ESE 5310: Digital Signal Processing

Lecture 24: April 18, 2023

Wavelet Transform

Penn ESE 5310 Spring 2023 – Khanna
Adapted from M. Lustig, EECS Berkeley

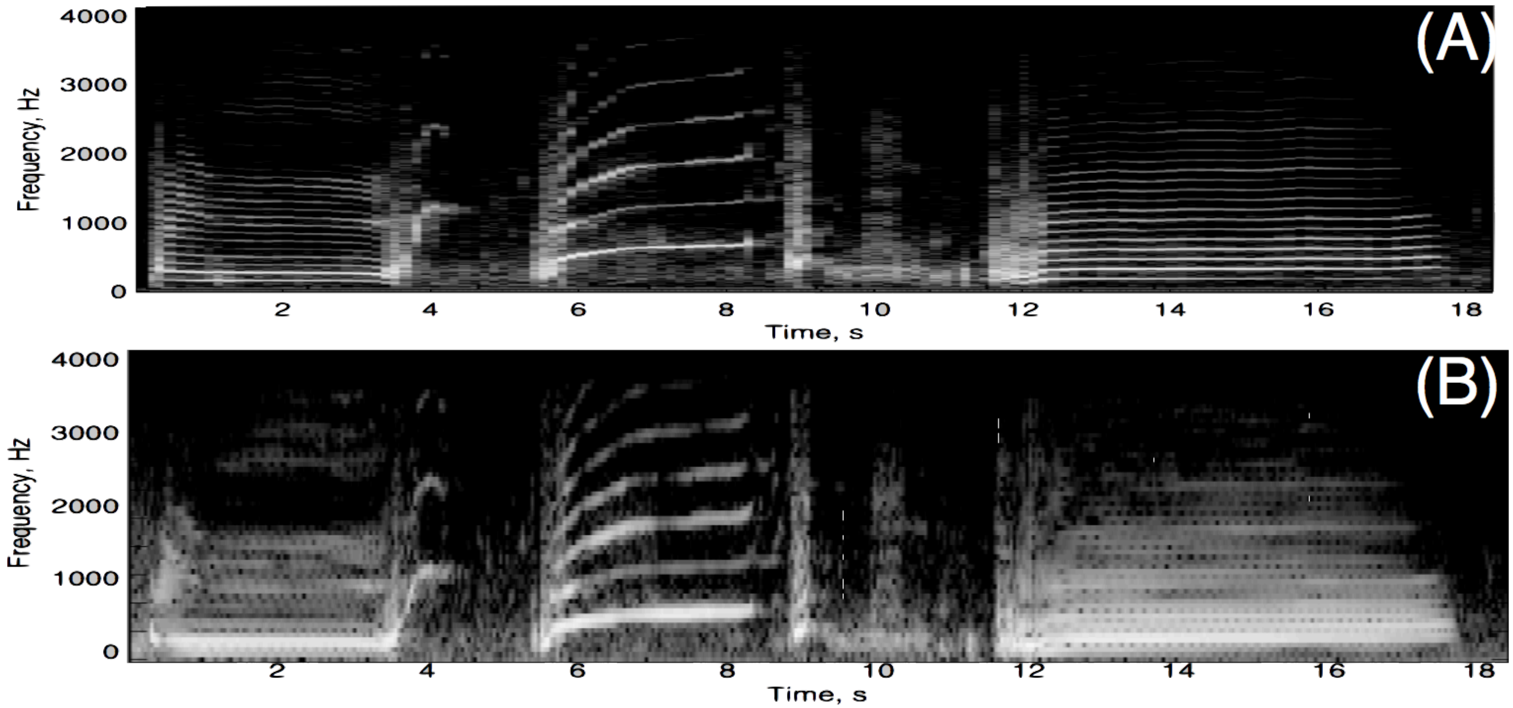
Discrete Time-Dependent Fourier Transform

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

$$X[rR,k] = X[rR,2\pi k / N) = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}$$

$$\boxed{X_r[k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}}$$

Spectrogram



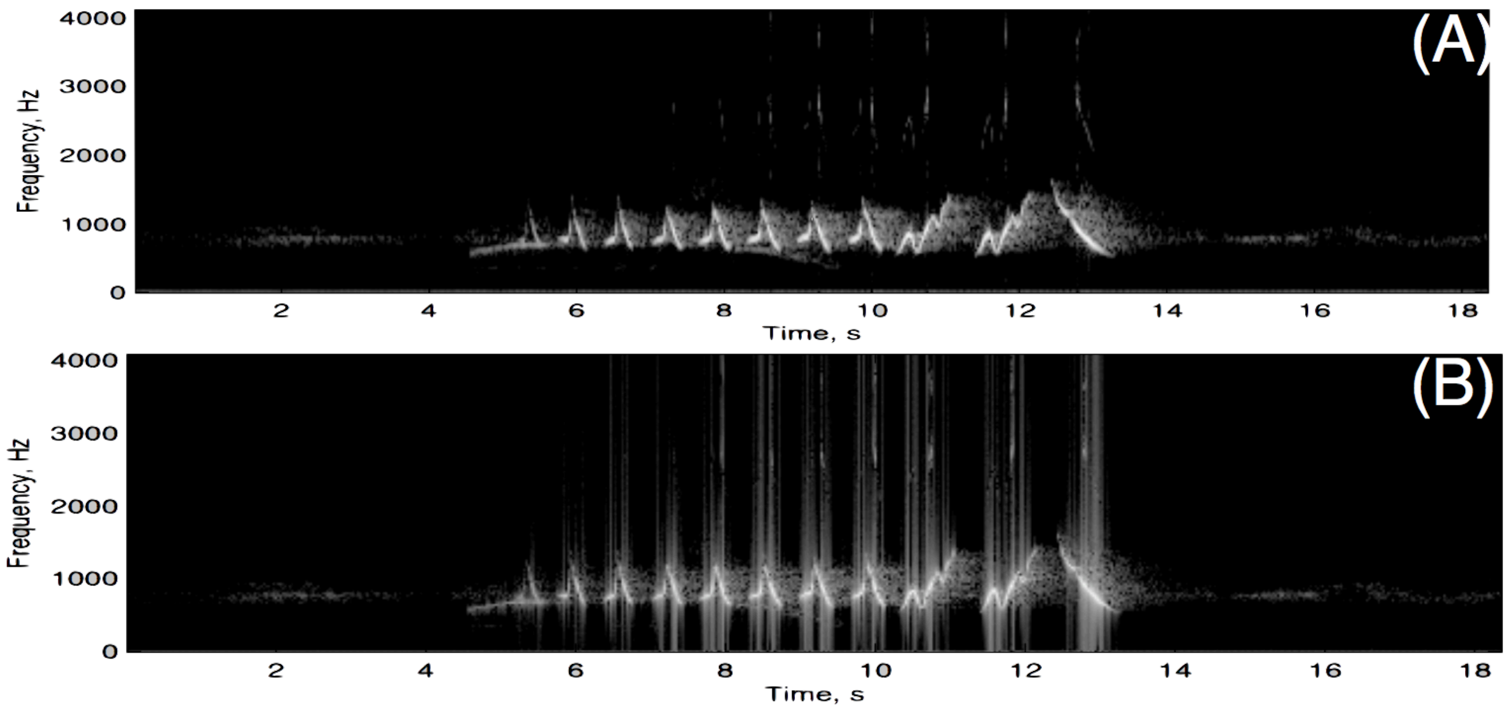
- What is the difference between the spectrograms?

a) Window size B<A

c) Window type is different

b) Window size B>A

Spectrogram



- What is the difference between the spectrograms?

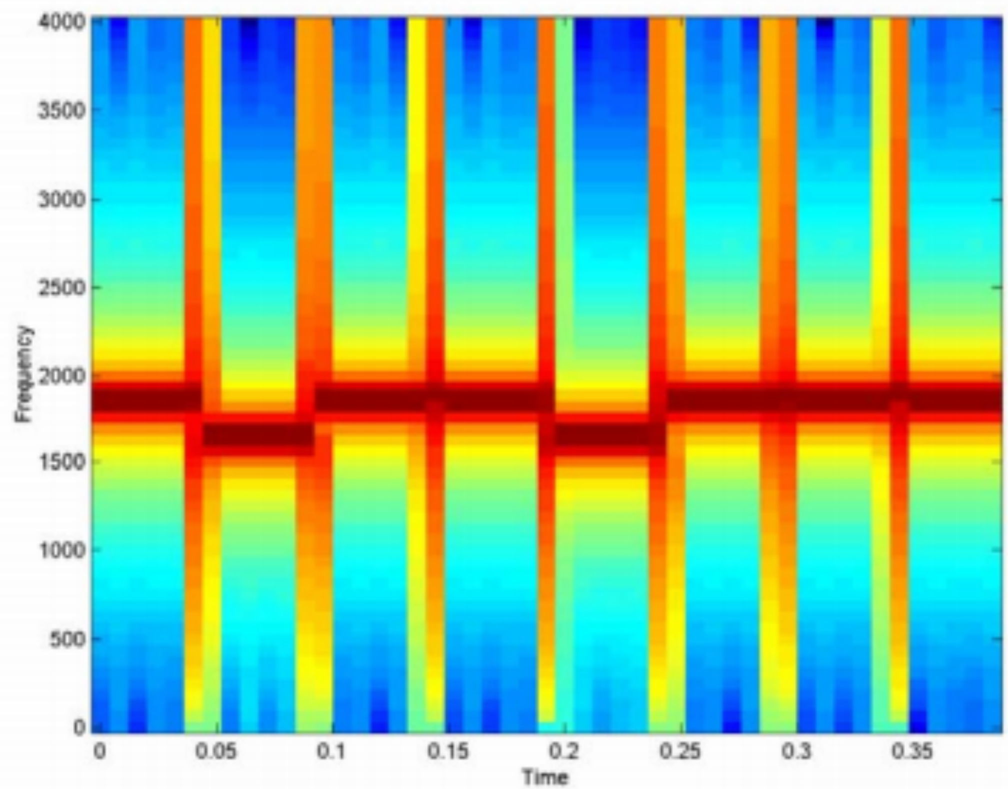
  a) Window size B<A

  b) Window size B>A

  c) Window type is different

Application – Frequency Shift Keying

- FSK Communications
  - Spectrogram transmitting 'H' (ASCII H = 01001000)



STFT Reconstruction

- If $R \leq L \leq N$, then we can recover x[n] block-by-block from $X_r[k]$
- For non-overlapping windows, R=L

$$x_r[m] = \frac{1}{N} \sum_{k=0}^{N-1} X_r[k] e^{j2\pi km/N}$$

STFT Reconstruction

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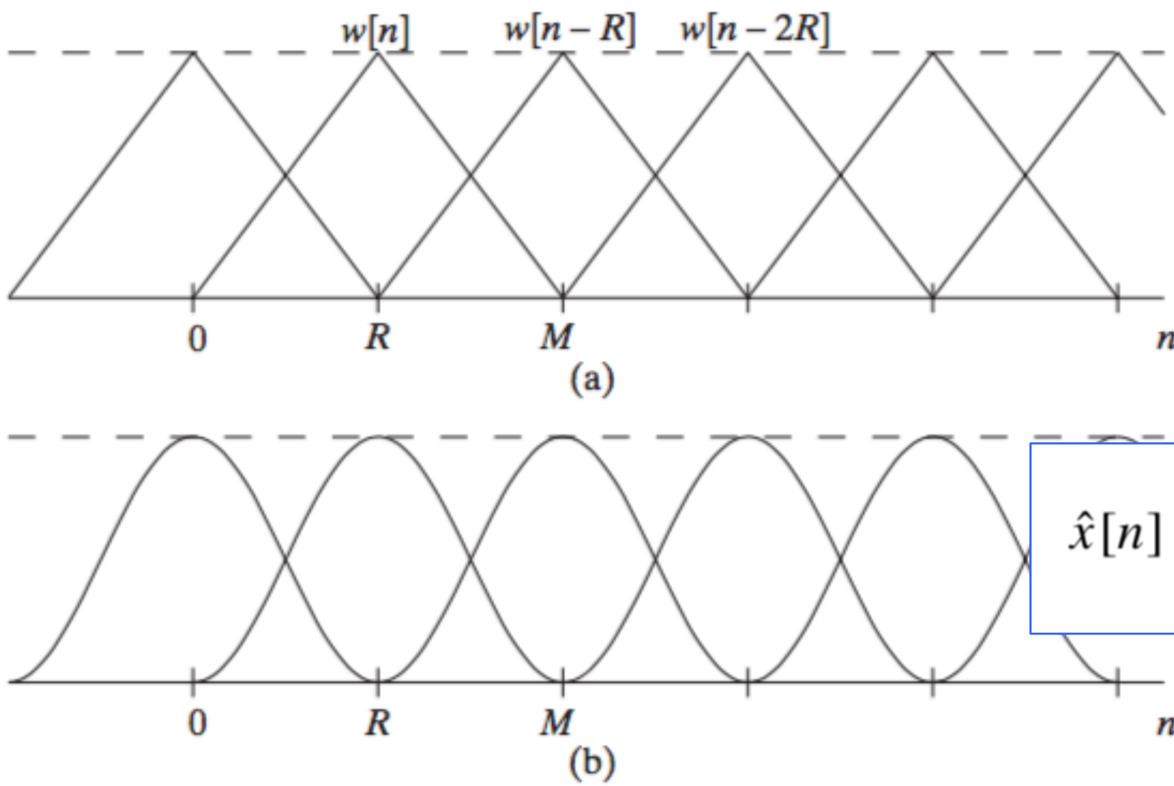
$$x_r[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_r[k] e^{j2\pi km/N}$$

$$x[n] = \frac{x_r[n-rR]}{w[n-rR]} \qquad \forall \quad rR \le n \le (r+1)R-1$$

SFTF Reconstruction with overlap

- Practically make R<L<N
- If we choose R, L, and N appropriately with window, the overlap-add will negate the window effects



$$\hat{x}[n] = \sum_{r=-\infty}^{\infty} x_r[n - rR].$$

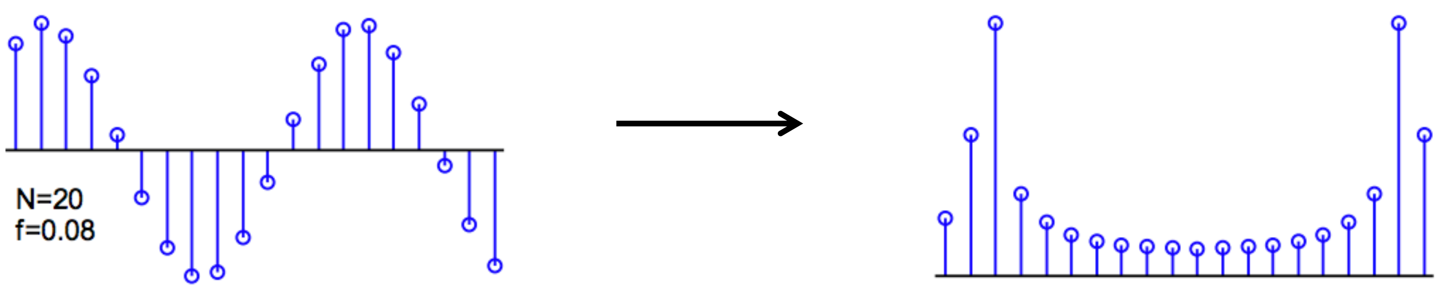
Discrete Cosine Transform

- Similar to the discrete Fourier transform (DFT), but using only real numbers
- Widely used in lossy compression applications (eg. Mp3, JPEG)
- Why use it?

DFT Problems

- For processing 1-D or 2-D signals (especially coding), a common method is to divide the signal into "frames" and then apply an invertible transform to each frame that compresses the information into few coefficients.
- The DFT has some problems when used for this purpose:
  - N real x[n] ↔ N complex X[k] : 2 real, N/2 − 1 conjugate pairs
  - DFT is of the periodic signal formed by replicating x[n]

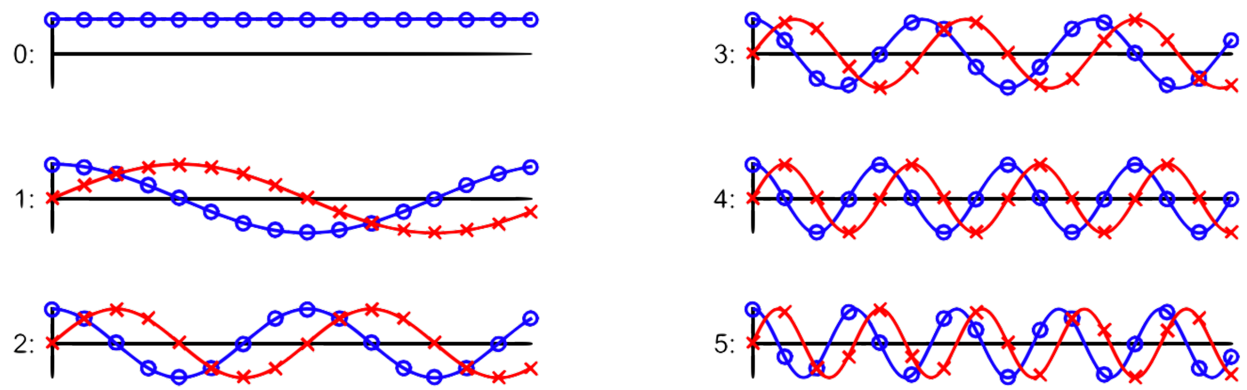
    ⇒ Spurious frequency components from boundary discontinuity



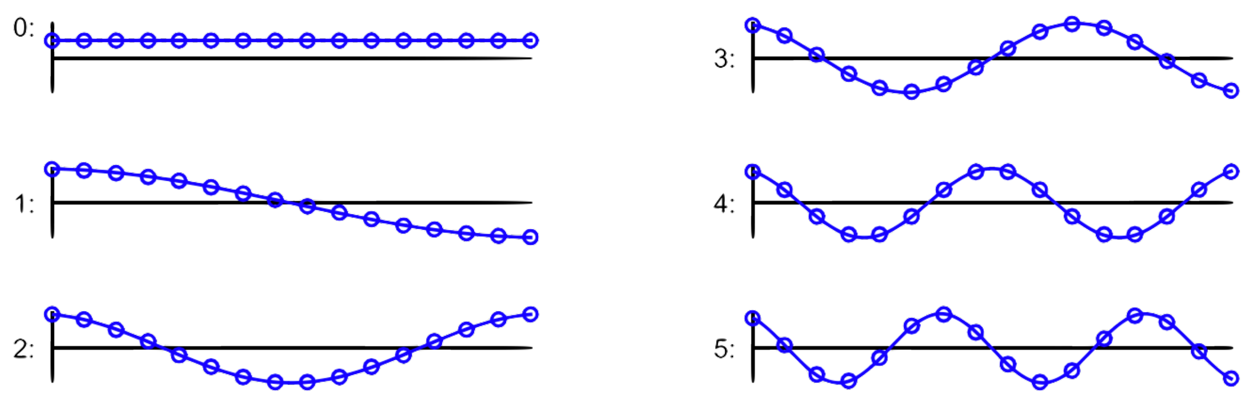
The Discrete Cosine Transform (DCT) overcomes these problems.

