ESE 5310: Digital Signal Processing

Lec 26: April 25, 2023

Review

Course Content

- Introduction
- Discrete Time Signals & Systems
- Discrete Time Fourier Transform
- Z-Transform
- Inverse Z-Transform
- Sampling of Continuous Time Signals
- Frequency Domain of Discrete Time Series
- Downsampling/Upsampling
- Data Converters, Sigma Delta Modulation
- Oversampling, Noise Shaping
- Frequency Response of LTI Systems
- Basic Structures for IIR and FIR Systems
- Design of IIR and FIR Filters
- Filter Banks
- Computation of the Discrete Fourier Transform
- Fast Fourier Transform
- Adaptive Filters
- Spectral Analysis
- Wavelet Transform
- Compressive Sampling

Digital Signal Processing

- Represent signals by a sequence of numbers
  - Sampling and quantization (or analog-to-digital conversion)
- Perform processing on these numbers with a digital processor
  - Digital signal processing
- Reconstruct analog signal from processed numbers
  - Reconstruction or digital-to-analog conversion

- Analog input → analog output
  - Eg. Digital recording music
- Analog input → digital output
  - Eg. Touch tone phone dialing, speech to text
- Digital input → analog output
  - Eg. Text to speech
- Digital input → digital output
  - Eg. Compression of a file on computer

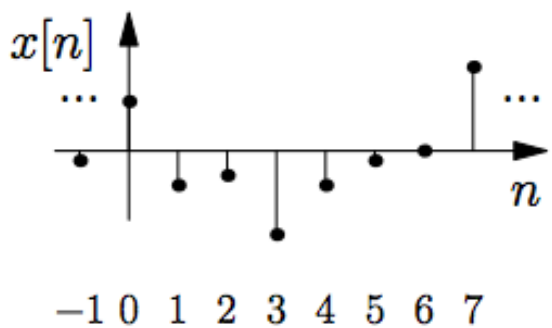
Discrete Time Signals and Systems

Signals are Functions

DEFINITION

A **signal** is a function that maps an independent variable to a dependent variable.

- Signal $x[n]$: each value of n produces the value $x[n]$
- In this course, we will focus on **discrete-time** signals:
  - Independent variable is an **integer**: $n \in \mathbb{Z}$ (will refer to as time)
  - Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

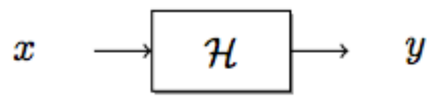


Discrete Time Systems

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

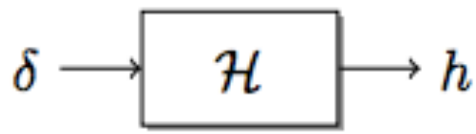
System Properties

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

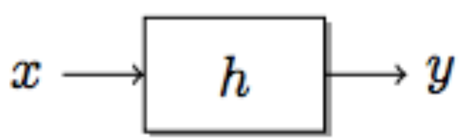
LTI Systems

> **DEFINITION**
>
> A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] \; = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

Discrete Time Fourier Transform

DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

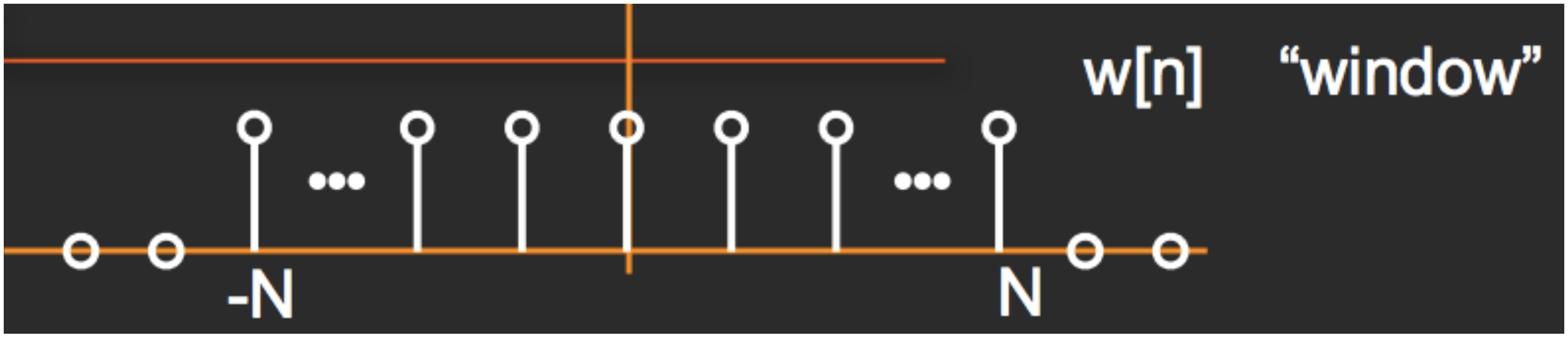
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk}$$

$$x[n] = \int_{-0.5}^{0.5} X(f)e^{j2\pi fn}df$$

Example: Window DTFT



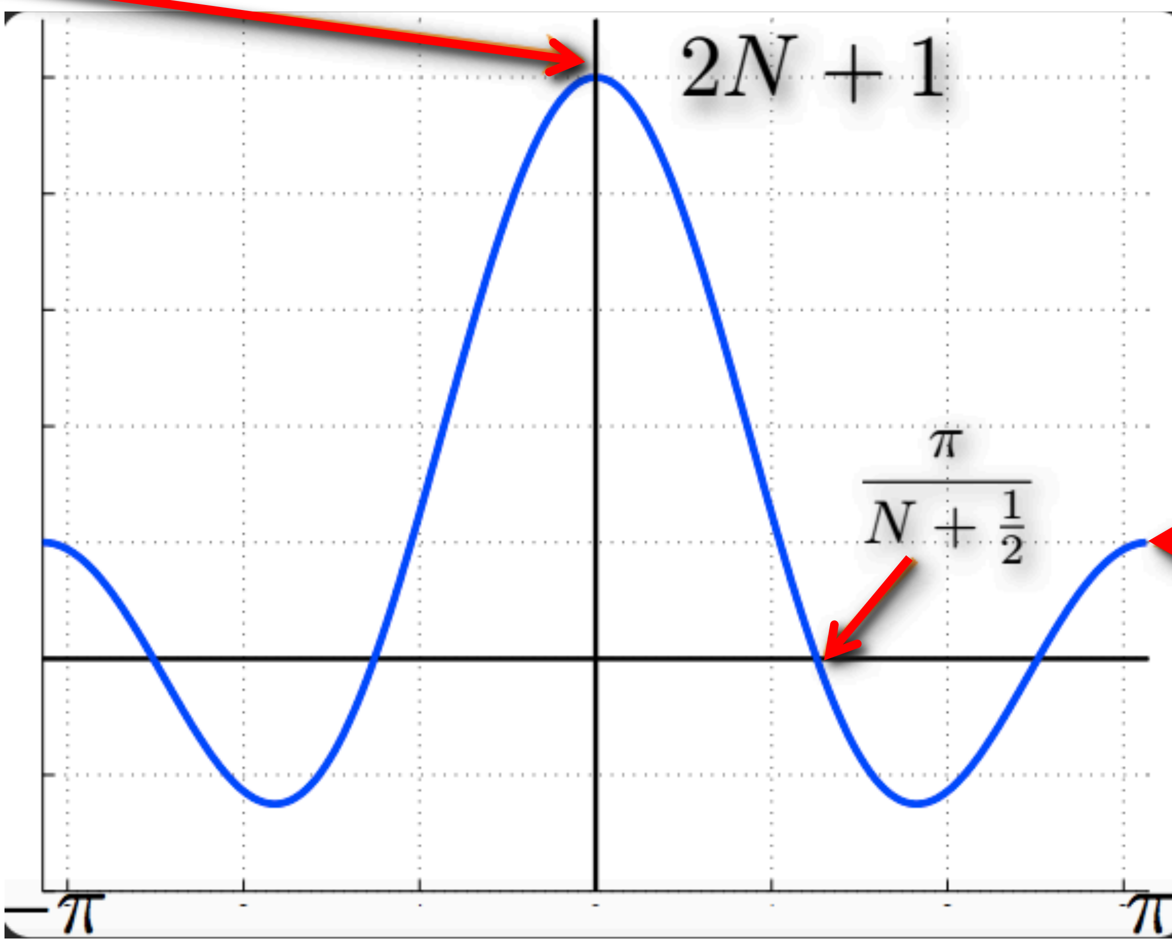
$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[k]e^{-j\omega k}$$

$$= \sum_{k=-N}^{N} e^{-j\omega k}$$

Example: Window DTFT

$$W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Also, Σx[n]

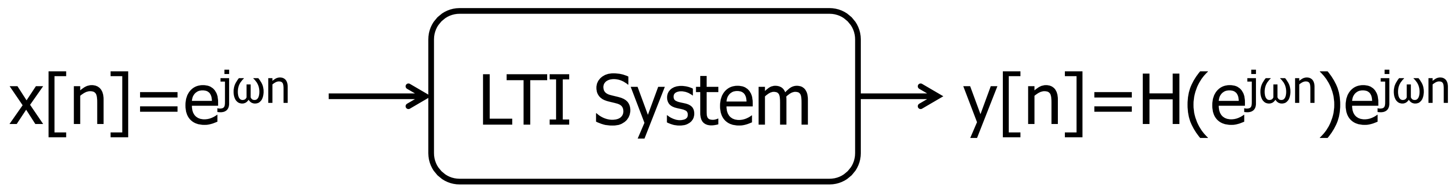


=1 why?

Plot for N=2

LTI System Frequency Response

- Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

z-Transform

- The z-transform generalizes the Discrete-Time Fourier Transform (DTFT) for analyzing infinite-length signals and systems
- Very useful for designing and analyzing signal processing systems
- Properties are very similar to the DTFT with a few caveats

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

Region of Convergence (ROC)

DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| < \infty$$

Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

  - Power series expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Difference Equation to z-Transform

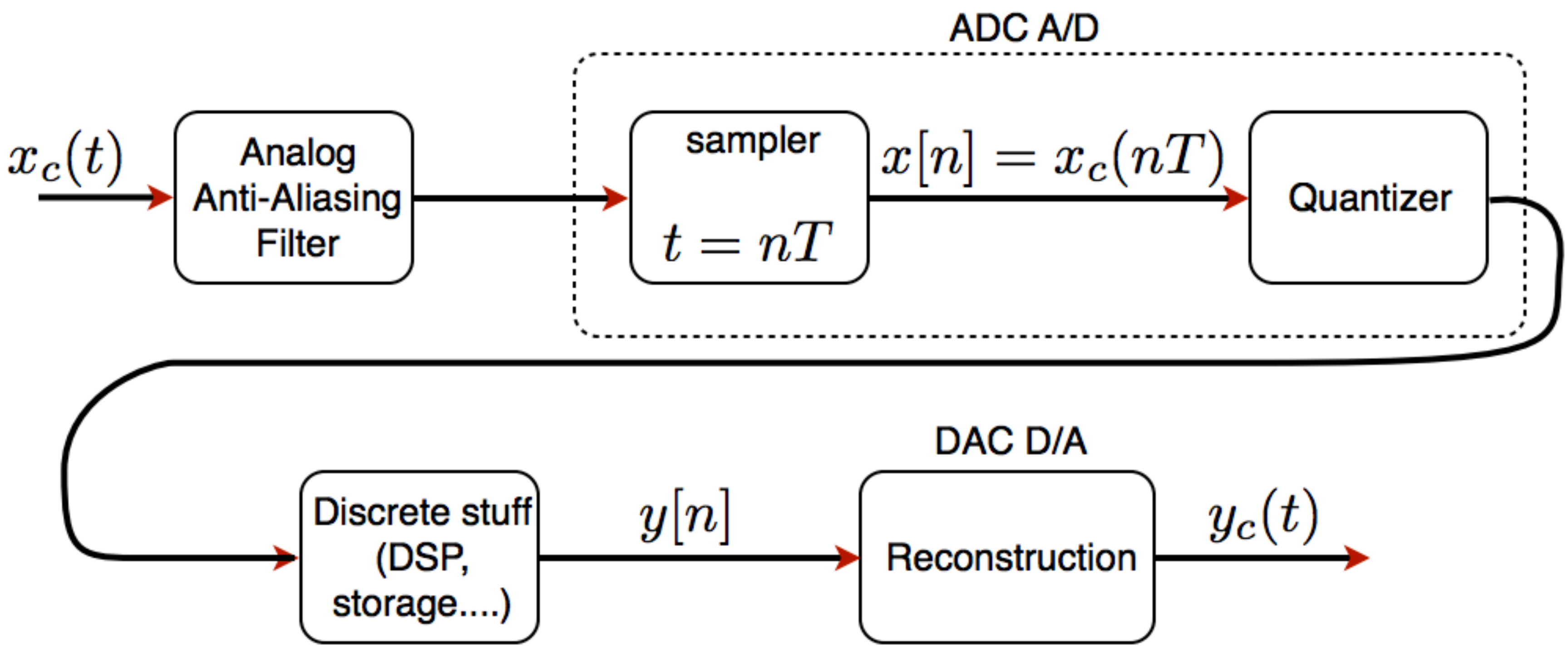
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

