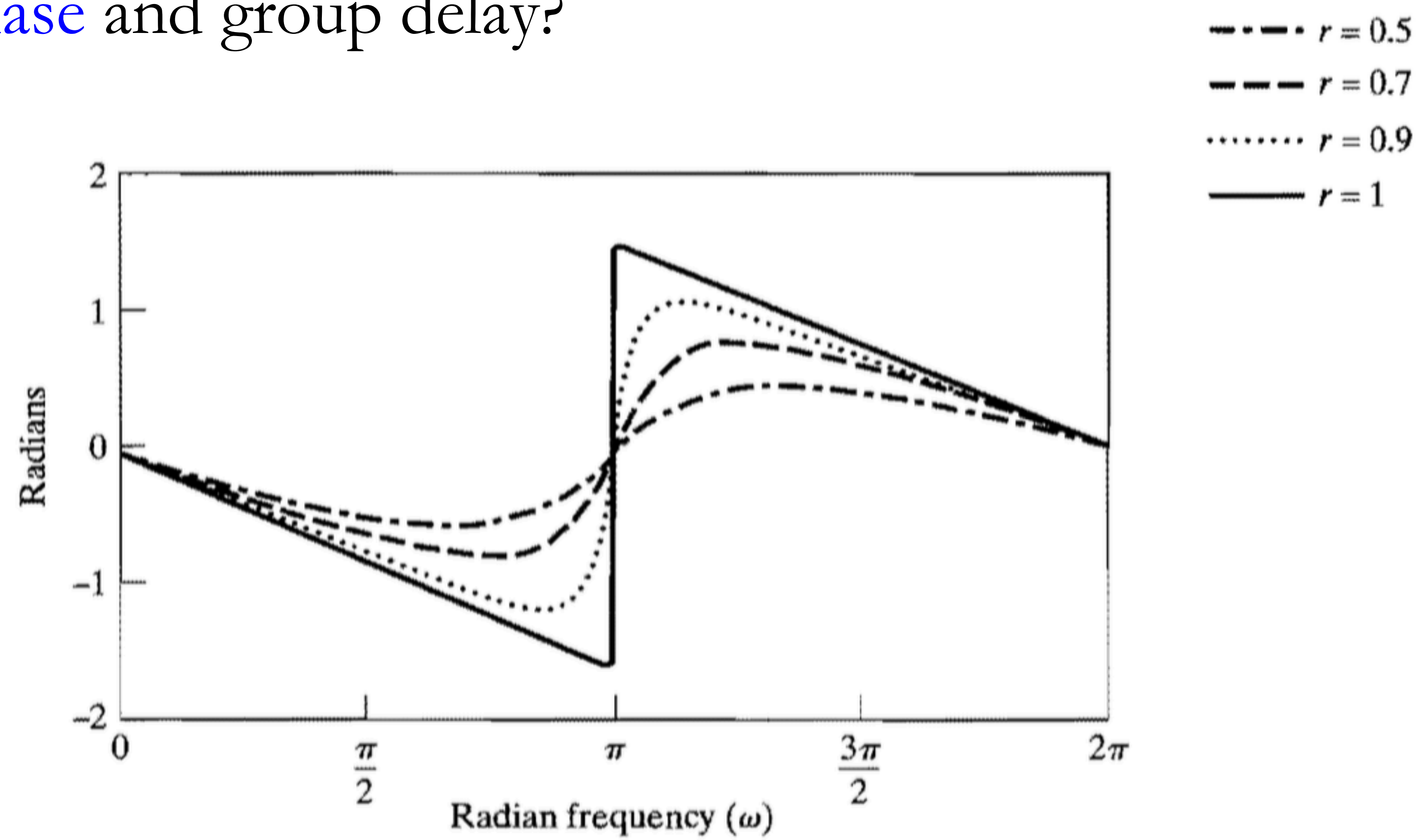

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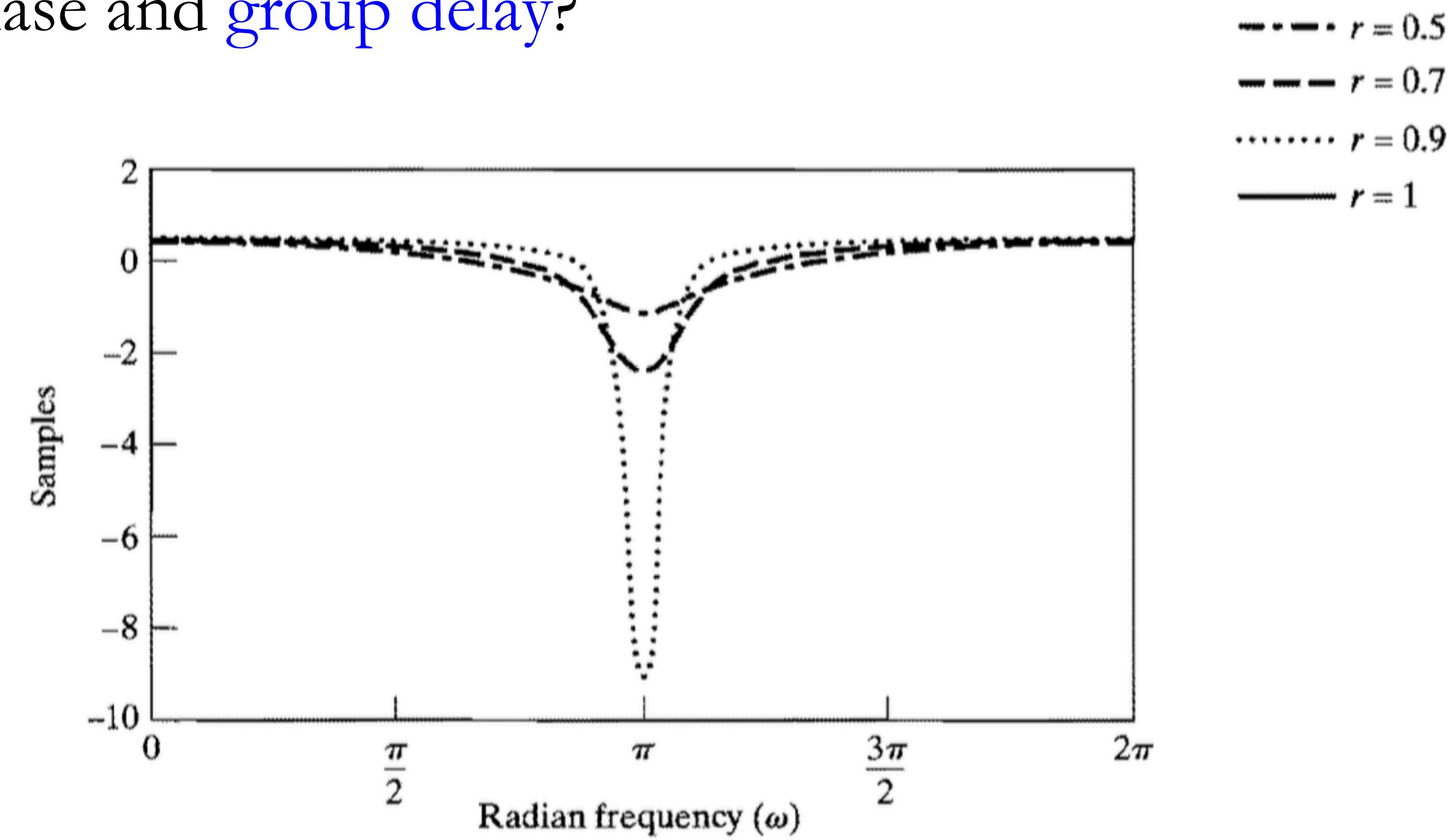


All Pass Systems

Stability and Causality

LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

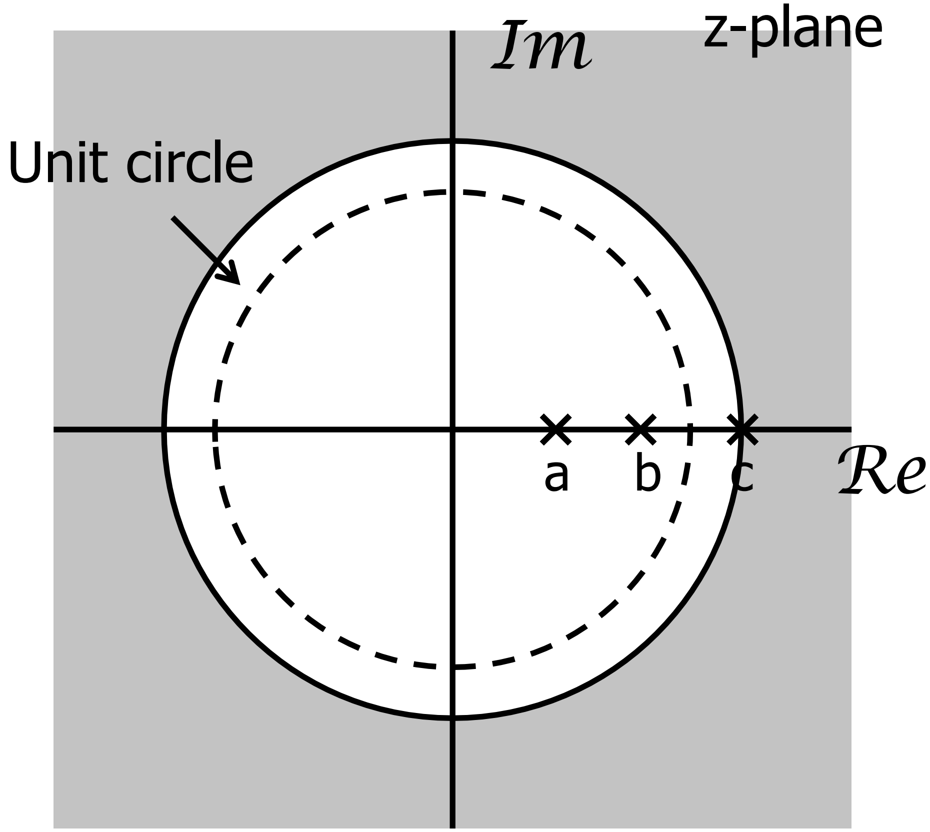
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities (i.e poles)
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

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Example: $y[n] = x[n] + 0.1y[n-1]$

If stable and causal, all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

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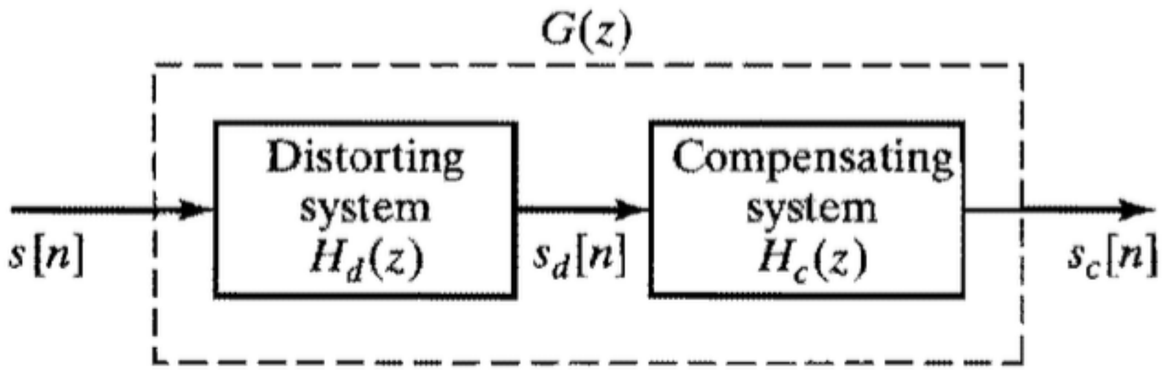
Distortion Compensation

- Have some distortion that we want to compensate for:



All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \text{ all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

