

ESE 5310: Digital Signal Processing

Lecture 19: March 28, 2023

Discrete Fourier Transform



Today

- ❑ Discrete Fourier Series
- ❑ Discrete Fourier Transform (DFT)
- ❑ DFT Properties
- ❑ Circular Convolution

Discrete Fourier Series

Reminder: Eigenvalue (DTFT)

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
 - Re and Im
 - Mag and Phase



DTFT

DTFT:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$



DTFT

DTFT:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

- If $x[n]$ is periodic, then finite set of frequencies in the finite range
 - Can represent with discrete Fourier series



Discrete Fourier Series

□ Definition:

- Consider N-periodic signal:

$$\tilde{x}[n + N] = \tilde{x}[n] \quad \forall n$$

- Frequency-domain also periodic in N:

$$\tilde{X}[k + N] = \tilde{X}[k] \quad \forall k$$

- “~” indicates periodic signal/spectrum



Discrete Fourier Series

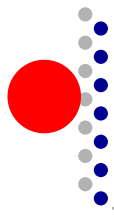
□ Define:

$$W_N \triangleq e^{-j2\pi/N}$$

□ DFS:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$



Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

□ Properties of W_N :

- $W_N^0 = W_N^N = W_N^{2N} = \dots = 1$
- $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$



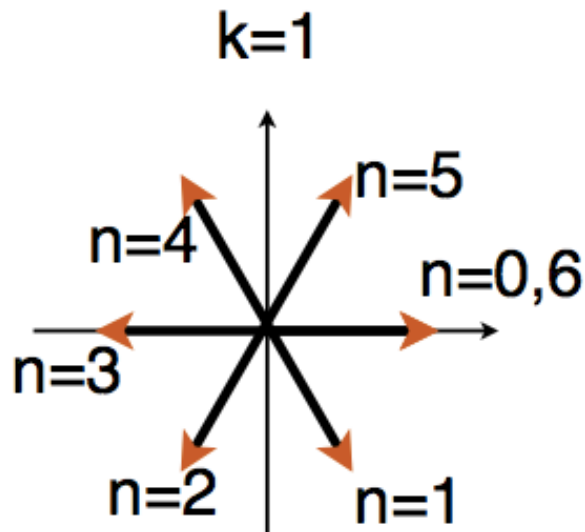
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□ Example: W_N^{kn} ($N=6$)



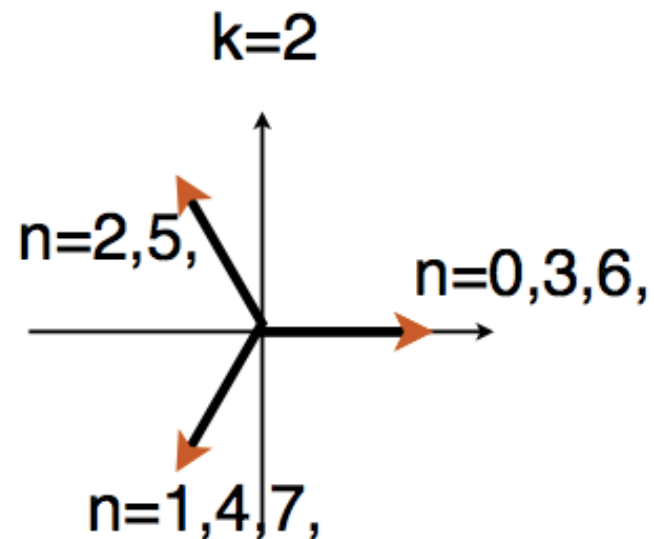
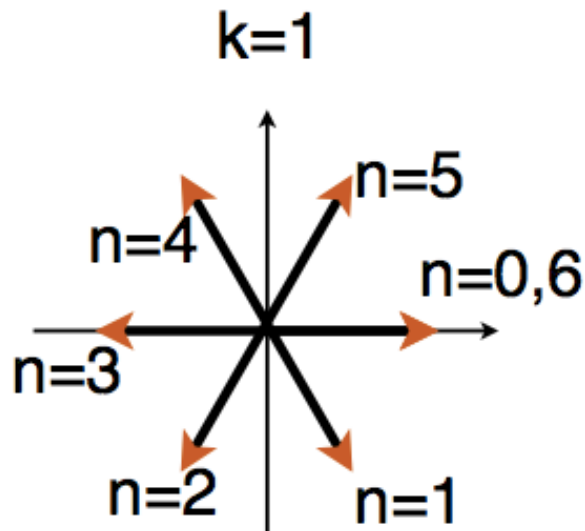
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□ Example: W_N^{kn} ($N=6$)





Discrete Fourier Transform

- By convention, work with one period:

$$x[n] \triangleq \begin{cases} \tilde{x}[n] & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] \triangleq \begin{cases} \tilde{X}[k] & 0 \leq k \leq N - 1 \\ 0 & \text{otherwise} \end{cases}$$

Same, but different!

Discrete Fourier Transform

□ The DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad \text{DFT, analysis}$$

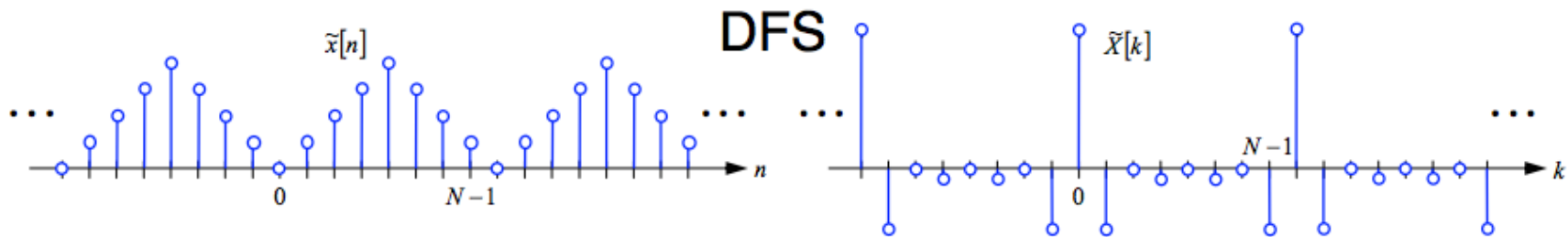
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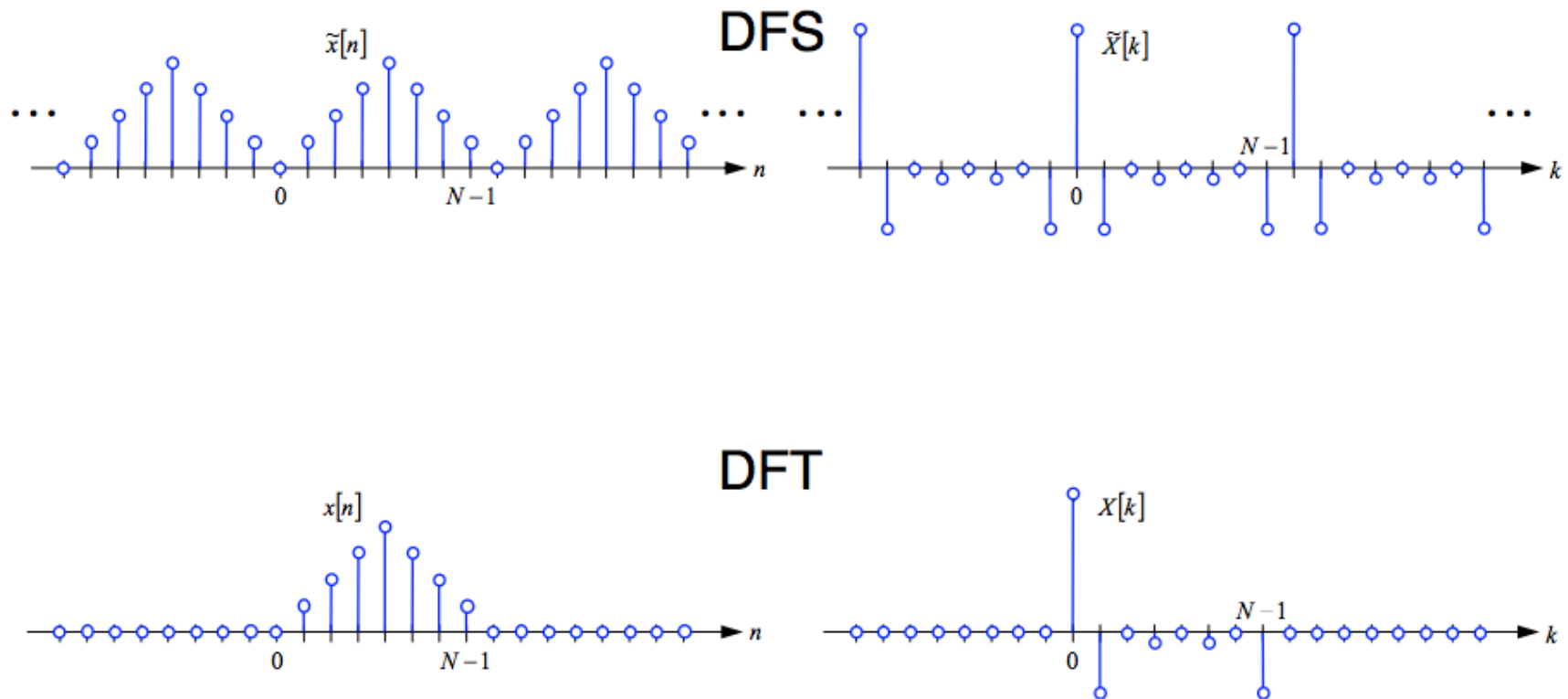


DFS vs. DFT





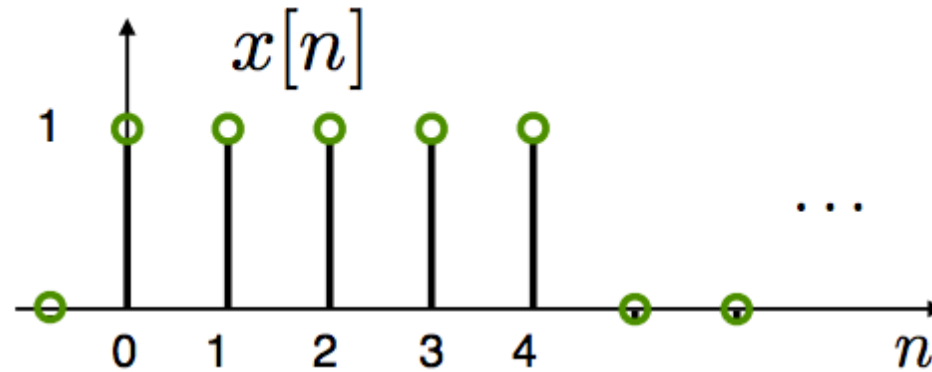
DFS vs. DFT





Example

$$W_N \triangleq e^{-j2\pi/N}$$



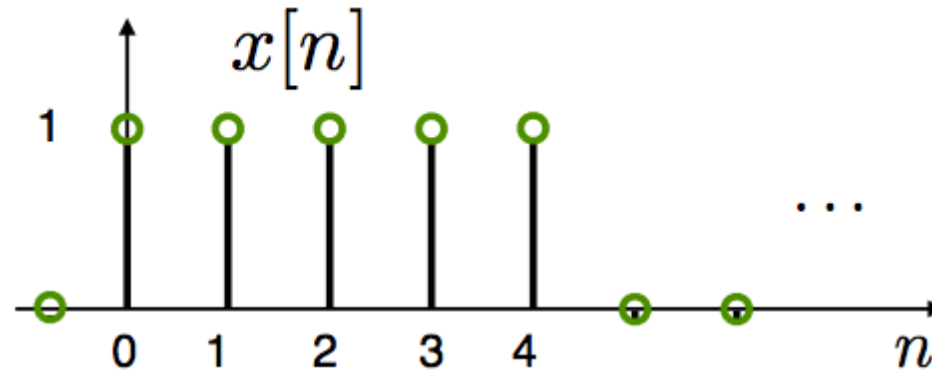
Take $N=5$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