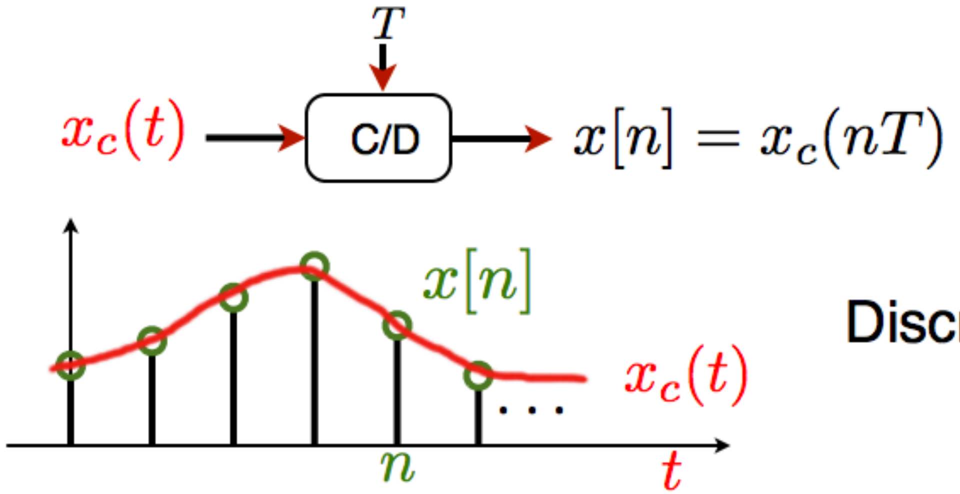
- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

Sampling and Reconstruction

DSP System


ADC A/D
$x_c(t)$
Analog Anti-Aliasing Filter
sampler
$t = nT$
$x[n] = x_c(nT)$
Quantizer
DAC D/A
Discrete stuff (DSP, storage....)
$y[n]$
Reconstruction
$y_c(t)$

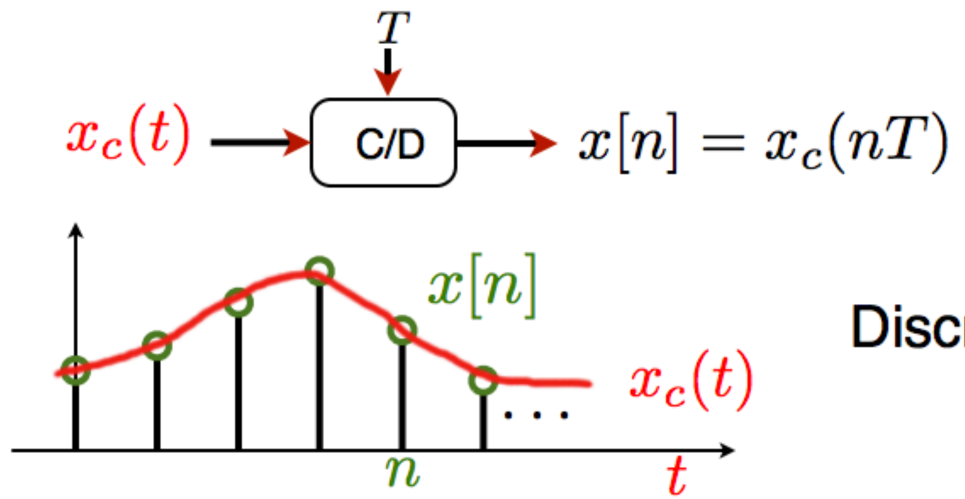
Ideal Sampling Model



Discrete and Continuous

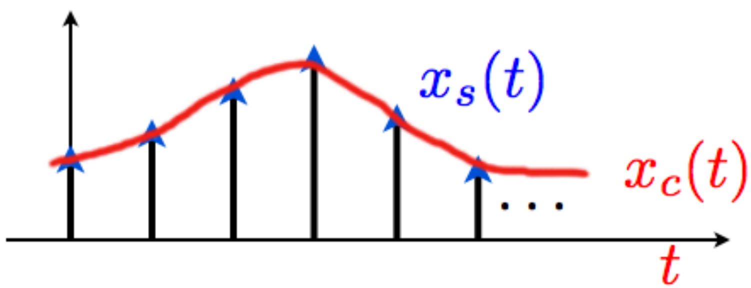
- Ideal continuous-to-discrete time (C/D) converter
  - T is the sampling period
  - f_s=1/T is the sampling frequency
  - Ω_s=2π/T

Ideal Sampling Model



Discrete and Continuous

define impulsive sampling:

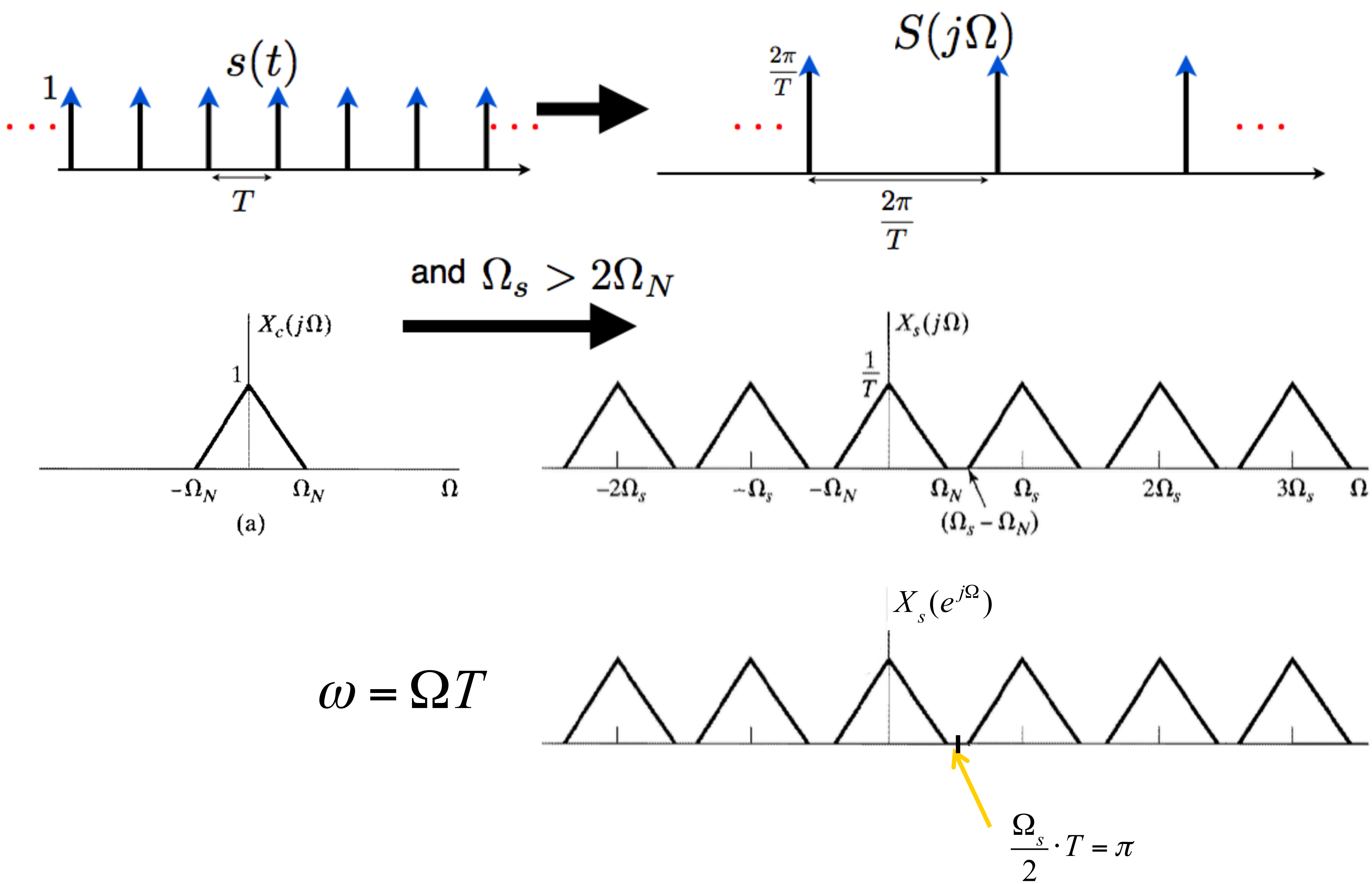


Continuous

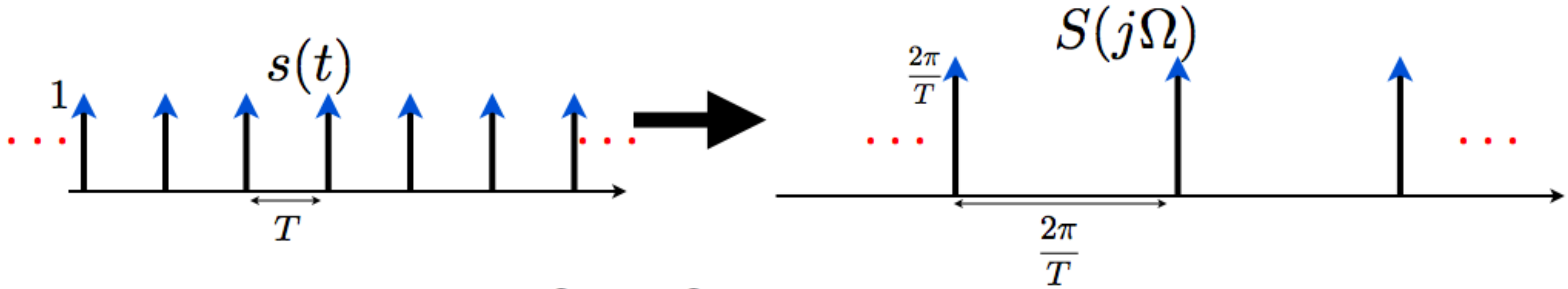
$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

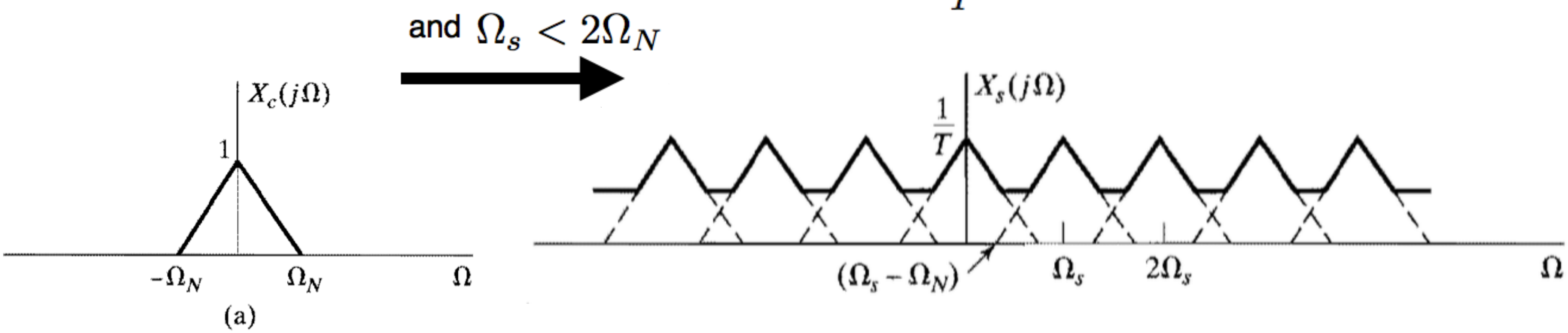
$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Frequency Domain Analysis