First Order All-Pass Filter

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

First Order All-Pass Filter

$$H(z)=\frac{z^{-1}-a^*}{1-az^{-1}}$$

$$\left|H(e^{j\omega})\right|=\frac{\left|e^{-j\omega}-a^*\right|}{\left|1-ae^{-j\omega}\right|}$$

$$=\frac{\left|e^{-j\omega}(1-a^*e^{j\omega})\right|}{\left|1-ae^{-j\omega}\right|}$$

$$=\frac{\left|1-a^*e^{j\omega}\right|}{\left|1-ae^{-j\omega}\right|}=1$$

General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

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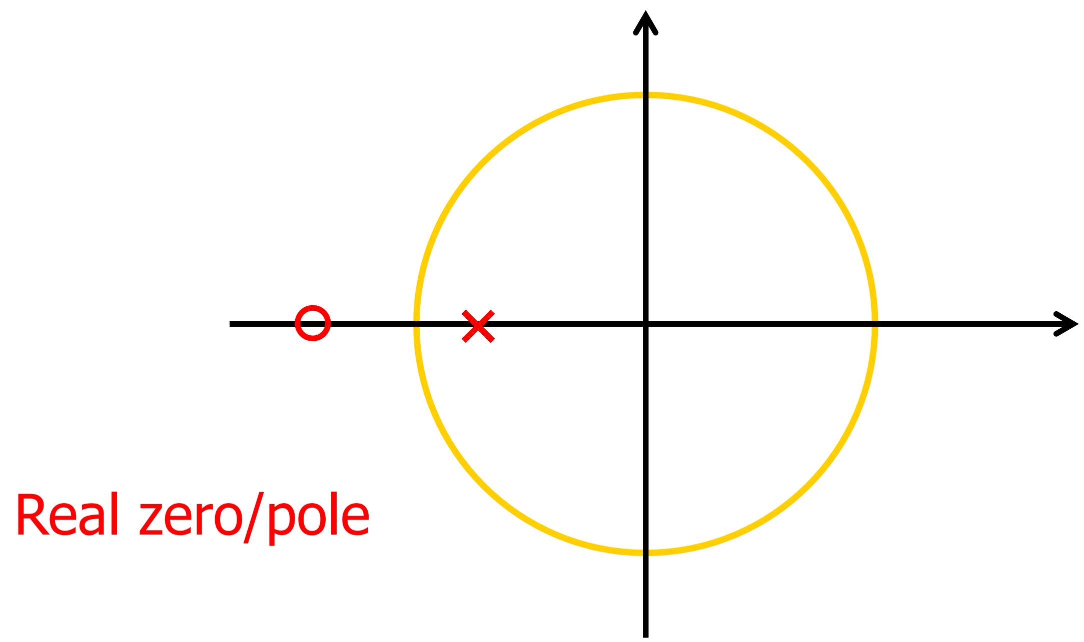

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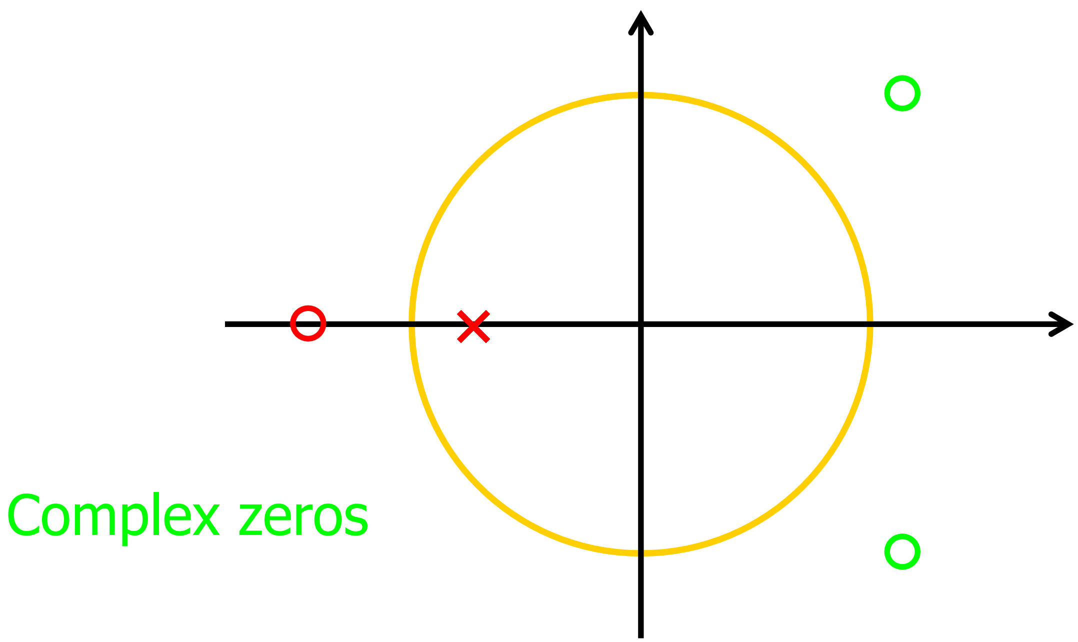

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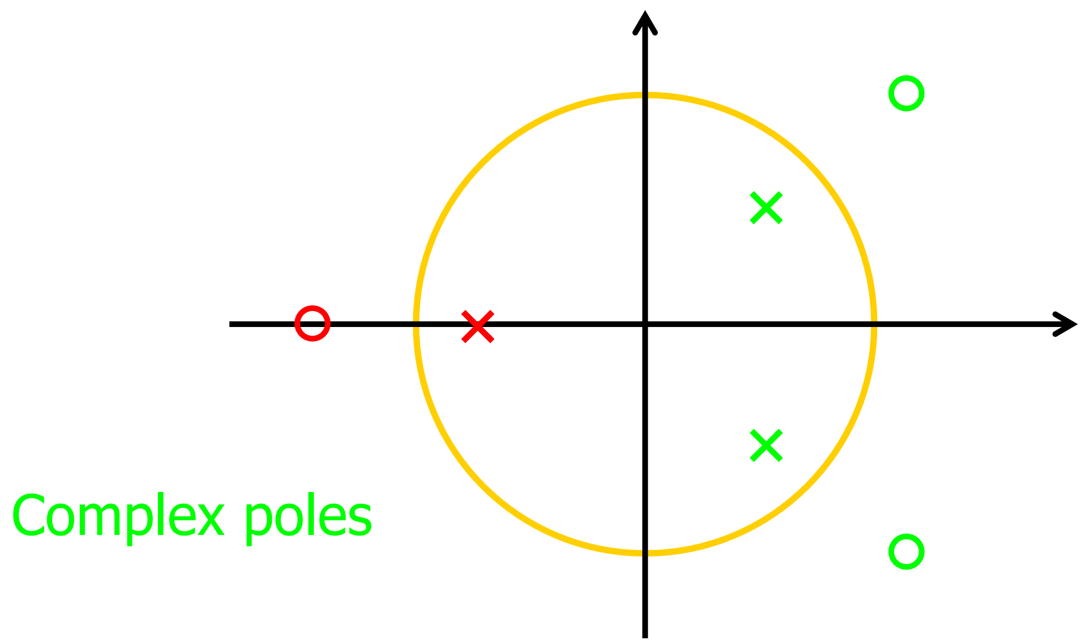

All Pass Filter Phase Response

- First order system

$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- phase

All Pass Filter Phase Response

- phase $\arg\left(\dfrac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$

All Pass Filter Phase Response

- phase
$$\arg\left(\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right)$$
$$= \arg\left(\frac{e^{-j\omega}(1 - re^{-j\theta}e^{j\omega})}{1 - re^{j\theta}e^{-j\omega}}\right)$$
$$= \arg(e^{-j\omega}) + \arg(1 - re^{-j\theta}e^{j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$
$$= -\omega - \arg(1 - re^{j\theta}e^{-j\omega}) - \arg(1 - re^{j\theta}e^{-j\omega})$$
$$= -\omega - 2\arg(1 - re^{j\theta}e^{-j\omega})$$

Group Delay Math

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- Look at each factor:

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)} \right)$$

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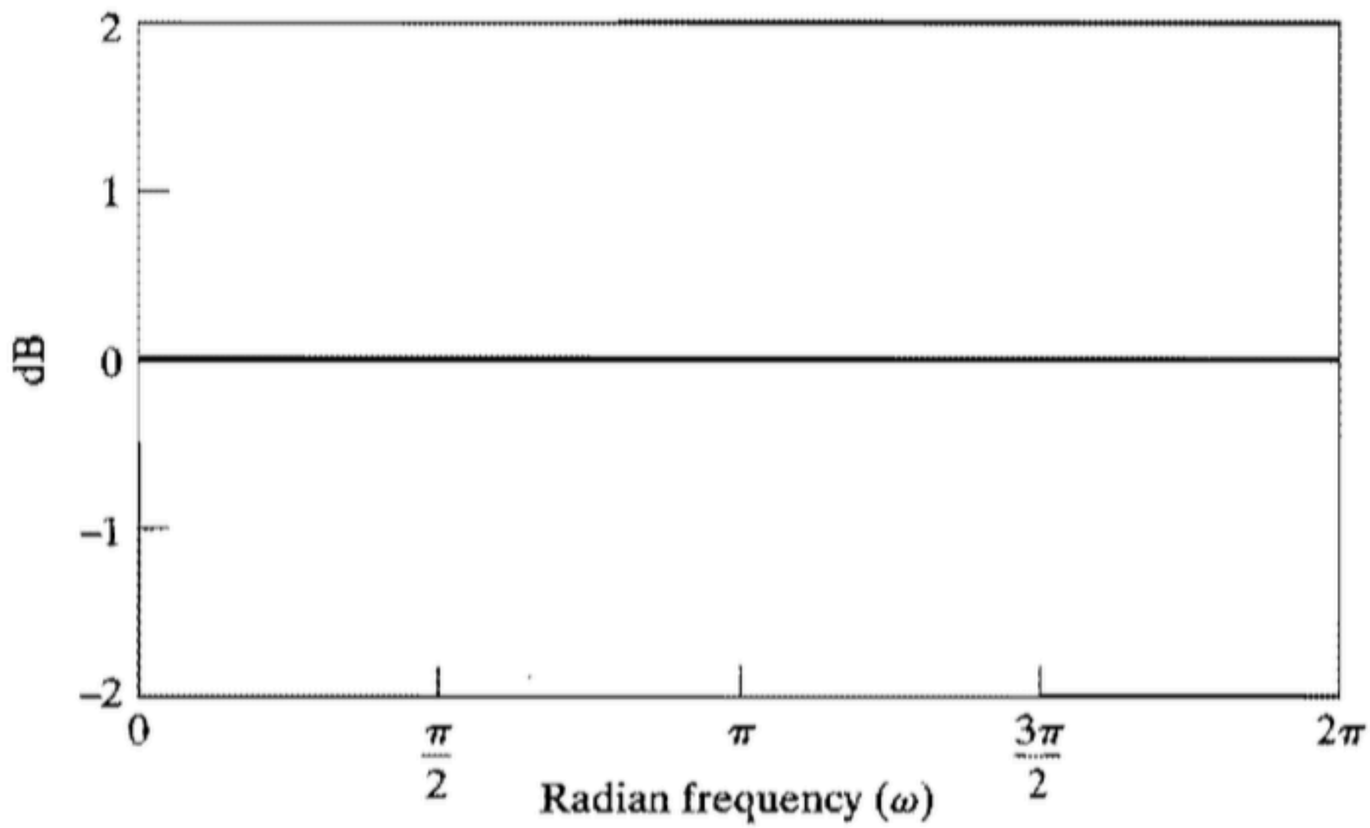
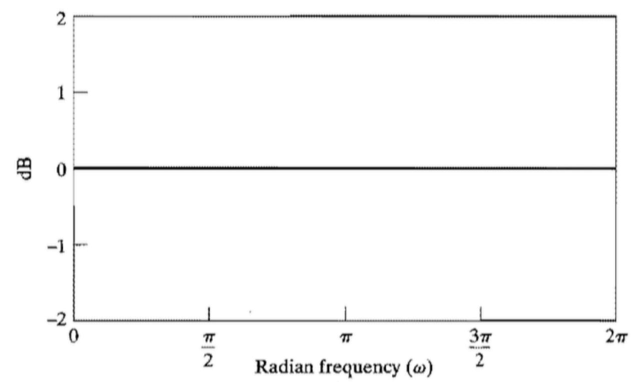
First Order Example