Basis Functions

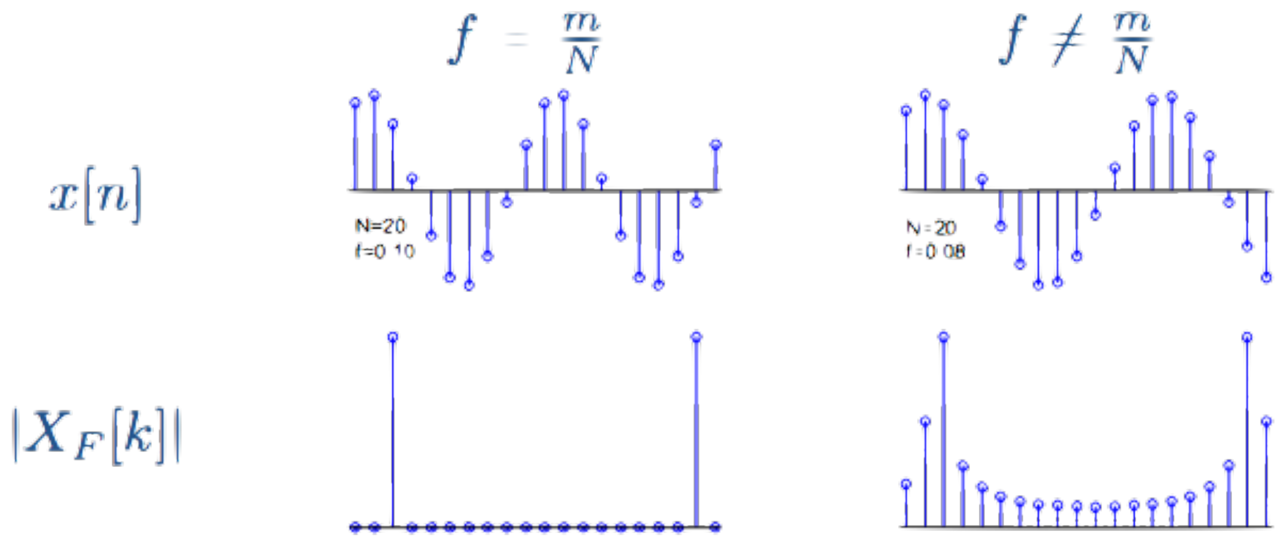
DFT basis functions: $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$



DCT basis functions: $x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$

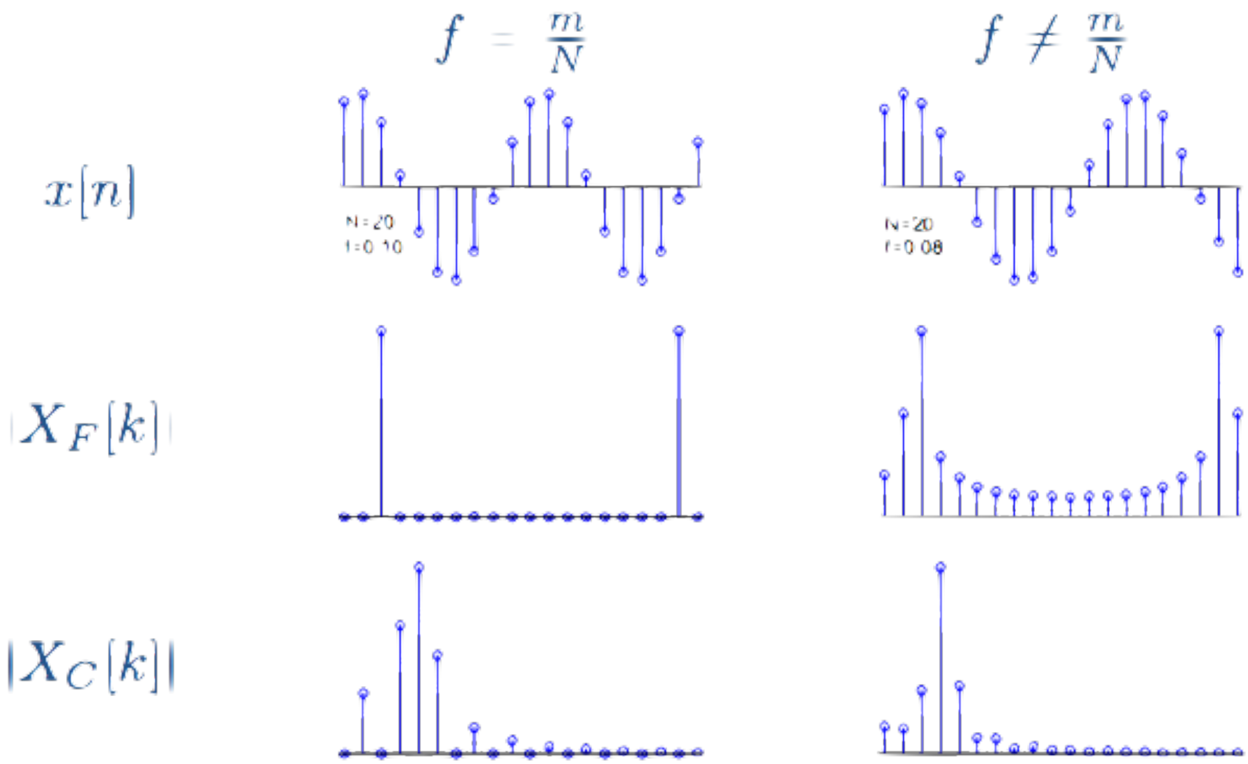


DFT of Sine Wave



DFT: Real→Complex; Freq range $[0, 1]$; Poorly localized unless $f = \frac{m}{N}$; $|X_F[k]| \propto k^{-1}$ for $Nf < k \ll \frac{N}{2}$

DCT of Sine Wave



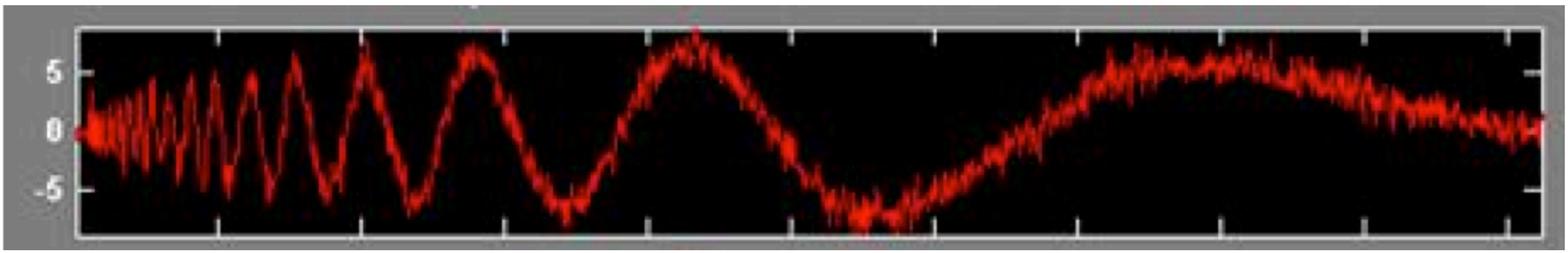
DFT: Real→Complex; Freq range $[0, 1]$; Poorly localized unless $f = \frac{m}{N}$; $|X_F[k]| \propto k^{-1}$ for $Nf < k \ll \frac{N}{2}$

DCT: Real→Real; Freq range $[0, 0.5]$; Well localized $\forall f$; $|X_C[k]| \propto k^{-2}$ for $2Nf < k < N$

Wavelet Transform

Motivation

- Some signals obviously have spectral characteristics that vary with time



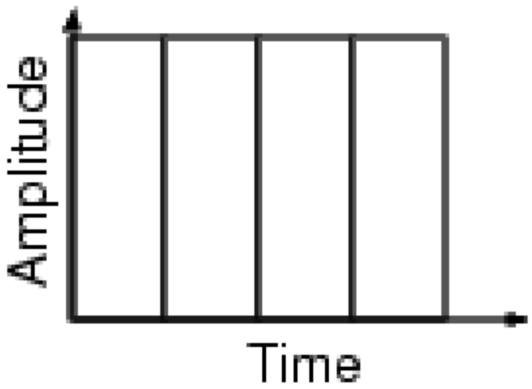
Criticism of Fourier Spectrum

- It's giving you the spectrum of the 'whole time-series'

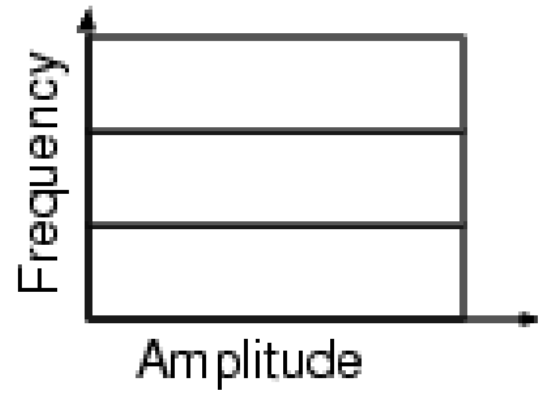
- Which is OK if the time-series is stationary. But what if it's not?

- We need a technique that can "march along" a time series and that is capable of:
  - Analyzing spectral content in different places
  - Detecting sharp changes in spectral character

Transform Comparison

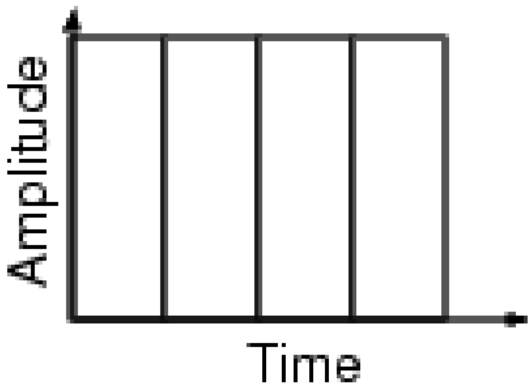


Time Domain (Shannon)

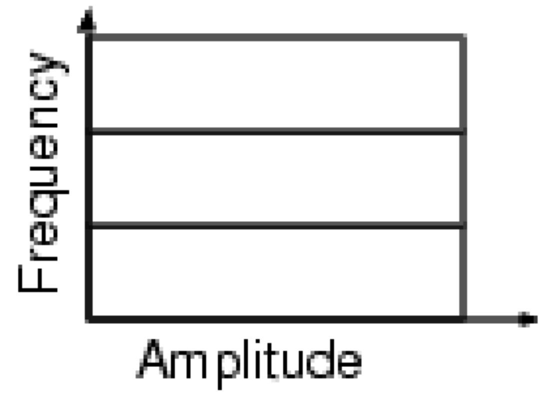


Frequency Domain (Fourier)

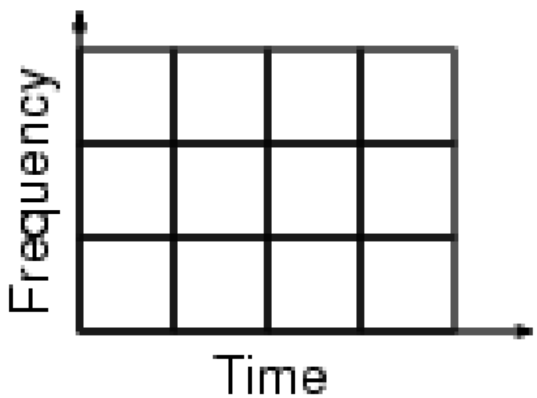
Transform Comparison



Time Domain (Shannon)



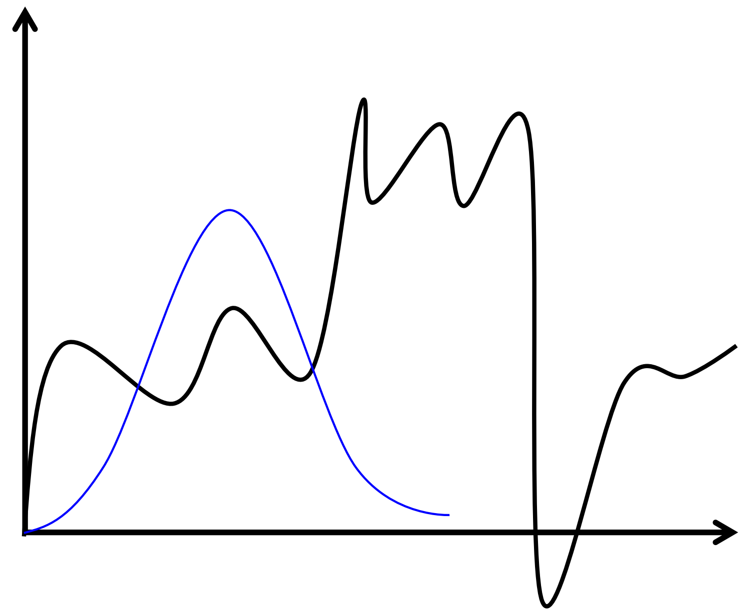
Frequency Domain (Fourier)



STFT (Gabor)

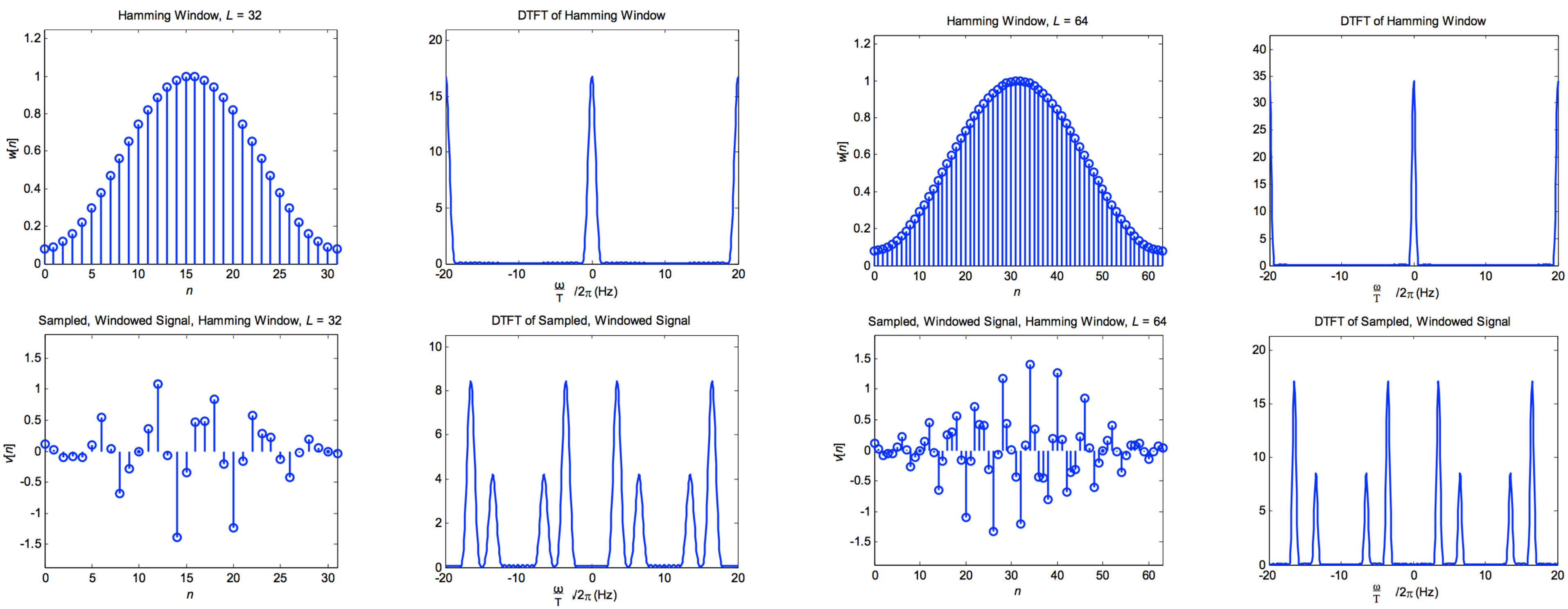
Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)