Example

$$W_N \triangleq e^{-j2\pi/N}$$



Take $N=5$

$$X[k] = \begin{cases} \sum_{n=0}^4 W_5^{nk} & k = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$
$$= 5\delta[k]$$

“5-point DFT”

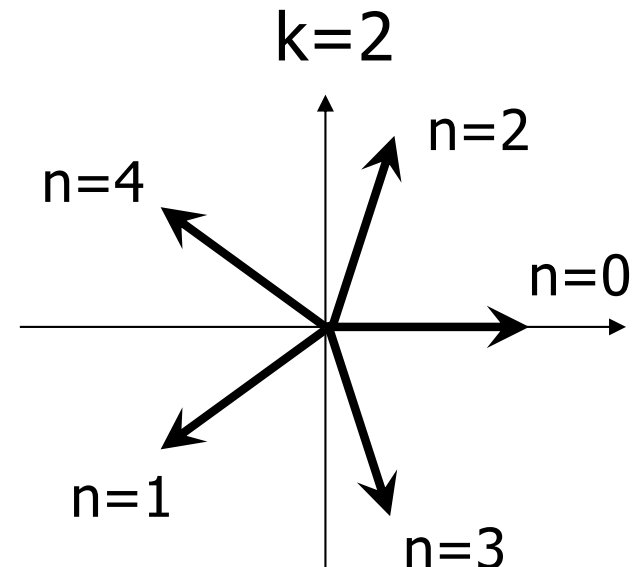
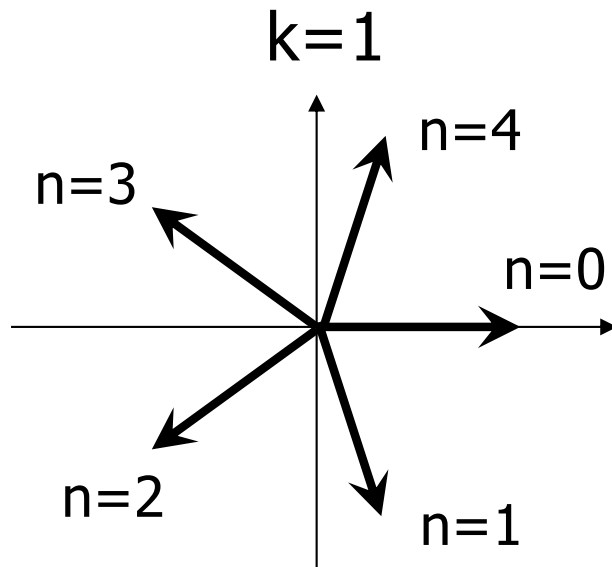
Discrete Fourier Series

$$W_N \triangleq e^{-j2\pi/N}$$

□ Properties of W_N :

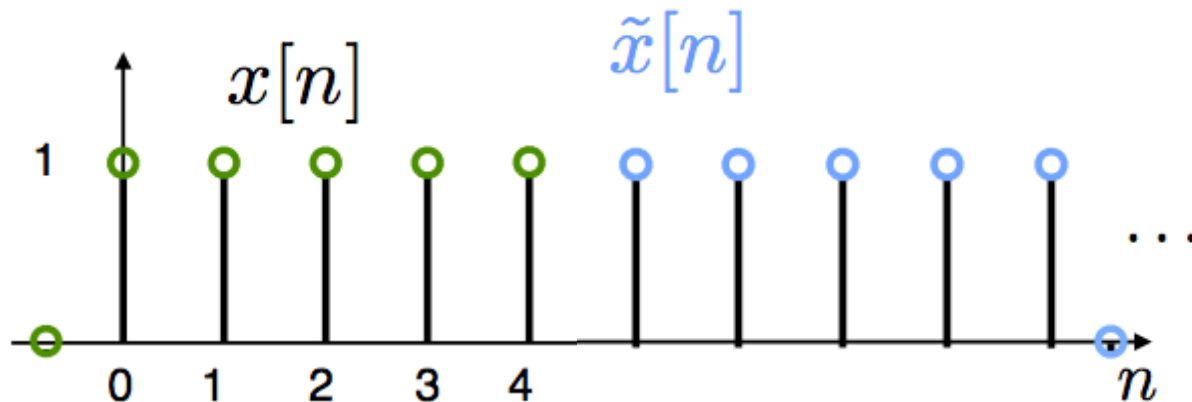
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- $W_N^{k+r} = W_N^k W_N^r$ and, $W_N^{k+N} = W_N^k$

□ Example: W_N^{kn} ($N=5$)



Example

$$W_N \triangleq e^{-j2\pi/N}$$



Take $N=5$

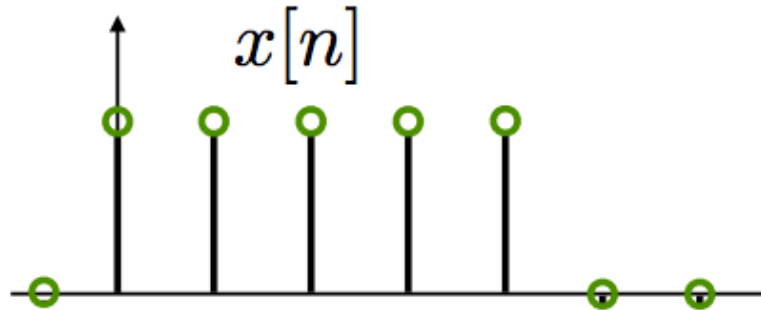
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“5-point DFT”

Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take $N=10$?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



Example

$$W_N \triangleq e^{-j2\pi/N}$$

- Q: What if we take $N=10$?
- A: $X[k] = \tilde{X}[k]$ where $\tilde{x}[n]$ is a period-10 seq.



$$X[k] = \begin{cases} \sum_{n=0}^9 W_{10}^{nk} & k = 0, 1, 2, \dots, 9 \\ 0 & \text{otherwise} \end{cases}$$

“10-point DFT”



Example

□ Now, sum from $n=0$ to 9

$$X[k] = \sum_{n=0}^9 x[n] W_{10}^{nk}$$



Example

- Now, sum from $n=0$ to 9

$$\begin{aligned} X[k] &= \sum_{n=0}^9 x[n] W_{10}^{nk} \\ &= \sum_{n=0}^4 W_{10}^{nk} \\ &= e^{-j \frac{4\pi}{10} k} \frac{\sin(\frac{\pi}{2} k)}{\sin(\frac{\pi}{10} k)} \end{aligned}$$



DFT vs. DTFT

□ For finite sequences of length N :

■ The N -point DFT of $x[n]$ is:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} \quad 0 \leq k \leq N-1$$

■ The DTFT of $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \quad -\infty < \omega < \infty$$



DFT vs. DTFT

- The DFT are samples of the DTFT at N equally spaced frequencies

$$X[k] = X(e^{j\omega})|_{\omega=k\frac{2\pi}{N}} \quad 0 \leq k \leq N - 1$$



DFT vs DTFT

□ Back to example

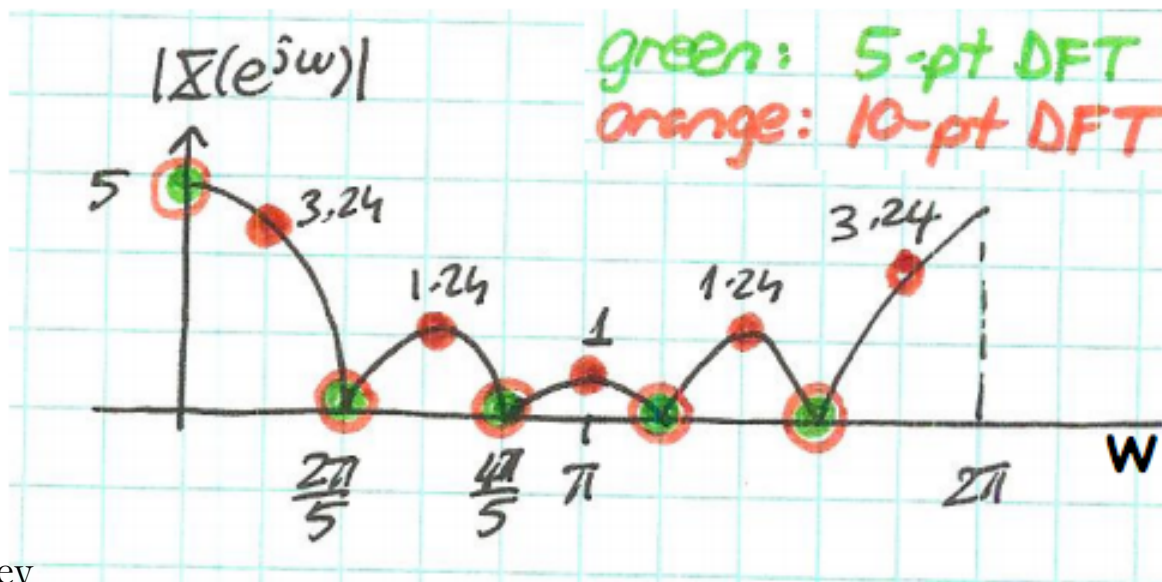
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DFT vs DTFT