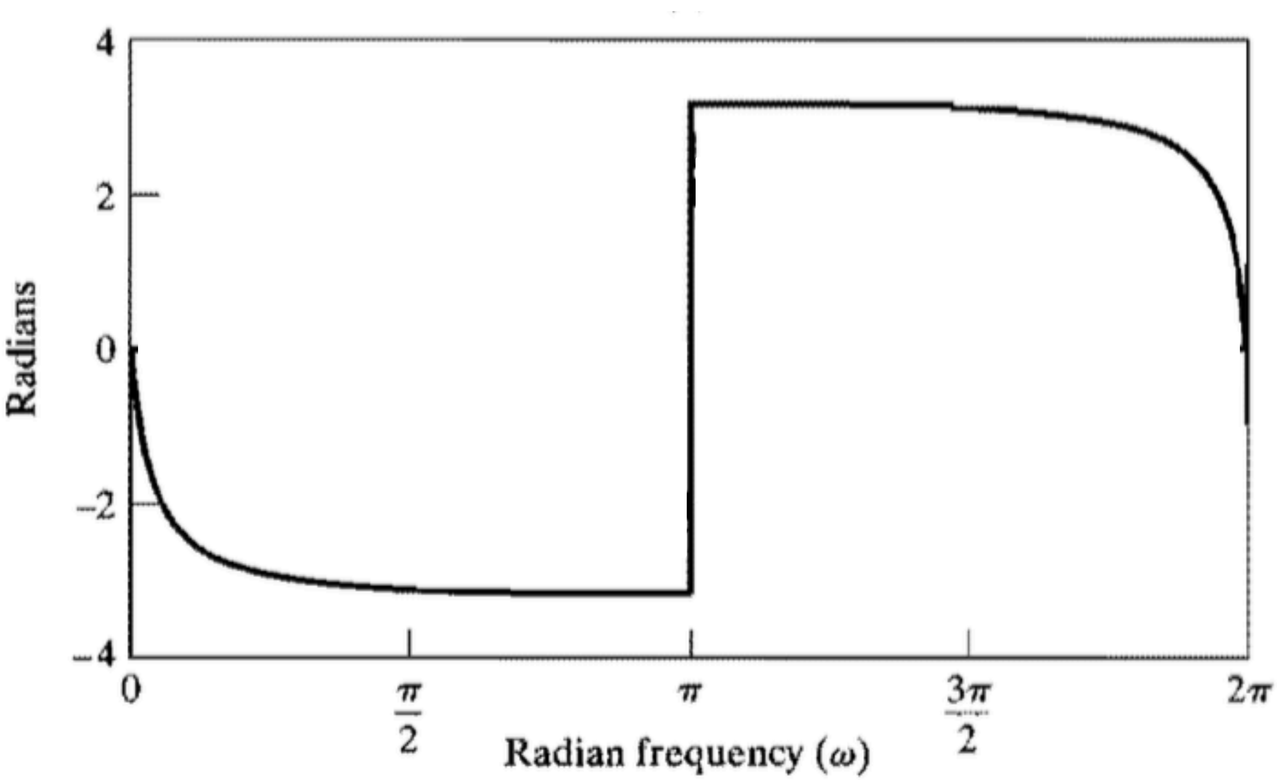
- Magnitude:



First Order Example

- Phase:



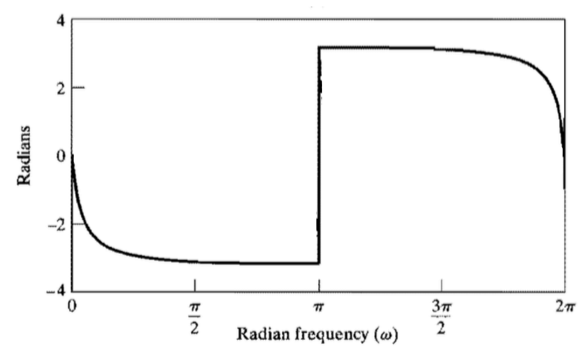


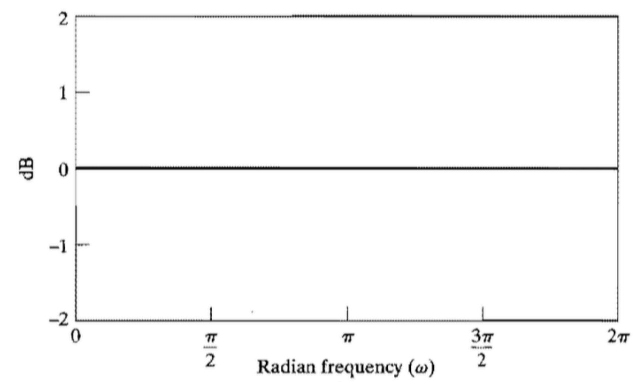
$z = 0.9$

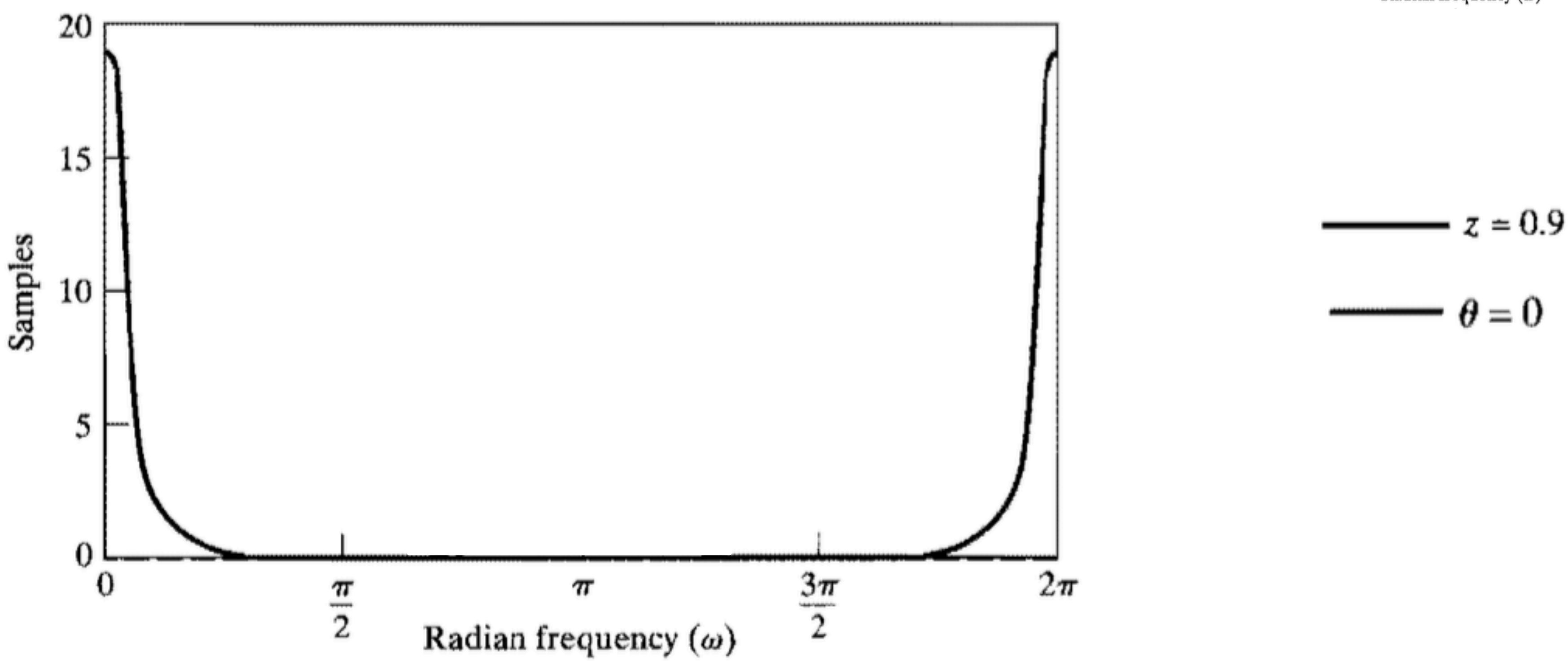
$\theta = 0$

First Order Example

- Group Delay:







All Pass Filter Phase Response

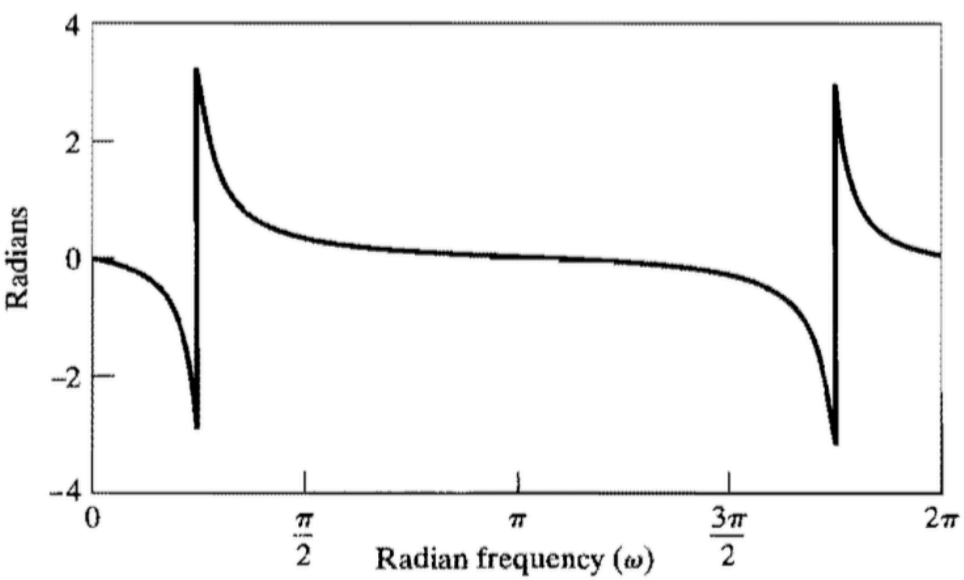
- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

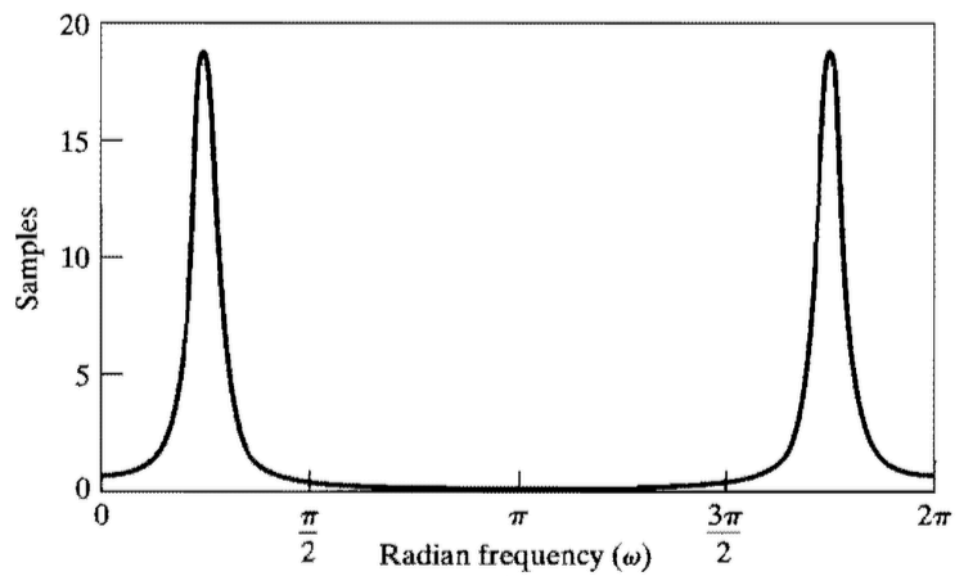
$$\angle\left[\frac{(e^{-j\omega}-re^{-j\theta})(e^{-j\omega}-re^{j\theta})}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

$$-2\arctan\left[\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right].$$

Second Order Example

- Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)





All-pass Example

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- Example:

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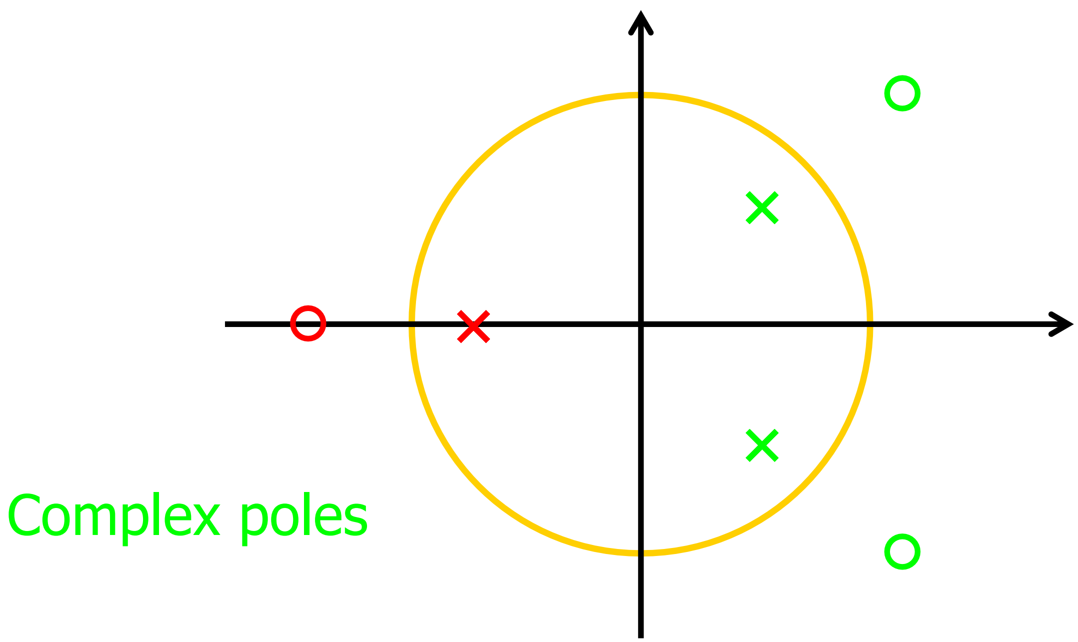
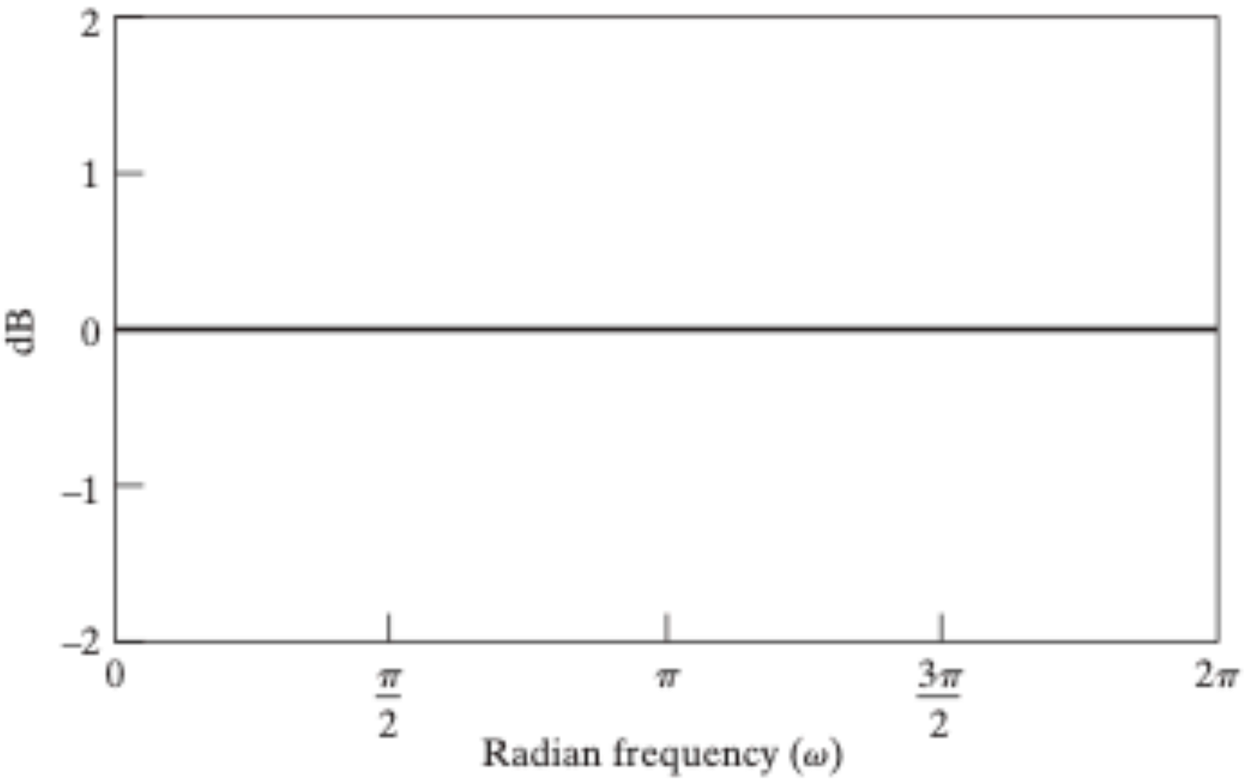
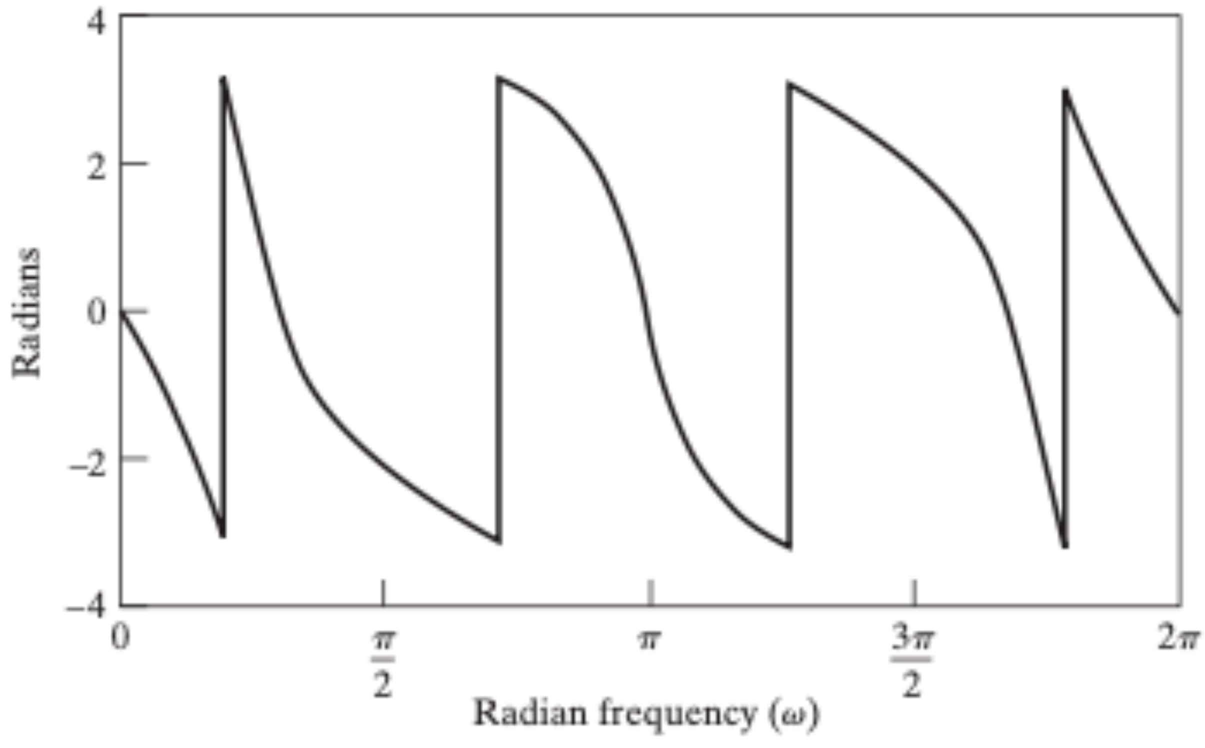


All-pass Example: Magnitude



All-pass Example: Phase



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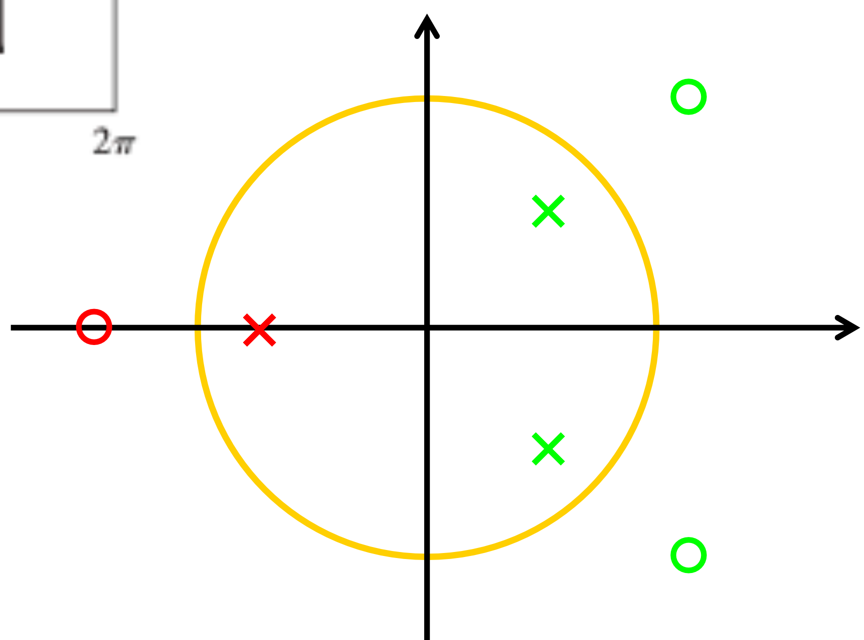
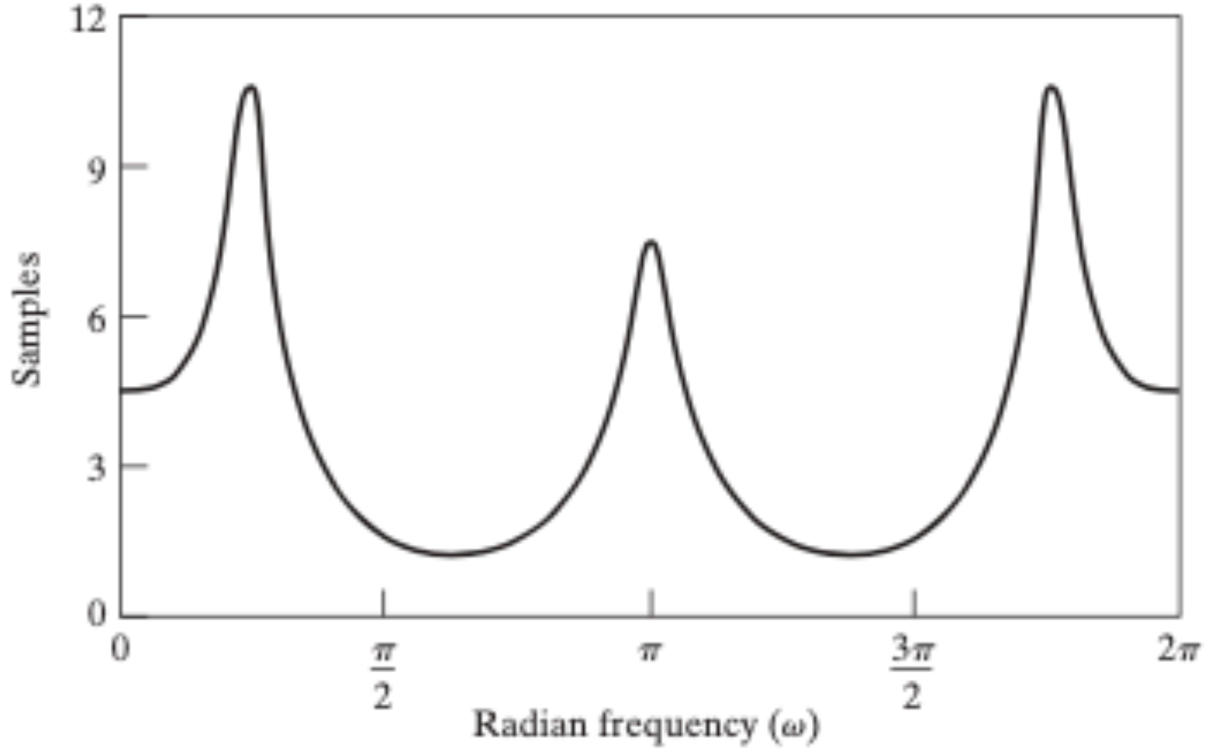