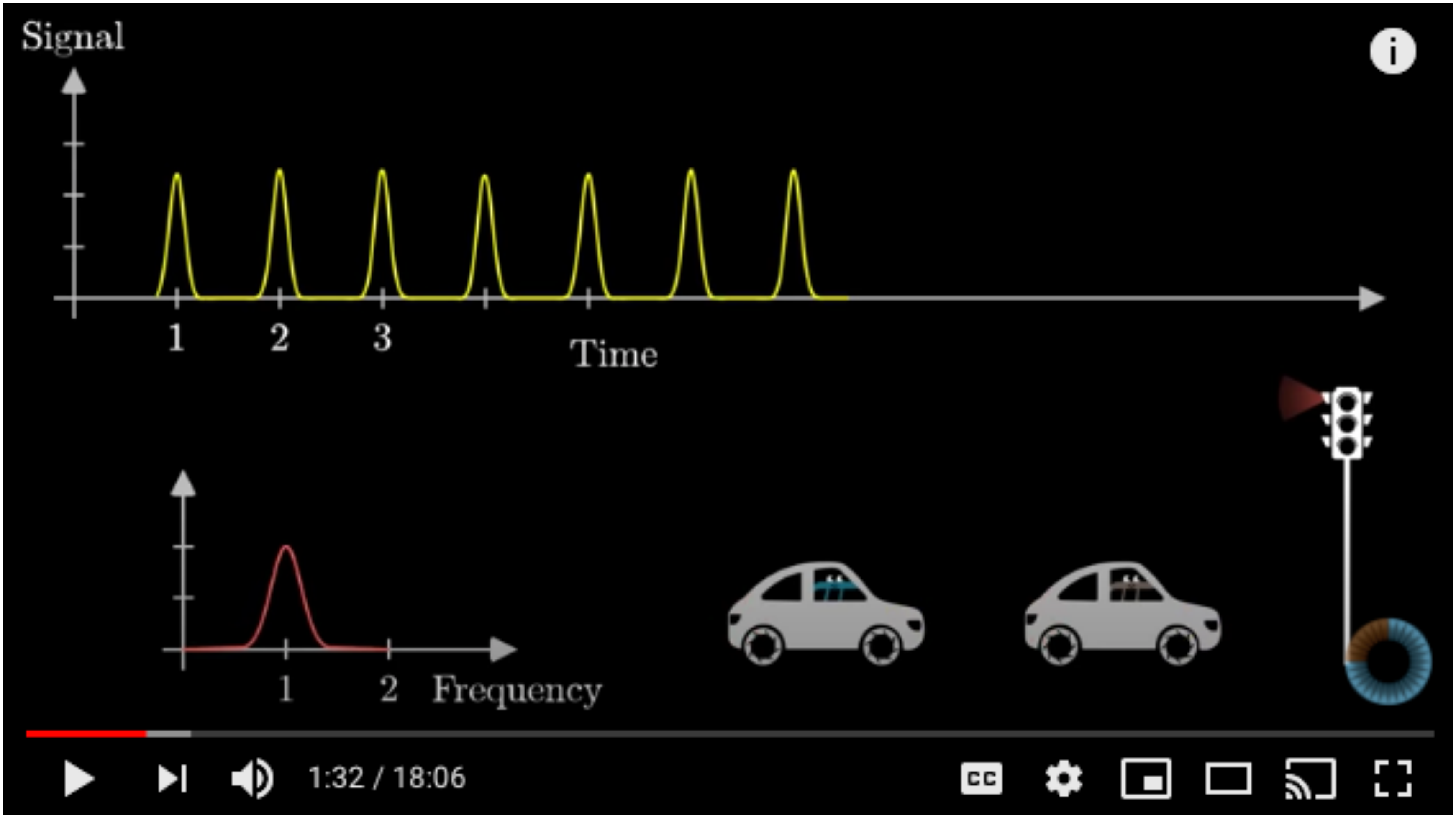


Windowed Sampled CT Signal Example

- As before, the sampling rate is $\Omega s/2\pi=1/T=20Hz$
- Hamming Window, L = 32 vs. L = 64



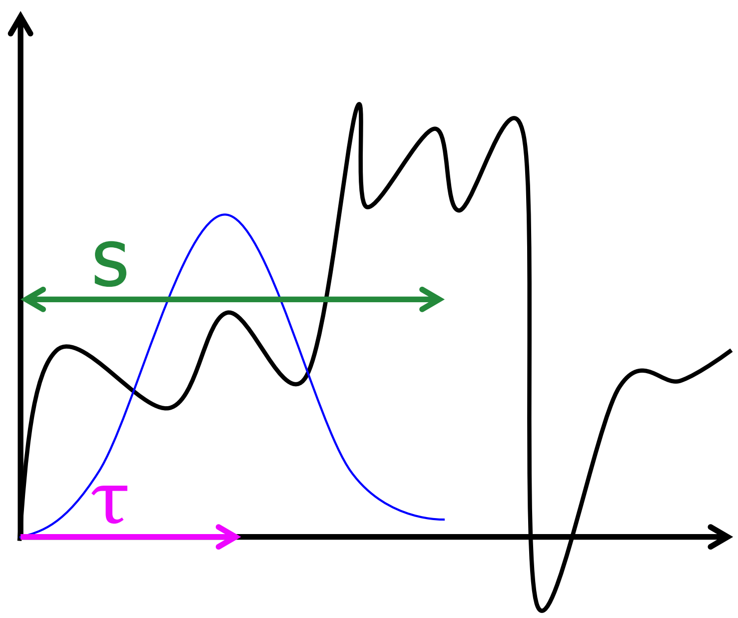
Uncertainty Principle



https://youtu.be/MBnnXbOM5S4?t=49

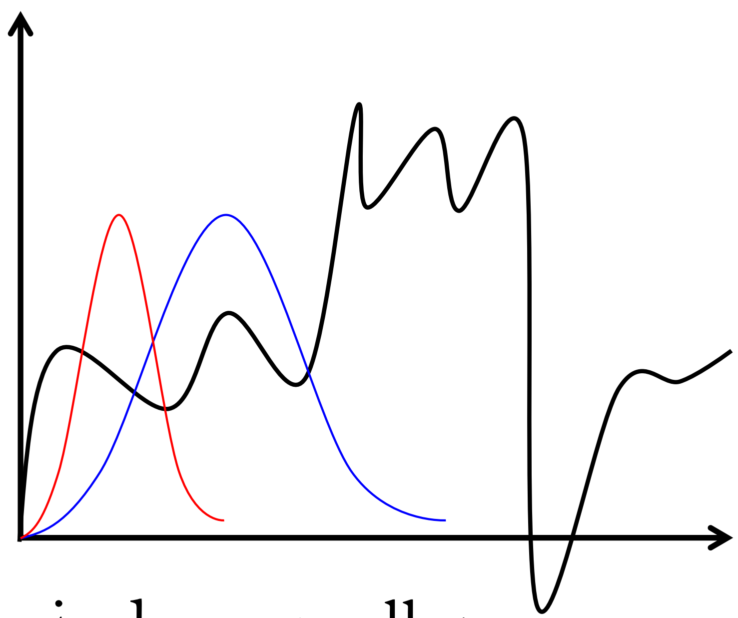
Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)



Discrete Time-Dependent FT

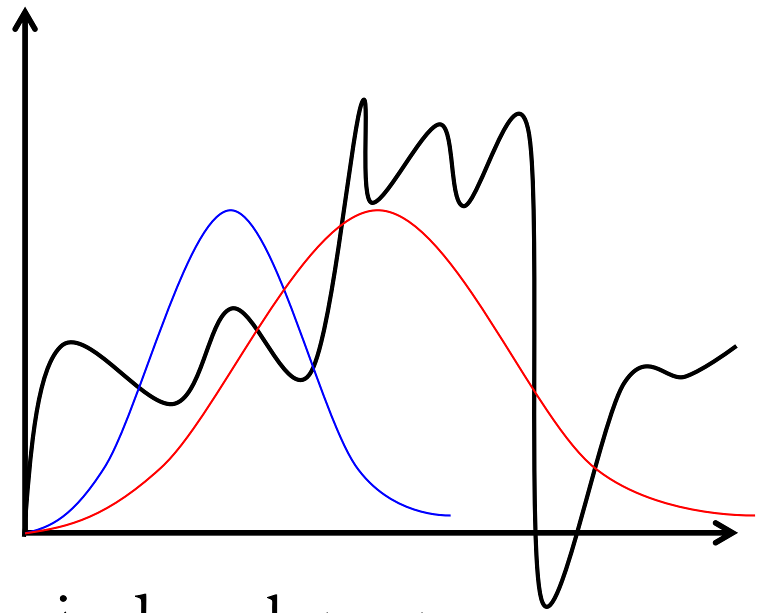
- Fixed window size, shift in time (Gabor)



- Make the window smaller
  - Better localization in time
  - Less spectral resolution

Discrete Time-Dependent FT

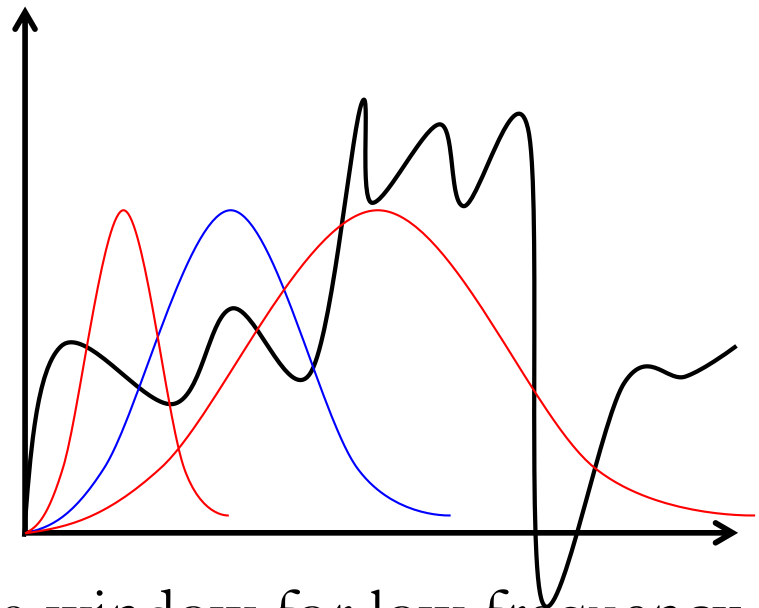
- Fixed window size, shift in time (Gabor)



- Make the window larger
  - Worse localization in time
  - More spectral resolution

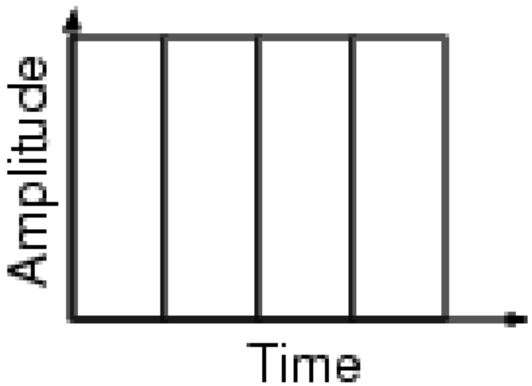
Discrete Time-Dependent FT

- Fixed window size, shift in time (Gabor)

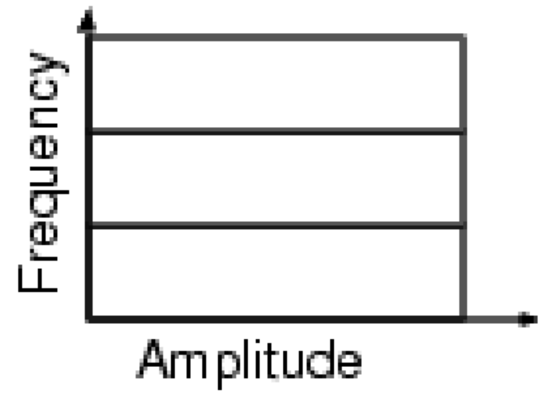


  - Use a big window for low frequency content that is not localized in time
  - Use a small window for high frequency content that is localized in time

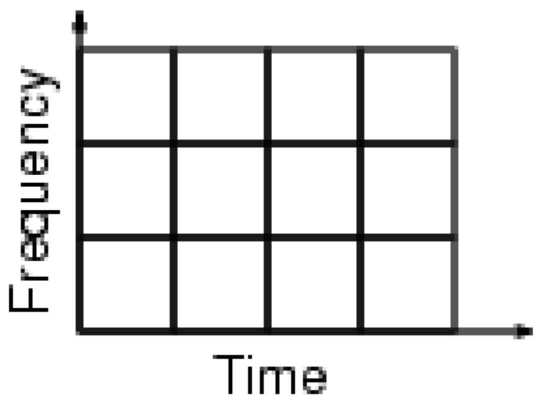
Transform Comparison



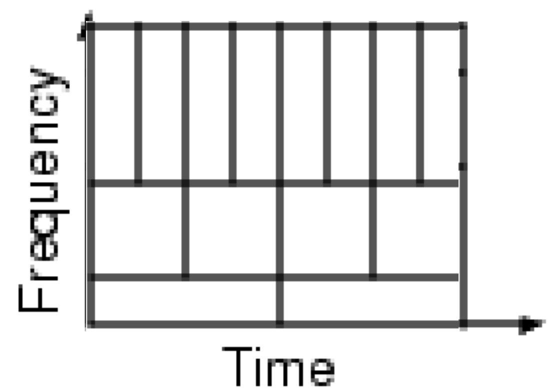
Time Domain (Shannon)



Frequency Domain (Fourier)



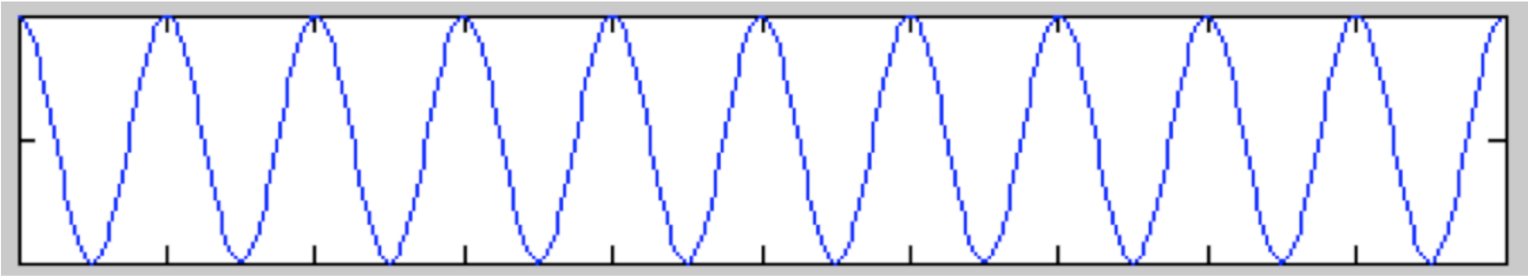
STFT (Gabor)



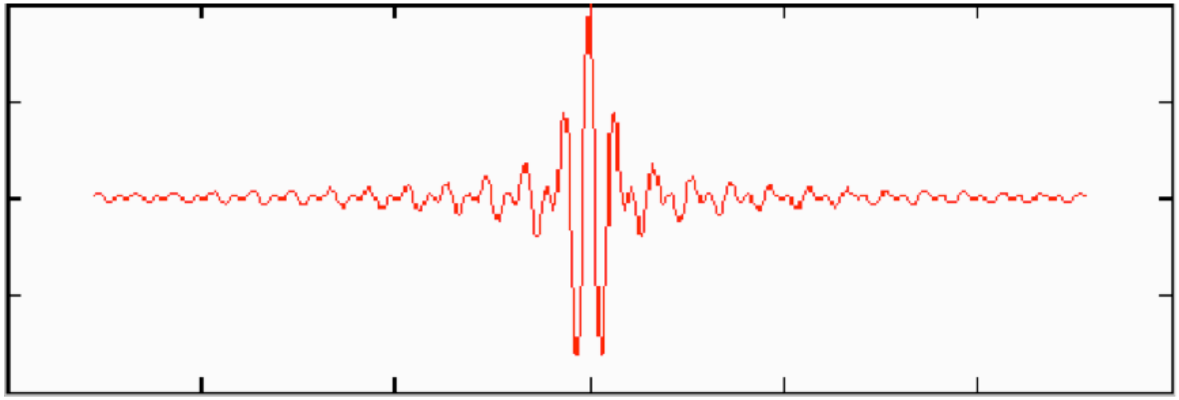
Wavelet Analysis

Fourier vs. Wavelet

- Fourier Analysis is based on an indefinitely long cosine wave of a specific frequency



- Wavelet Analysis is based on a short duration wavelet of a specific center frequency