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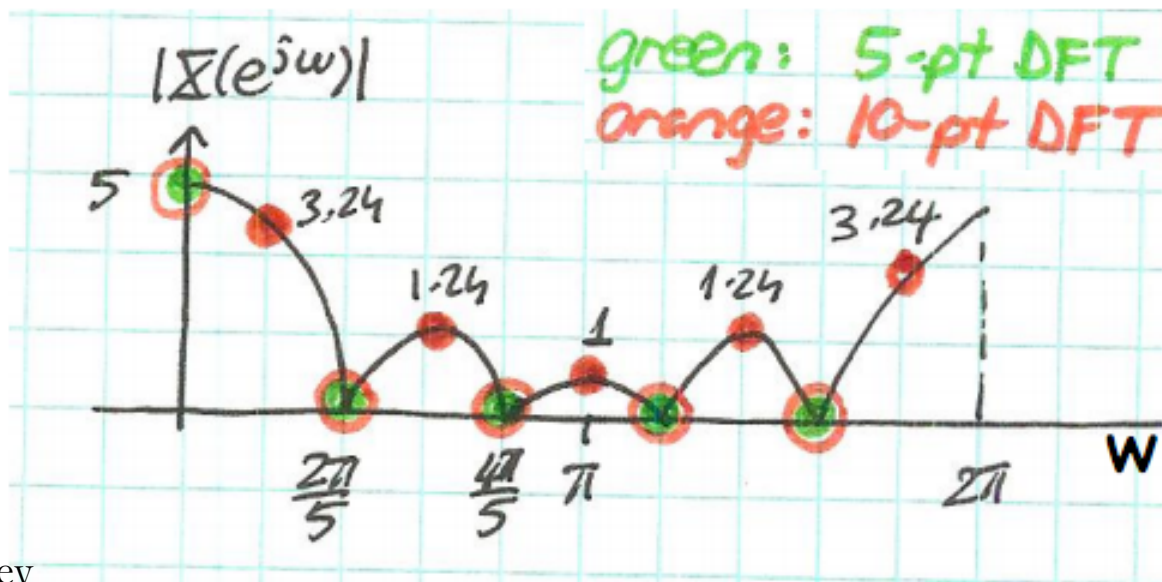


DFT vs DTFT

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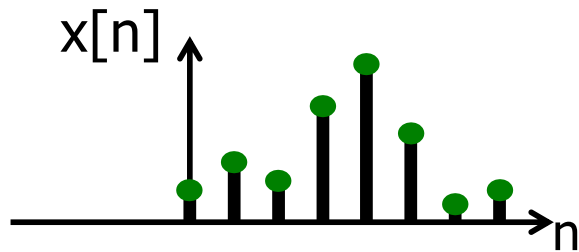
Use `fftshift`
to center
around dc





DFT Intuition

Time

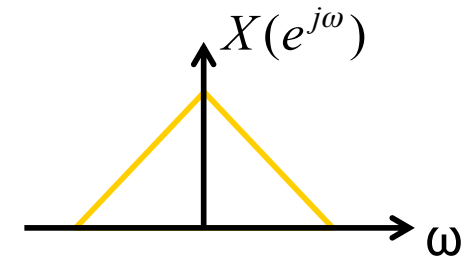


Transform

DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

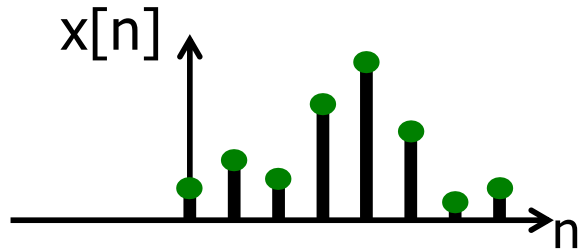
Frequency





DFT Intuition

Time

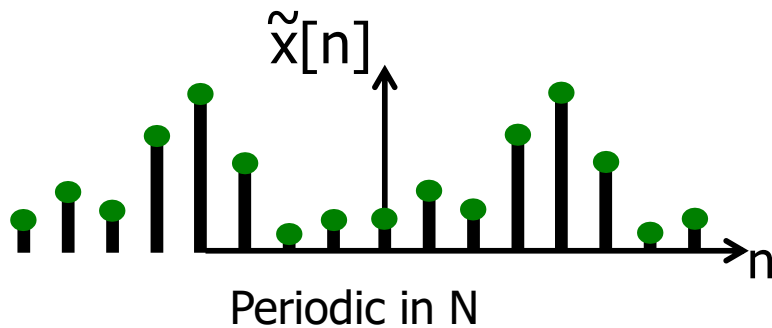
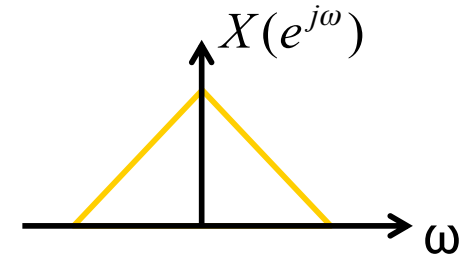


Transform

DTFT

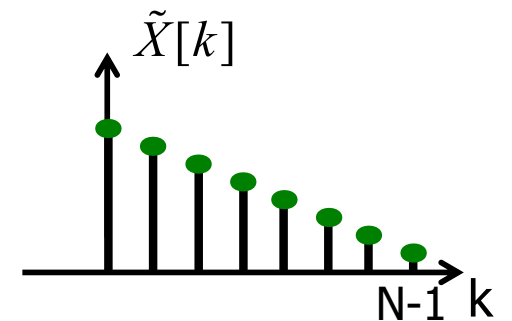
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Frequency



DFS

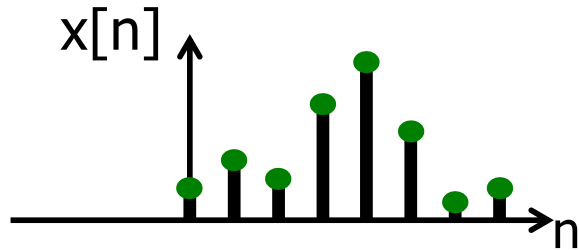
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DFT Intuition

Time

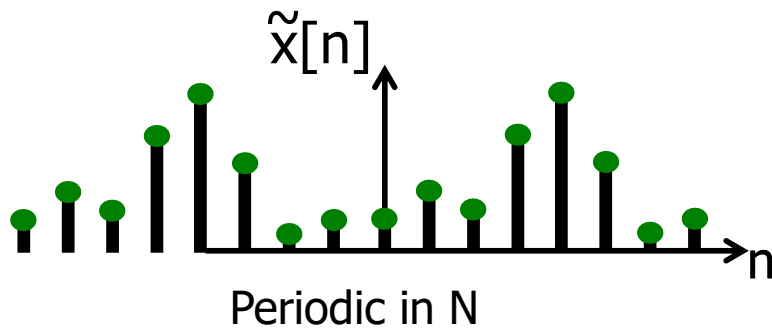
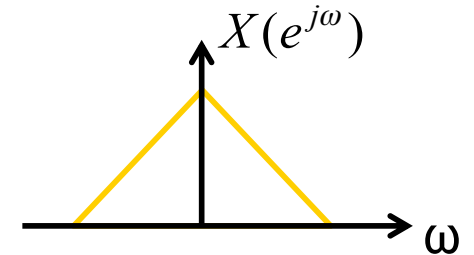


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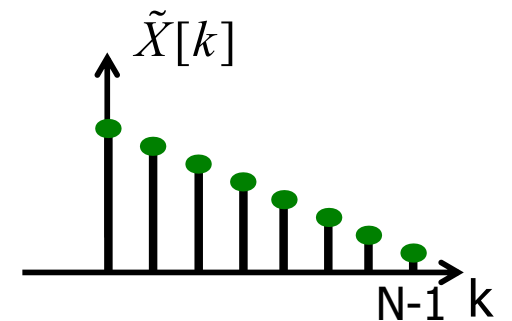
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Frequency



DFS

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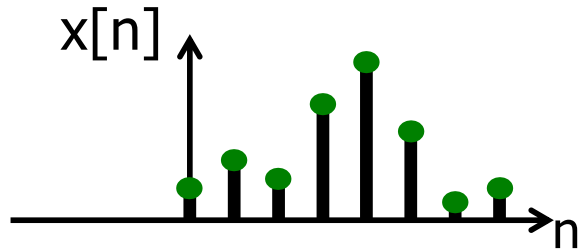


$$W_N = e^{-j\frac{2\pi}{N}}$$



DFT Intuition

Time

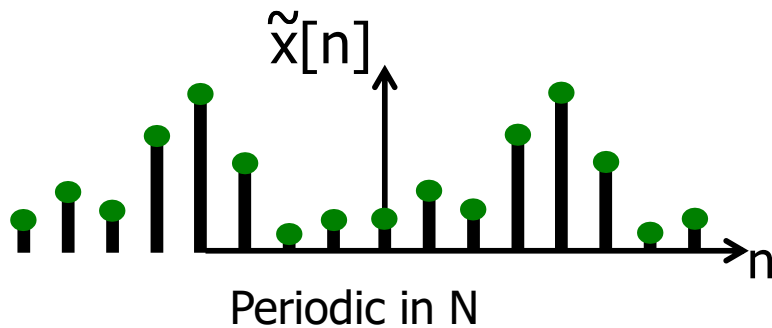
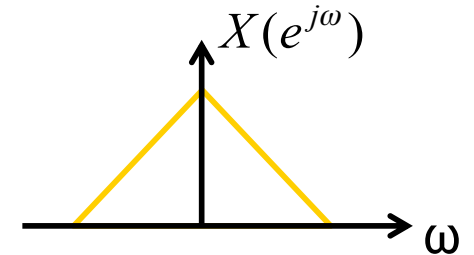


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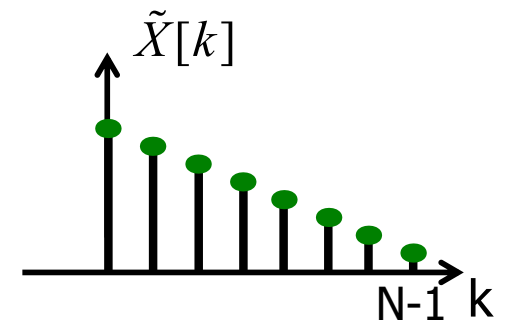
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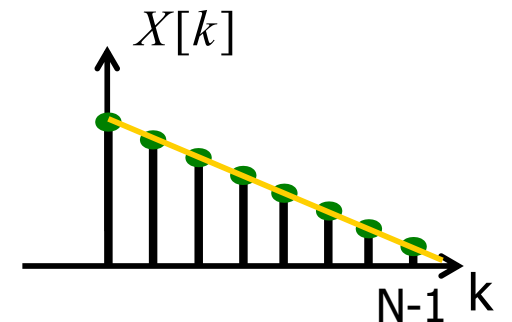
DFS

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DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$





DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?