All-pass Example: Group Delay



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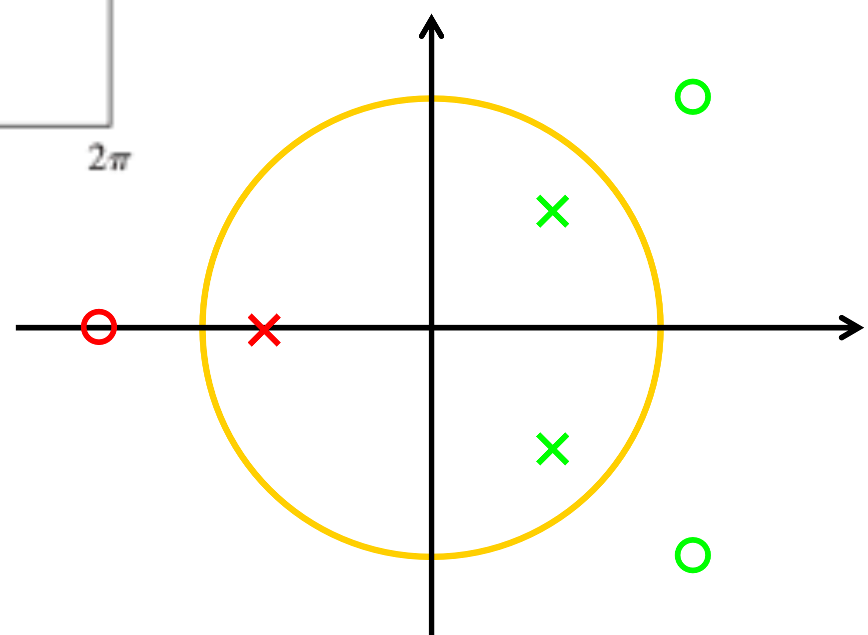


All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

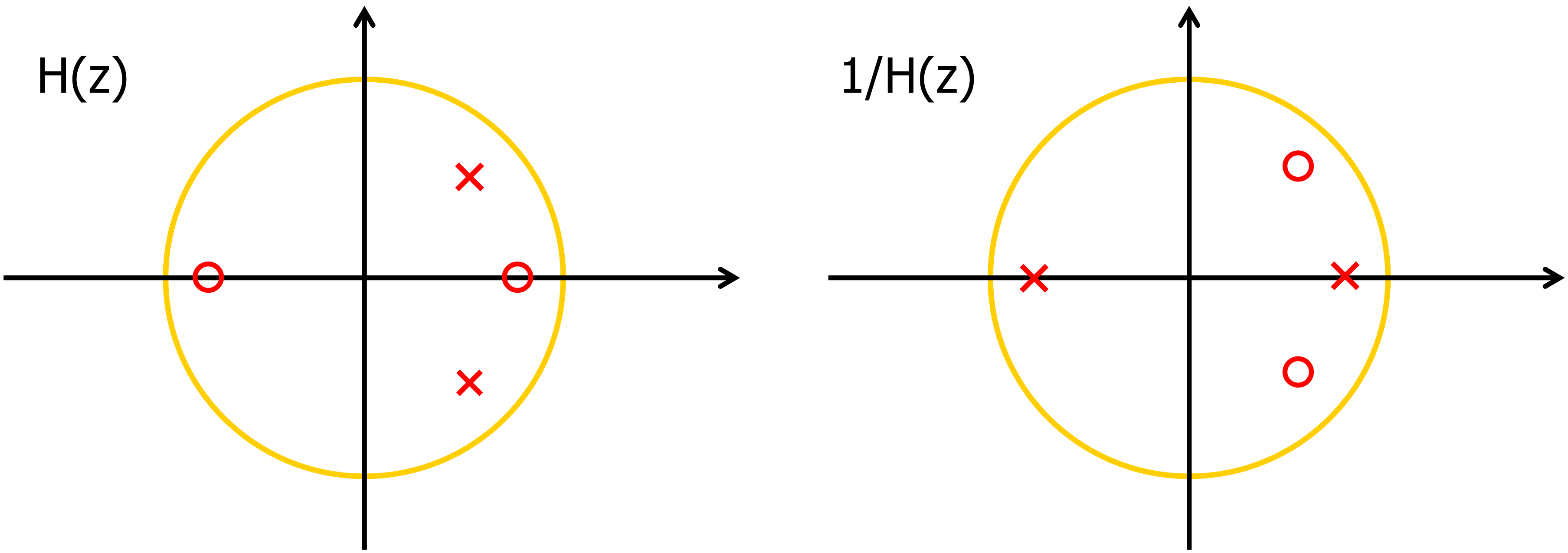
Minimum Phase Systems

Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)

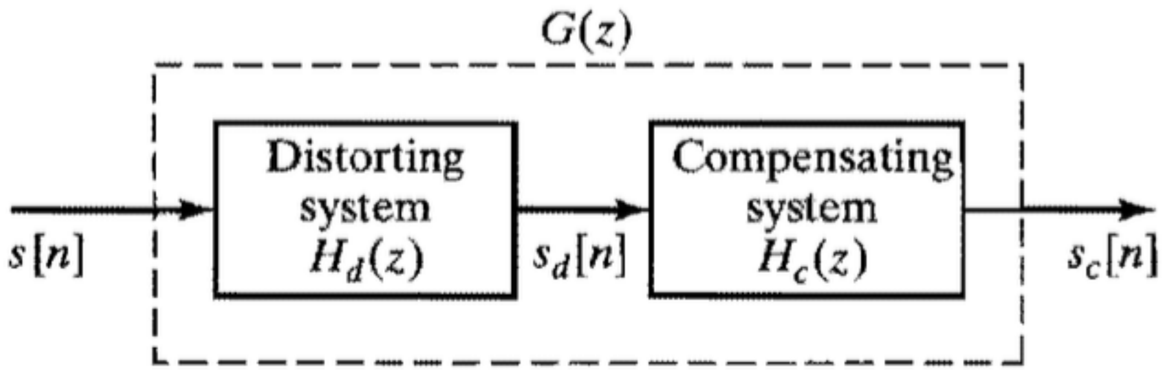
Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



Distortion Compensation

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- What if not min phase?

All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

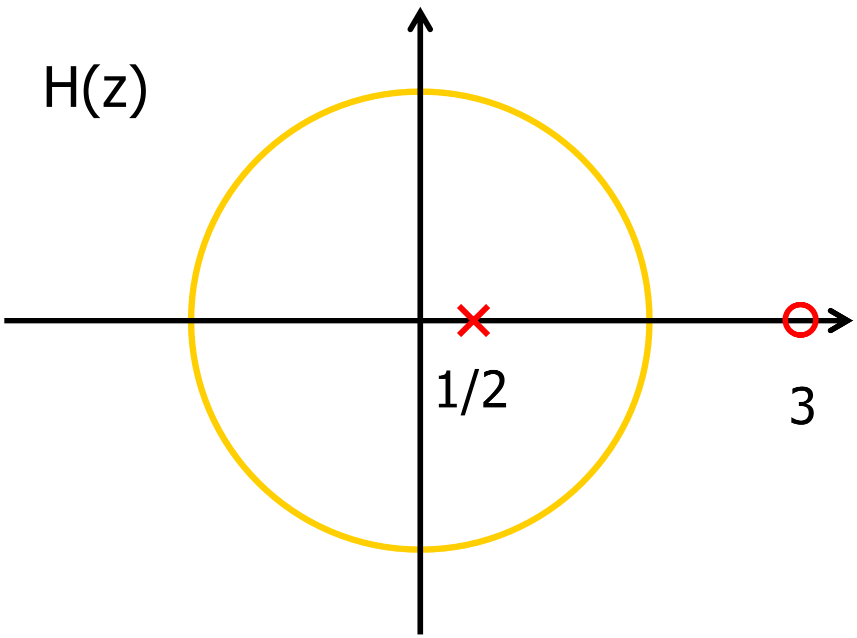
$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
  - (1) First construct H_{ap} with all zeros outside unit circle
  - (2) Compute

$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$



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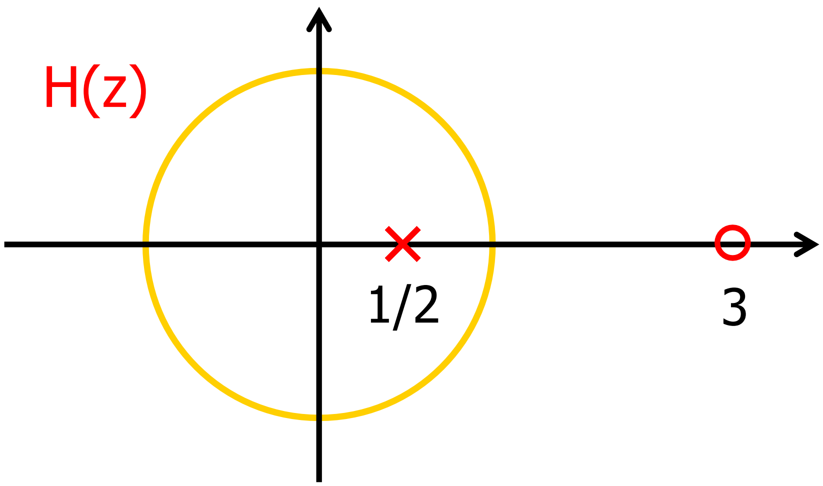
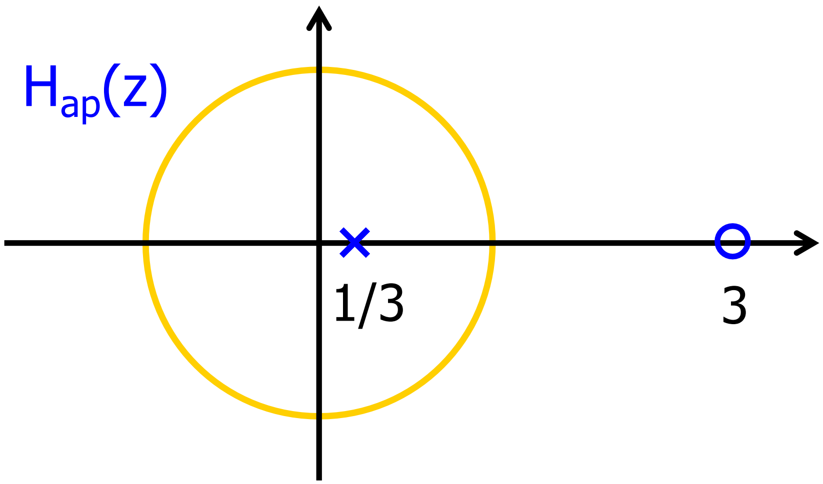


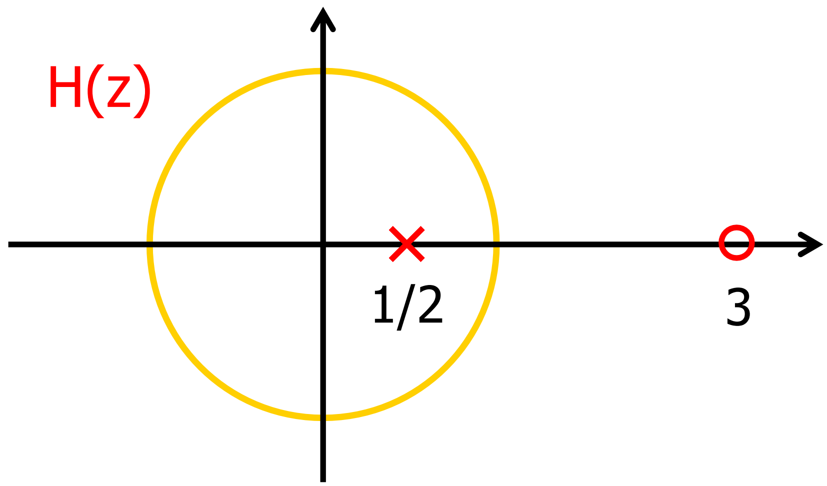
- Set $H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$