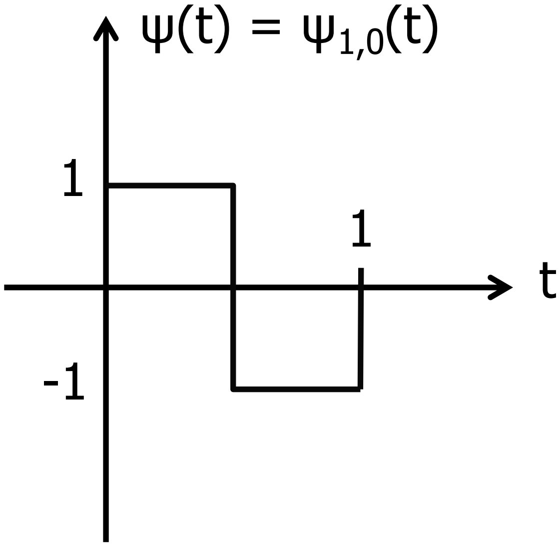


Wavelet Transform

- All wavelets derived from *mother* wavelet

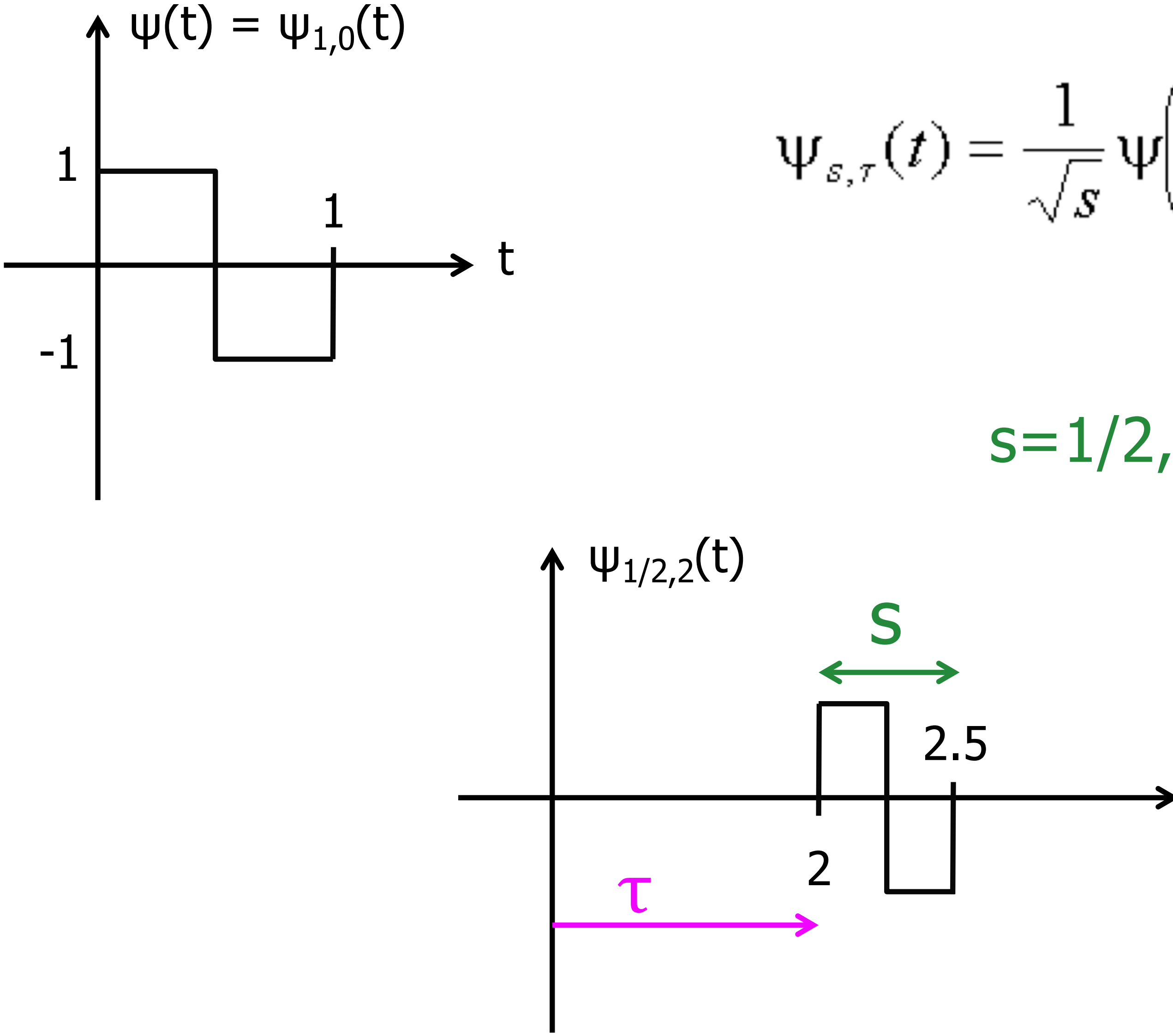
$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

Example: Haar Wavelet



$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

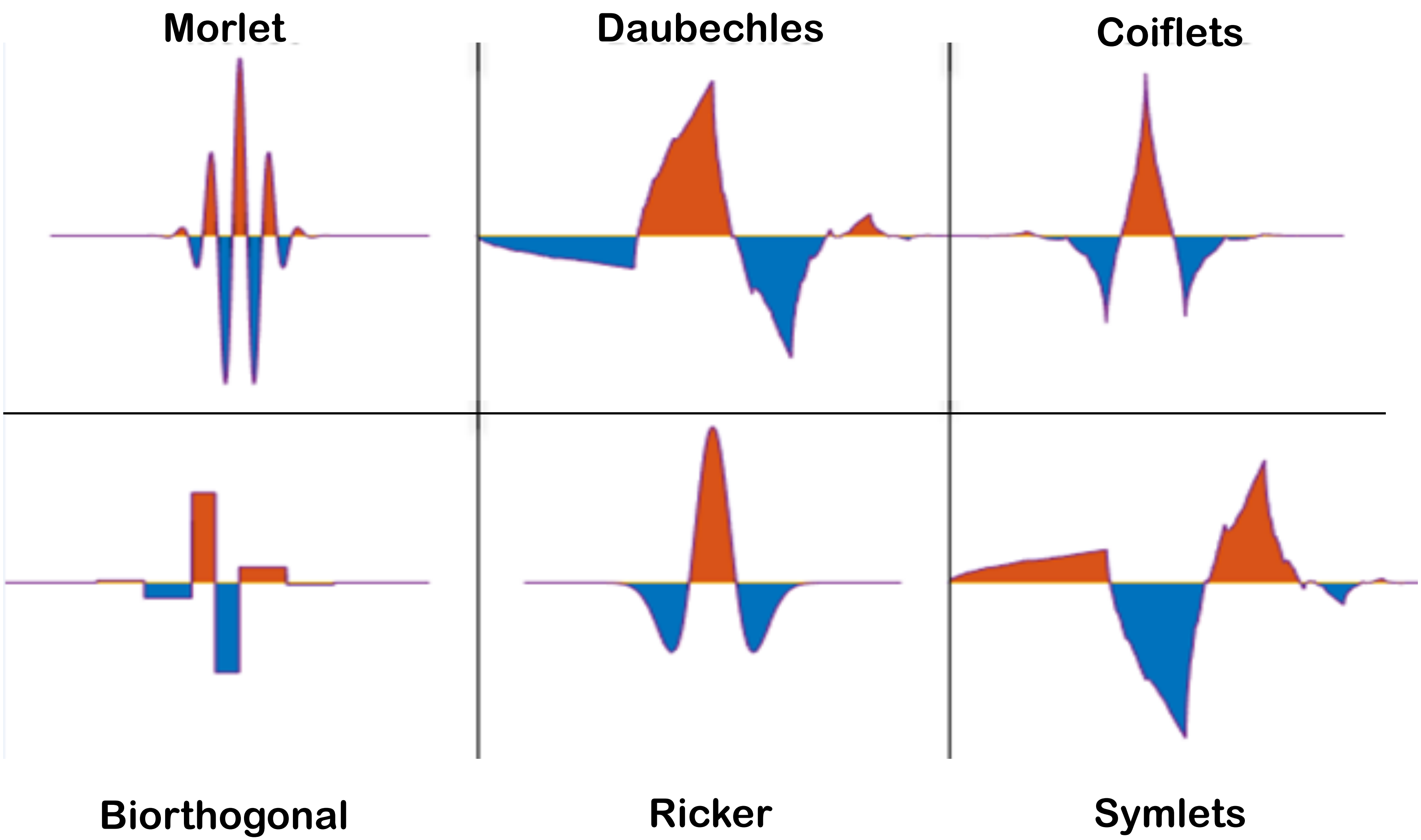
Example: Haar Wavelet



$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

$s=1/2$, $\tau=2$

Examples of Wavelets



Ingrid Daubechies

- Defined worldwide standard for image compression
- https://www.fi.edu/en/laureates/ingrid-daubechies#:~:text=Her%20contributions%20have%20revolutionized%20and,the%20JPEG2000%20image%20processing%20standard.

Wavelet – Scaled and Shifted

normalization

shift in time

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\,\psi\!\left(\frac{t-\tau}{s}\right)$$

change in scale

wavelet with
scale, s and
translation, τ

Mother wavelet

Continuous Wavelet Transform

time-series

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)\,dt$$

coefficient of wavelet
with
scale, s and time, τ

wavelet with
scale, s, and shift, τ

Inverse Wavelet Transform

- Build up a time-series as sum of wavelets of different scales, s, and positions, t

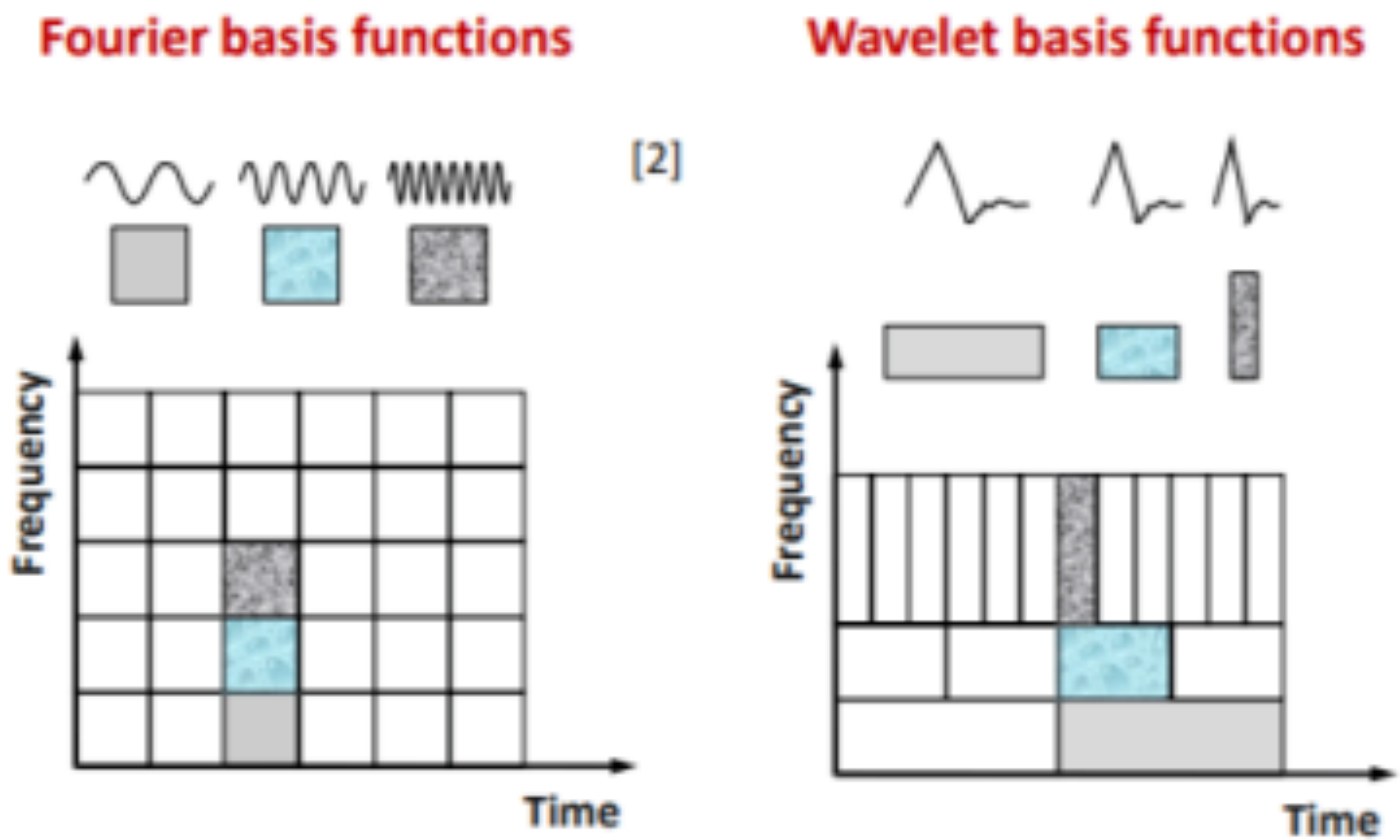
$$f(t) = \int\int \gamma(s,\tau)\psi_{s,\tau}(t)\,d\tau\,ds$$

time-series

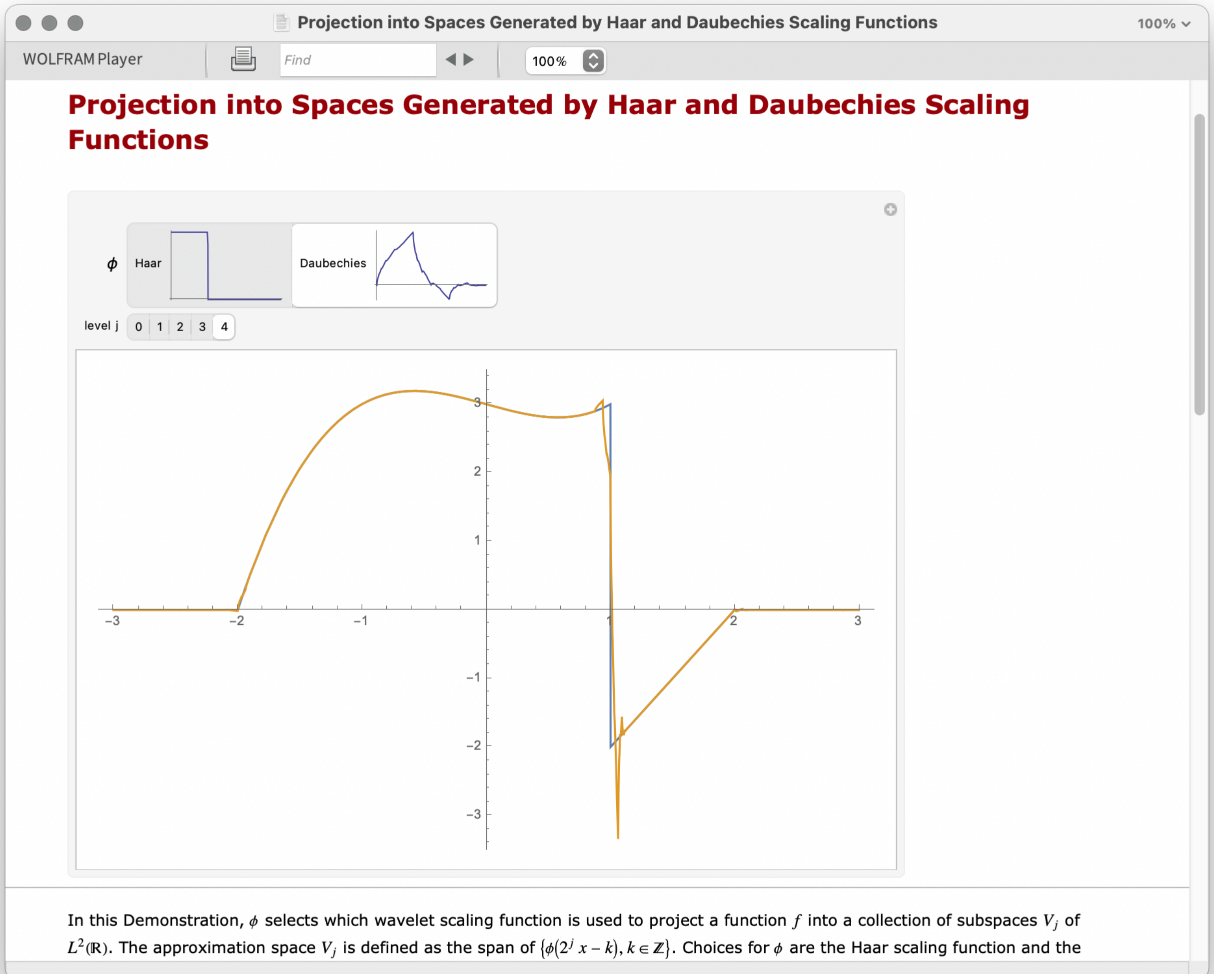
coefficients of wavelets

wavelet with scale, s and time, τ

Wavelet Basis Functions



Wavelet reconstruction demo



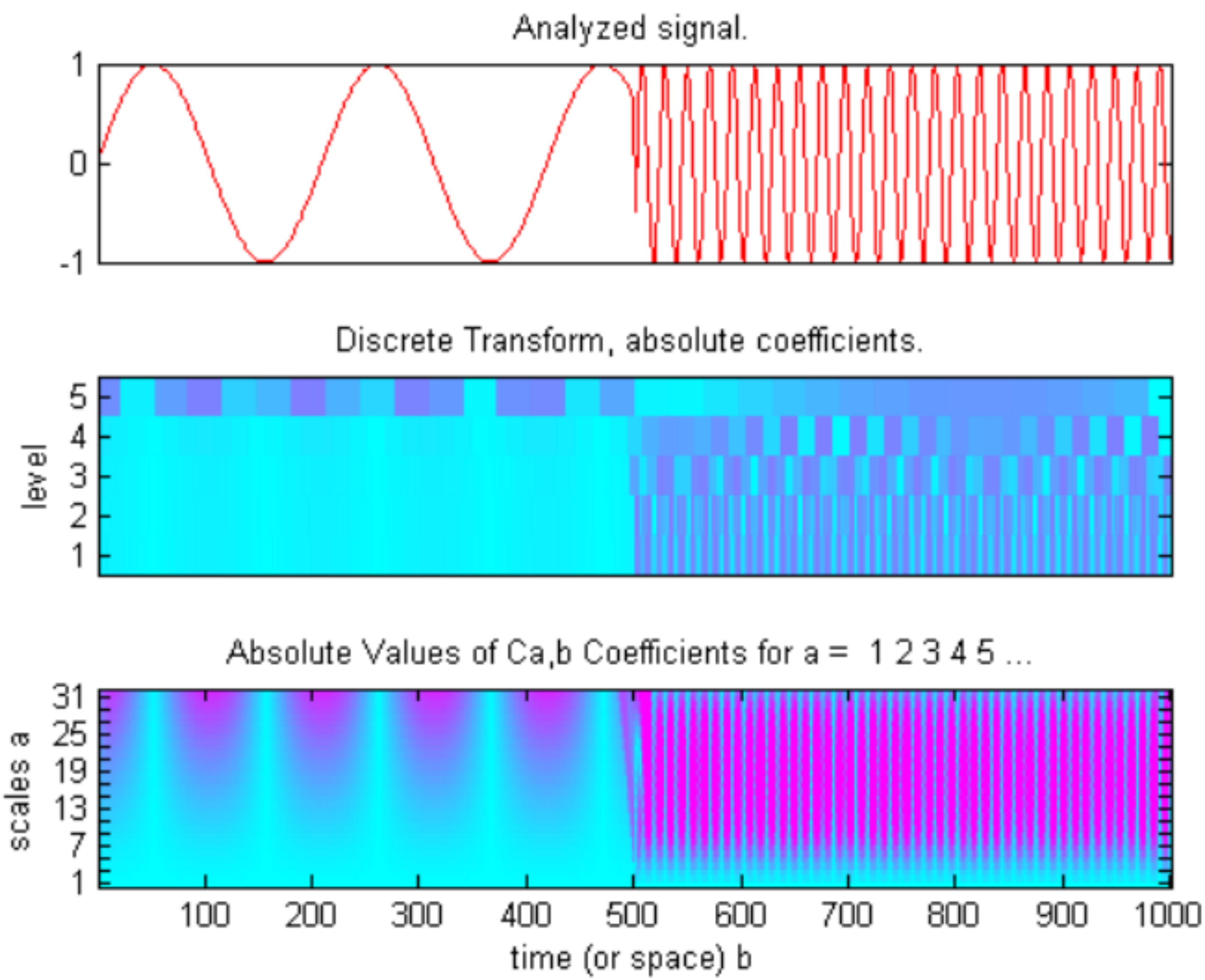
https://demonstrations.wolfram.com/ProjectionIntoSpacesGeneratedByHaarAndDaubechiesScalingFunct/