DFT and Inverse DFT

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$$N \cdot x^*[n] = N \left(\mathcal{DFT}^{-1} \{X[k]\} \right)^*$$

Reminder: Discrete Fourier Transform

□ The DFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \quad \text{Inverse DFT, synthesis}$$

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□ It is understood that,

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DFT and Inverse DFT

- Use the DFT to compute the inverse DFT. How?

$$\begin{aligned} N \cdot x^*[n] &= N \left(\mathcal{DFT}^{-1} \{X[k]\} \right)^* \\ &= N \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right)^* \\ &= \sum_{k=0}^{N-1} X^*[k] W_N^{kn} \\ &= \mathcal{DFT} \{X^*[k]\}. \end{aligned}$$



DFT and Inverse DFT

□ So

$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$

DFT and Inverse DFT

□ So

$$\mathcal{DFT} \{X^*[k]\} = N (\mathcal{DFT}^{-1} \{X[k]\})^*$$

$$\mathcal{DFT}^{-1} \{X[k]\} = \frac{1}{N} (\mathcal{DFT} \{X^*[k]\})^*$$

□ Implement IDFT by:

- Take complex conjugate
- Take DFT
- Multiply by 1/N
- Take complex conjugate



DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

DFT:

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \dots & W_N^{0n} & \dots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \dots & W_N^{kn} & \dots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \dots & W_N^{(N-1)n} & \dots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$



DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

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IDFT:

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DFT as Matrix Operator

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

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DFT as Matrix Operator

- Can write compactly as

$$\begin{aligned}\mathbf{X} &= \mathbf{W}_N \mathbf{x} \\ \mathbf{x} &= \frac{1}{N} \mathbf{W}_N^* \mathbf{X}\end{aligned}$$



Properties of the DFT

- ❑ Properties of DFT inherited from DFS
- ❑ Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$



Properties of the DFT

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- Linearity

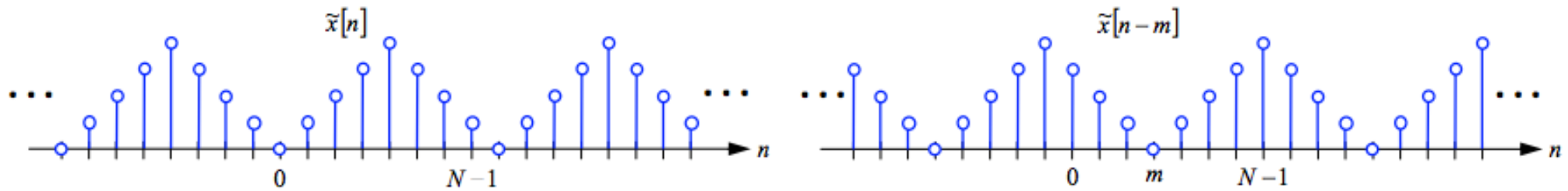
$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

- Circular Time Shift

$$x[((n - m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$

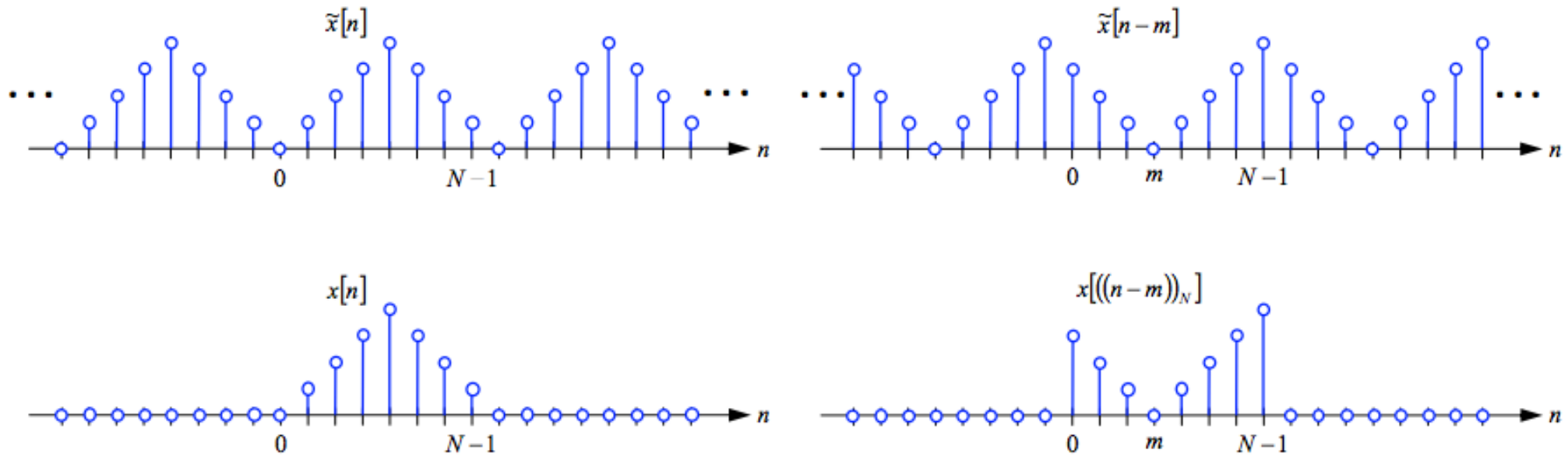


Circular Shift





Circular Shift





Properties of DFT

- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

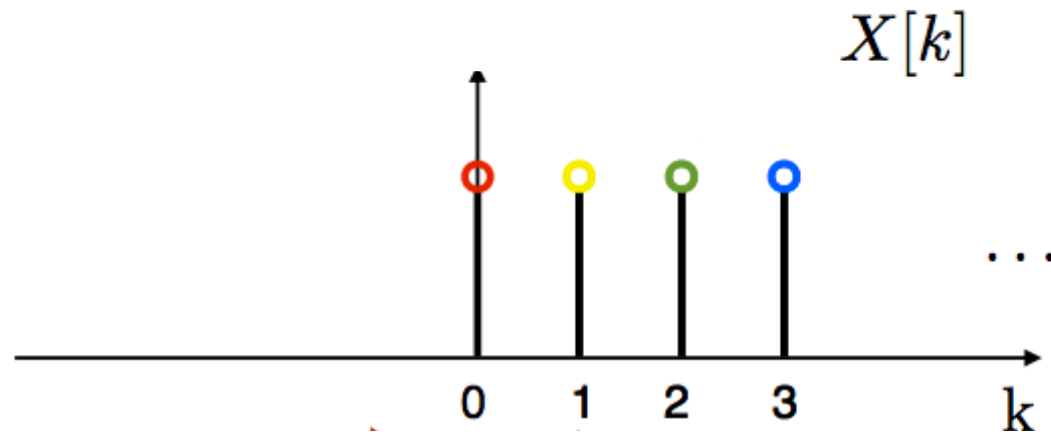
$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

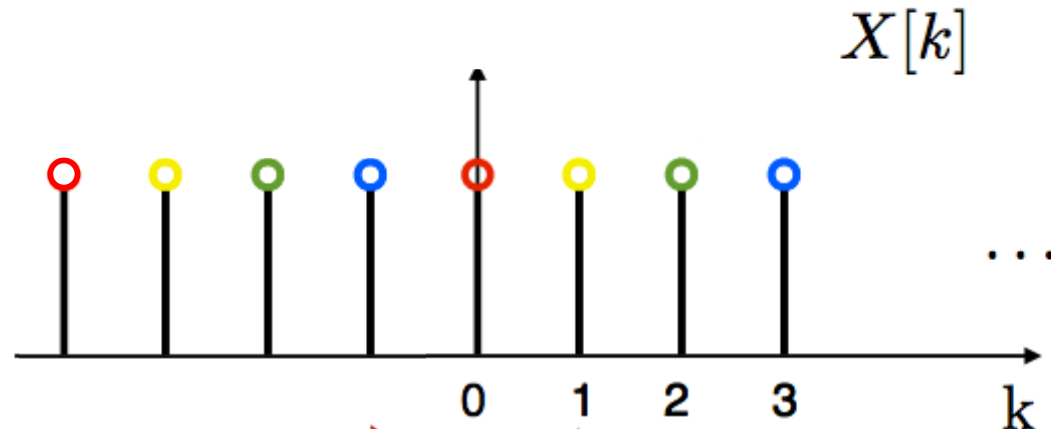
4-point DFT
–Symmetry



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Example: Conjugate Symmetry

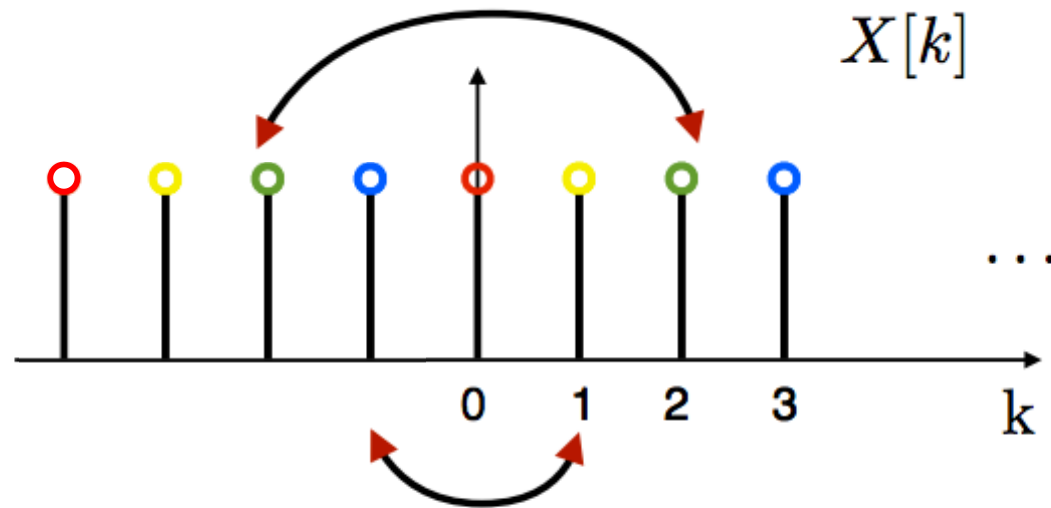
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$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

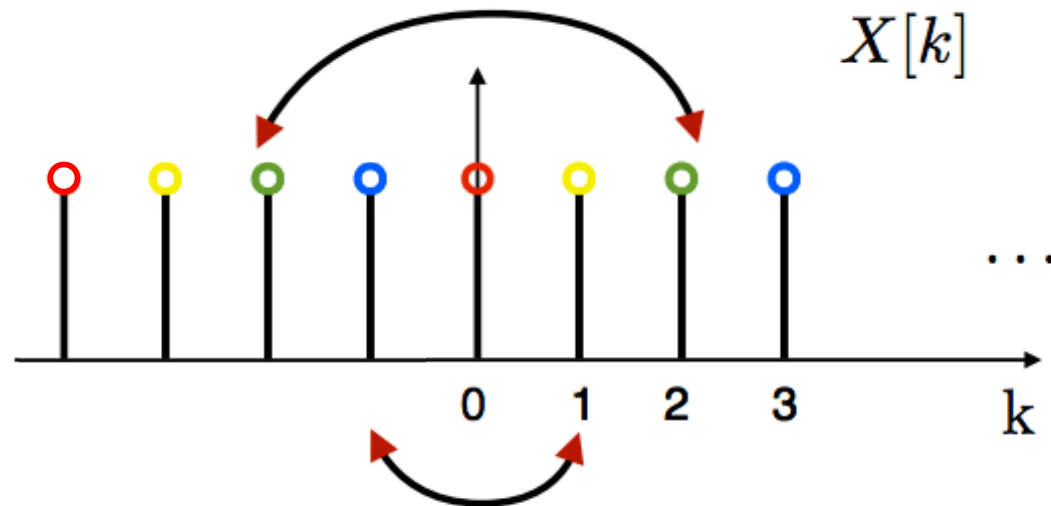
4-point DFT
–Symmetry



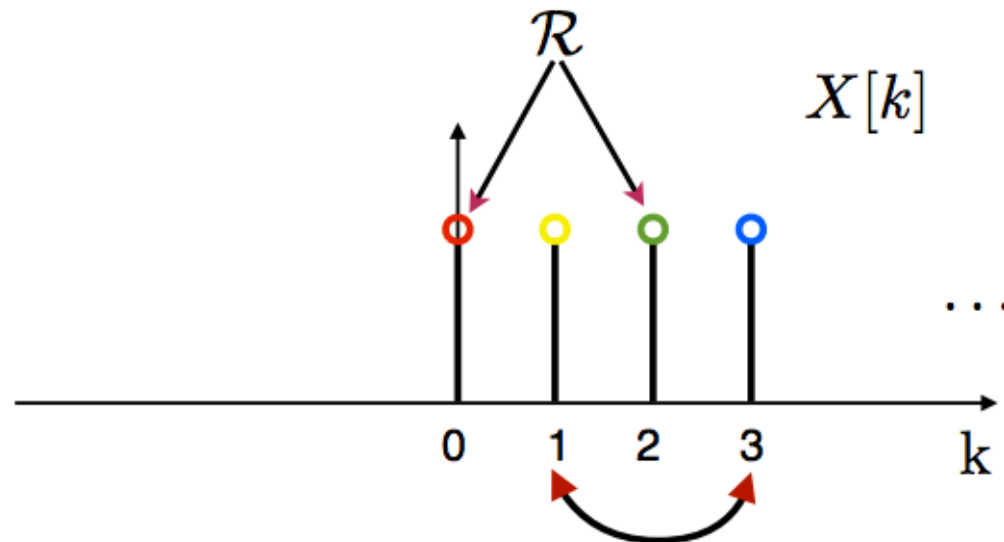
$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