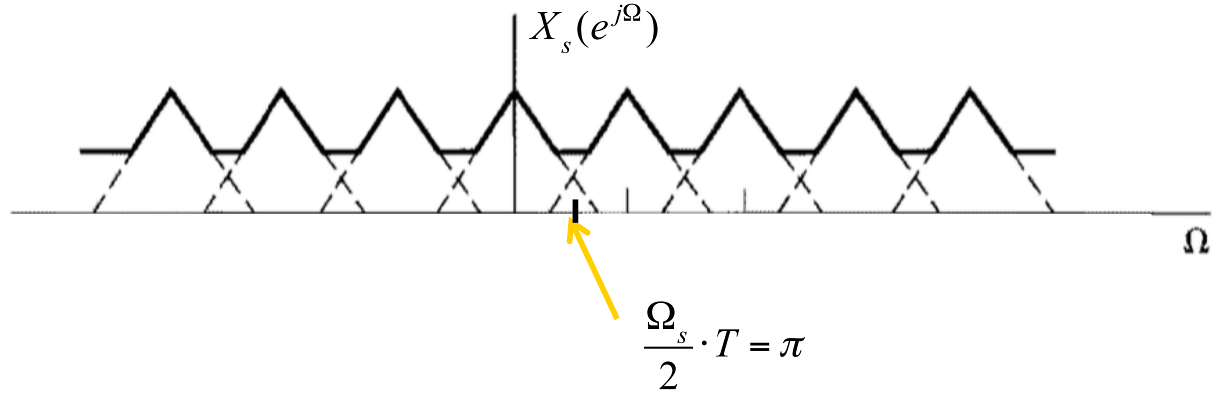


Frequency Domain Analysis



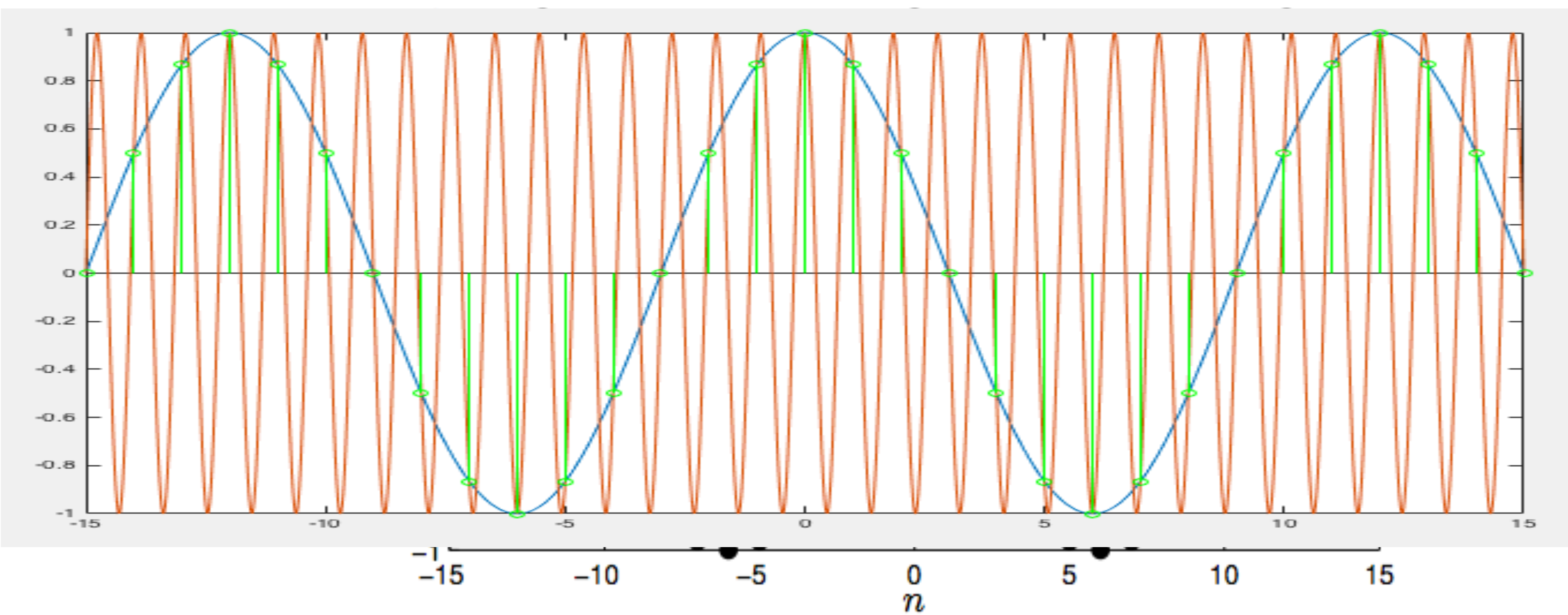


$$\omega = \Omega T$$



Aliasing Example

- $x_1[n] = \cos\left(\frac{\pi}{6}n\right)$

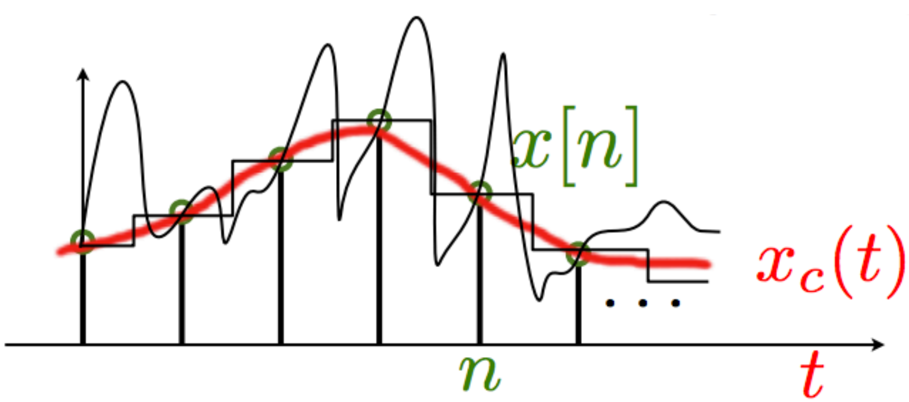


Reconstruction of Bandlimited Signals

- Nyquist Sampling Theorem: Suppose $x_c(t)$ is bandlimited. I.e.

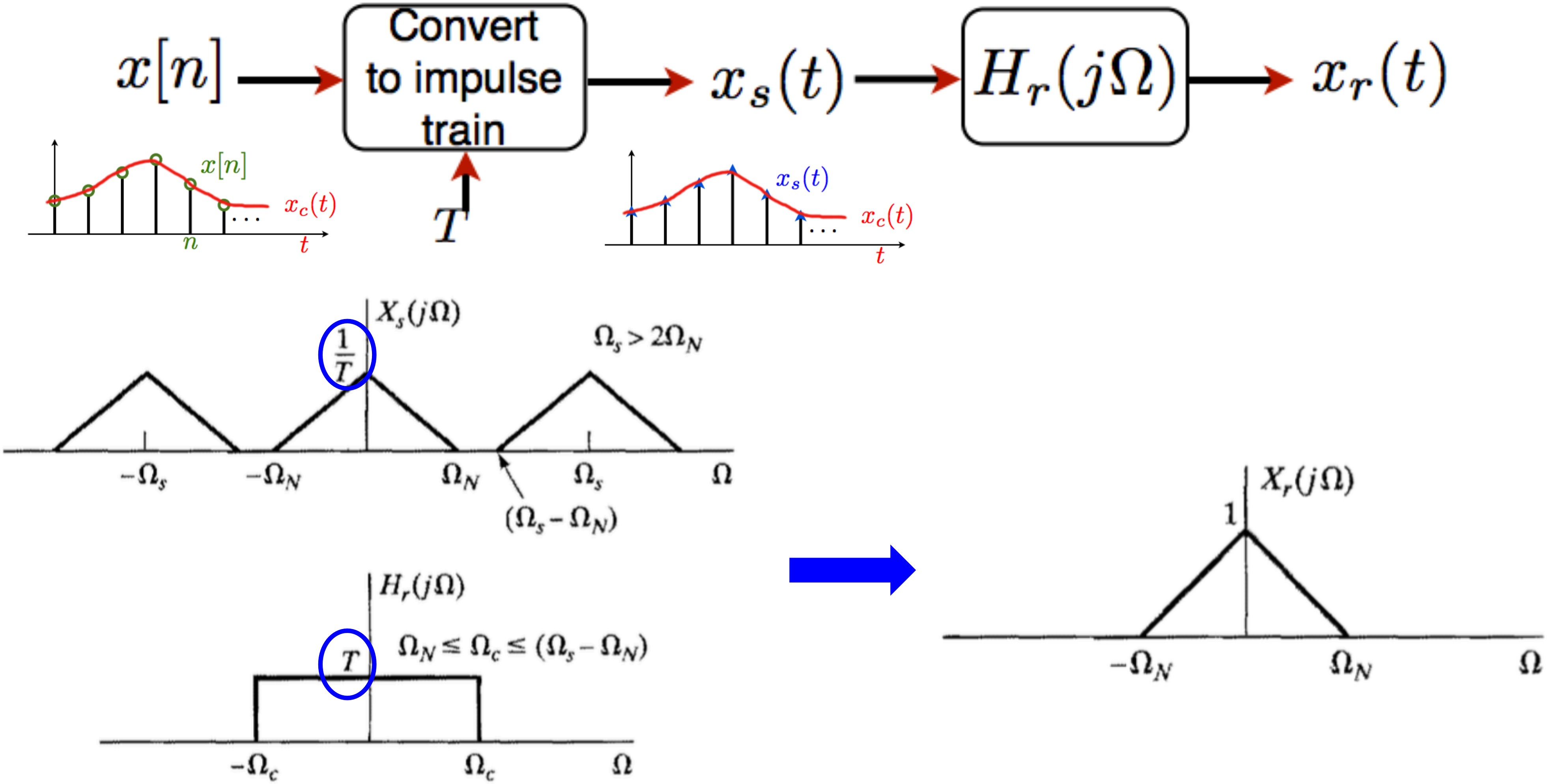
$$X_c(j\Omega) = 0 \quad \forall \quad |\Omega| \geq \Omega_N$$

- If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n]=x_c(nT)$
- Bandlimitedness is the key to uniqueness



Mulitiple signals go through the samples, but only one is bandlimited within our sampling band

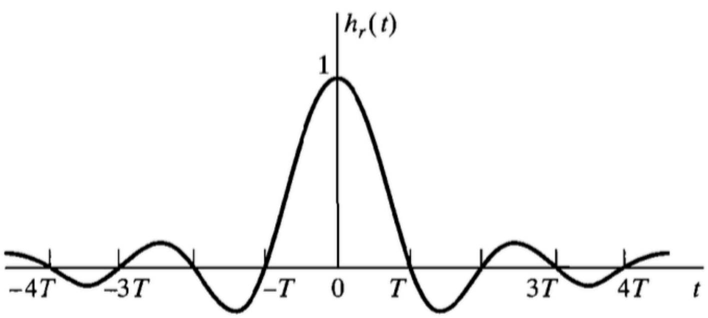
Reconstruction in Frequency Domain



Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t - nT) \right) * h_r(t)$$

$$= \sum_n x[n] h_r(t - nT)$$



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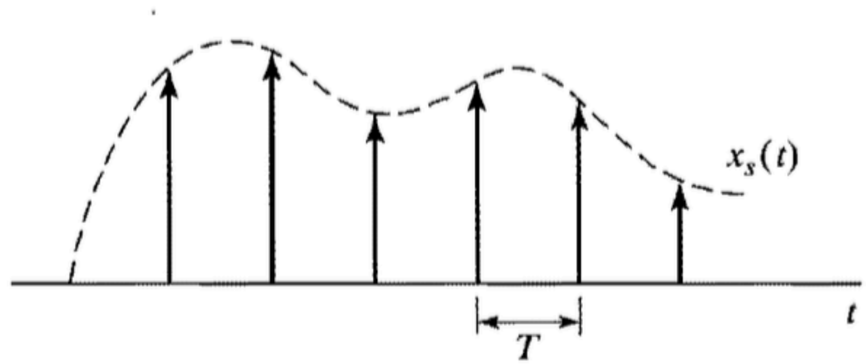

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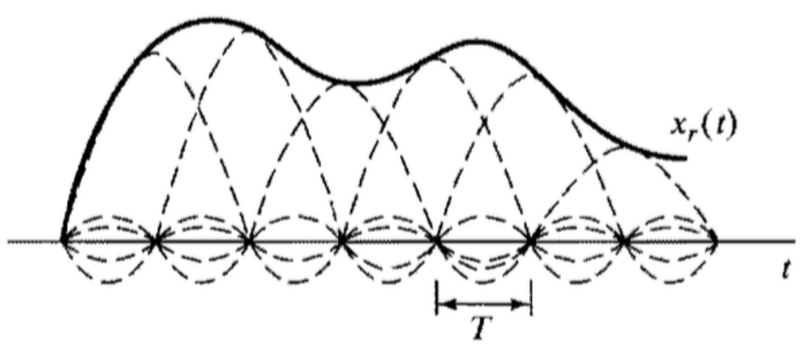
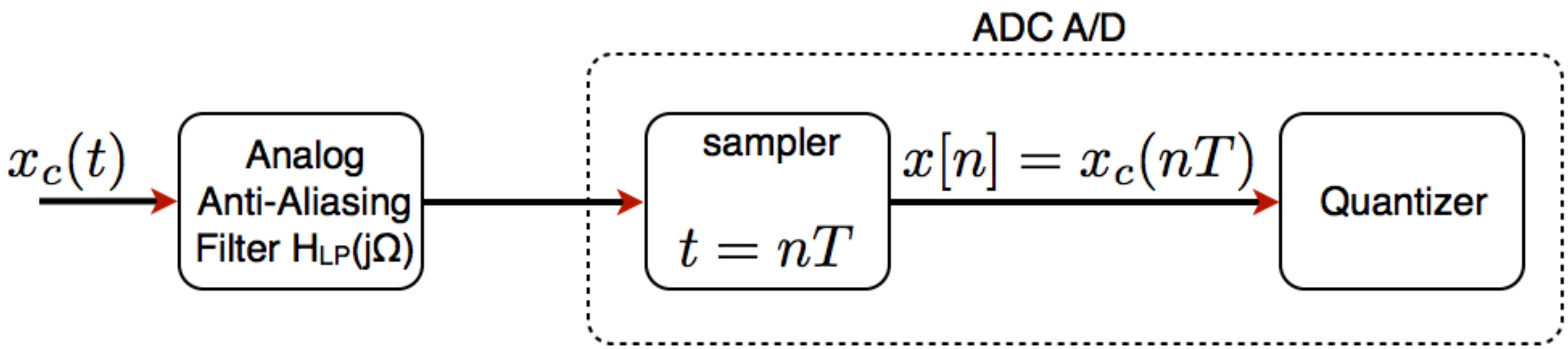
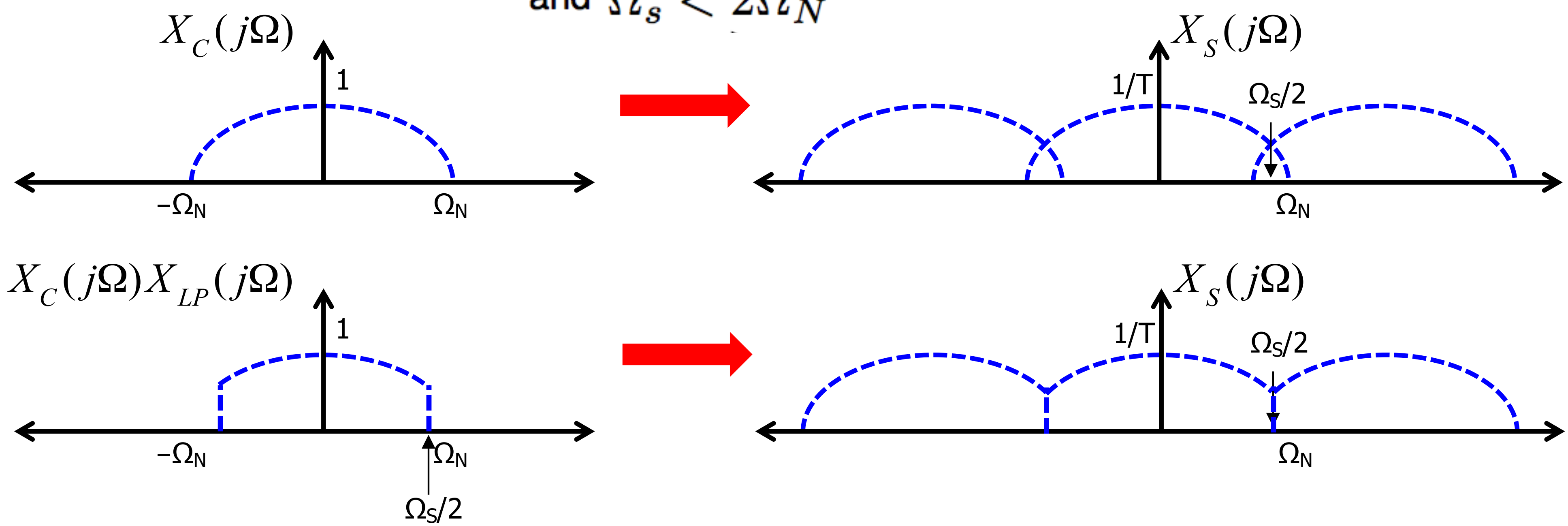