Discrete wavelets:

- Scale wavelets only by integer powers of 2
  - $s_j = 2^j$
- And shifting by integer multiples of s_j for each successive scale
  - $\tau_{j,k} = k2^j$
- Then $\gamma(s_j, \tau_{j,k}) = \gamma_{jk}$
  - where $j = 1, 2, \ldots\infty$, and $k = -\infty \ldots -2, -1, 0, 1, 2, \ldots \infty$

$$\gamma_{j,k} = \frac{1}{\sqrt{2^j}} \int f(t) \Psi\left(\frac{t - k2^j}{2^j} \right) dt$$

DWT vs CWT



Analyzed signal.
1
0
-1
Discrete Transform, absolute coefficients.
level
5
4
3
2
1
Absolute Values of Ca,b Coefficients for a = 1 2 3 4 5 ...
scales a
31
25
19
13
7
1
100
200
300
400
500
600
700
800
900
1000
time (or space) b

Wavelet Transform

- Determining the wavelet coefficients for a fixed scale, *s*, can be thought of as a filtering operation

$$\gamma(s,\tau) = \int f(t)\Psi_{s,\tau}(t)dt$$

$$\gamma_s(\tau) = \int f(t)\Psi_s(t-\tau)dt = f(\tau) * \Psi_s(-\tau)$$

- where

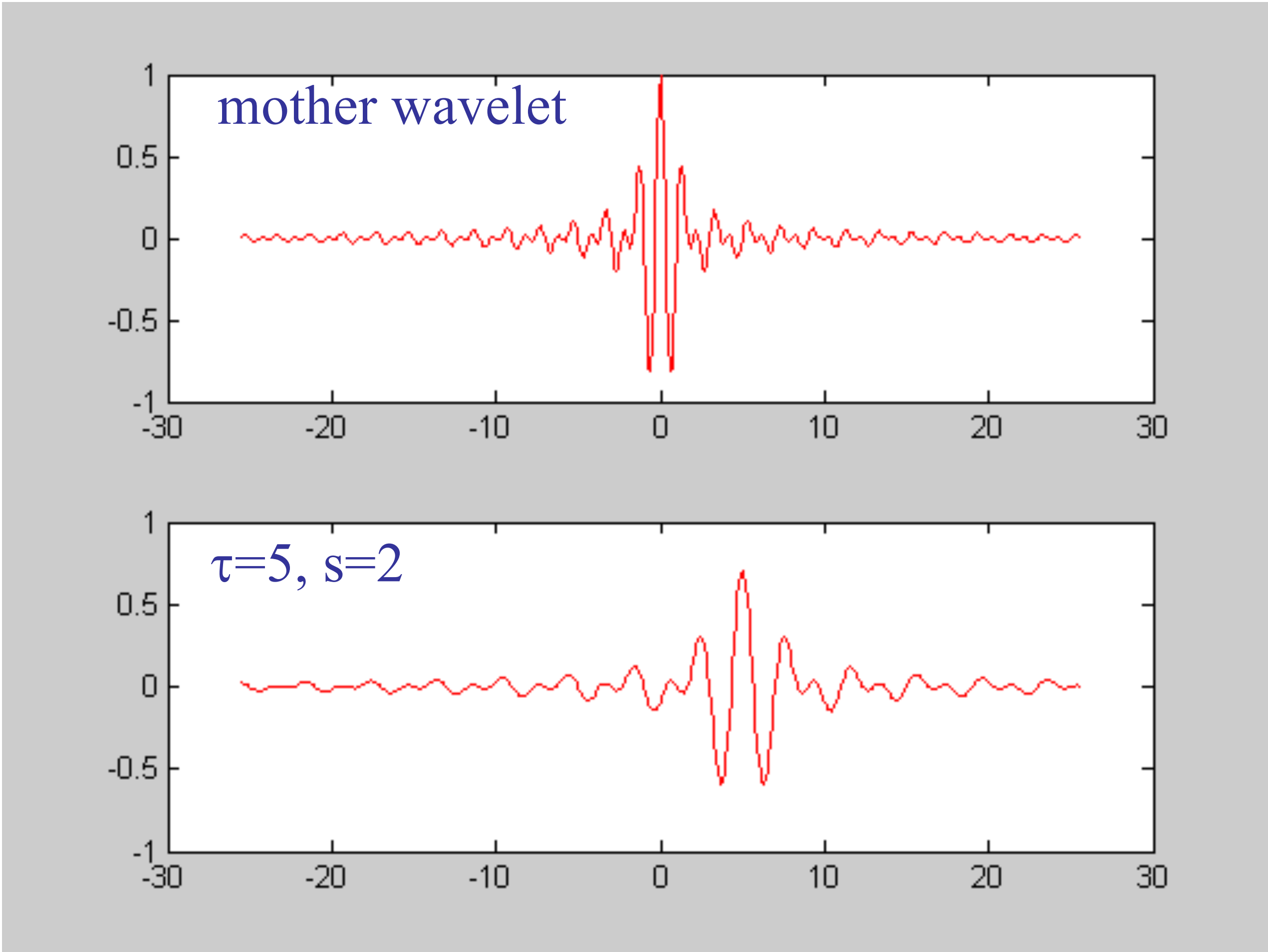
If wavelet is even,
$\Psi(-\tau) = \Psi(\tau)$

$$\Psi_s(t) = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

Shannon Wavelet

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

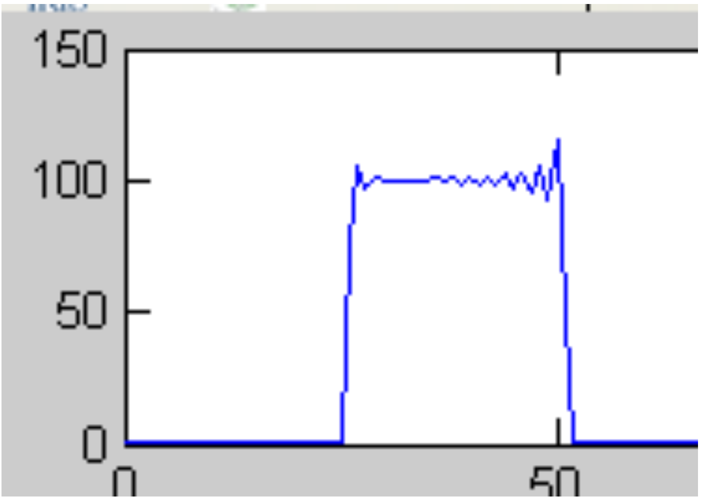
- $\Psi(t) = 2\ \mathrm{sinc}(2t) - \mathrm{sinc}(t)$



time

Fourier spectrum of Shannon Wavelet

$\Psi_s(j\Omega)$



frequency, Ω

- Wavelet coefficients are a result of bandpass filtering

Discrete Wavelet Transform

- The coefficients of Ψ is just the band-pass filtered time-series, where Ψ is the wavelet, now viewed as the impulse response of a bandpass filter.
  - Discrete wavelet → $s = 2^j$

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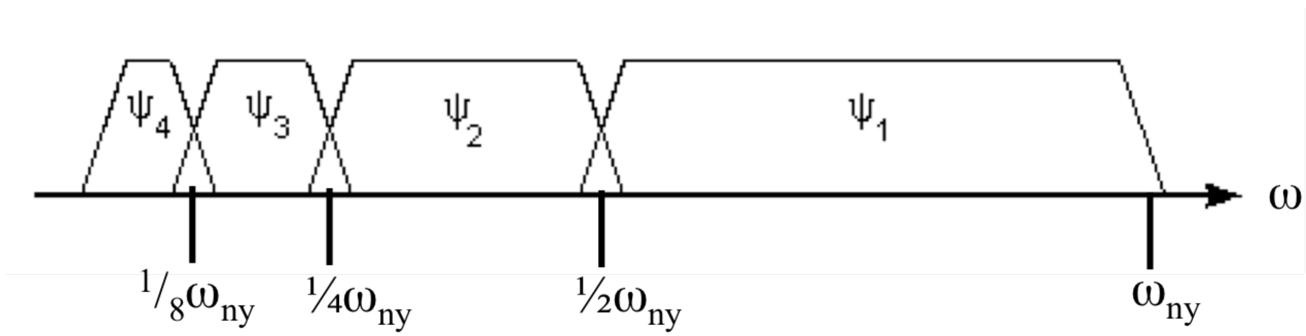

Discrete Wavelet Transform

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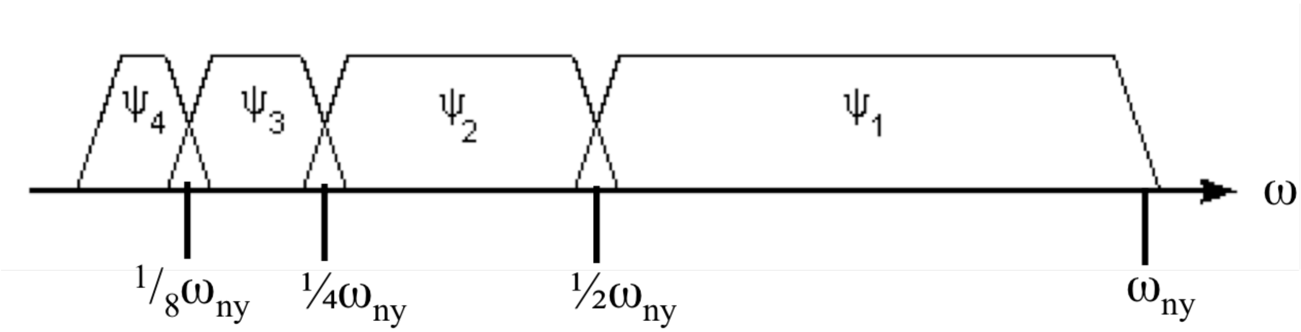
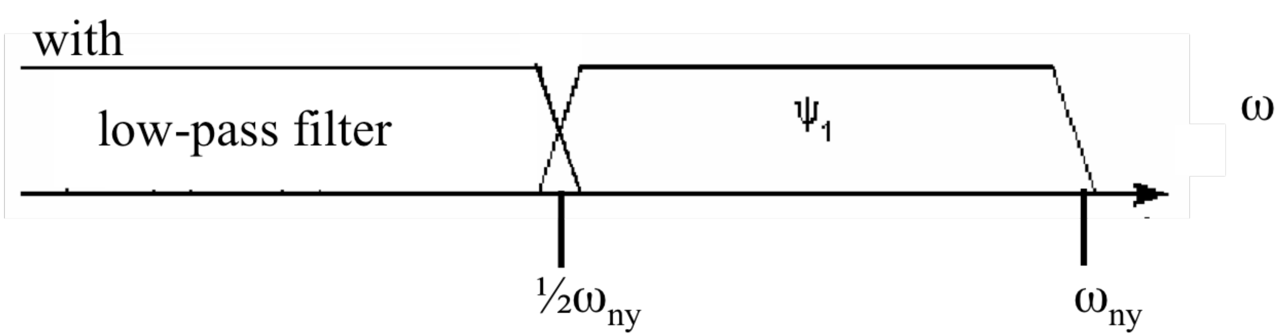
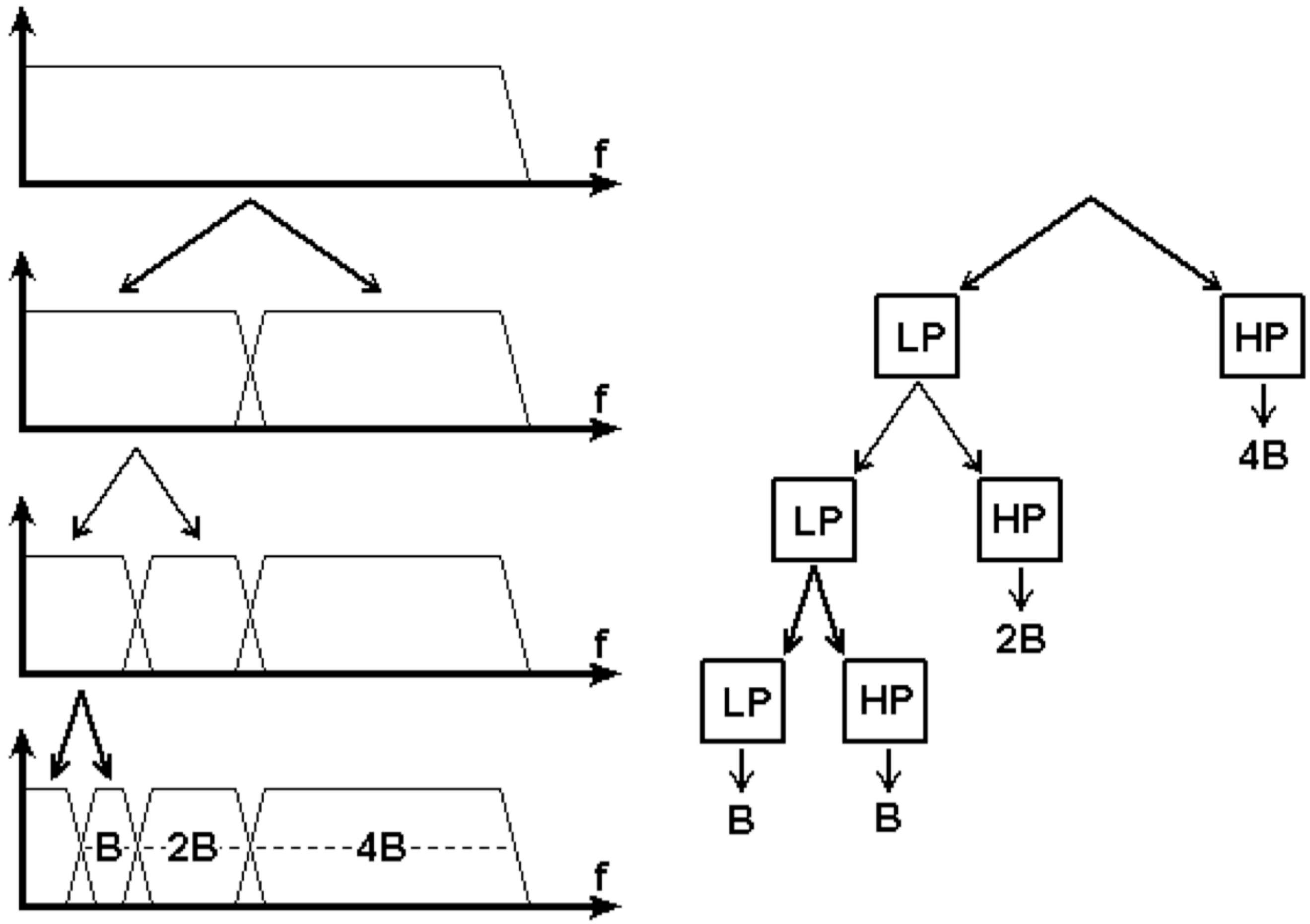