4-point DFT
–Symmetry



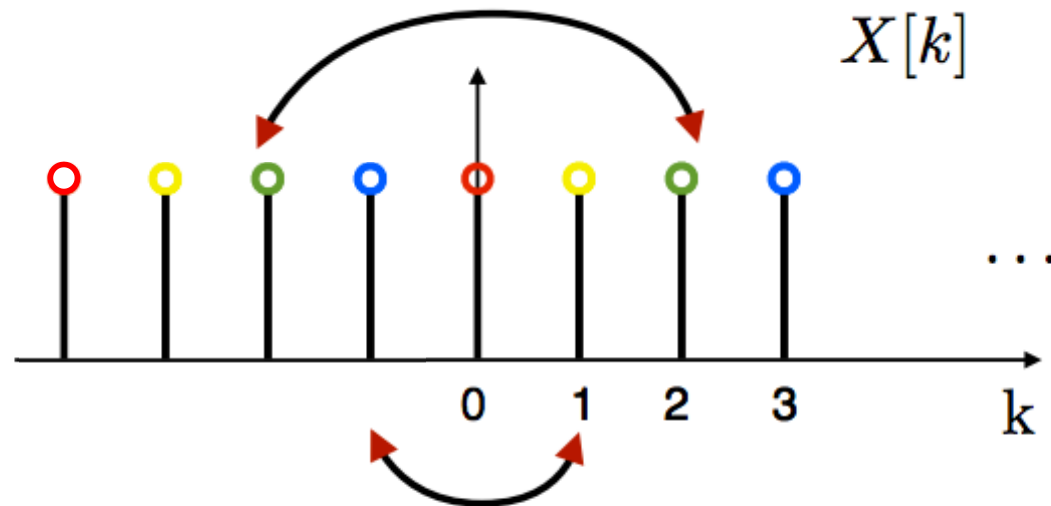
4-point DFT
–Symmetry



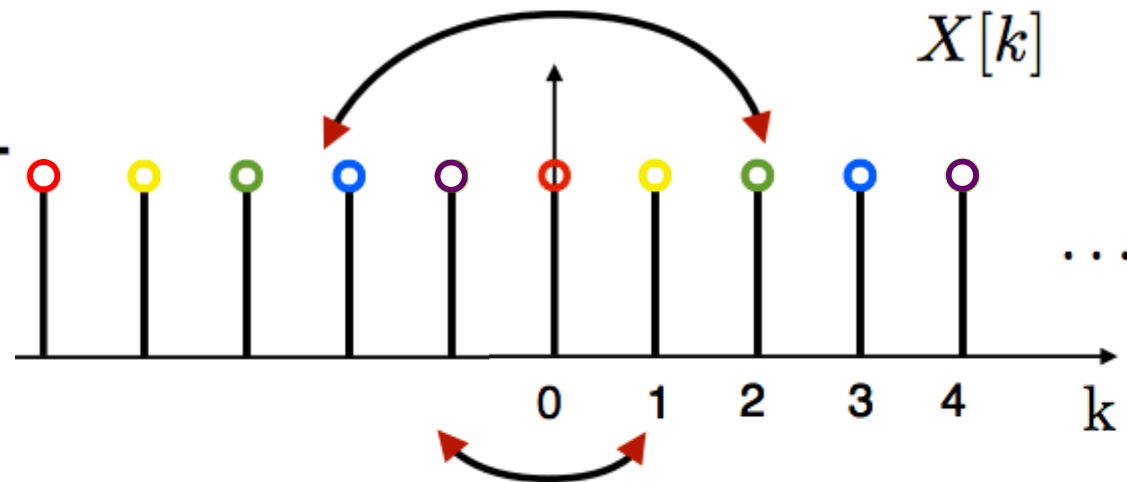
$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Example: Conjugate Symmetry

4-point DFT
–Symmetry



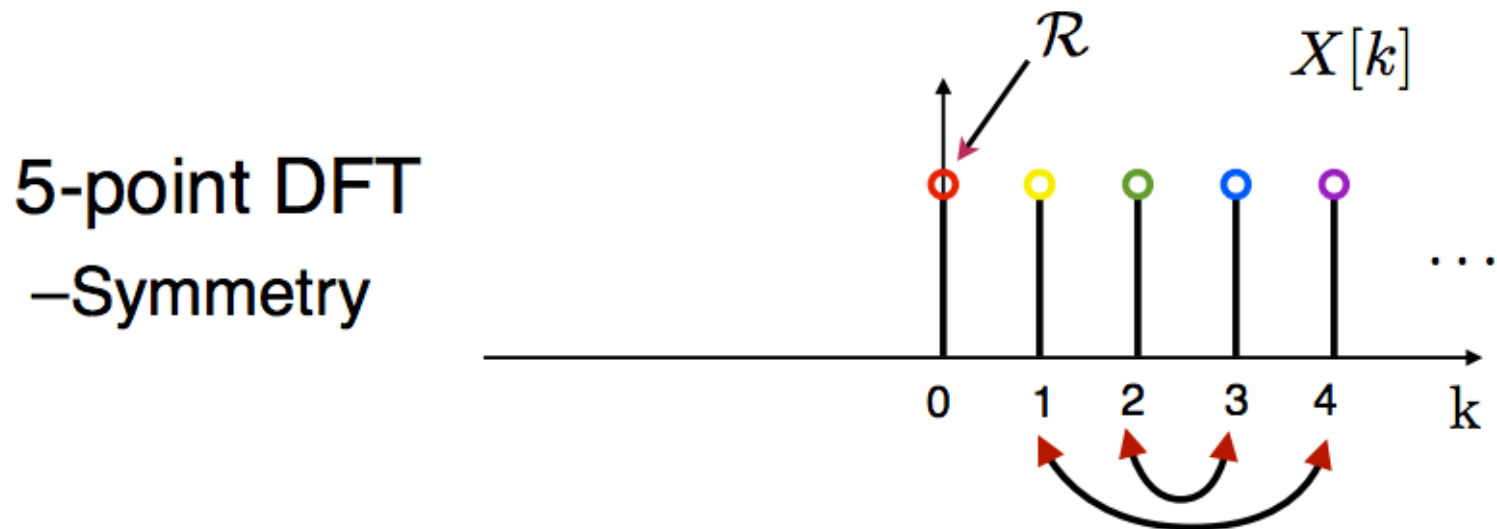
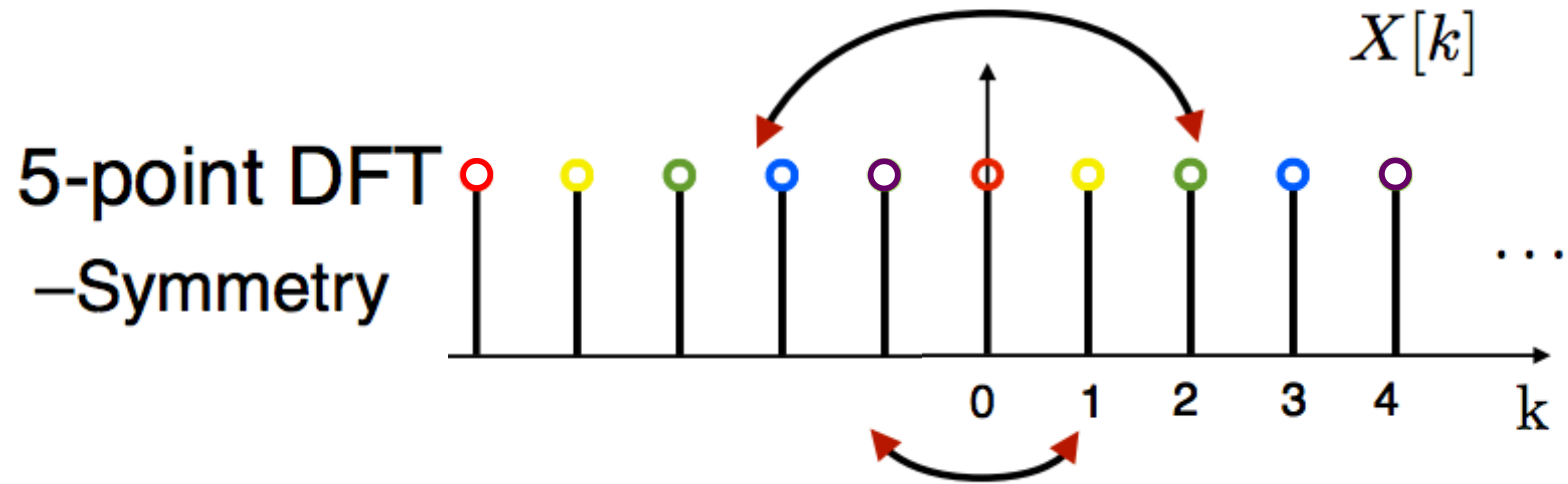
5-point DFT
–Symmetry