The sum of "sincs" gives $x_r(t)$ → unique signal that is bandlimited by sampling bandwidth

Anti-Aliasing Filter



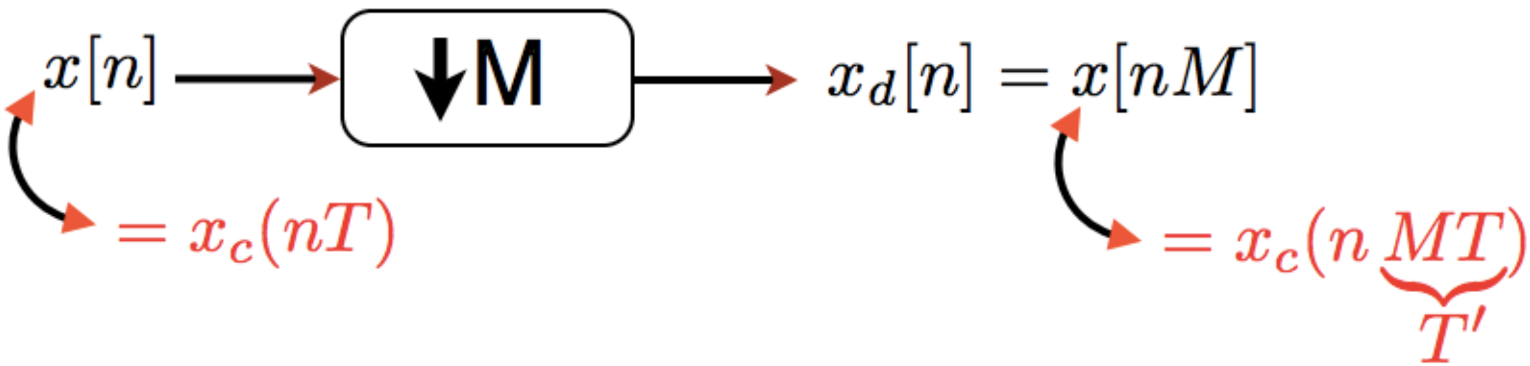
and $\Omega_s < 2\Omega_N$



Rate Re-Sampling

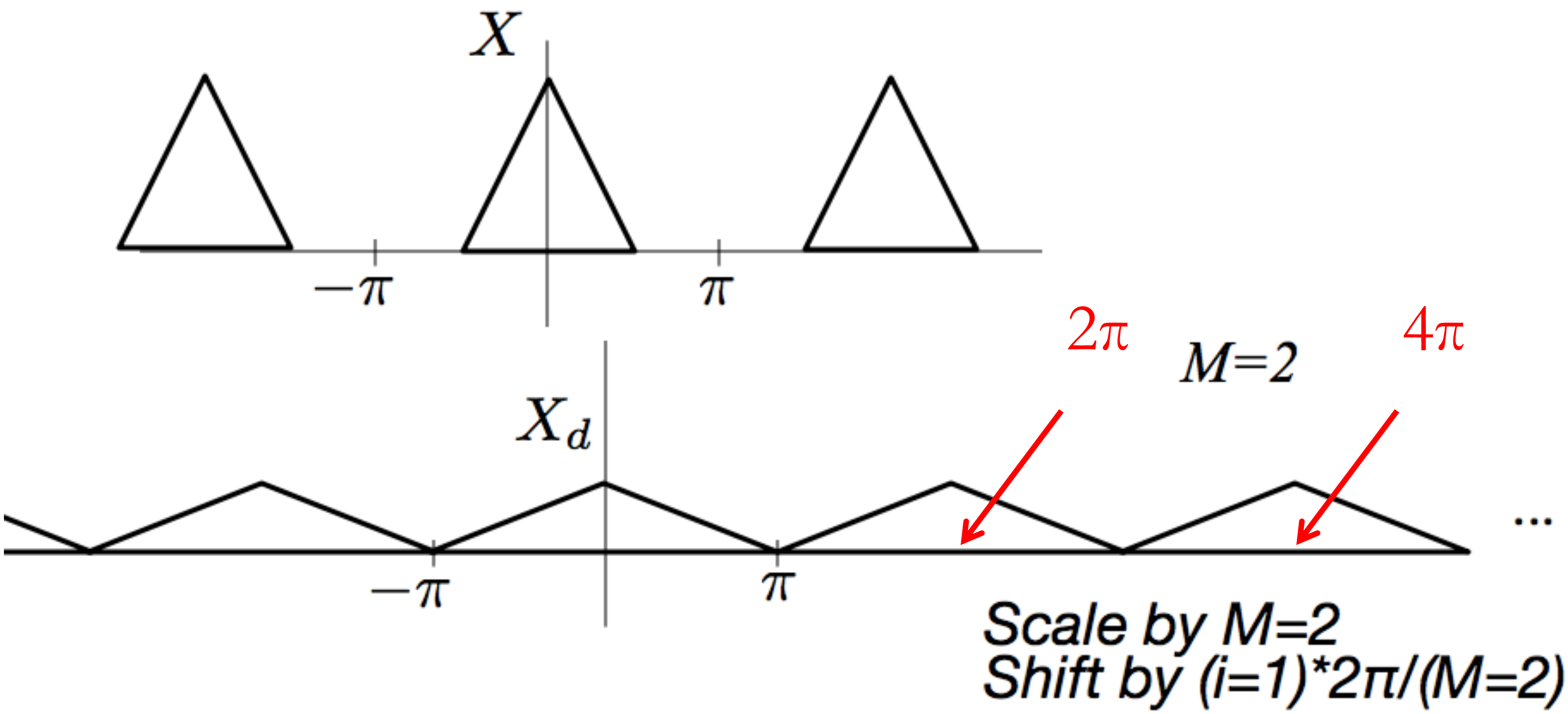
Downsampling

- Definition: Reducing the sampling rate by an integer number



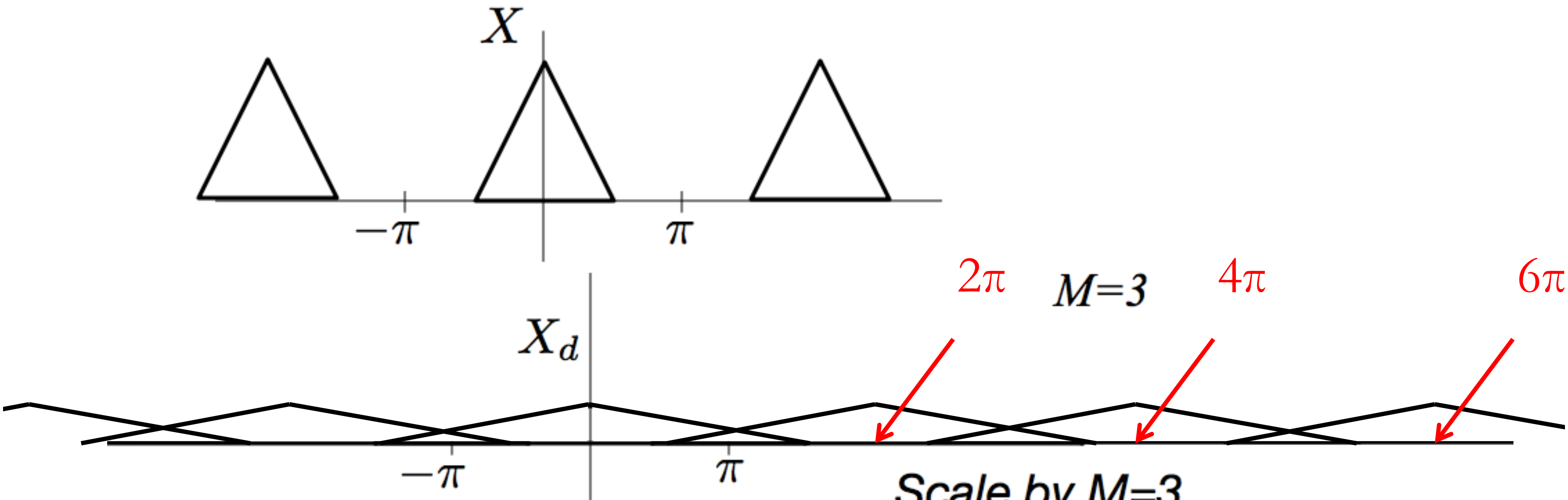
Example: M=2

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



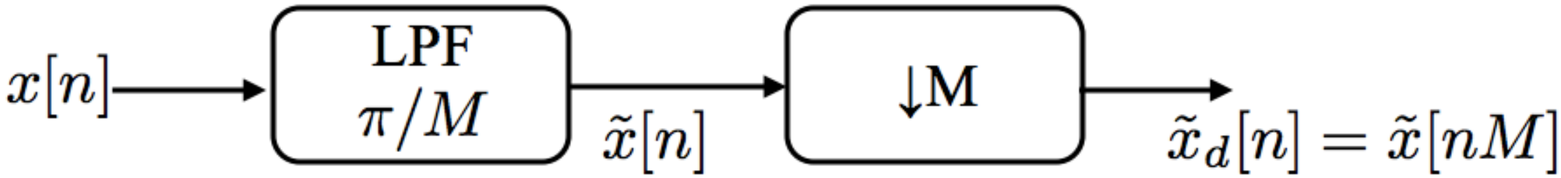
Example: M=3

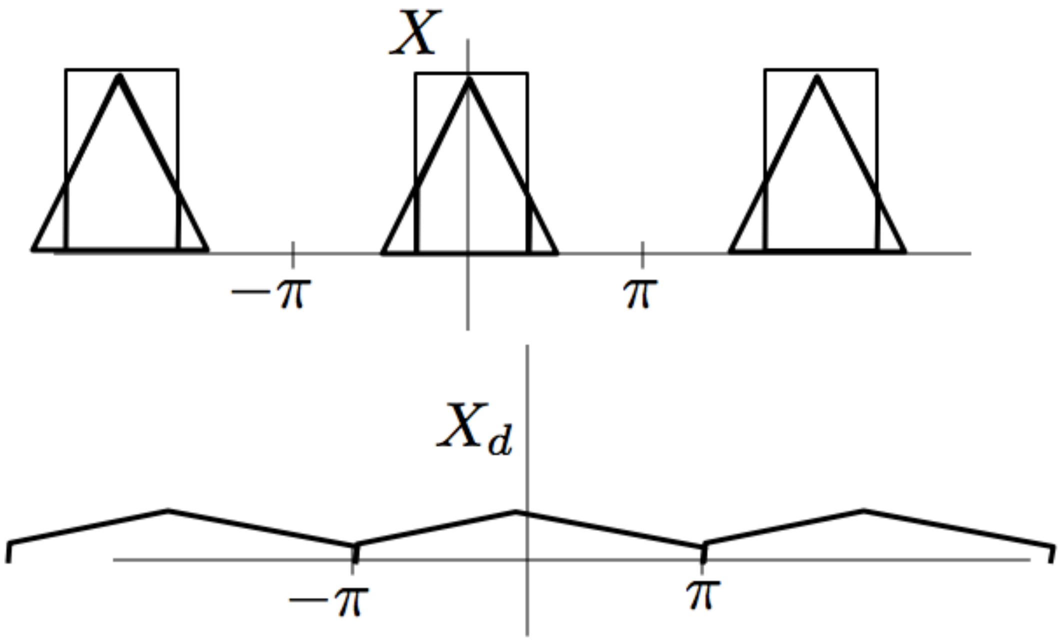
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\left(\frac{w}{M} - \frac{2\pi}{M}i\right)}\right)$$