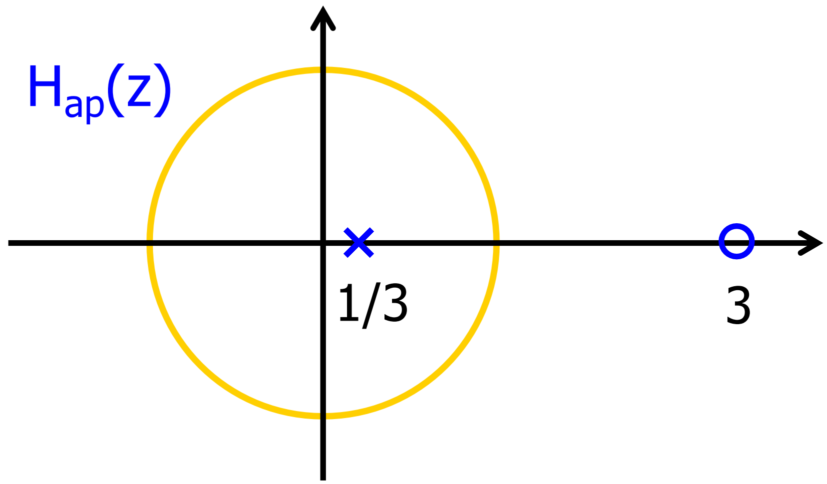


Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z) = \dfrac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$



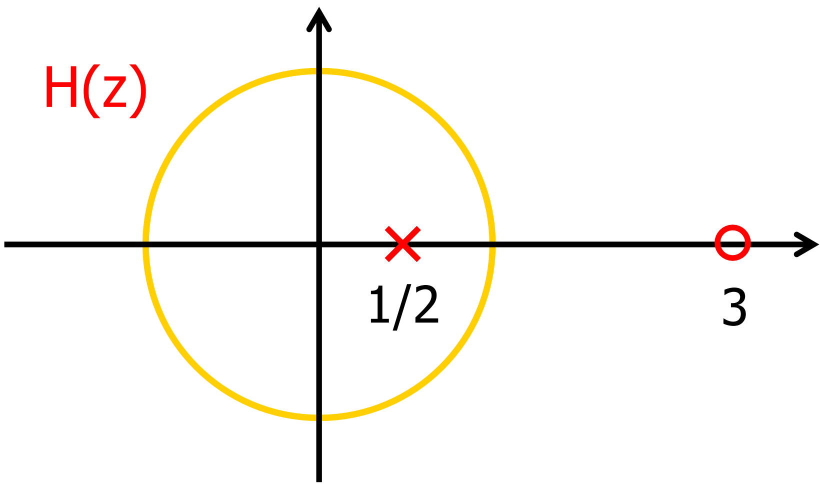


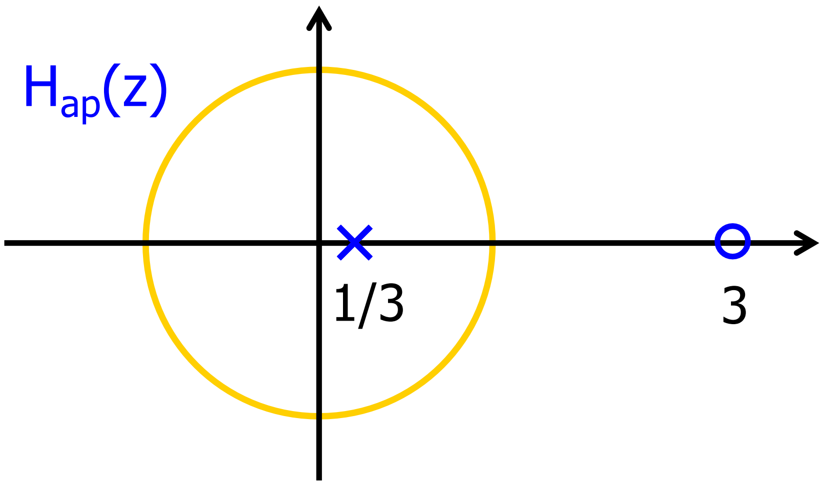
Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z) = \dfrac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$

$$H_{\min}(z) = -3\frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$





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- Set $H_{ap}(z) = \dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

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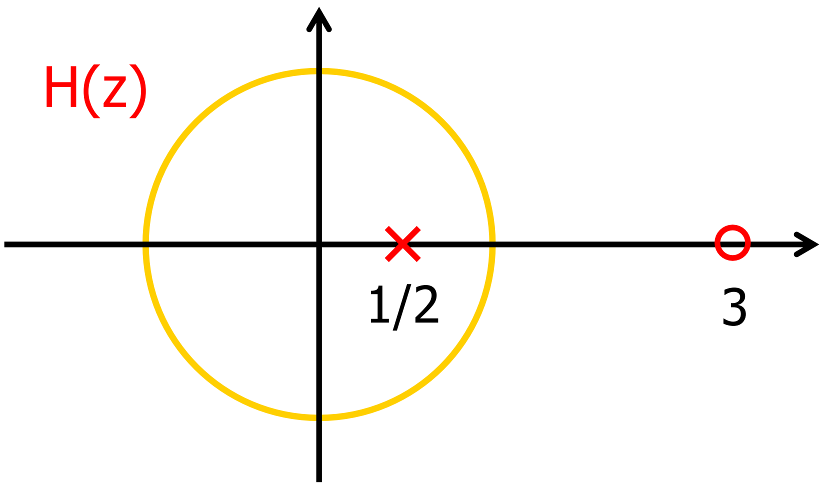
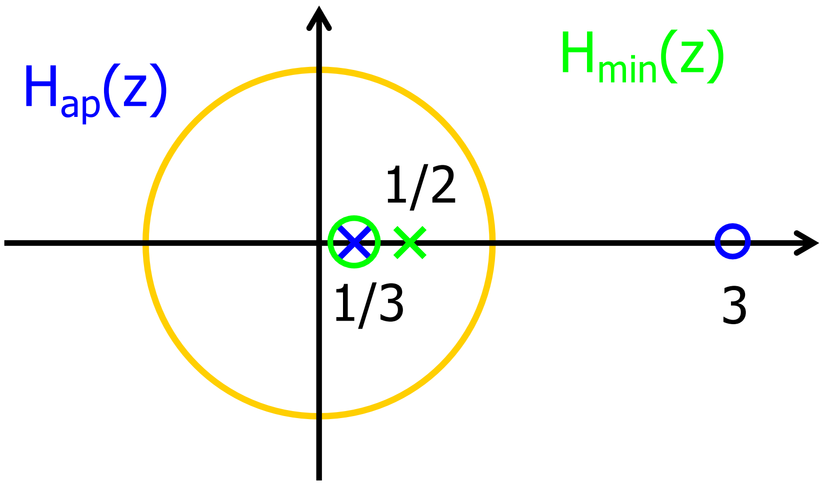
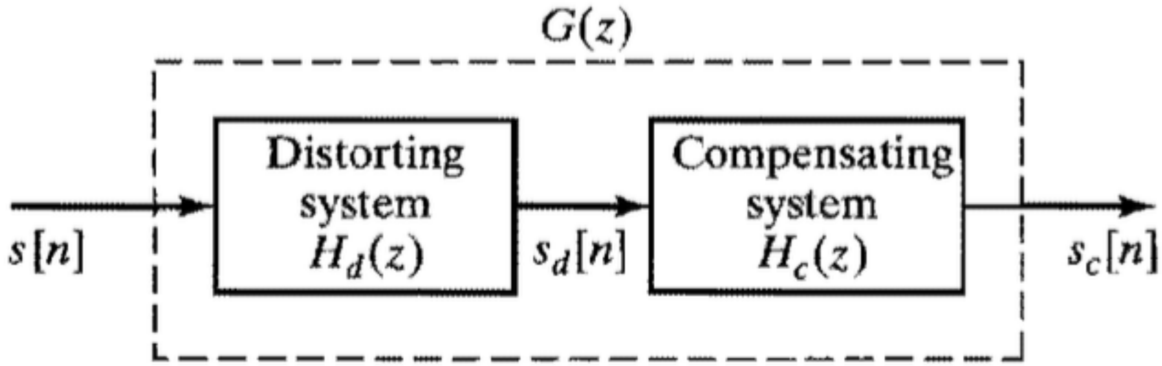




Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal

Min-Phase Decomposition Purpose

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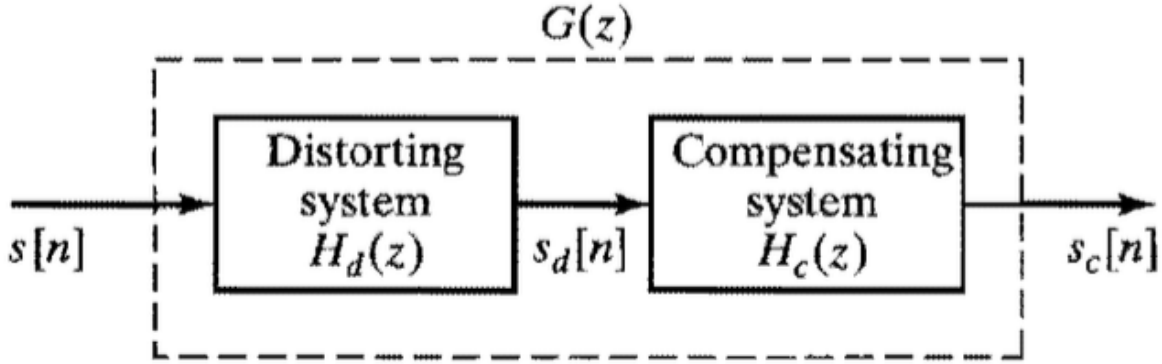
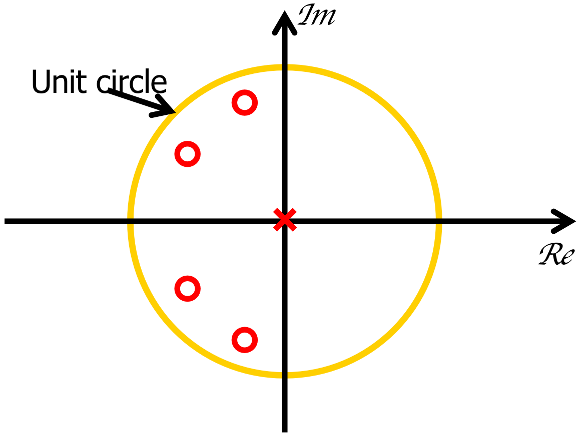


- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

Minimum Phase to Max Phase

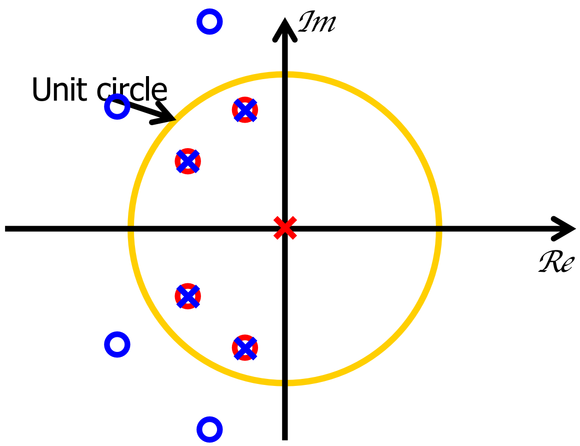
Min phase



Minimum Phase to Max Phase

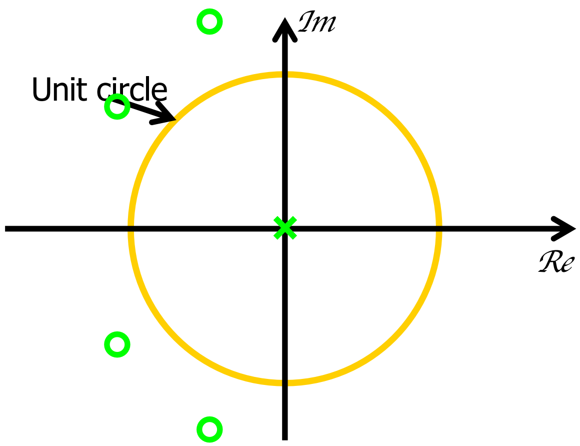
Min phase

x All Pass



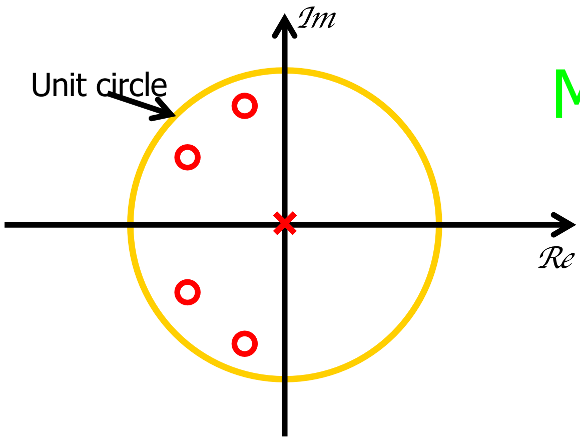
Minimum Phase to Max Phase

Max phase

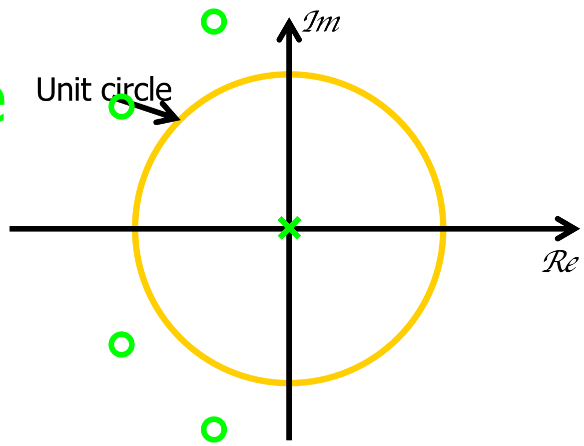


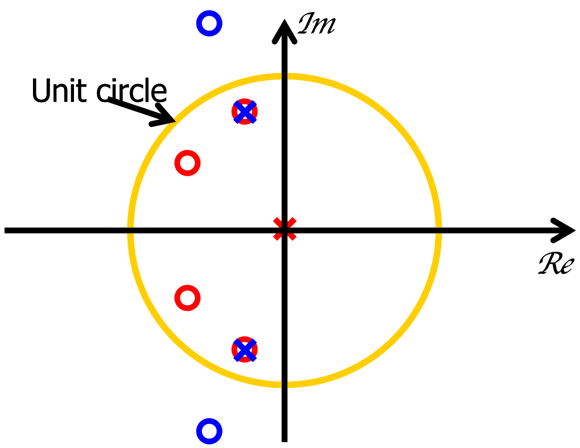
Minimum Phase

Min phase



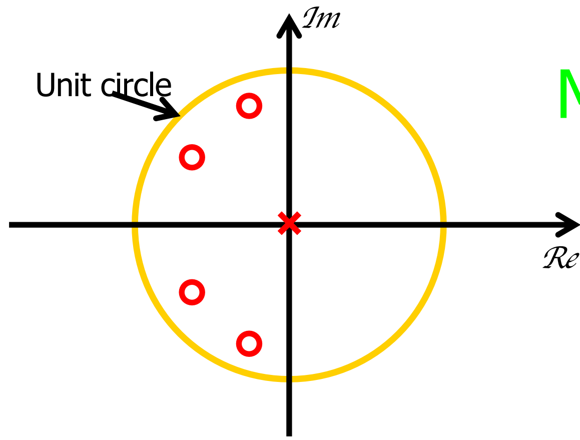
Max phase



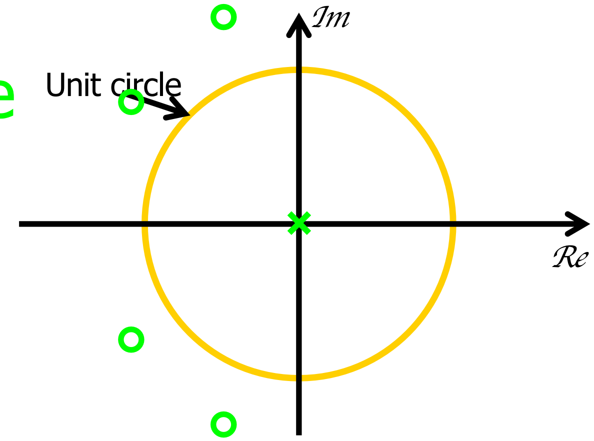


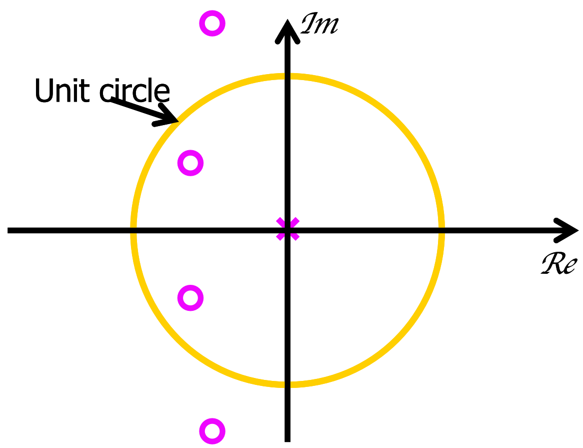
Minimum Phase

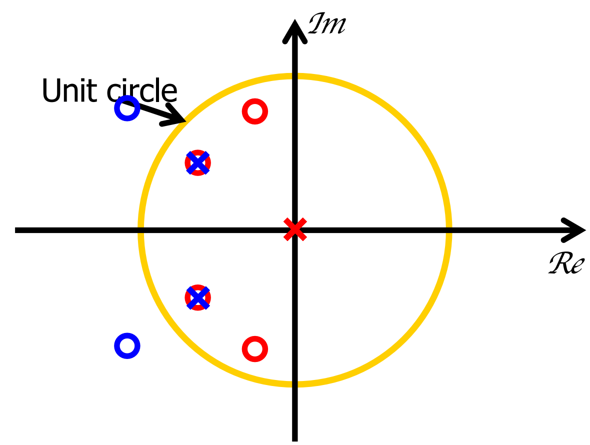
Min phase



Max phase

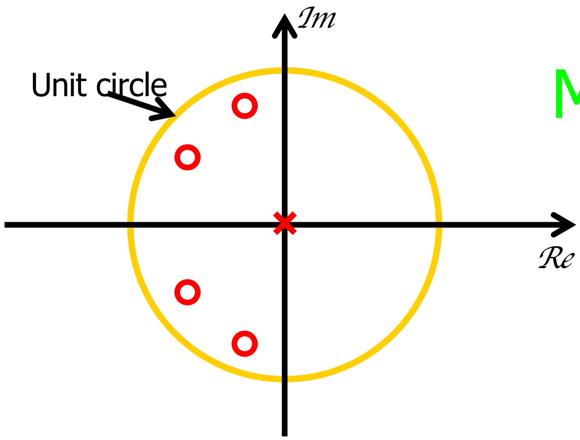




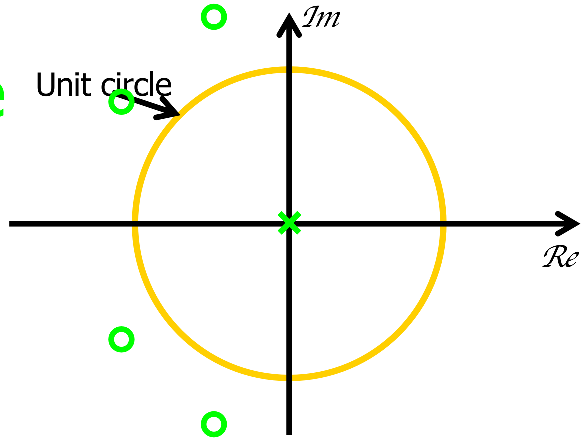


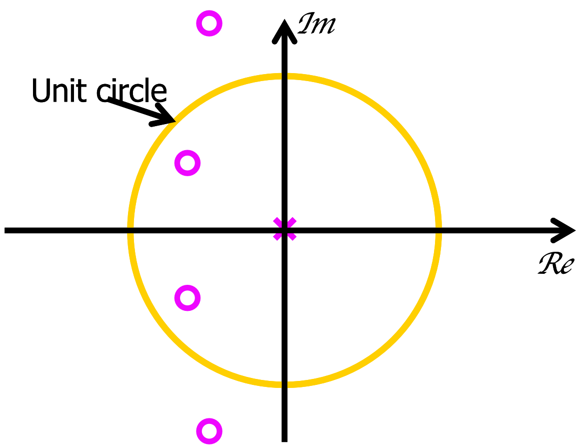
Minimum Phase

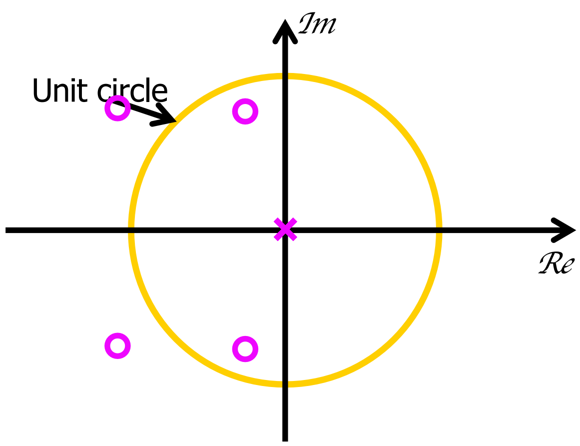
Min phase



Max phase







Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$

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  - $\arg[H_{ap}(e^{j\omega})] \le 0$
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- $\arg[H_{max}(e^{j\omega})] = \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

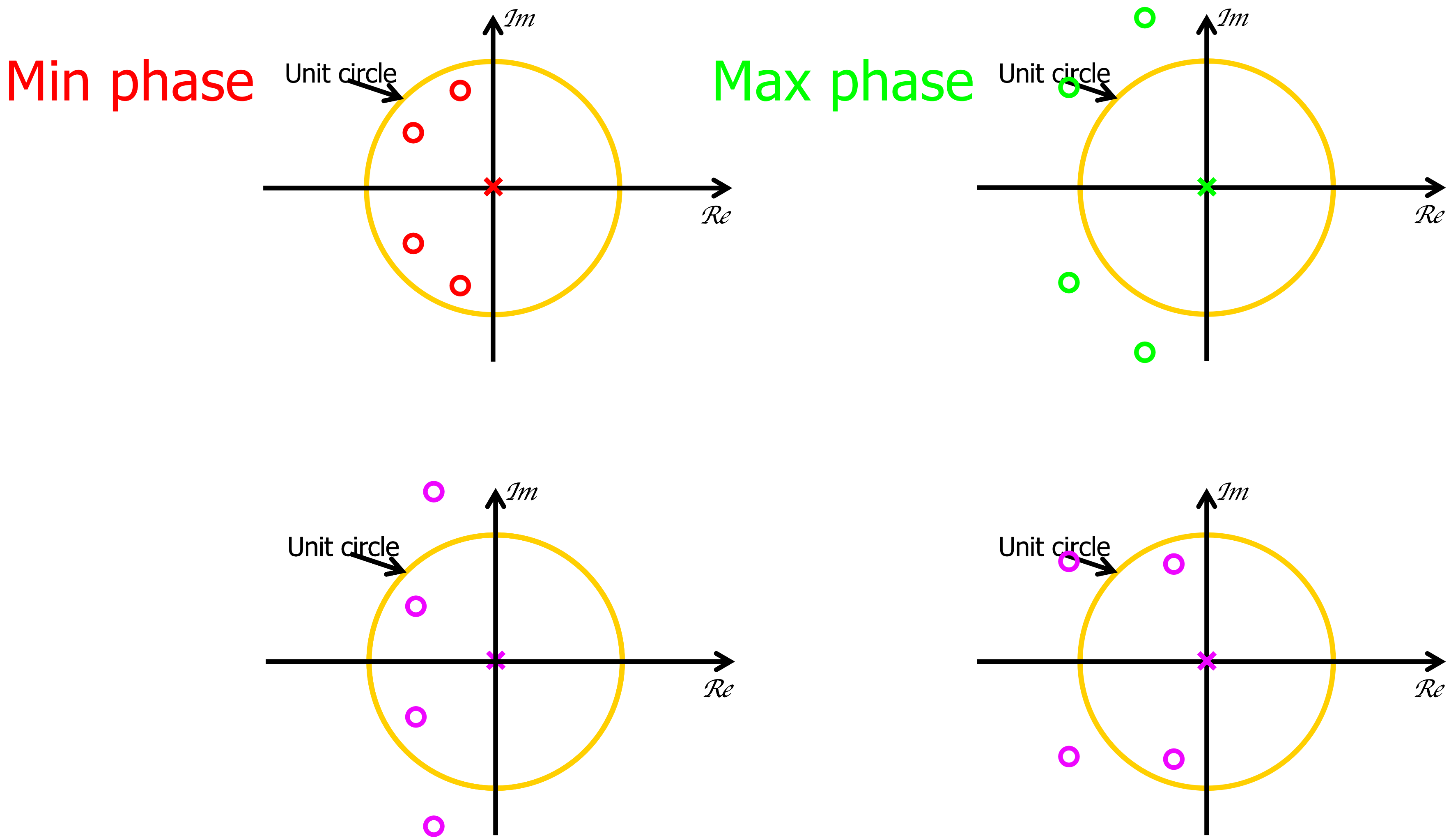
$$= \arg[H_{min}(e^{j\omega})] + \quad \le 0$$