$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$



Example





Properties of the DFS/DFT

Discrete Fourier Series			Discrete Fourier Transform		
Property	<i>N</i> -periodic sequence	<i>N</i> -periodic DFS	Property	<i>N</i> -point sequence	<i>N</i> -point DFT
	$\tilde{x}[n]$ $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}[k]$ $\tilde{X}_1[k], \tilde{X}_2[k]$		$x[n]$ $x_1[n], x_2[n]$	$X[k]$ $X_1[k], X_2[k]$
Linearity	$a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$	Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Duality	$\tilde{X}[n]$	$N\tilde{x}[-k]$	Duality	$X[n]$	$Nx[((-k))_N]$
Time Shift	$\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$	Circular Time Shift	$x[((n-m))_N]$	$W_N^{km}X[k]$
Frequency Shift	$W_N^{-ln}\tilde{x}[n]$	$\tilde{X}[k-l]$	Circular Frequency Shift	$W_N^{-ln}x[n]$	$X[((k-l))_N]$
Periodic Convolution	$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$	$\tilde{X}_1[k]\tilde{X}_2[k]$	Circular Convolution	$\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
Multiplication	$\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} \tilde{X}_1[l]\tilde{X}_2[k-l]$	Multiplication	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} X_1[l]X_2[((k-l))_N]$
Complex Conjugation	$\tilde{x}^*[n]$	$\tilde{X}^*[-k]$	Complex Conjugation	$x^*[n]$	$X^*[((-k))_N]$



Properties (Continued)

Time-Reversal and Complex Conjugation	$\tilde{x}^*[-n]$	$\tilde{X}^*[k]$	Time-Reversal and Complex Conjugation	$x^*[((-n))_N]$	$X^*[k]$
Real Part	$\text{Re}\{\tilde{x}[n]\}$	$\tilde{X}_{ep}[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$	Real Part	$\text{Re}\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}(X[k] + X^*[((-k))_N])$
Imaginary Part	$j \text{Im}\{\tilde{x}[n]\}$	$\tilde{X}_{op}[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$	Imaginary Part	$j \text{Im}\{x[n]\}$	$X_{op}[k] = \frac{1}{2}(X[k] - X^*[((-k))_N])$
Even Part	$\tilde{x}_{ep}[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\text{Re}\{\tilde{X}[k]\}$	Even Part	$x_{ep}[n] = \frac{1}{2}(x[n] + x^*[((-n))_N])$	$\text{Re}\{X[k]\}$
Odd Part	$\tilde{x}_{op}[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j \text{Im}\{\tilde{X}[k]\}$	Odd Part	$x_{op}[n] = \frac{1}{2}(x[n] - x^*[((-n))_N])$	$j \text{Im}\{X[k]\}$
Symmetry for Real Sequence	$\tilde{x}[n] = \tilde{x}^*[n]$	$\tilde{X}[k] = \tilde{X}^*[-k]$ $\begin{cases} \text{Re}\{\tilde{X}[k]\} = \text{Re}\{\tilde{X}^*[-k]\} \\ \text{Im}\{\tilde{X}[k]\} = -\text{Im}\{\tilde{X}^*[-k]\} \end{cases}$ $\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \angle \tilde{X}[k] = -\angle \tilde{X}^*[-k] \end{cases}$	Symmetry for Real Sequence	$x[n] = x^*[n]$	$X[k] = X^*[((-k))_N]$ $\begin{cases} \text{Re}\{X[k]\} = \text{Re}\{X^*[((-k))_N]\} \\ \text{Im}\{X[k]\} = -\text{Im}\{X^*[((-k))_N]\} \end{cases}$ $\begin{cases} X[k] = X^*[((-k))_N] \\ \angle X[k] = -\angle X^*[((-k))_N] \end{cases}$
Parseval's Identity	$\sum_{n=0}^{N-1} \tilde{x}_1[n] \tilde{x}_2^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}_1[k] \tilde{X}_2^*[k]$ $\sum_{n=0}^{N-1} \tilde{x}[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] ^2$		Parseval's Identity	$\sum_{n=0}^{N-1} x_1[n] x_2^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_1[k] X_2^*[k]$ $\sum_{n=0}^{N-1} x[n] ^2 = \frac{1}{N} \sum_{k=0}^{N-1} X[k] ^2$	



Circular Convolution

□ Circular Convolution:

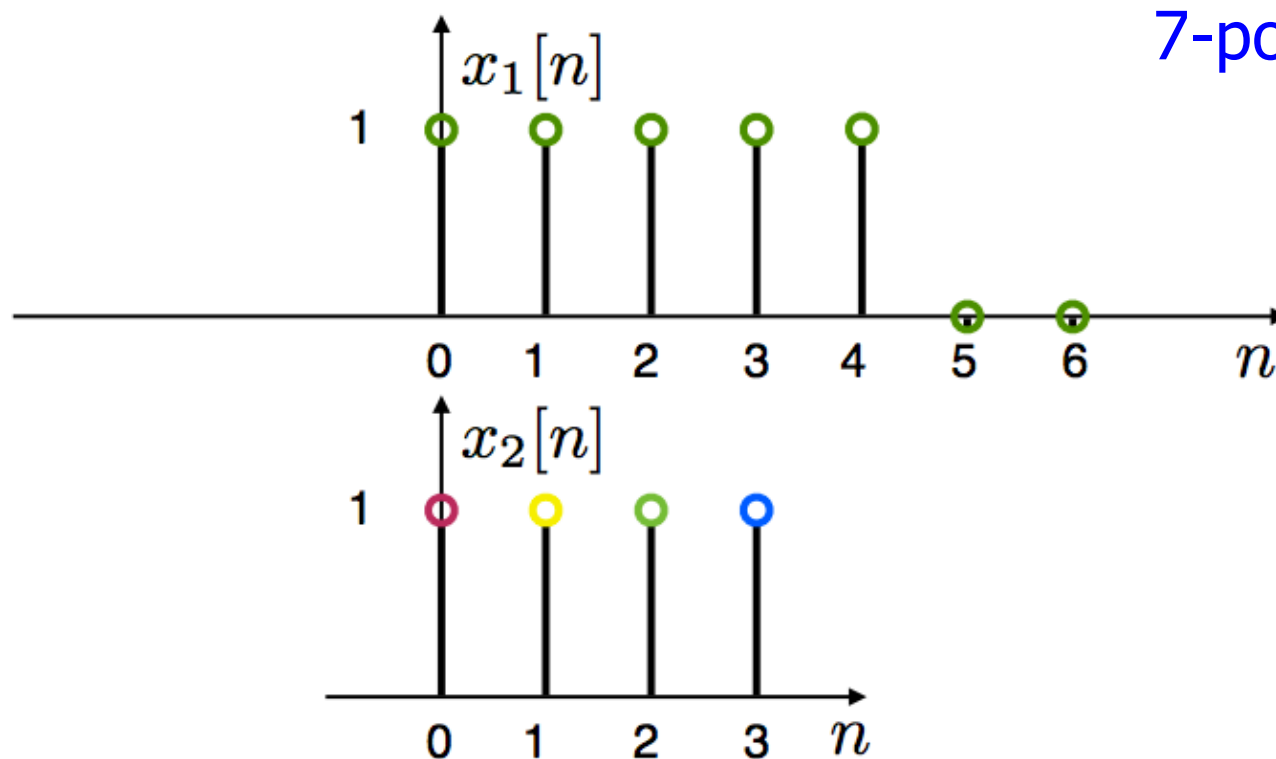
$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

For two signals of length N

Note: Circular convolution is commutative

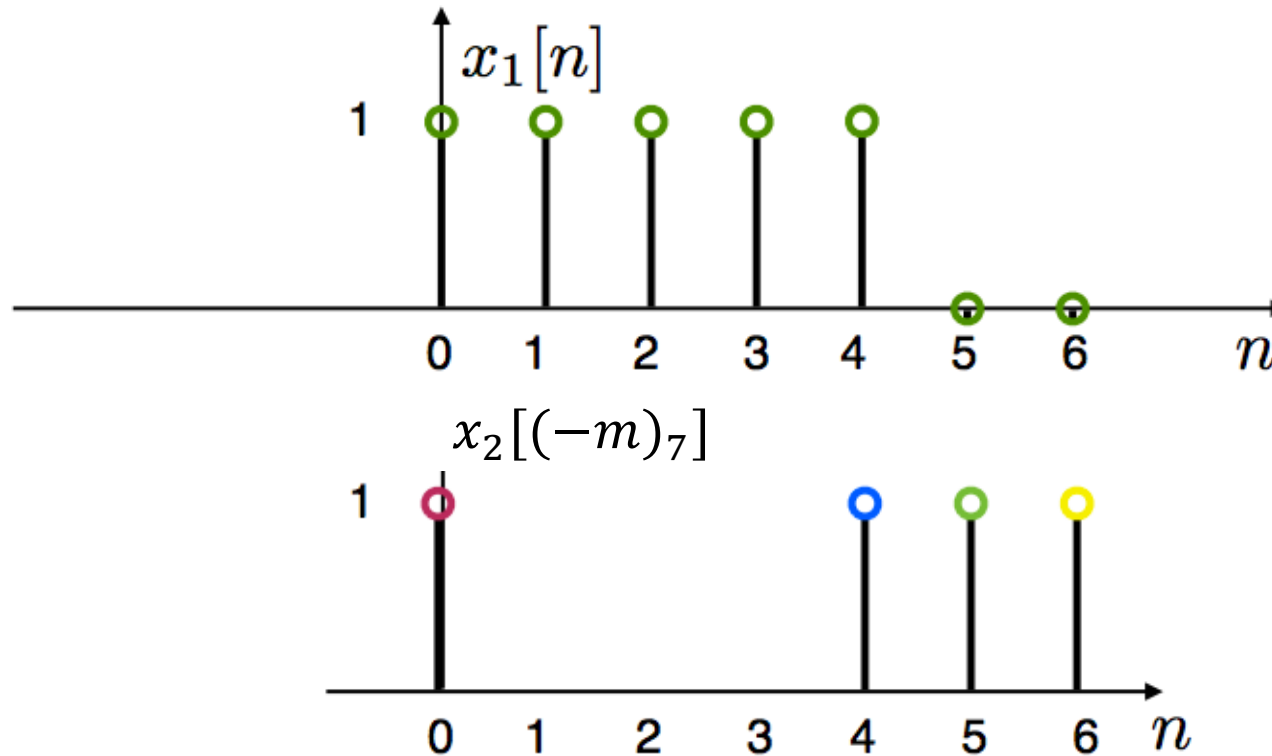
$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