Scale by M=3
*Shift by (i=1)*2π/(M=3)*
*Shift by (i=2)*2π/(M=3)*

Example: M=3 w/ Anti-aliasing





M=3

Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

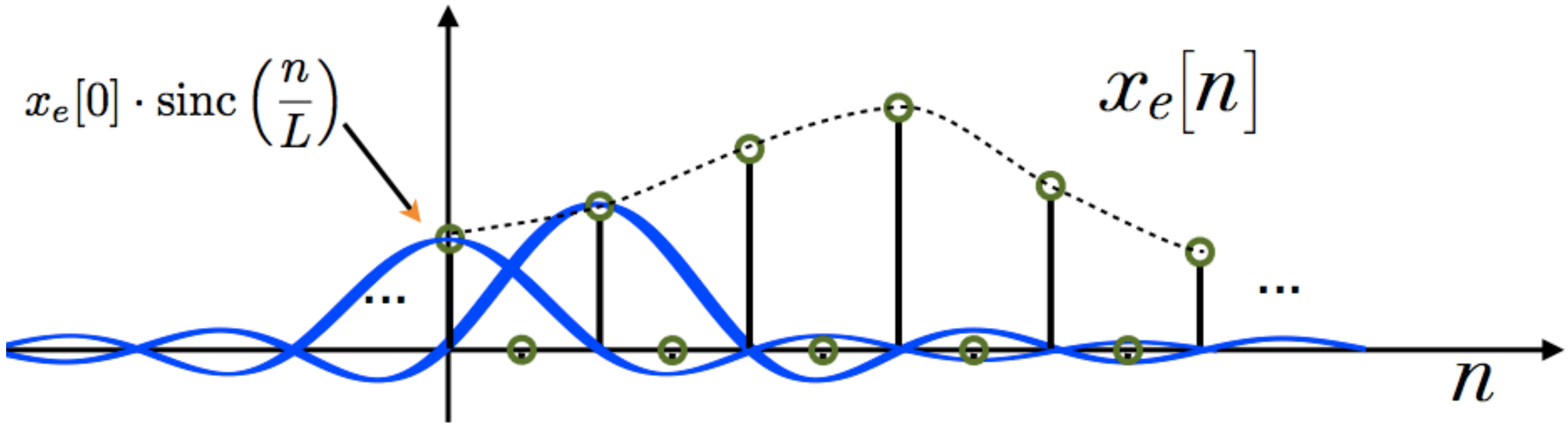
$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \qquad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

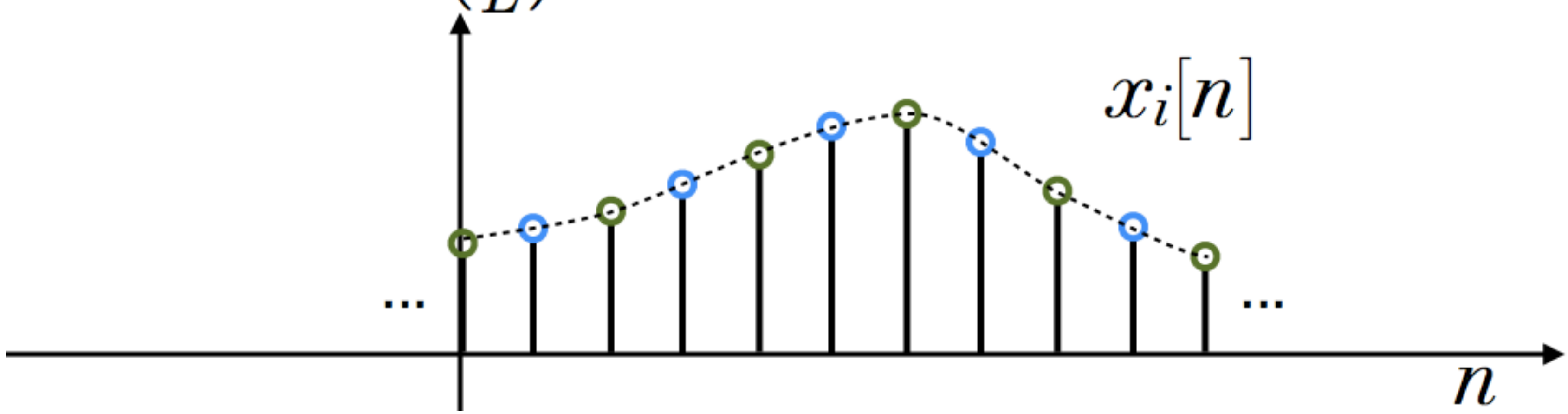
(1) Generate: $x_e[n] = \begin{cases} x[n/L] & n = 0, \ \pm L, \ \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$

Upsampling

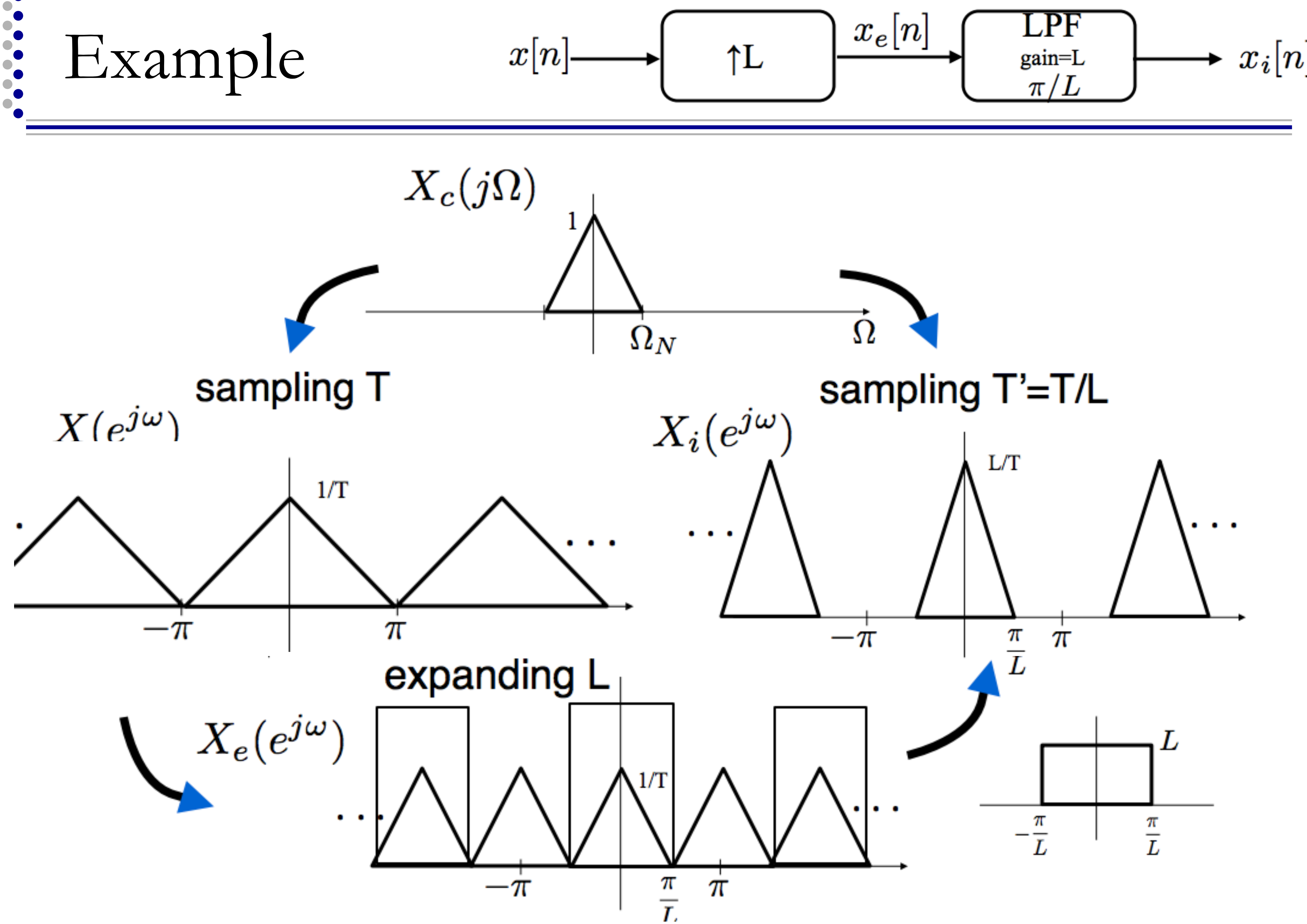
(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:



$$x_i[n] = x_e[n] * \text{sinc}\left(\frac{n}{L}\right)$$



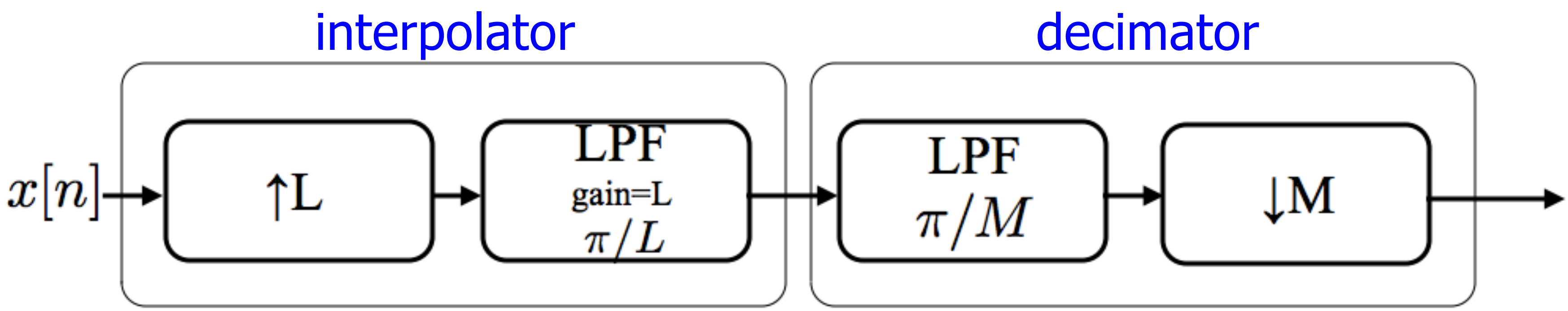
Example



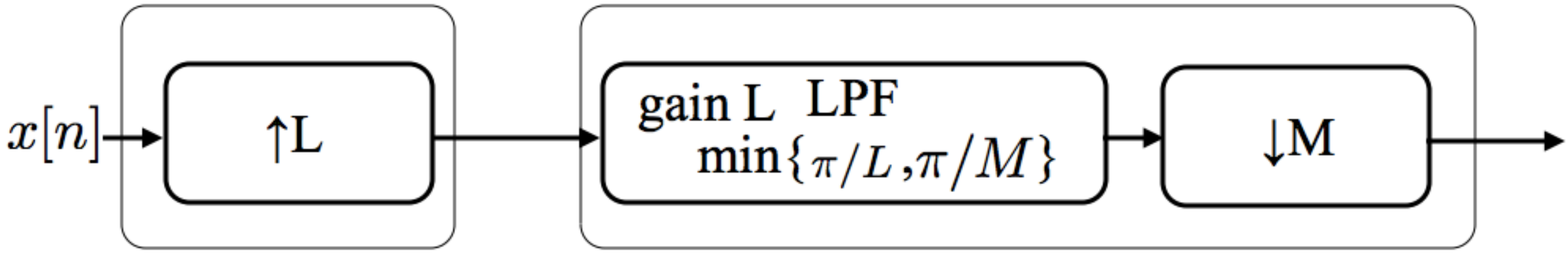
$x[n]$
↑L
$x_e[n]$
LPF
gain=L
π/L
$x_i[n]$
$X_c(j\Omega)$
1
Ω_N
Ω
sampling T
sampling T'=T/L
$X(e^{j\omega})$
1/T
$-\pi$
π
$X_i(e^{j\omega})$
L/T
$-\pi$
$\frac{\pi}{L}$
π
expanding L
$X_e(e^{j\omega})$
1/T
$-\pi$
$\frac{\pi}{L}$
π
L
$-\frac{\pi}{L}$
$\frac{\pi}{L}$

Non-integer Sampling

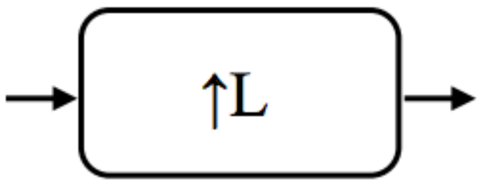
- T'=TM/L
  - Upsample by L, then downsample by M



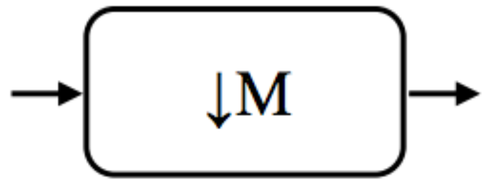
Or,



Interchanging Operations



"expander"



"compressor"

<u>not LTI!</u>

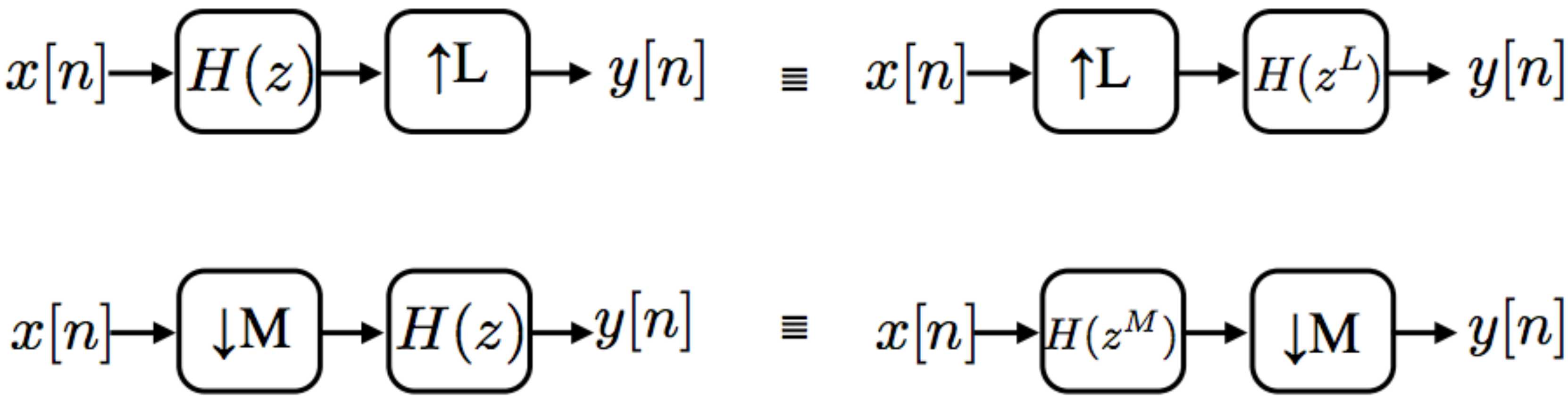
Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Summary