Minimum Group Delay Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

- $$\begin{aligned}\text{grd}[H_{max}(e^{j\omega})] &= \text{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})] \\ &= \text{grd}[H_{min}(e^{j\omega})] + \text{grd}[H_{ap}(e^{j\omega})] \\ &= \text{grd}[H_{min}(e^{j\omega})] + \quad \geq 0\end{aligned}$$

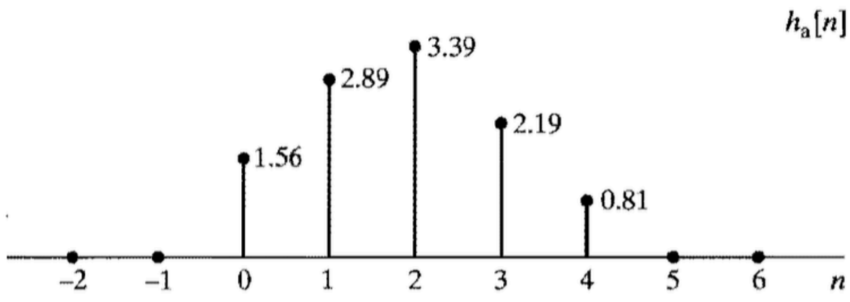
Minimum Phase



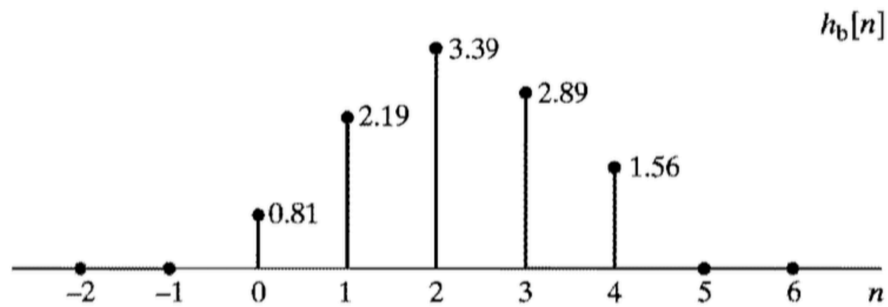
Minimum Energy-Delay Property

Min phase

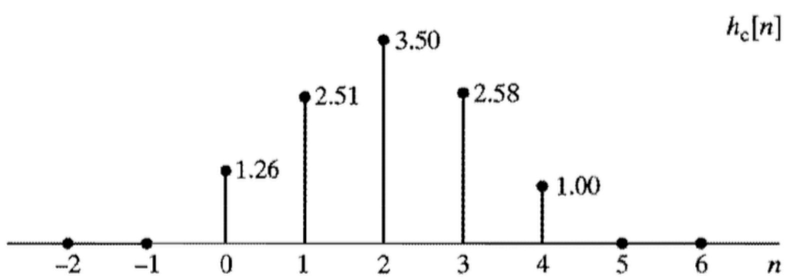
Max phase



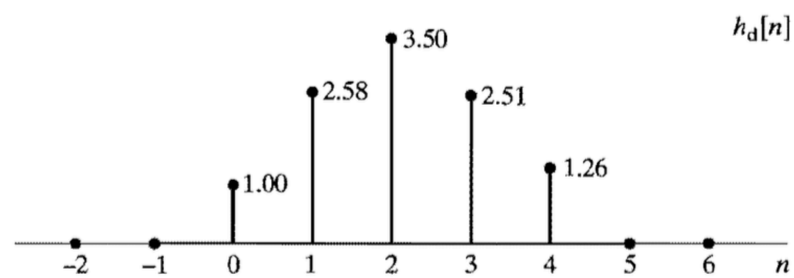
(a)



(b)



(c)



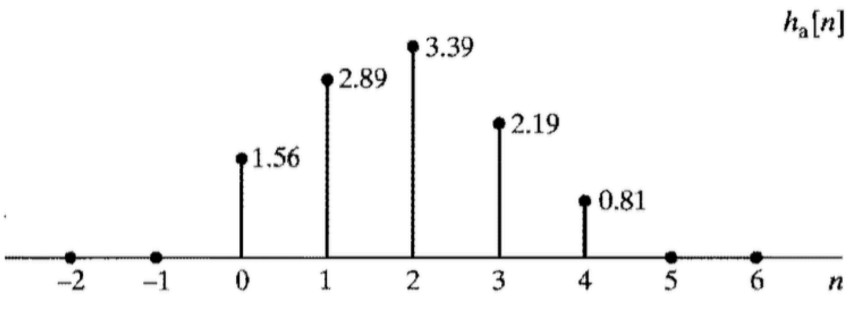
(d)

Total energy: $E = \sum_{n=0}^{\infty} |h[n]|^2$

Minimum Energy-Delay Property

Min phase

Max phase



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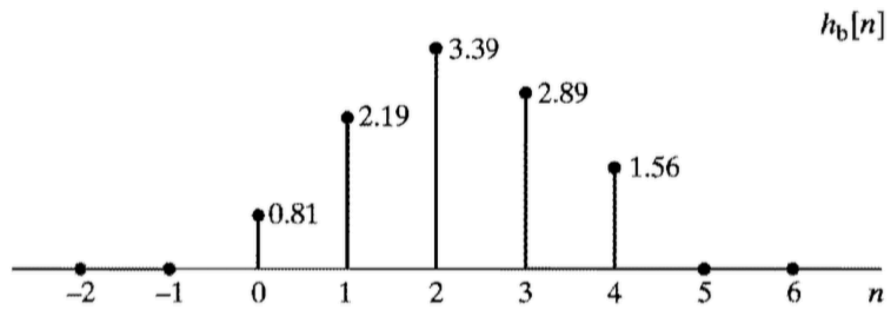

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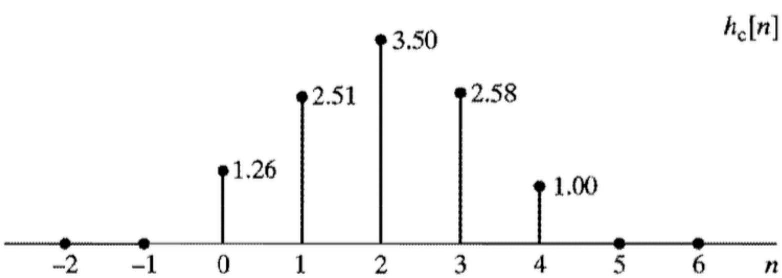

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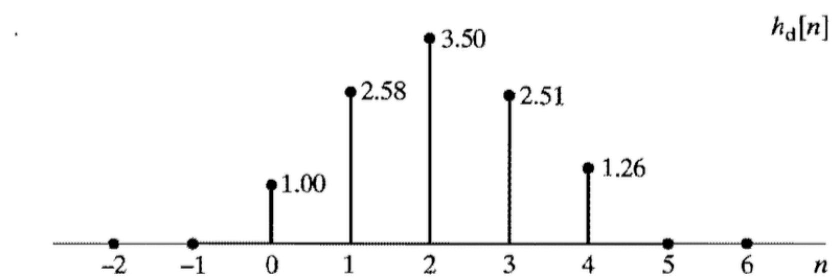
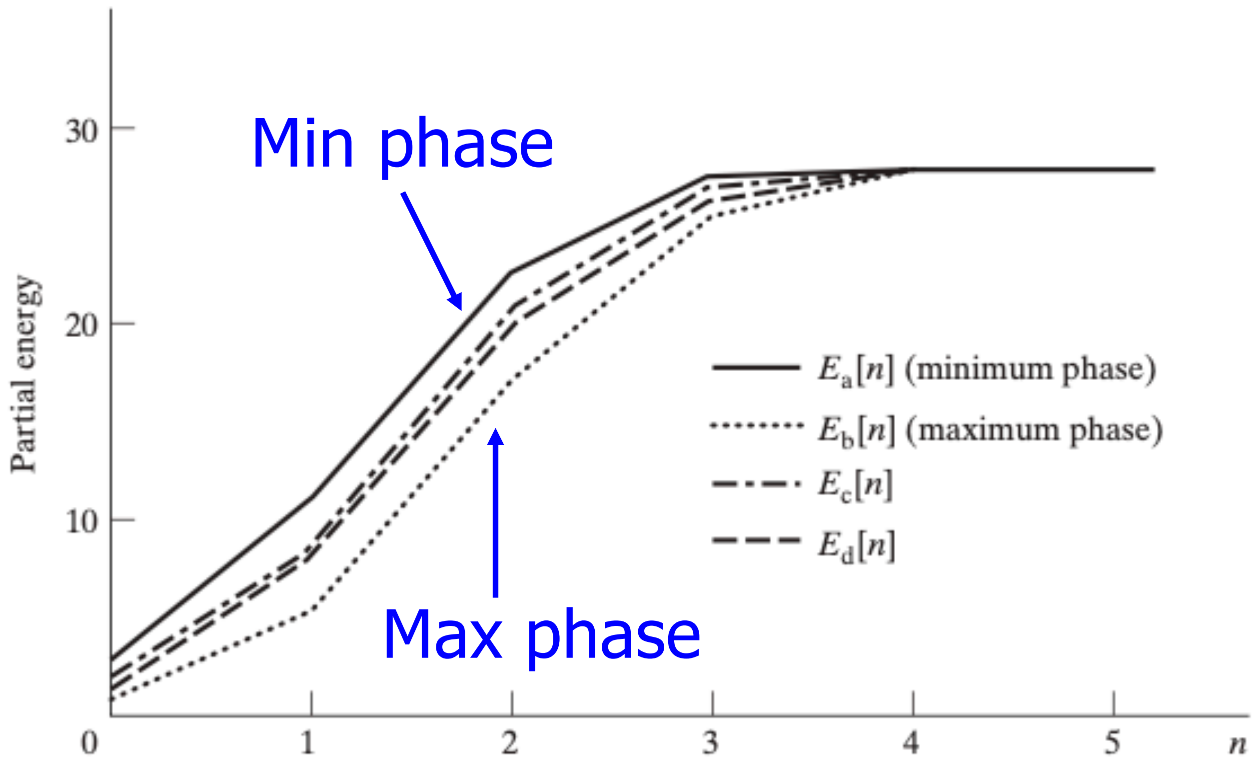


(d)

Partial energy: $E[n] = \sum_{m=0}^{n} |h[m]|^2$

Minimum Energy-Delay Property



Partial energy: $E[n] = \sum_{m=0}^{n} |h[m]|^2$

Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- Review: LTI Stability and Causality
  - If all poles inside unit circle
- All Pass Systems
  - Used for delay compensation
- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property

Admin

- HW 6 out
  - Due Sunday 3/12
  - After spring break, same length as 1-week homework