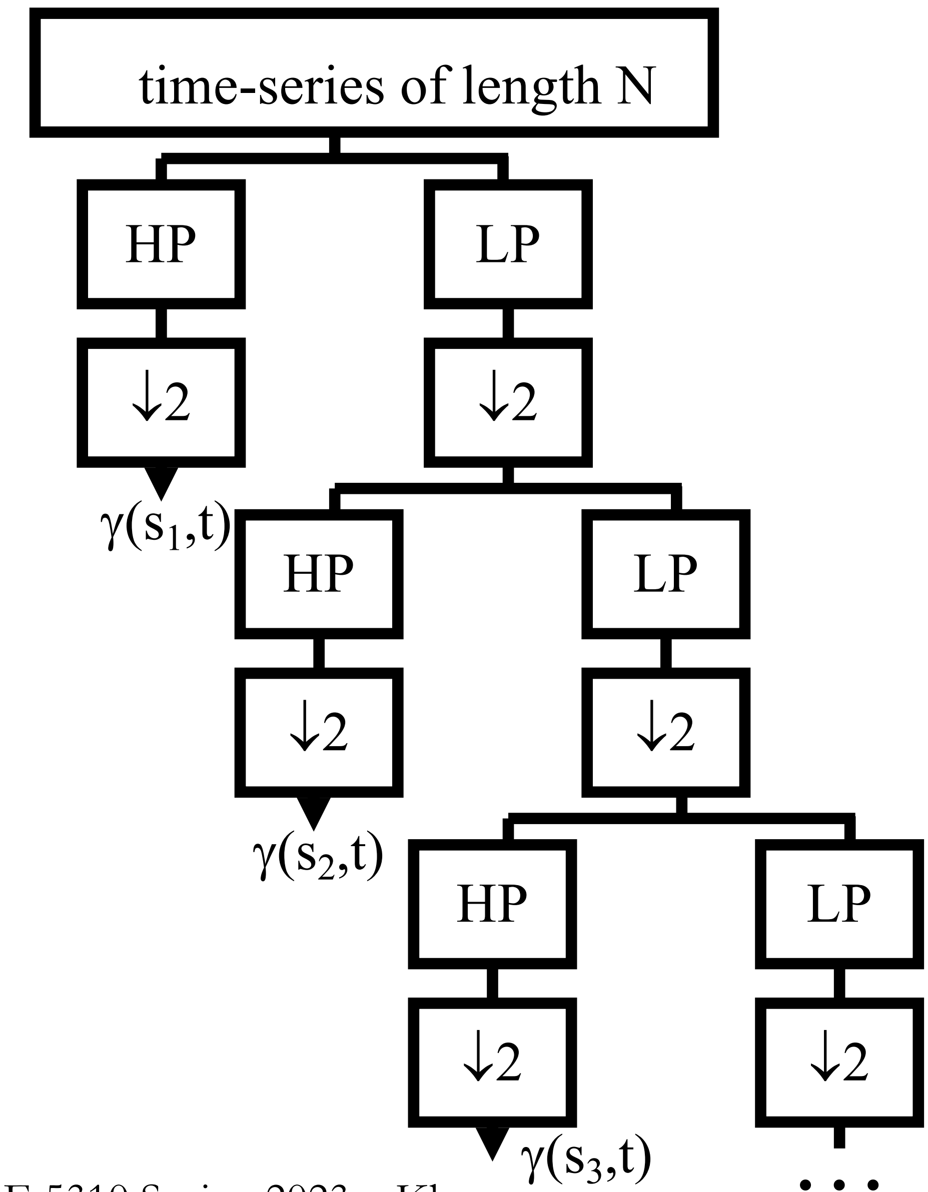


Digital Wavelet as Multirate Filter Bank

- Repeat recursively!