Filter and expander

Expander and expanded filter*

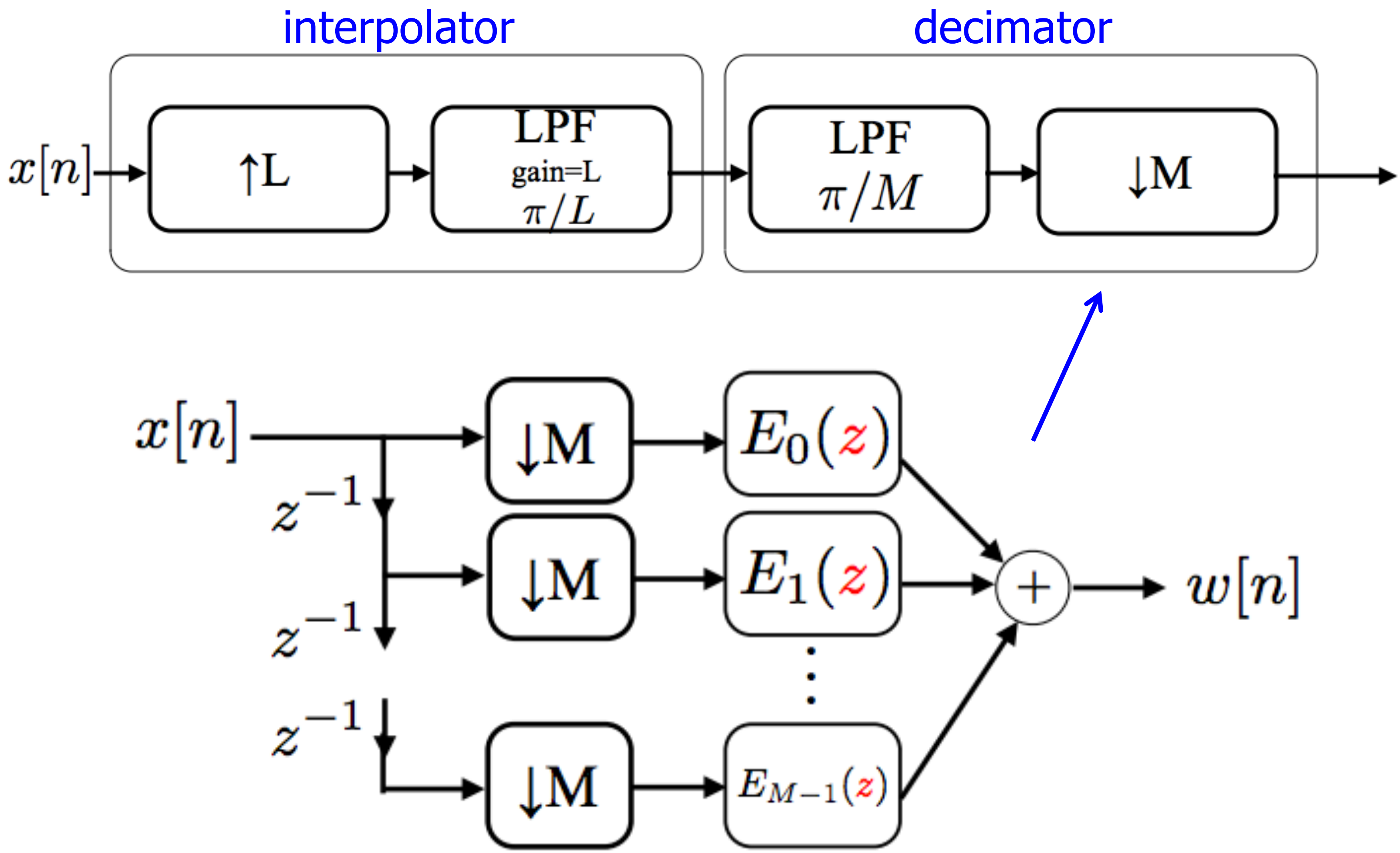


Compressor and filter

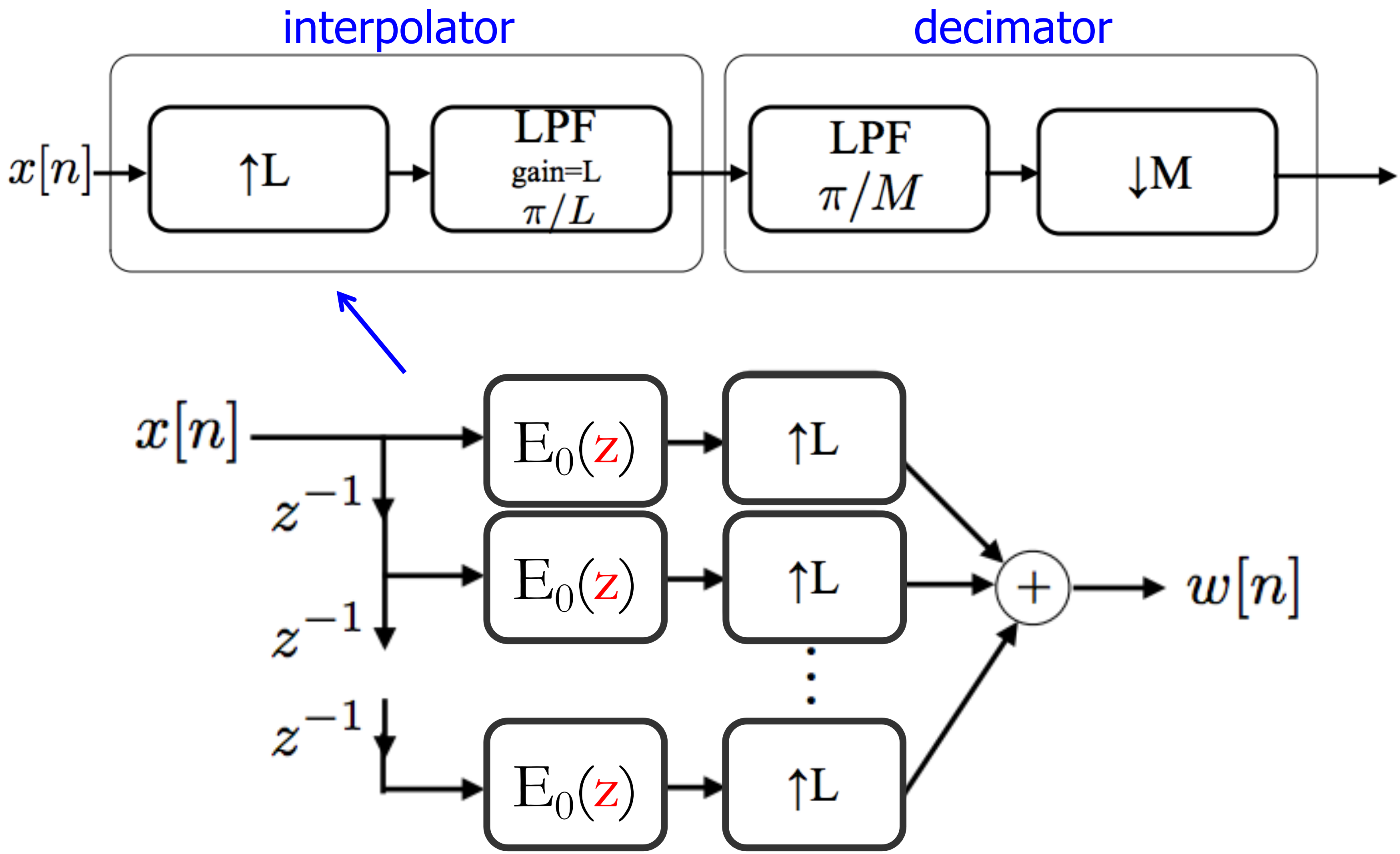
Expanded filter* and compressor

*Expanded filter = expanded impulse response, compressed freq response

Polyphase Implementation of Decimator



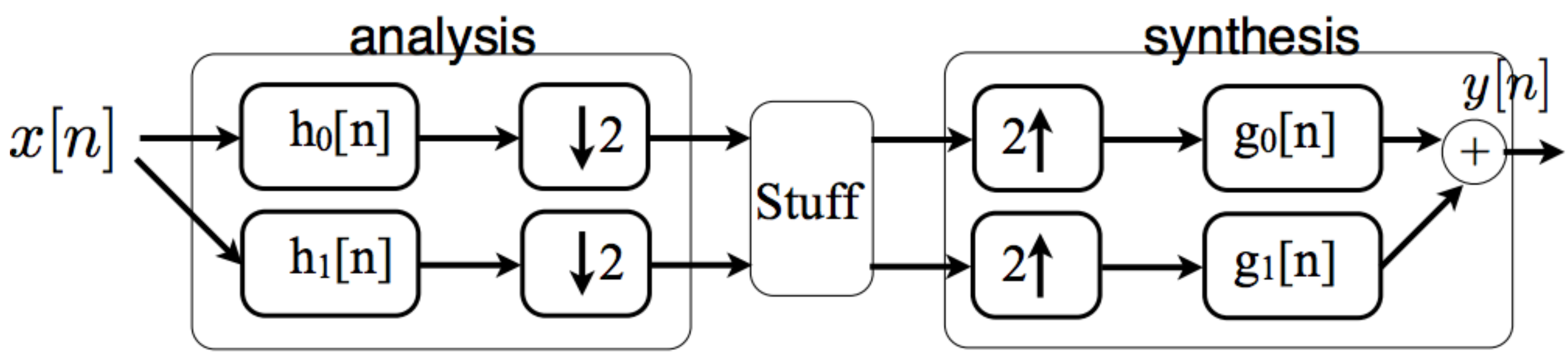
Polyphase Implementation of Interpolation



Multi-Rate Filter Banks

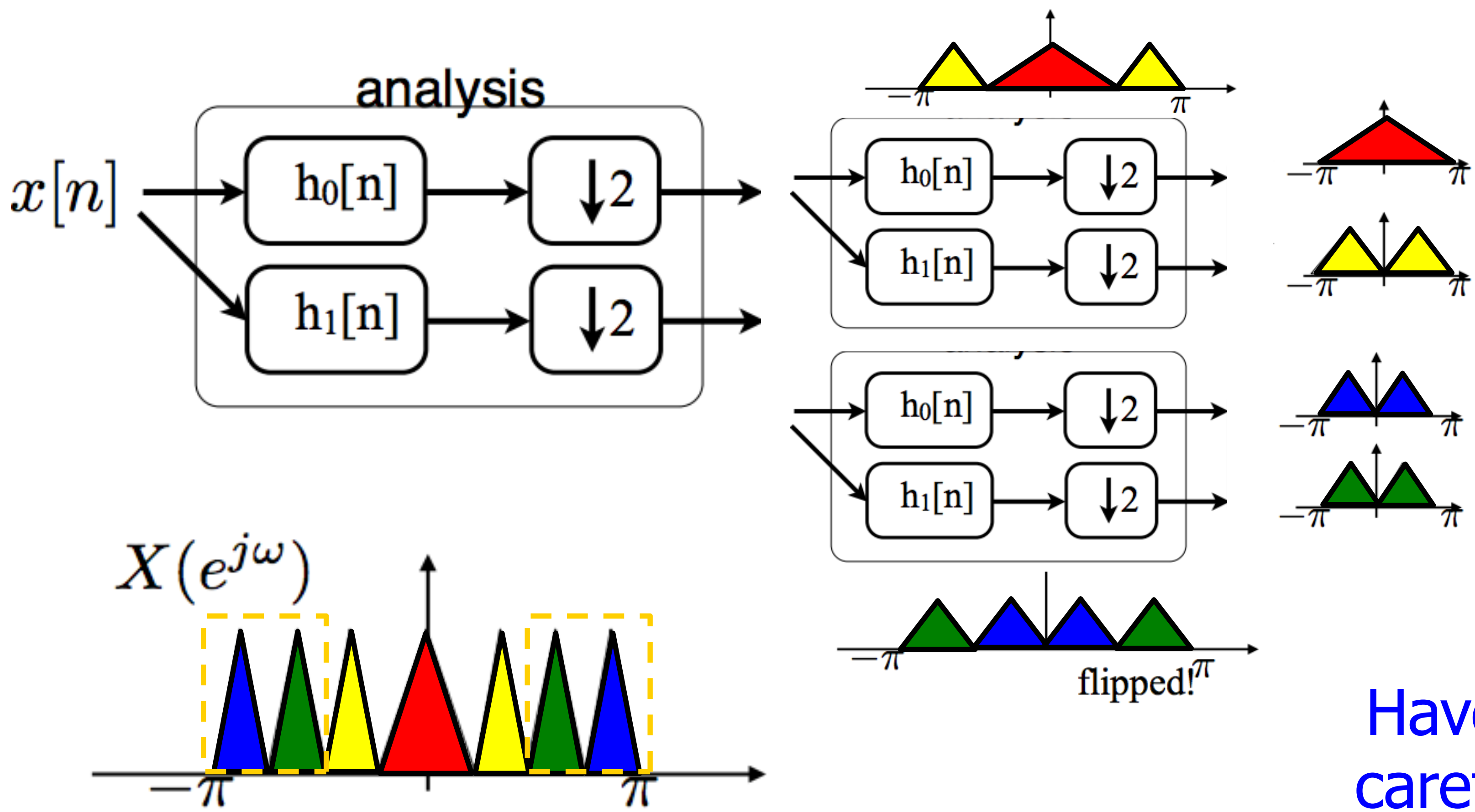
- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