Compute Circular Convolution Sum



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

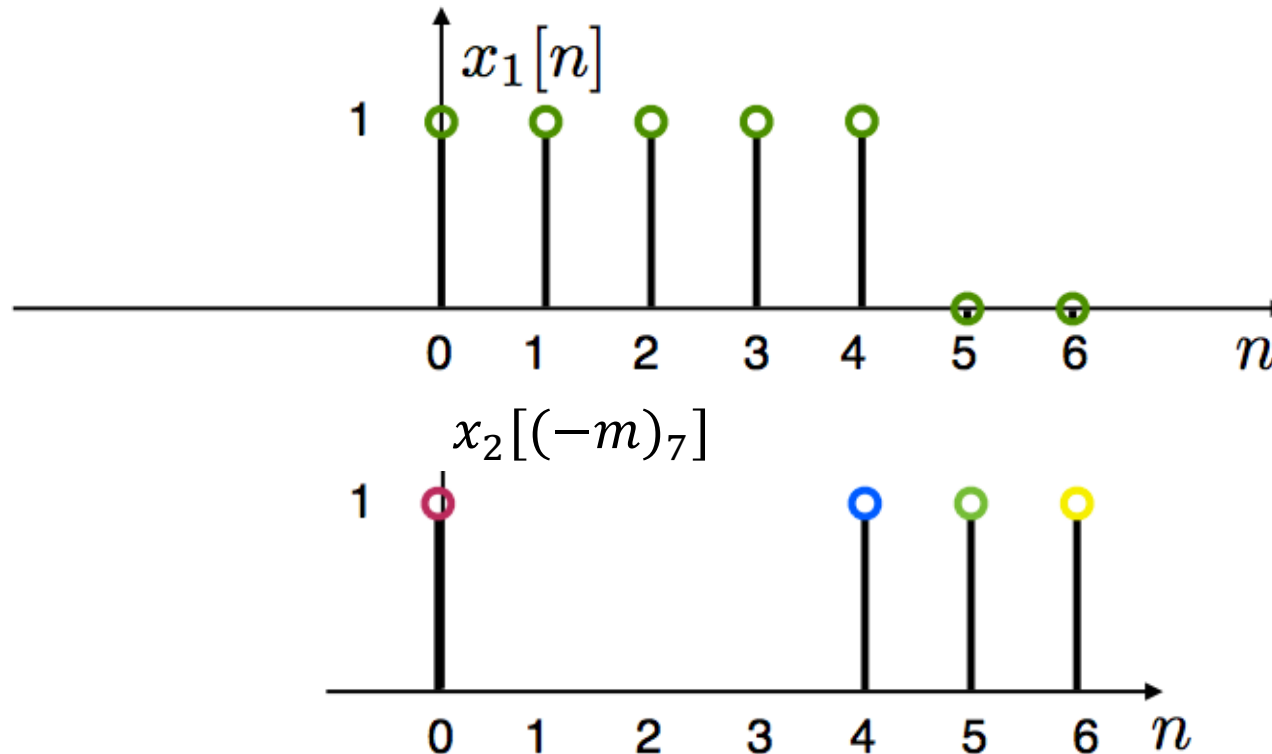
Compute Circular Convolution Sum



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum

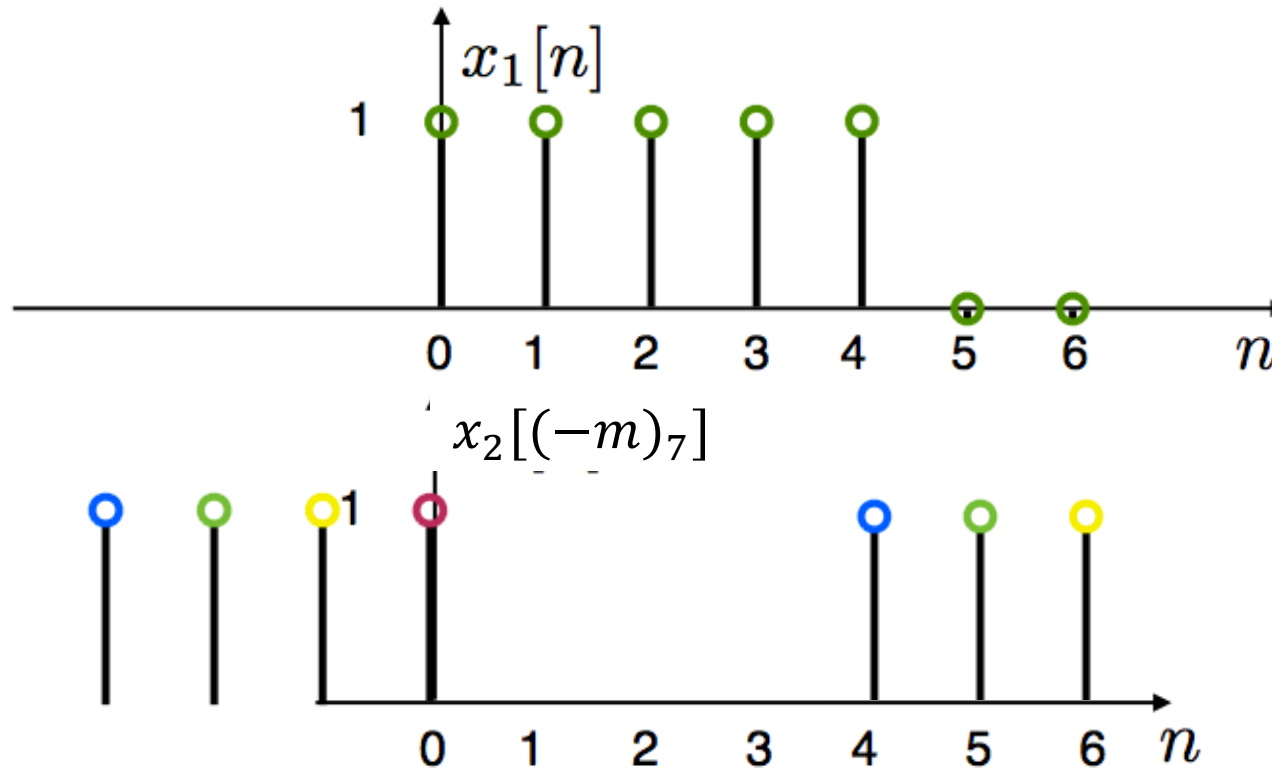
$$y[0]=2$$



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum

$$y[0]=2$$

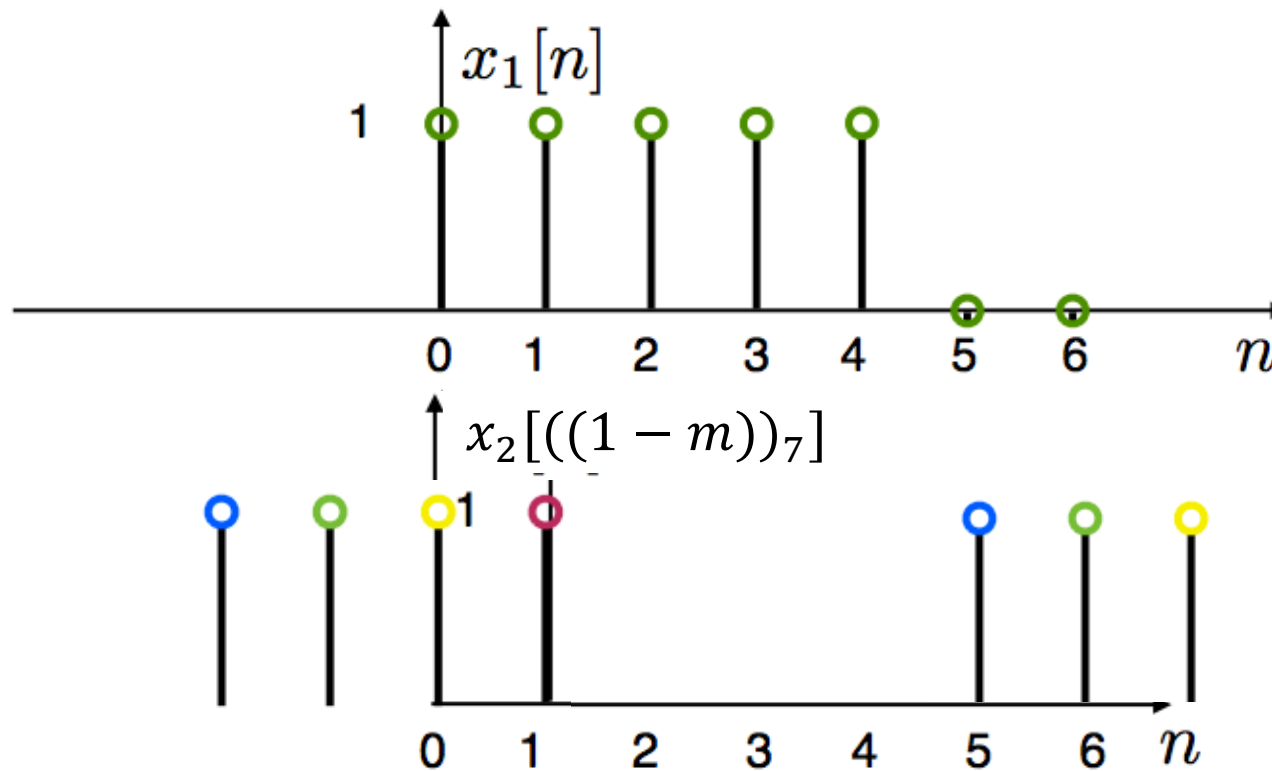


$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum

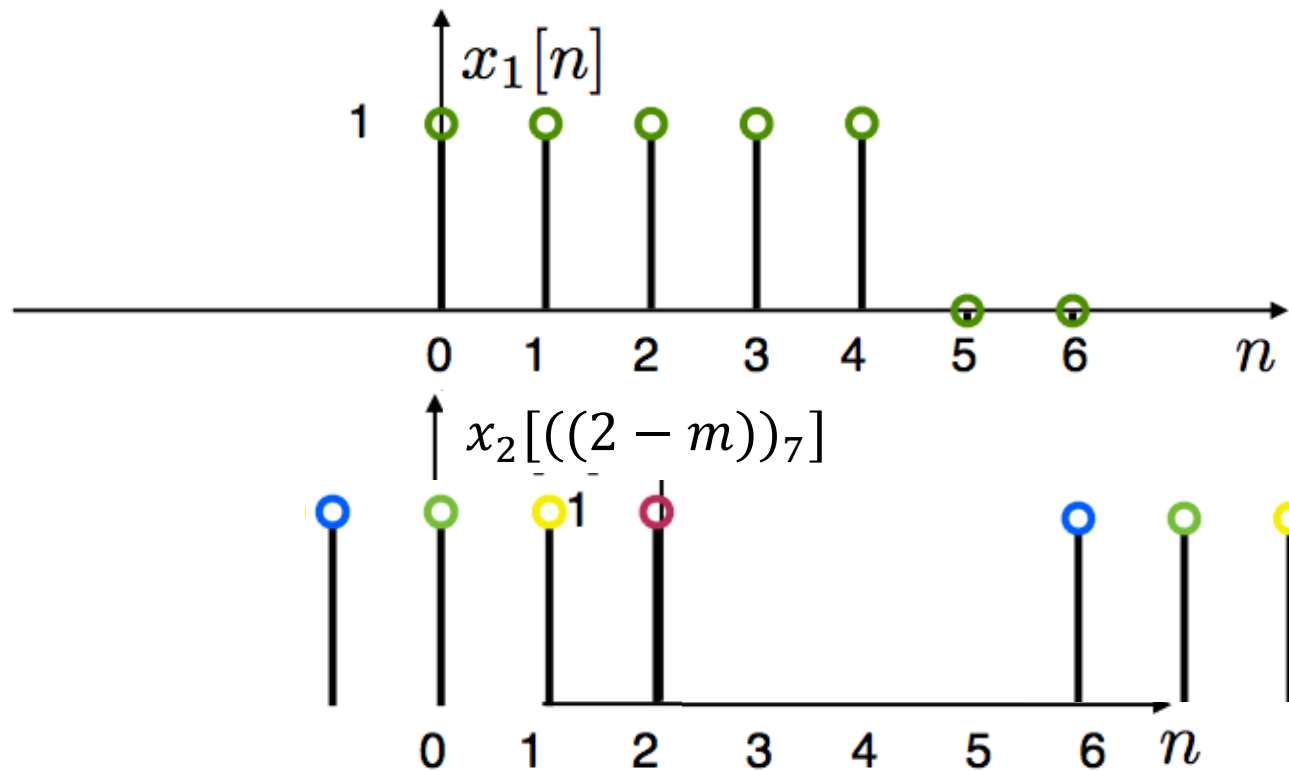
$$y[0]=2$$

$$y[1]=2$$



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

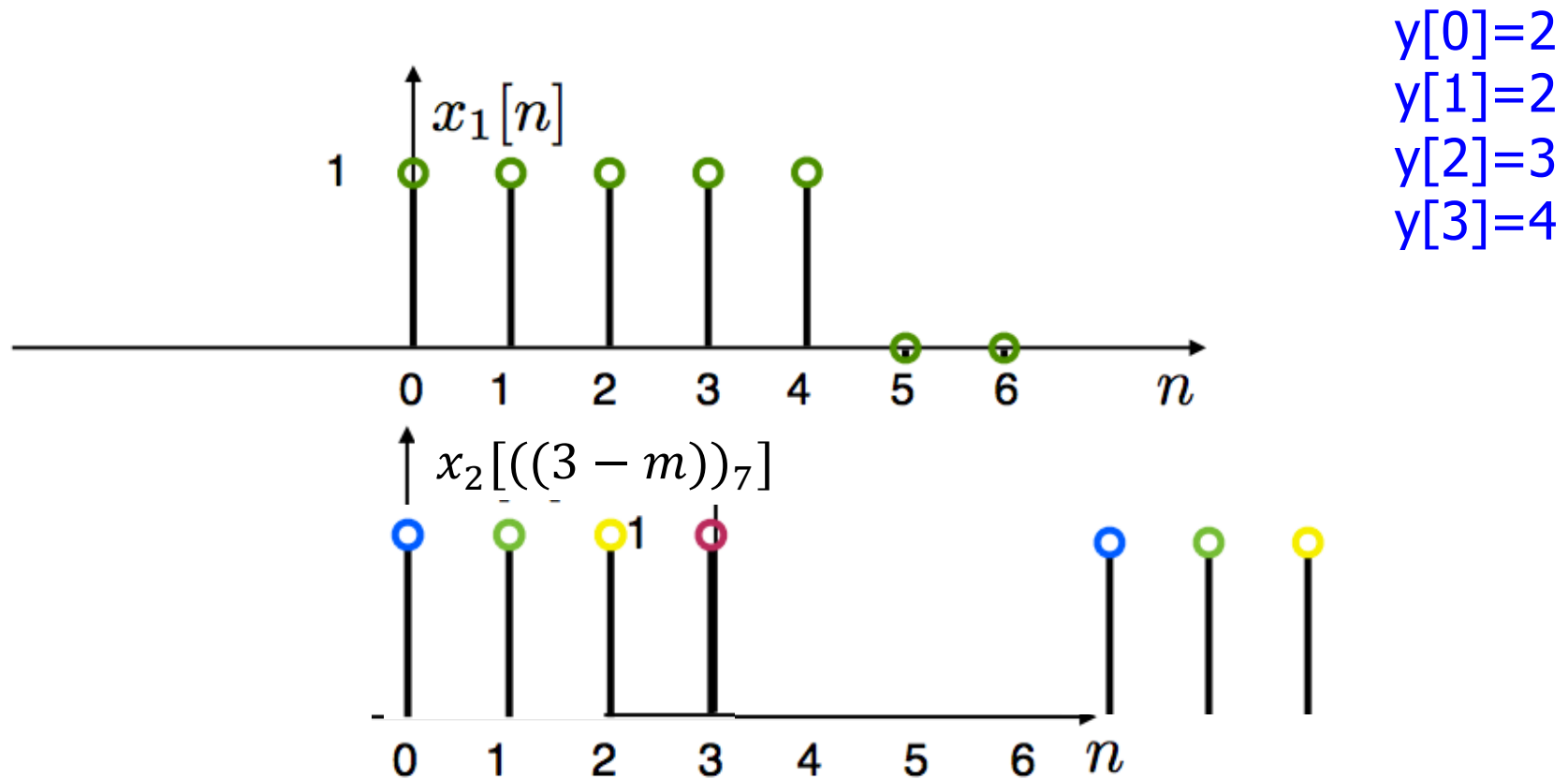
Compute Circular Convolution Sum



$$\begin{aligned} y[0] &= 2 \\ y[1] &= 2 \\ y[2] &= 3 \end{aligned}$$

$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$

Compute Circular Convolution Sum

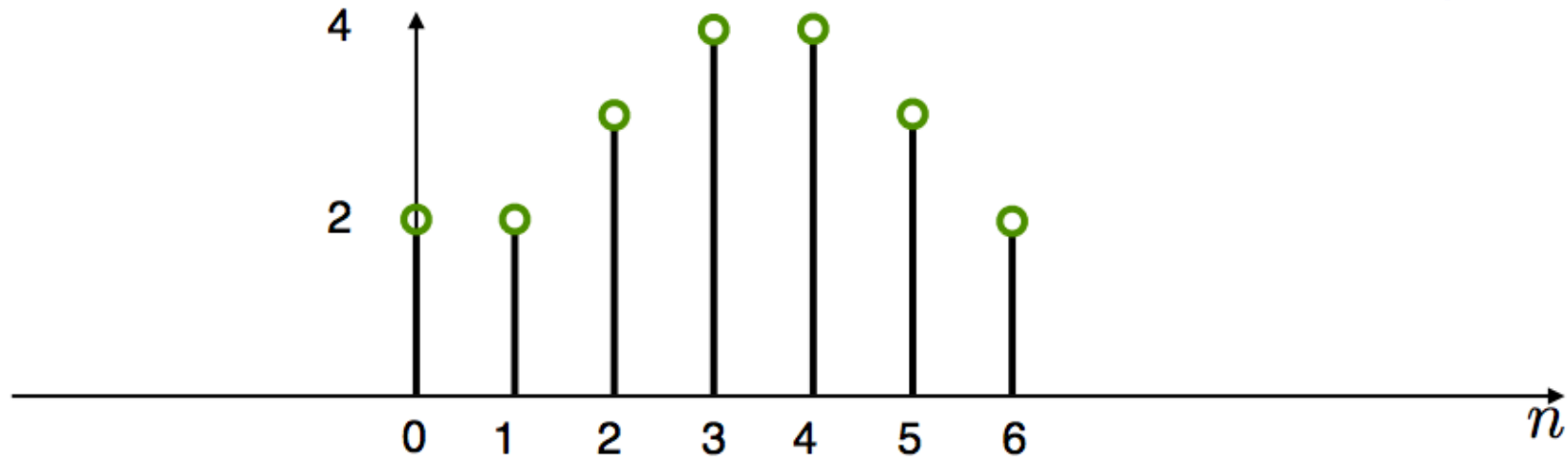


$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$



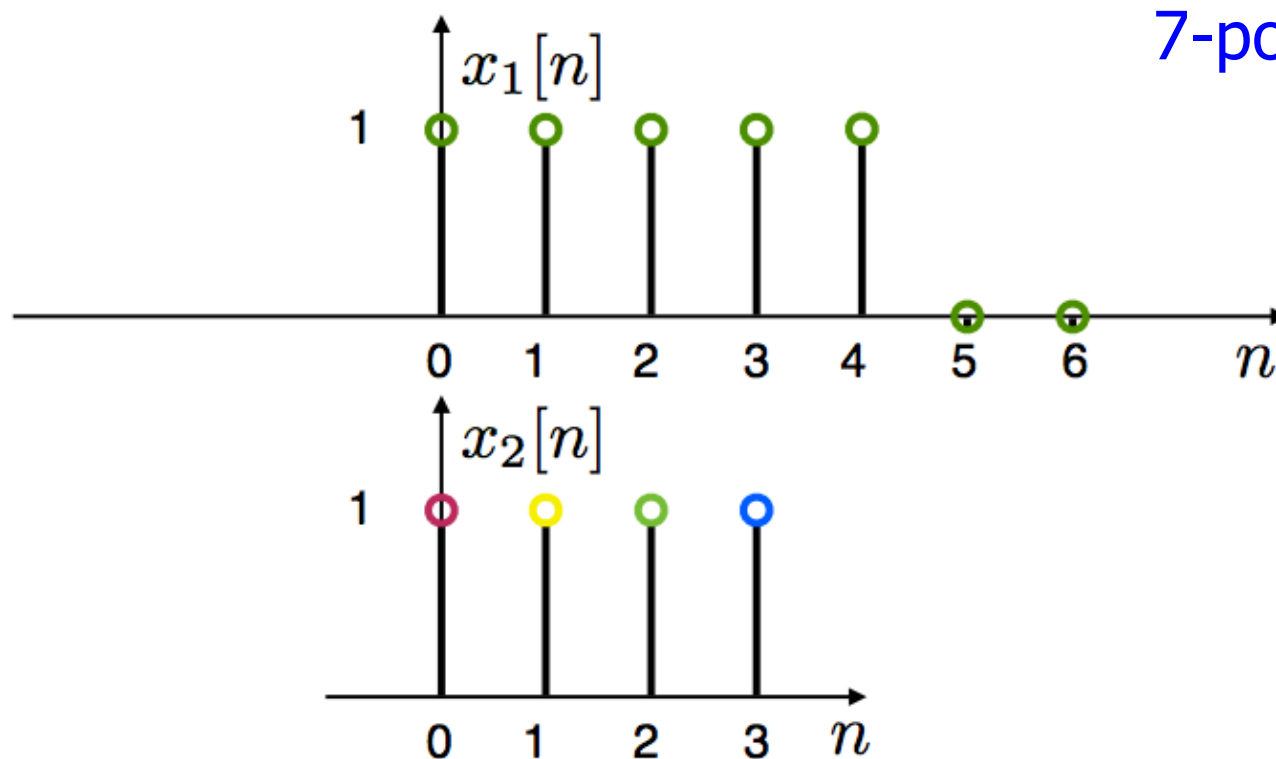
Result

$y[0]=2$
 $y[1]=2$
 $y[2]=3$
 $y[3]=4$



$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N]$$

Compute Circular Convolution Sum



$$x_1[n] \circledast x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m] x_2[((n - m))_N]$$



Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

- Very useful!! (for linear convolutions with DFT)



Multiplication

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledast X_2[k]$$



Linear Convolution

- Next....
 - Using DFT, circular convolution is easy
 - But, linear convolution is useful, not circular
 - So, show how to perform linear convolution with circular convolution
 - Use DFT to do linear convolution



Big Ideas

- ❑ Discrete Fourier Transform (DFT)
 - For finite signals assumed to be zero outside of defined length
 - N-point DFT is sampled DTFT at N points
 - Useful properties allow easier linear convolution
- ❑ DFT Properties
 - Inherited from DFS, but circular operations!



Admin

- ❑ Project 1
 - Due tonight midnight
- ❑ HW 7 out now