Digital Wavelet as Multirate Filter Bank



$\gamma(s_1,t)$: N/2 coefficients

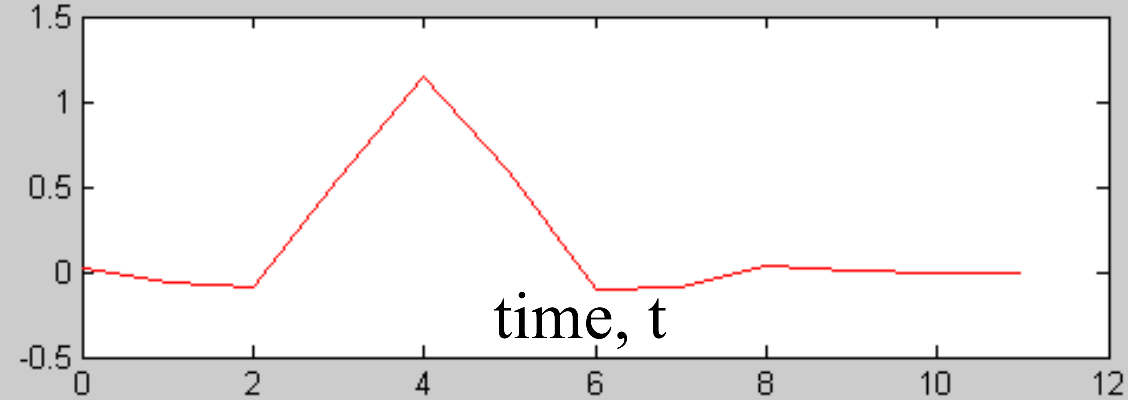
$\gamma(s_2,t)$: N/4 coefficients

$\gamma(s_2,t)$: N/8 coefficients

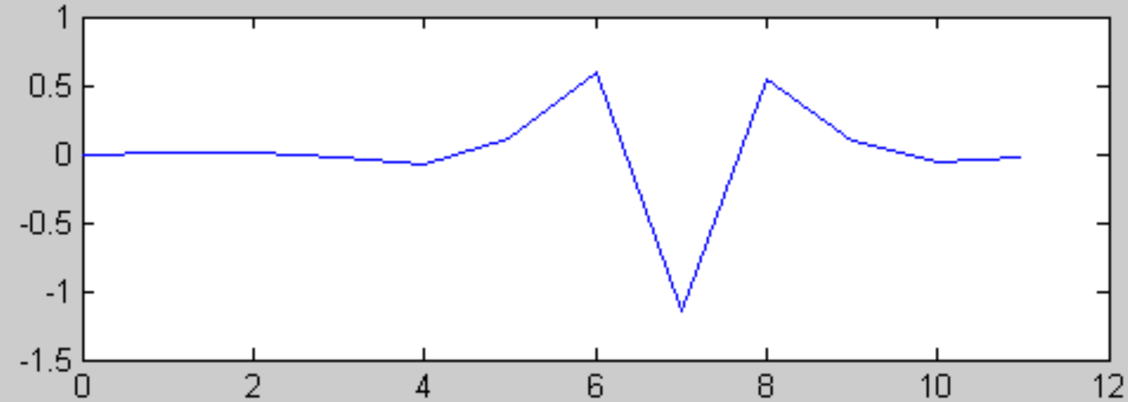
Total: N coefficients

Impulse Responses

Coiflet low pass filter



Coiflet high-pass filter

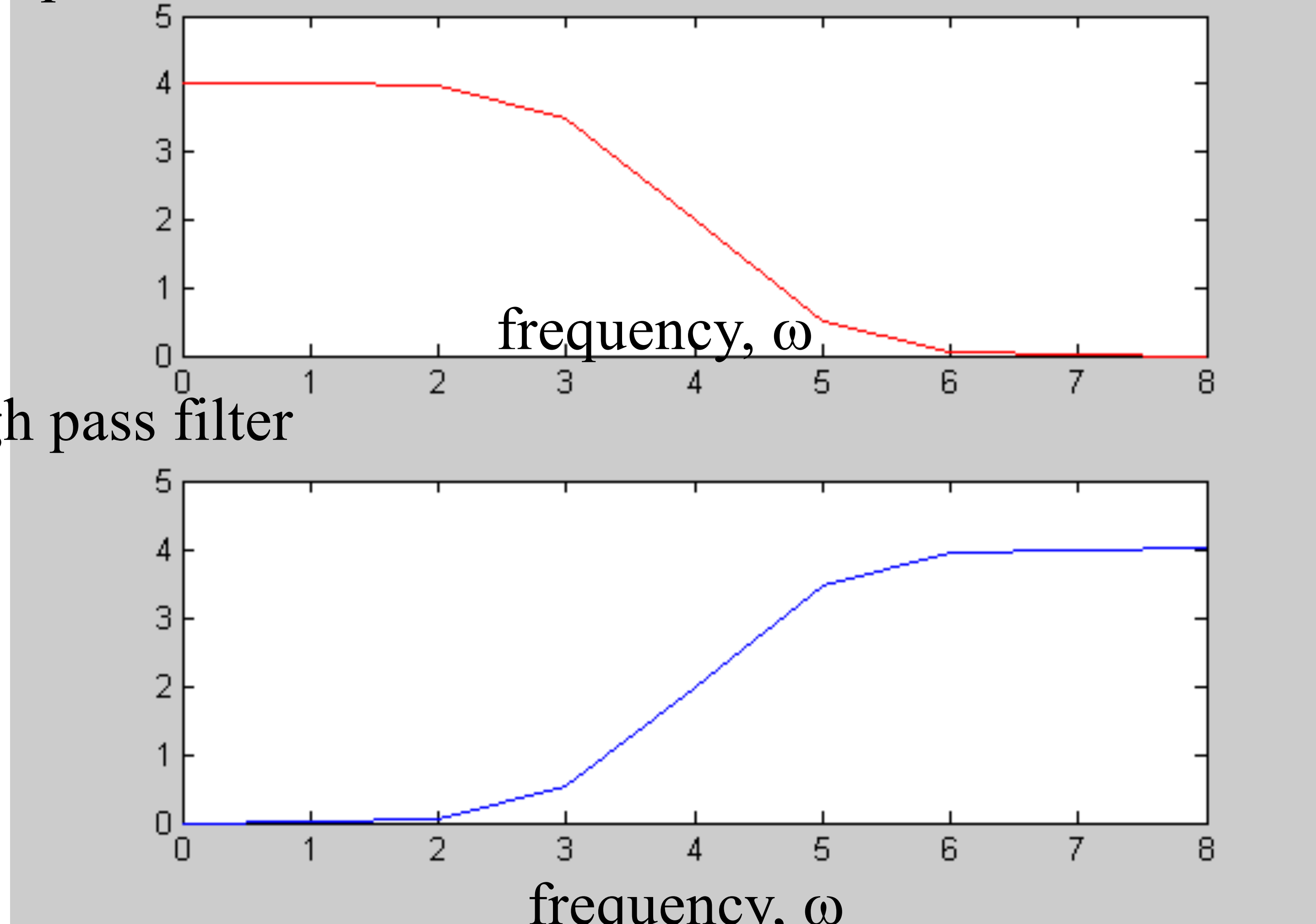


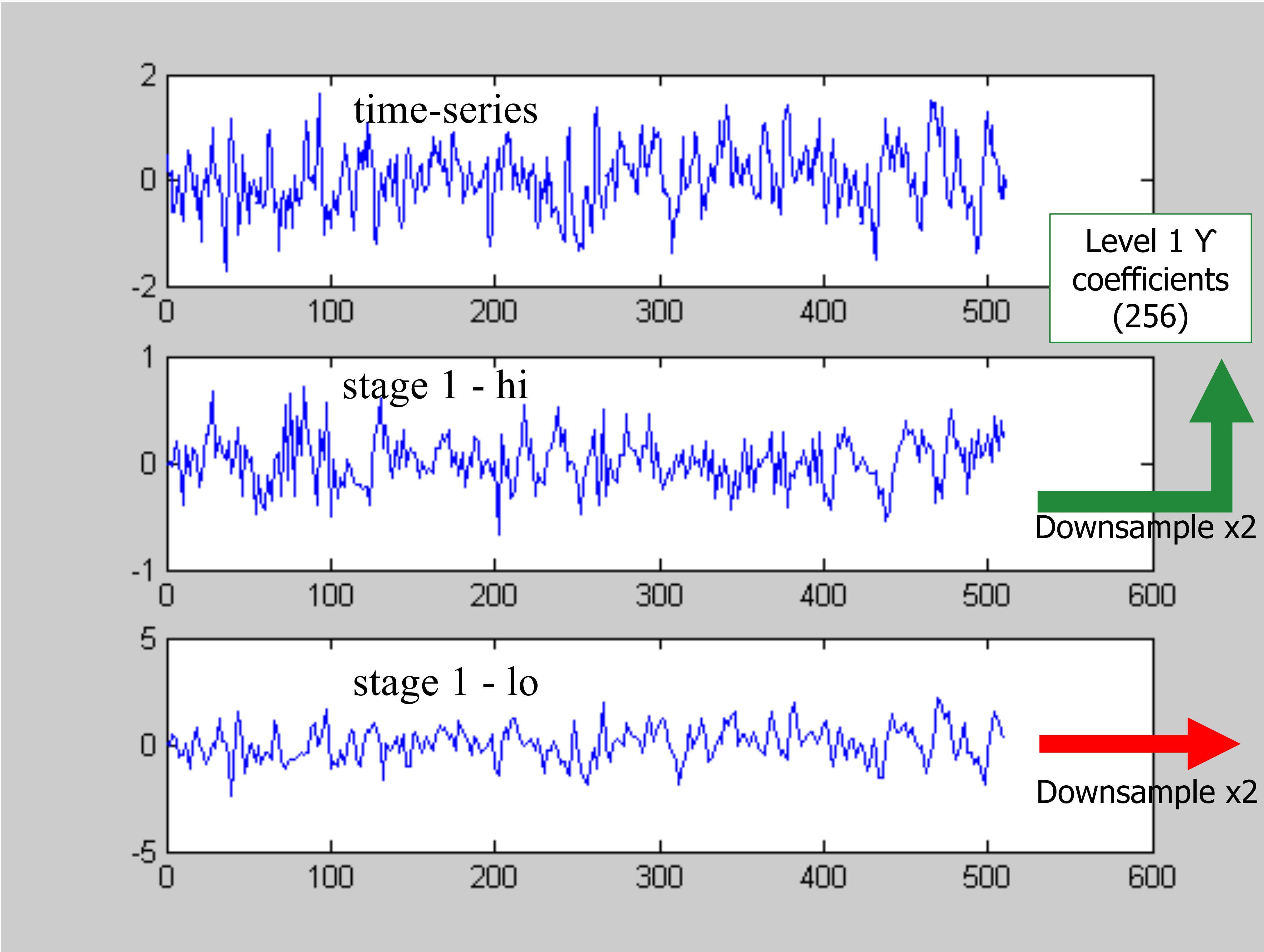
time, t

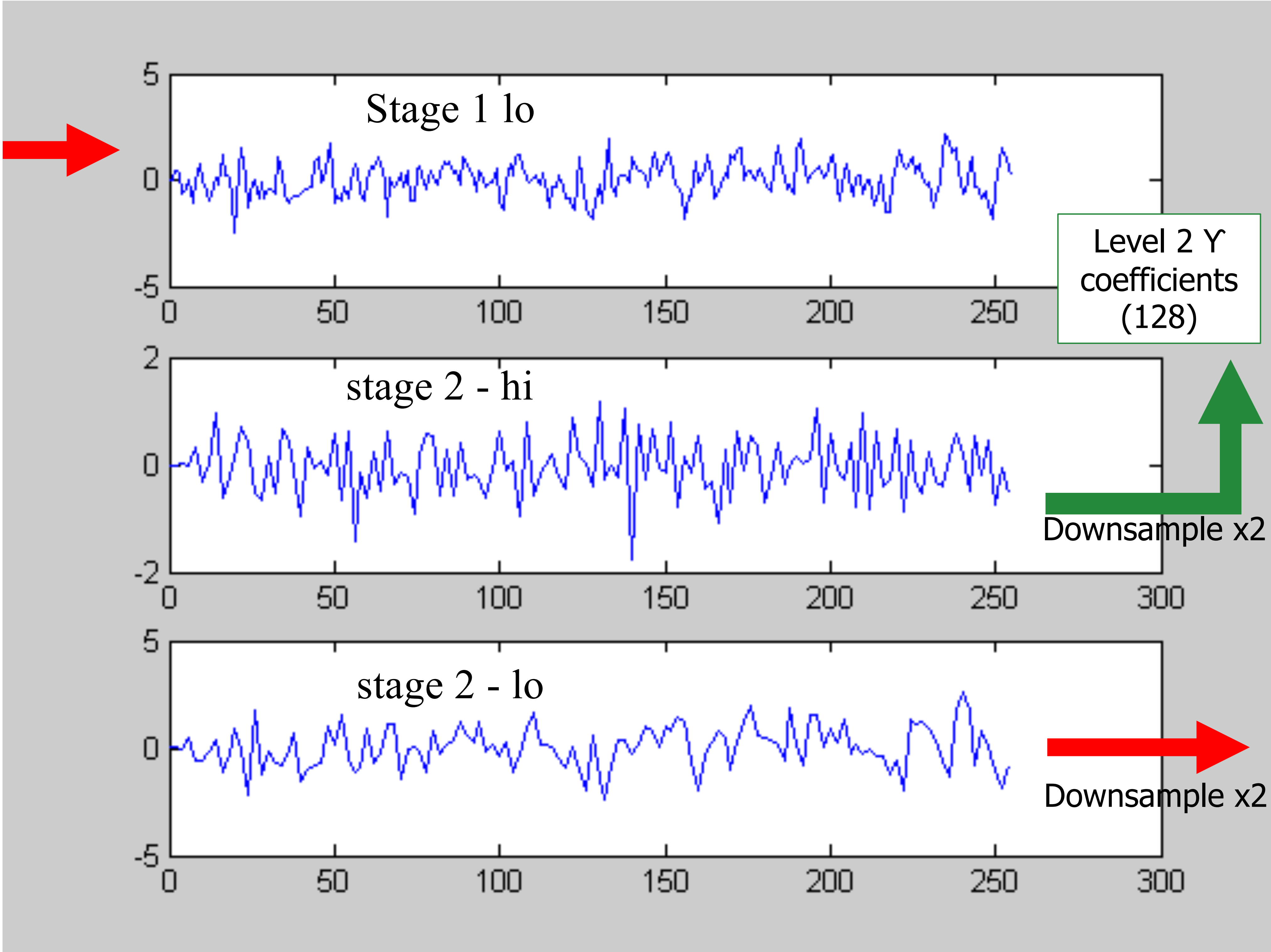
Filter Responses

Spectrum of low pass filter

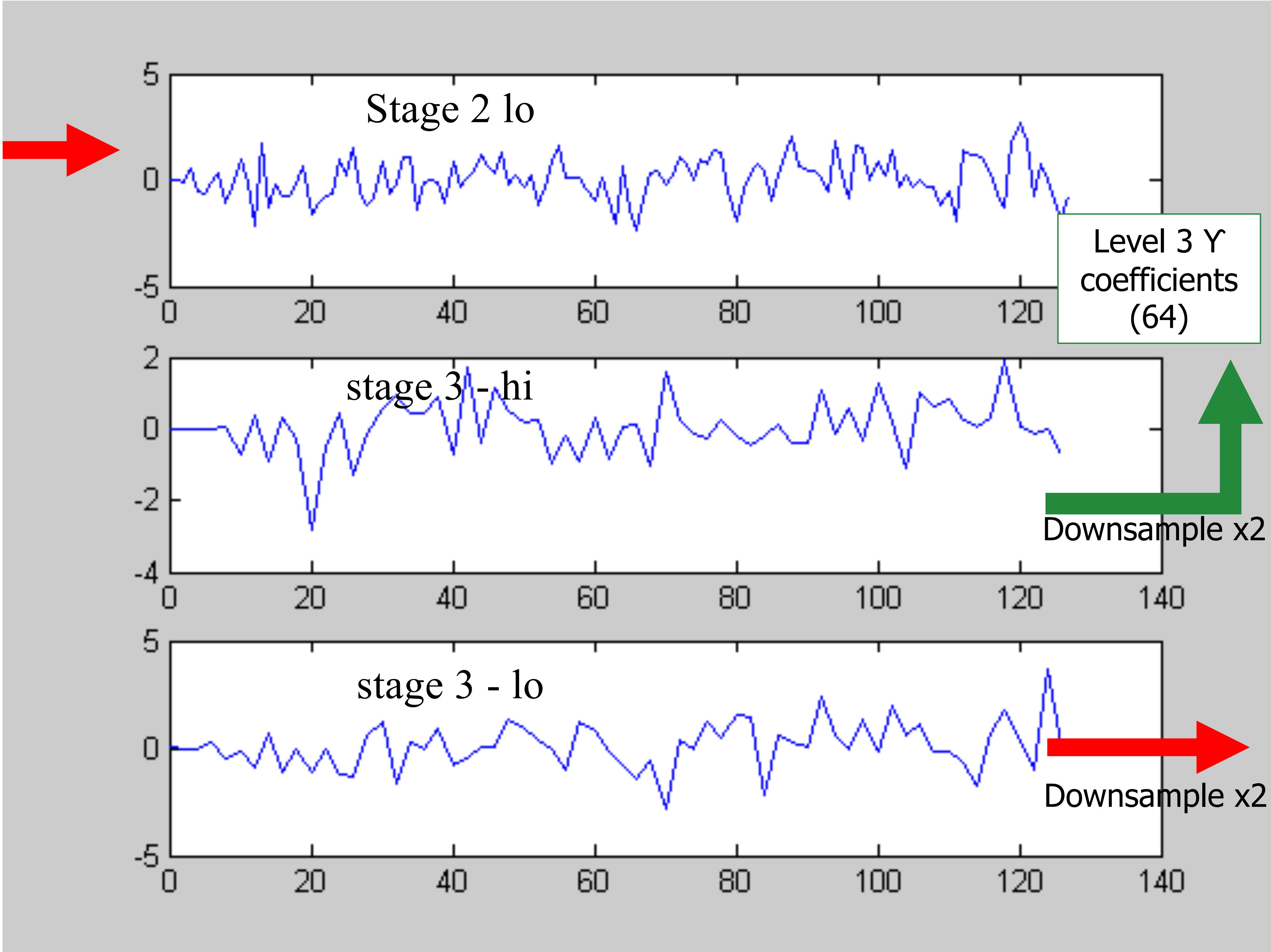
Spectrum of high pass filter

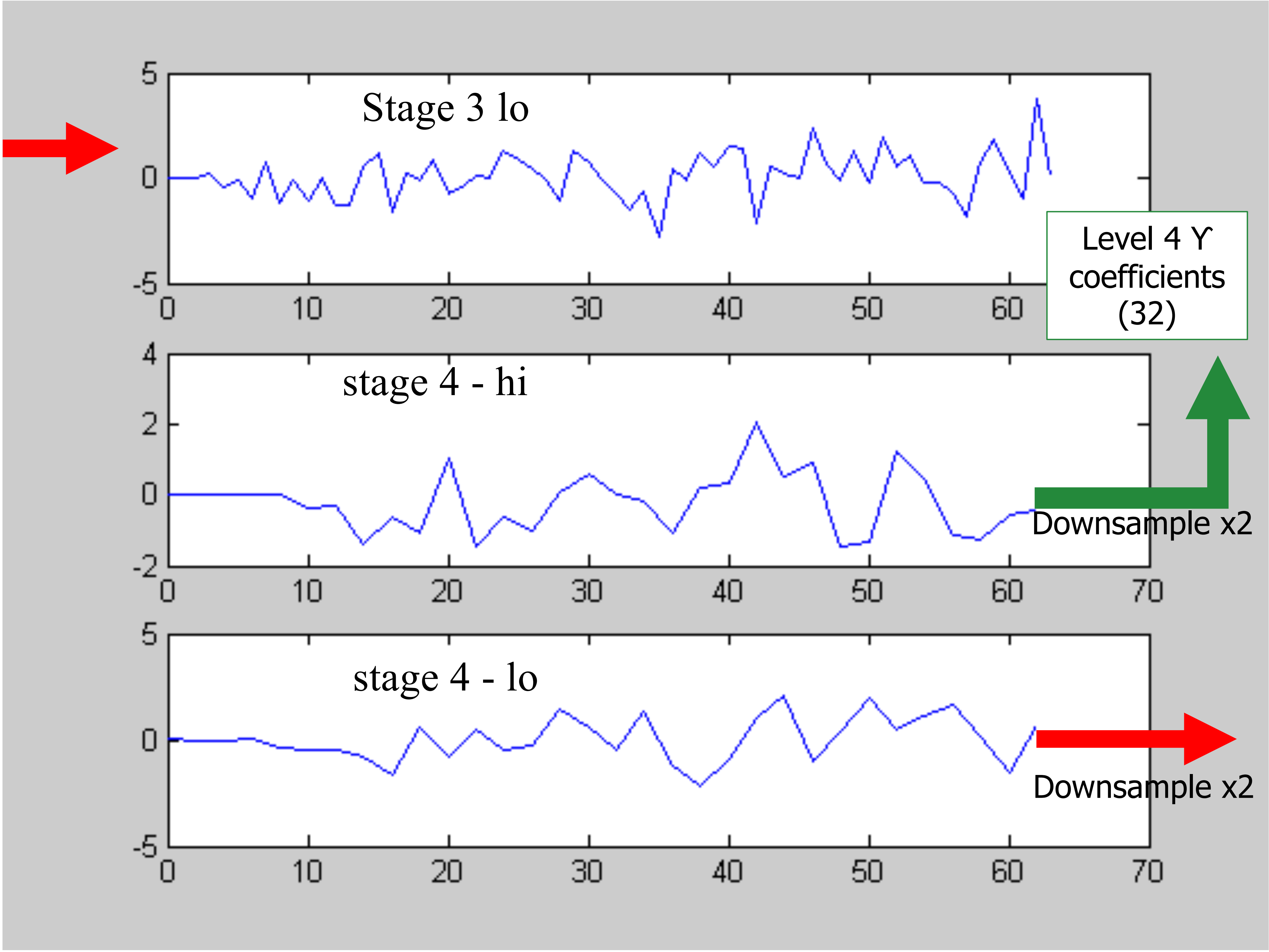


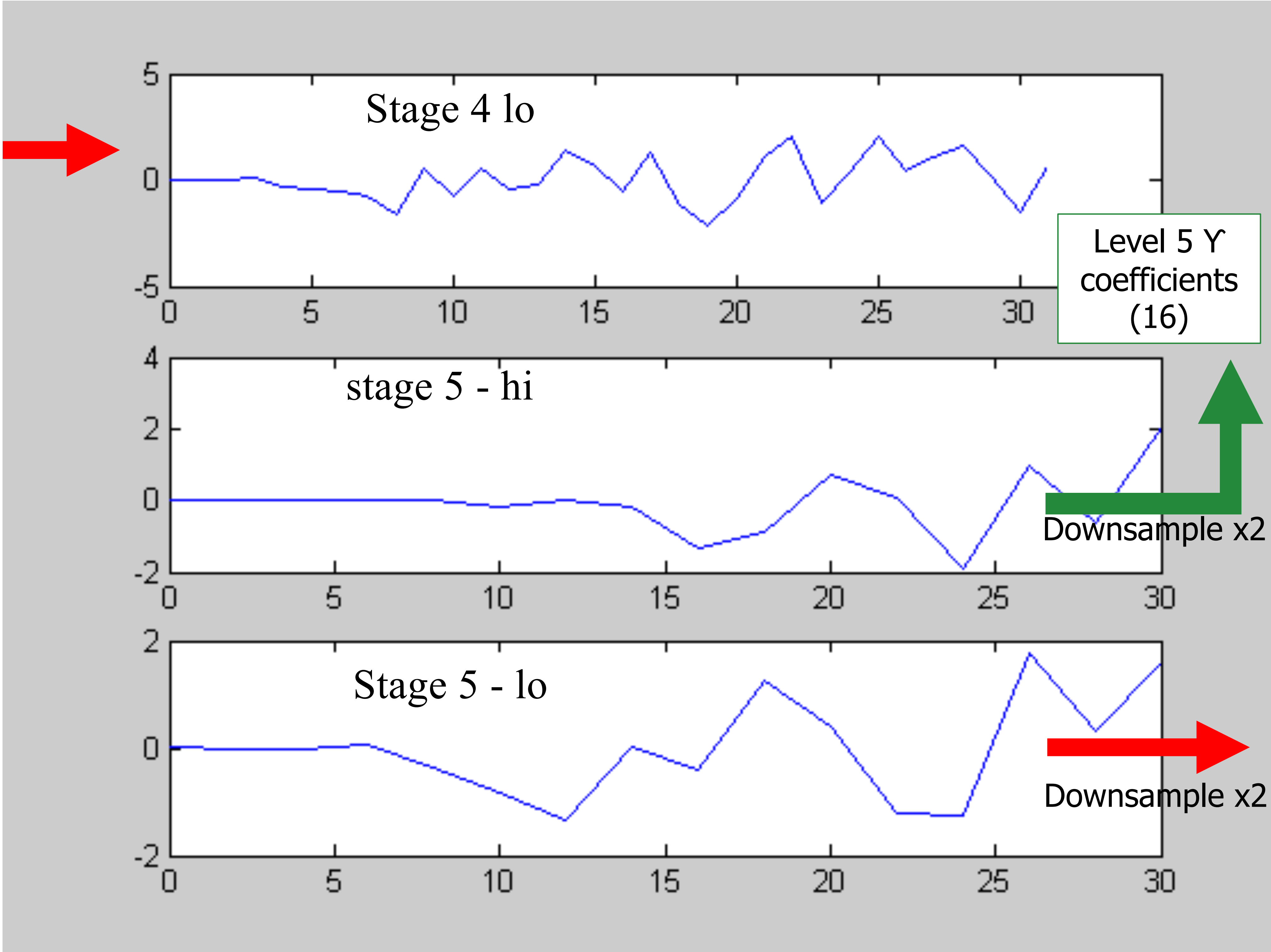

time-series
stage 1 - hi
stage 1 - lo
Level 1 Υ coefficients (256)
Downsample x2
Downsample x2
2
0
-2
1
0
-1
5
0
-5
0
100
200
300
400
500
600

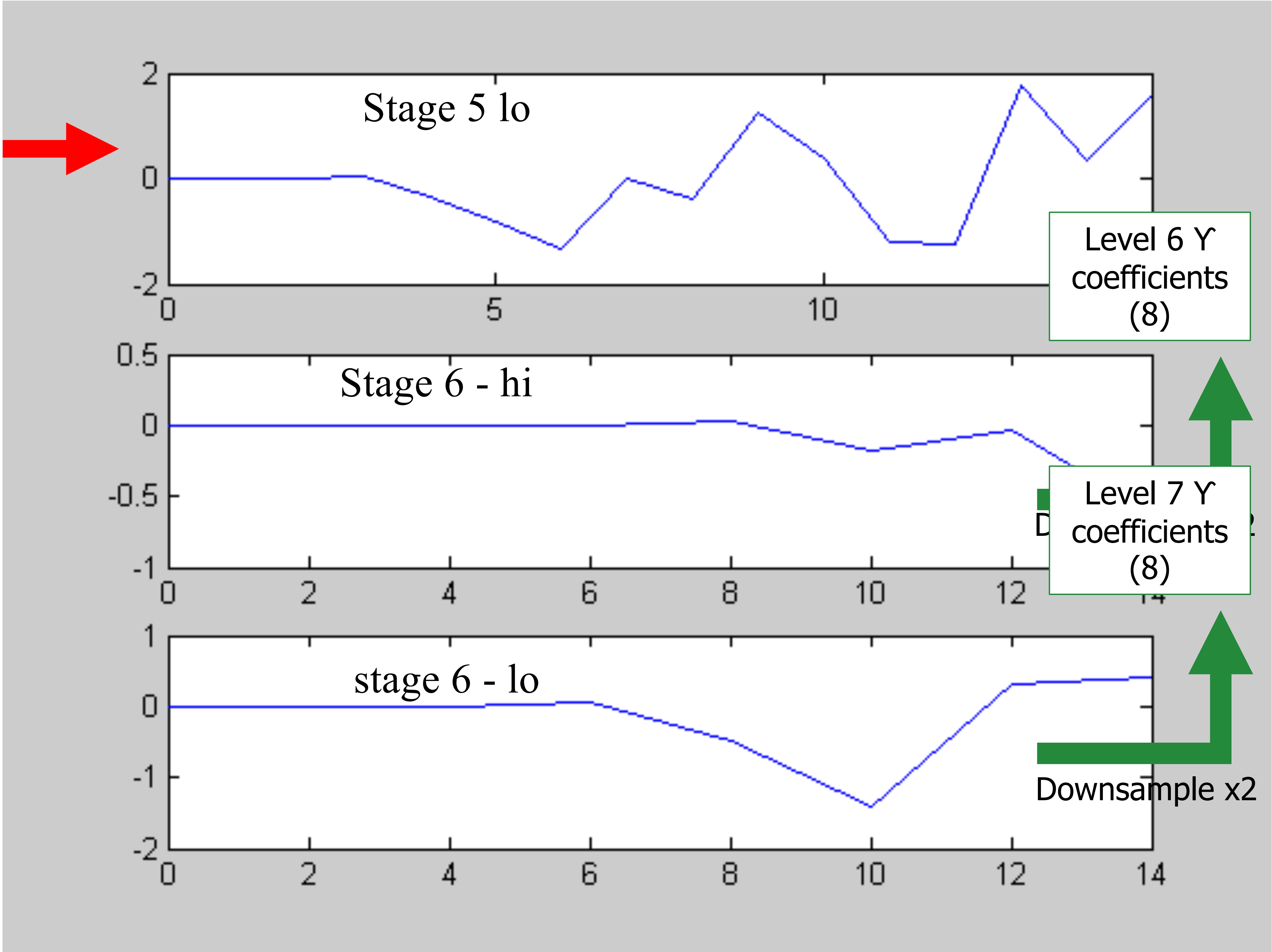


Stage 1 lo
stage 2 - hi
stage 2 - lo
Level 2 Y
coefficients
(128)
Downsample x2
Downsample x2
5
0
-5
2
-2
0
50
100
150
200
250
300


Stage 2 lo
5
0
-5
0
20
40
60
80
100
120
Level 3 Y
coefficients
(64)
stage 3 - hi
2
0
-2
-4
140
Downsample x2
stage 3 - lo
Downsample x2

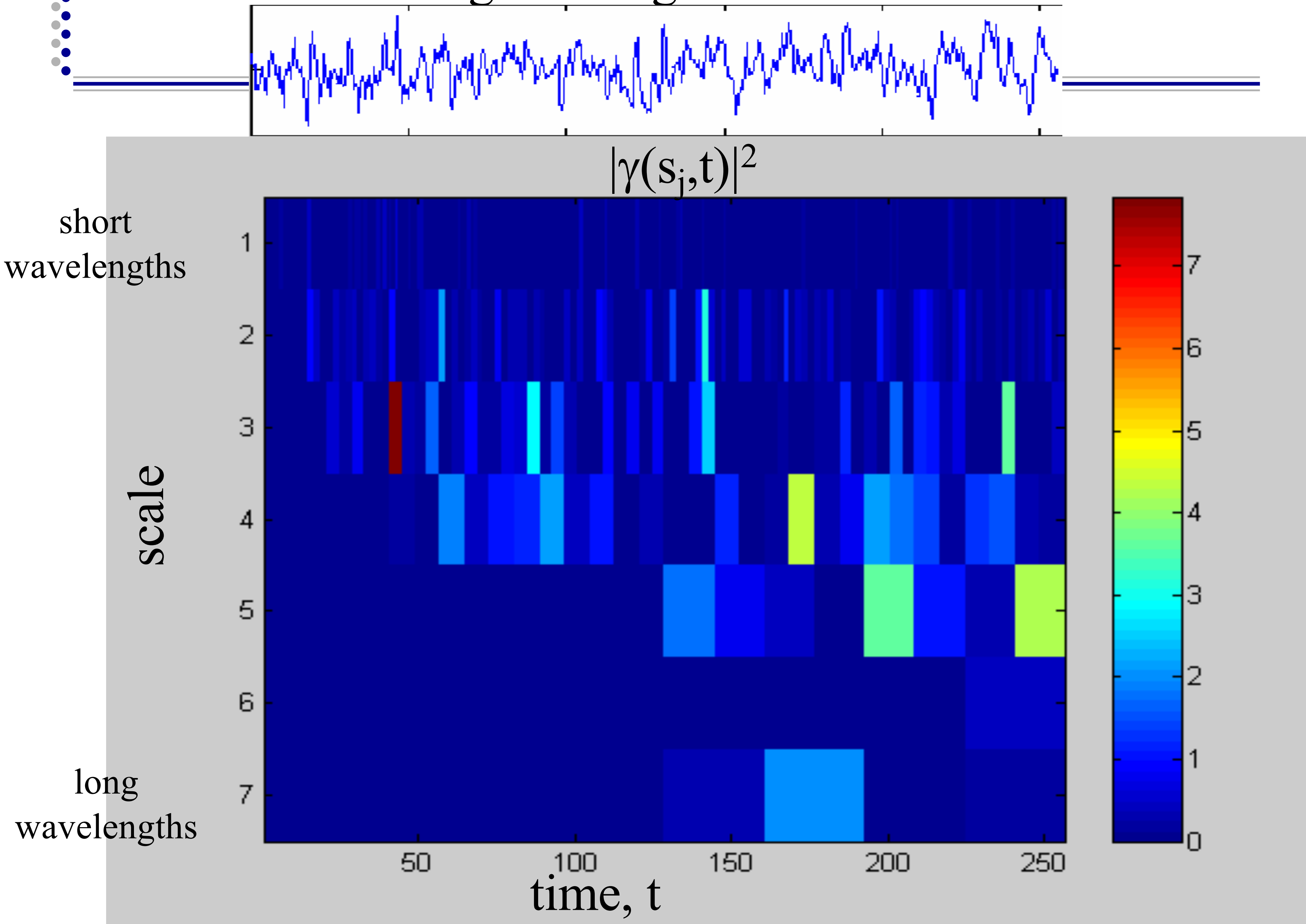

Stage 3 lo
stage 4 - hi
stage 4 - lo
Level 4 Y`
coefficients
(32)
Downsample x2
Downsample x2