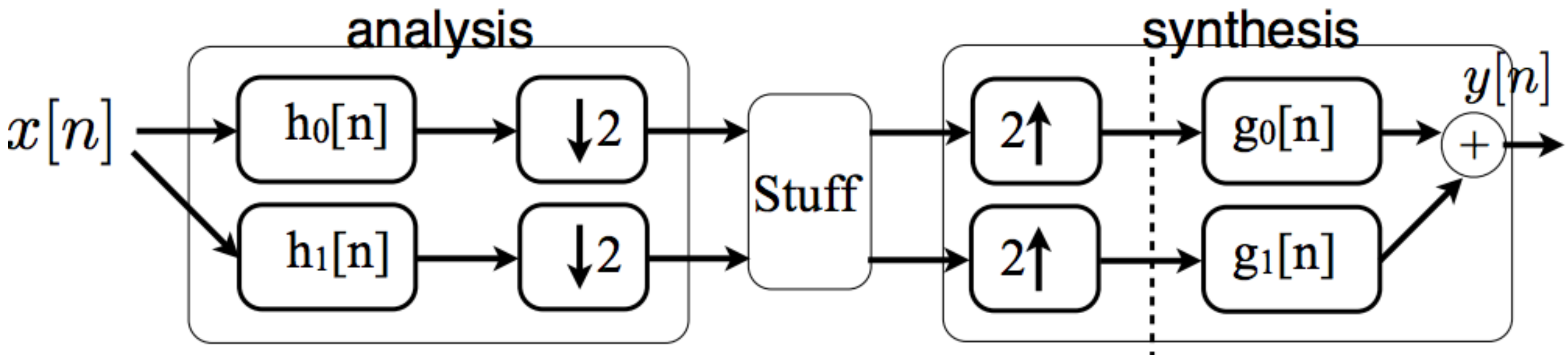
Multi-Rate Filter Banks

- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$



Have to be careful with order!

Perfect Reconstruction non-Ideal Filters

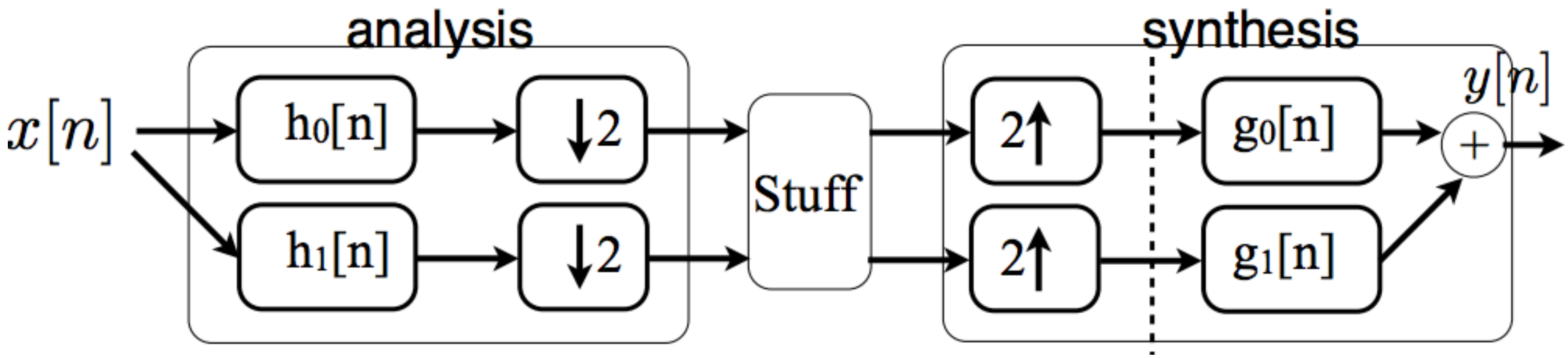


$$Y(e^{j\omega}) = \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right]X(e^{j\omega})$$
$$+\frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right]X(e^{j(\omega-\pi)})$$

aliasing

need to cancel!

Quadrature Mirror Filters



Quadrature mirror filters

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

Frequency Response of Systems

Frequency Response of LTI System

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) X\left(e^{j\omega}\right)$$

- We can define a magnitude response

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right| \left|X\left(e^{j\omega}\right)\right|$$

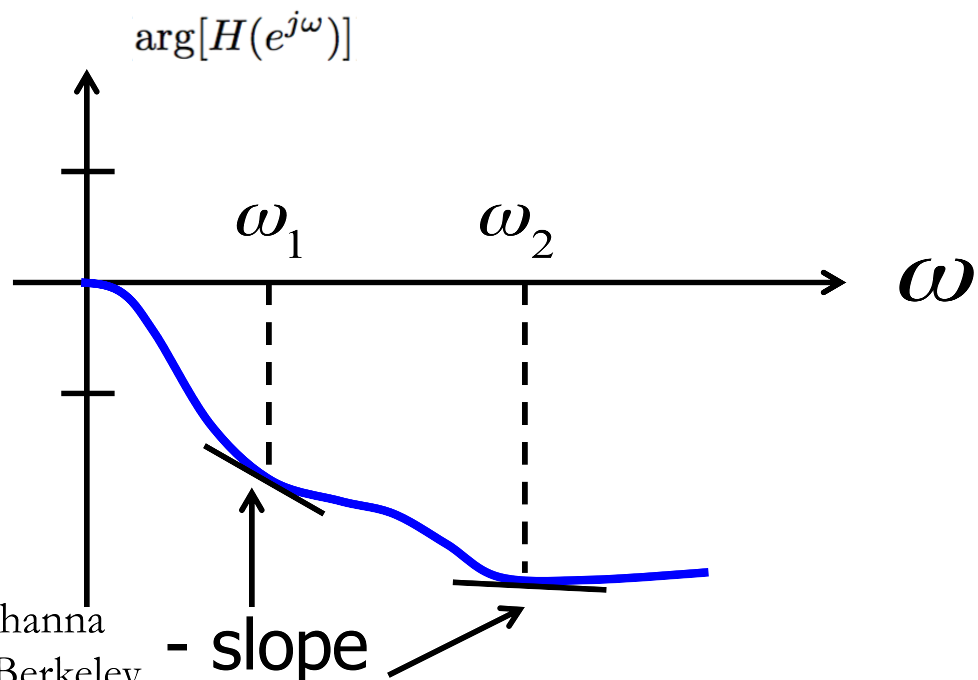
- And a phase response

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X\left(e^{j\omega}\right)$$

Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\mathrm{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$



LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

Stable and causal if all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

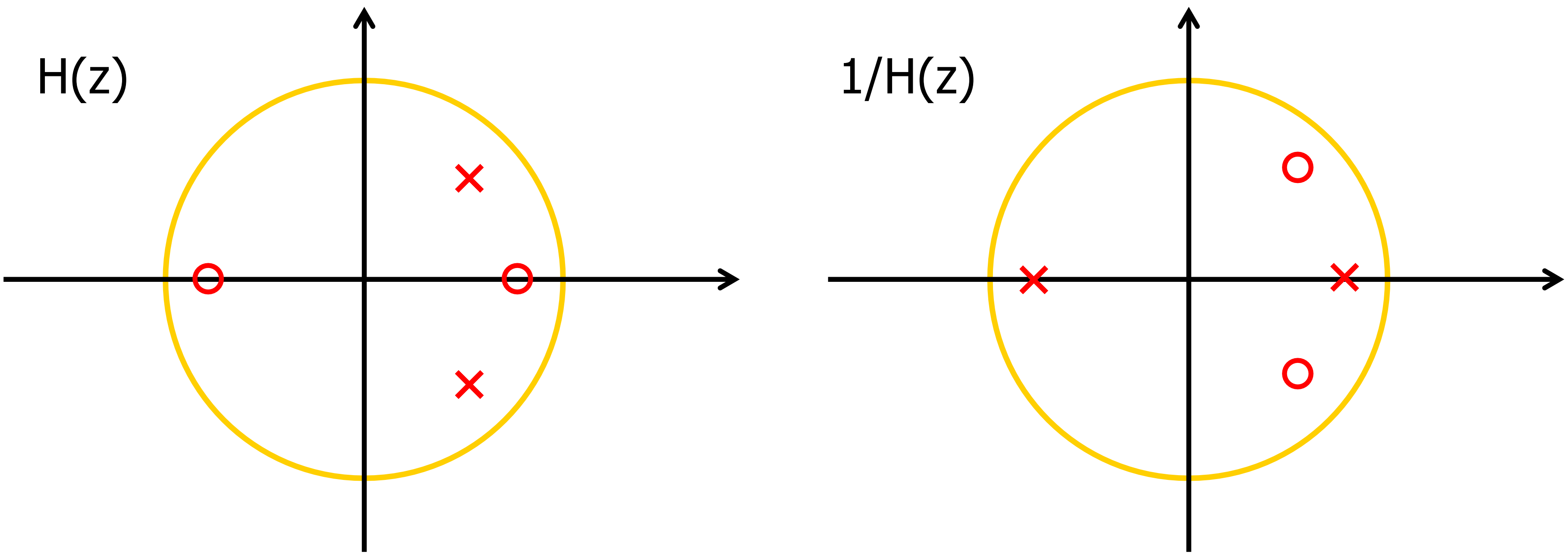
General All-Pass Filter

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

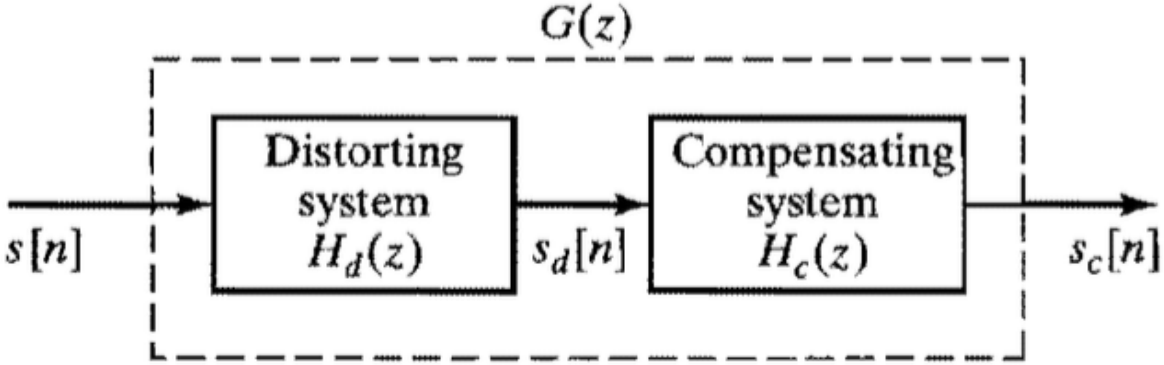
Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)
  - All poles and zeros inside unit circle



Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z)$ → $H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

Generalized Linear Phase

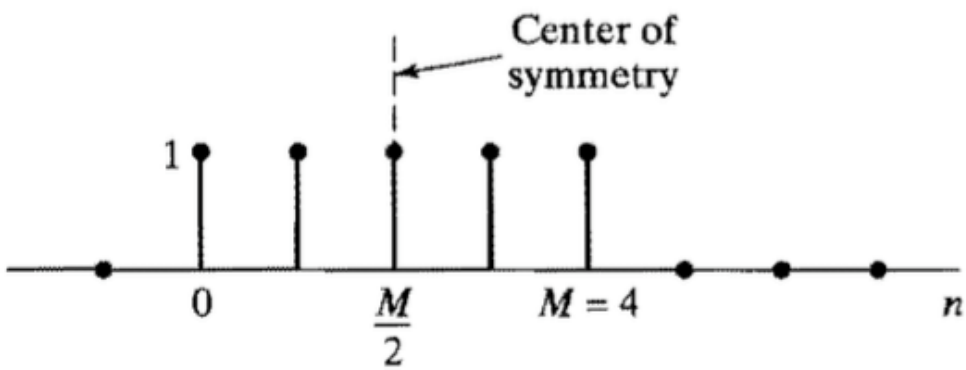
- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

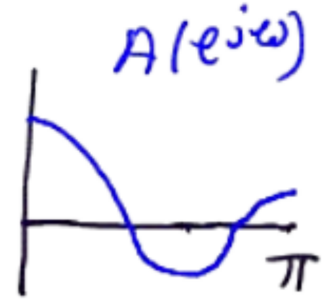
$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha + j\beta}, \quad |\omega| < \pi$$

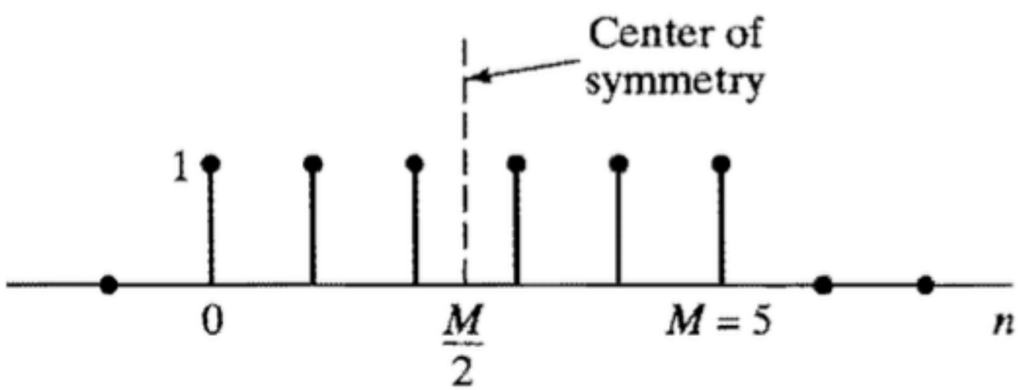
- Where $A(\omega)$ is a real function.