Stage 4 lo
stage 5 - hi
Stage 5 - lo
Level 5 Y` coefficients (16)
Downsample x2
Downsample x2



Stage 5 lo
Stage 6 - hi
stage 6 - lo
Level 6 Y
coefficients
(8)
Level 7 Y
coefficients
(8)
Downsample x2

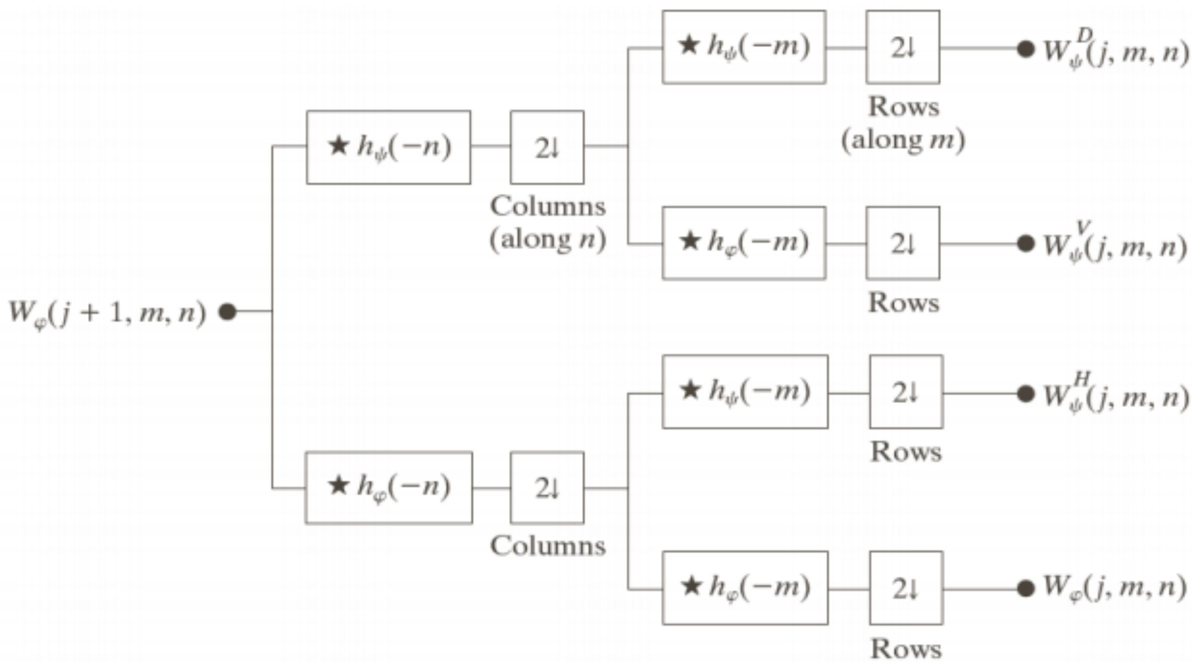
Putting it all together …



Expanding to Two Dimensions

- In two dimensions, a 2D scaling function $\phi(x, y)$ and 3 2D wavelet functions $\psi^H(x, y)$, $\psi^V(x, y)$, $\psi^D(x, y)$ are required
- We can create these from the 1D scaling and wavelet functions:
  - $\phi(x, y) = \phi(x)\phi(y)$
  - $\psi^H(x, y) = \psi(x)\phi(y)$
  - $\psi^V(x, y) = \phi(x)\psi(y)$
  - $\psi^D(x, y) = \psi(x)\psi(y)$

Expanding to Two Dimensions



Expanding to Two Dimensions

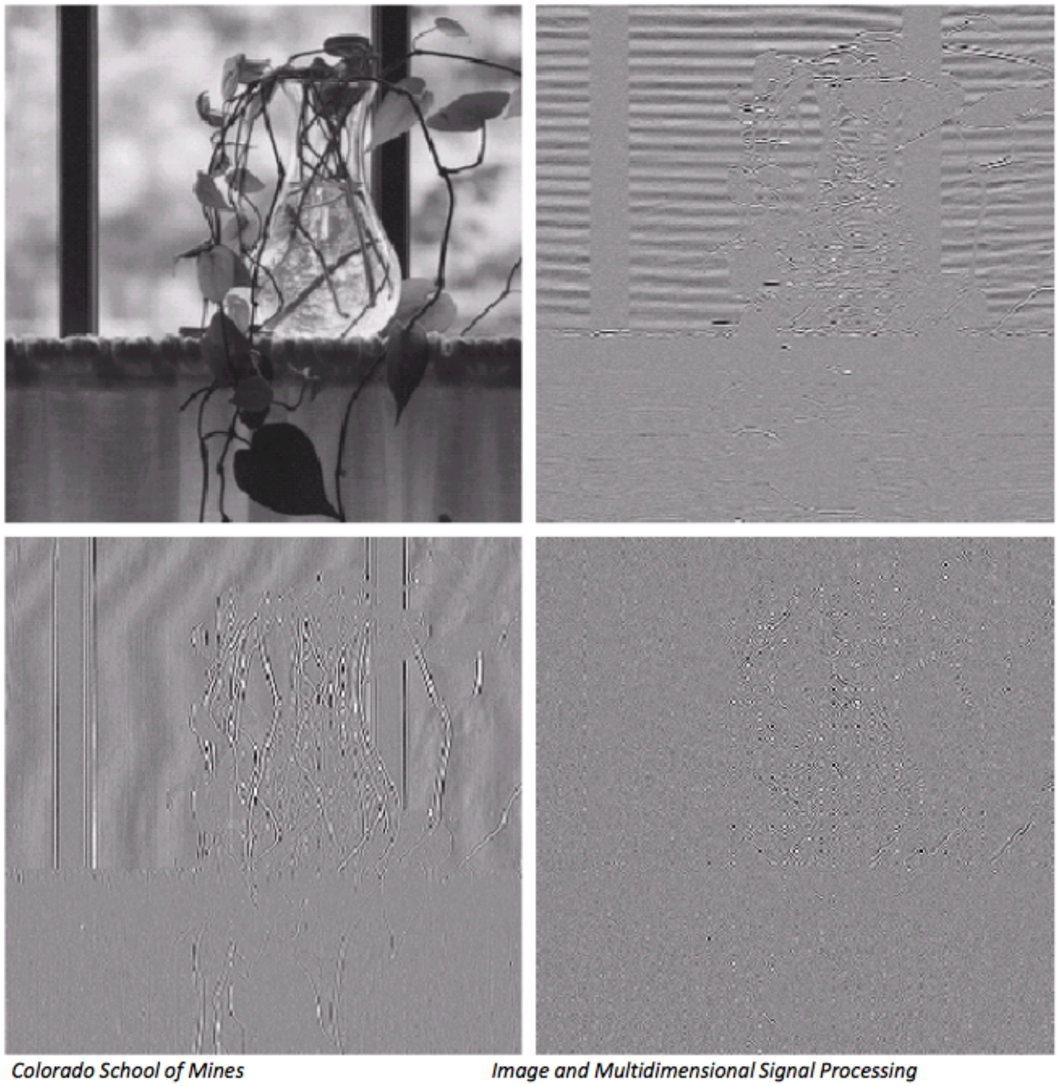


FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

a	d^V
d^H	d^D

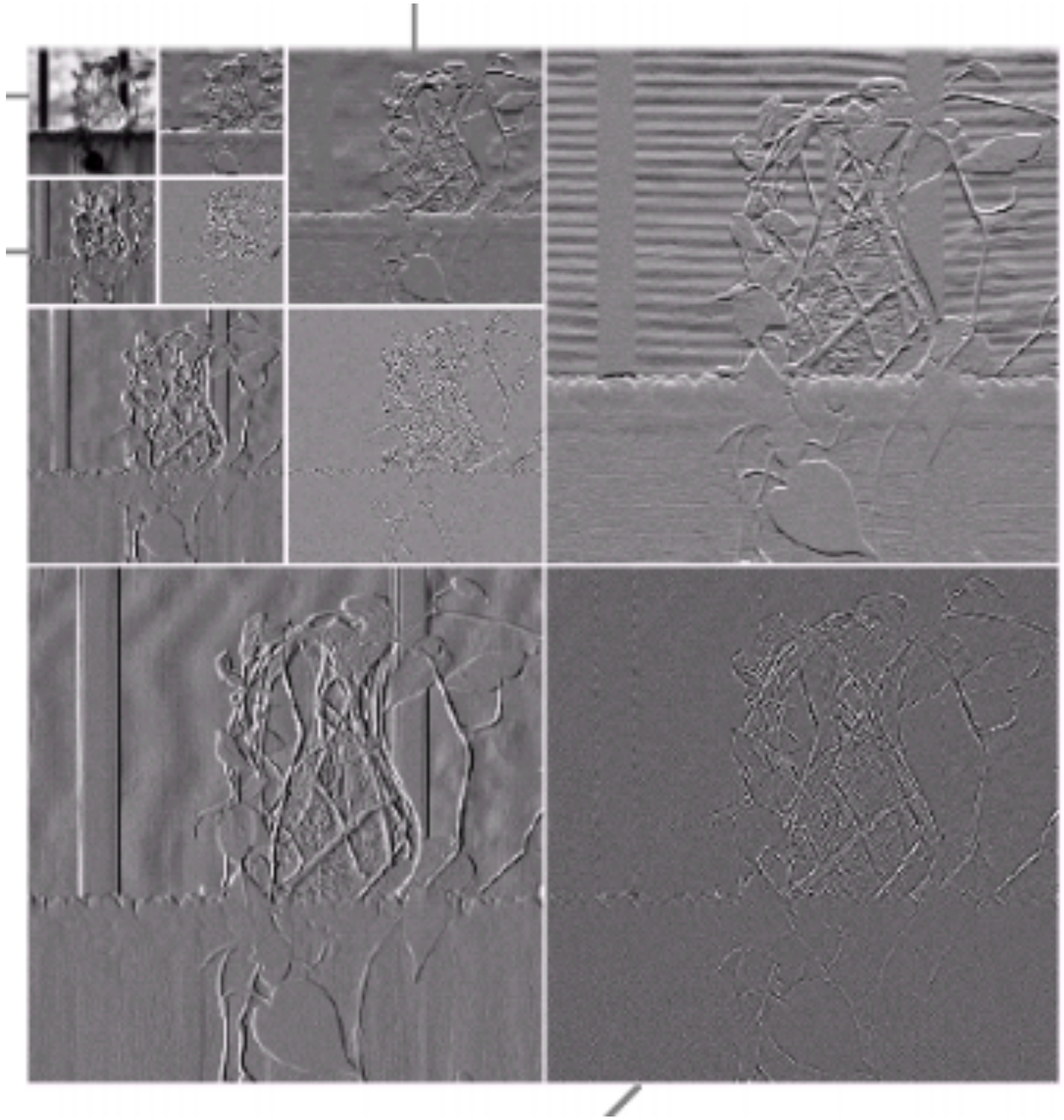
a(m,n): approximation

d^V(m,n): detail in vertical

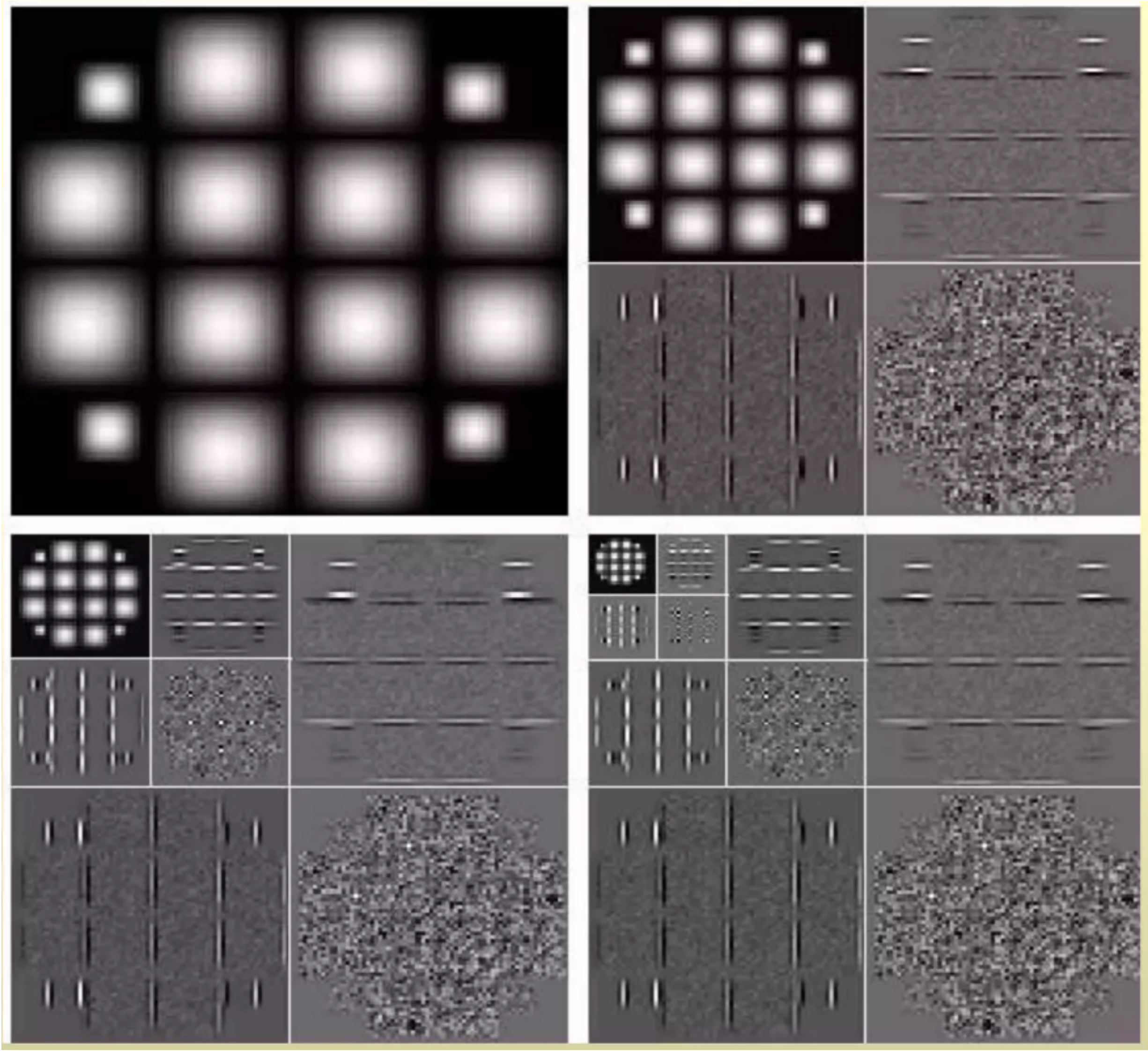
d^H(m,n): detail in horizontal

d^D(m,n): detail in diagonal

Expanding to Two Dimensions



Expanding to Two Dimensions



Big Ideas

- Wavelet transform
  - Capture temporal data with fewer coefficients than STFT
  - Use scaling and translation to get different resolution at different levels

Admin

- Project 2
  - Due 4/26
- Final Exam – 5/1
  - Covers lec 1-23*
    - Doesn't include lecture 12 (Data converters and noise shaping)
  - All old exams online
    - Disclaimers: old exams had different coverage for different years

Admin

- TA Office hour schedule for Zhihan and Jiyue
  - https://edstem.org/us/courses/33619/discussion/2954005
  - 18th (Tue):
    - Jiyue He, 10-noon
    - Jiyue He, 7-9 pm
  - 19th (Wed):
    - Jiyue He, 7-9 pm
  - 24th (Mon):
    - Zhihan Xu, 10-11:30 am
  - 25th (Tue):
    - Zhihan Xu, 10-11:30 am
    - Zhihan Xu, 7-8:30 pm
- Shuang and my office hours as usual