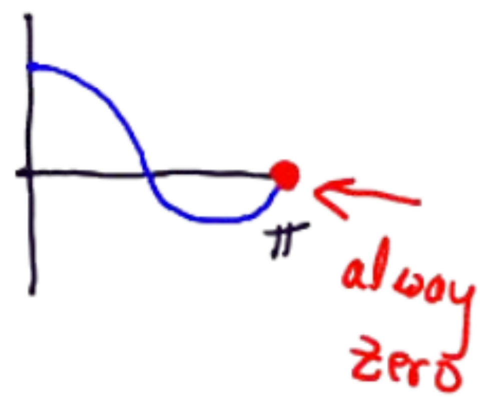
- What is the group delay?

FIR GLP: Type I and II

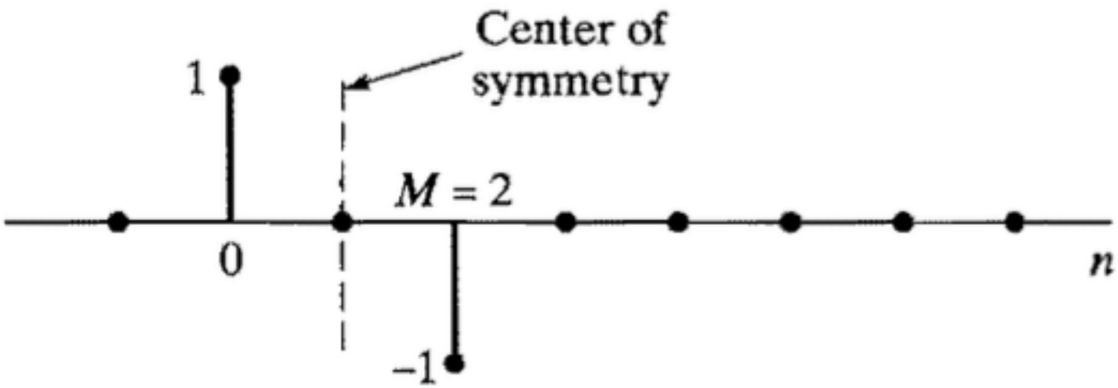


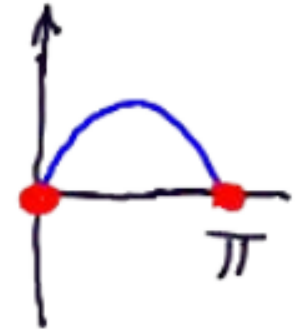


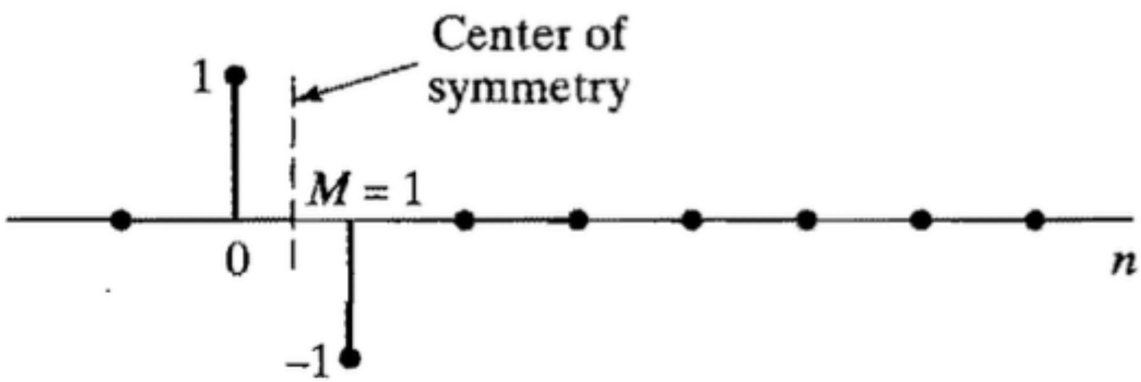


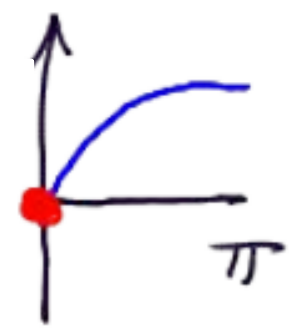


FIR GLP: Type III and IV









Zeros of GLP System

- FIR GLP System Function

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

- If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

- If zero is real and not on unit circle (θ=0)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

FIR Filter Design

FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$, find an impulse response

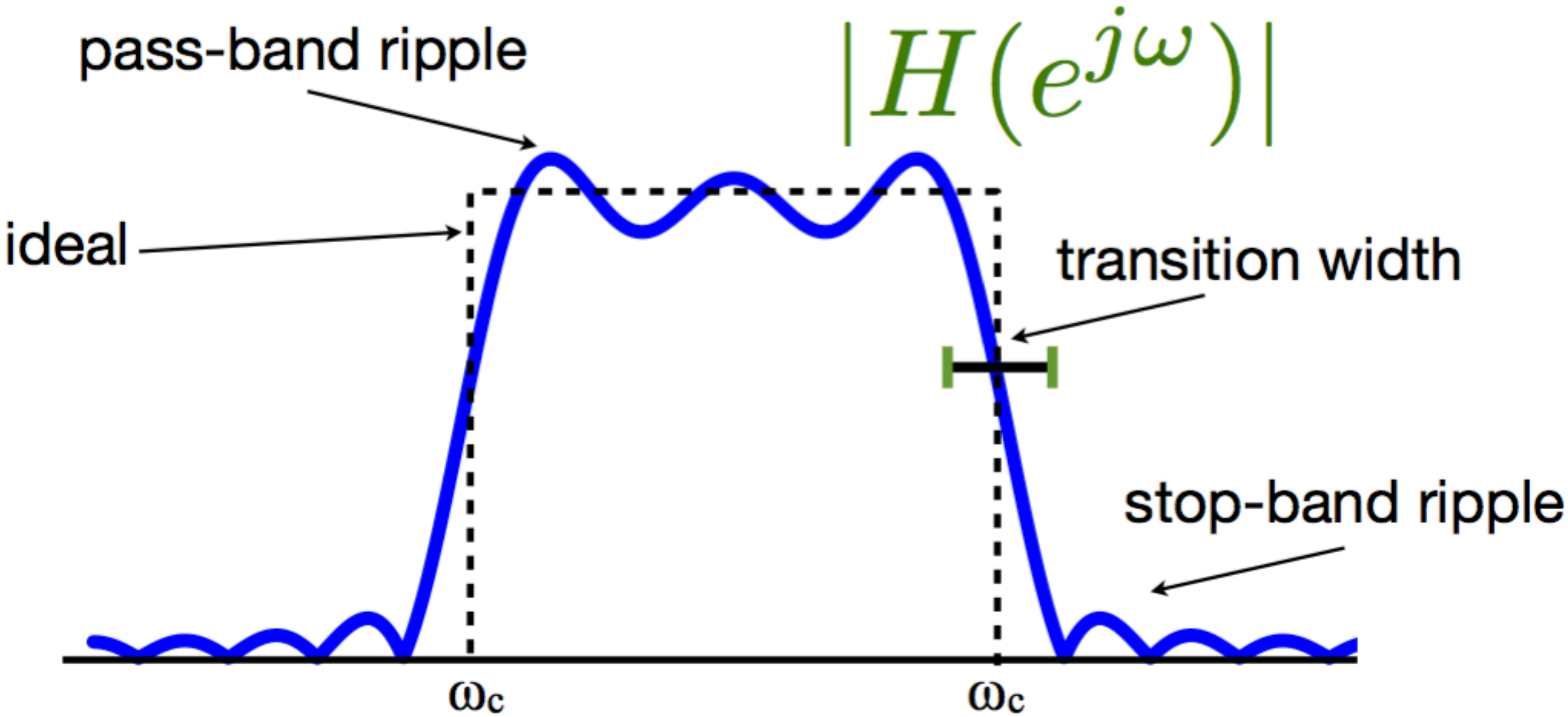
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

ideal

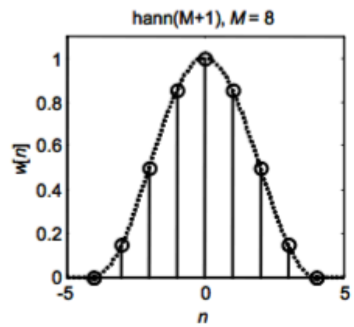
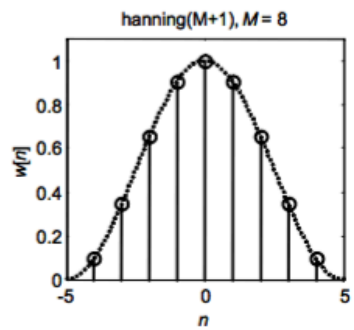
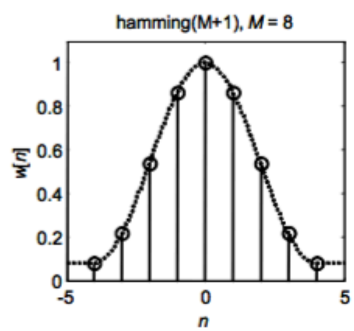
- Obtain the M^{th} order causal FIR filter by truncating/windowing it

$$h[n] = \left\{ \begin{array}{ll} h_d[n]w[n] & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{array} \right\}$$

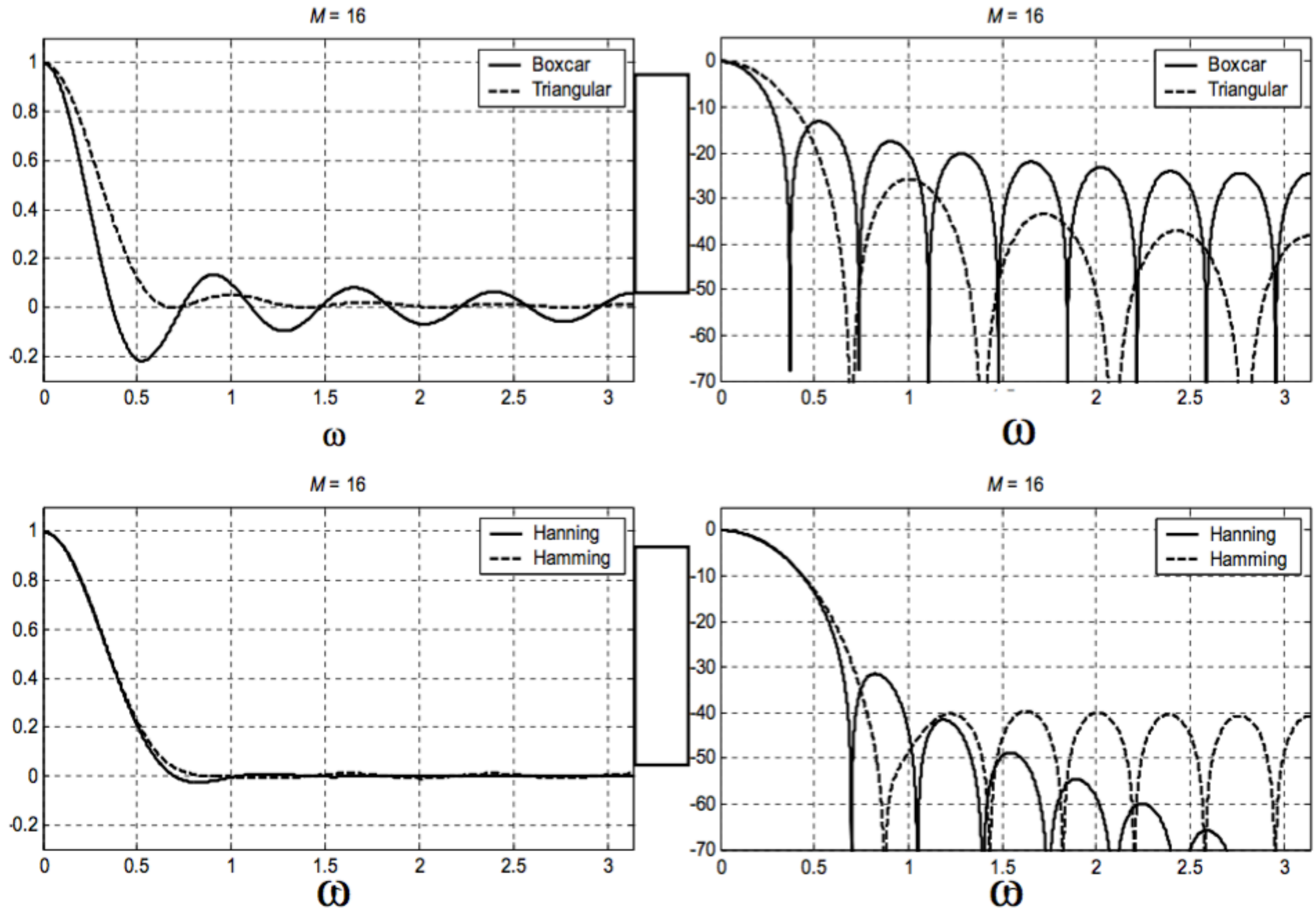
FIR Design by Windowing



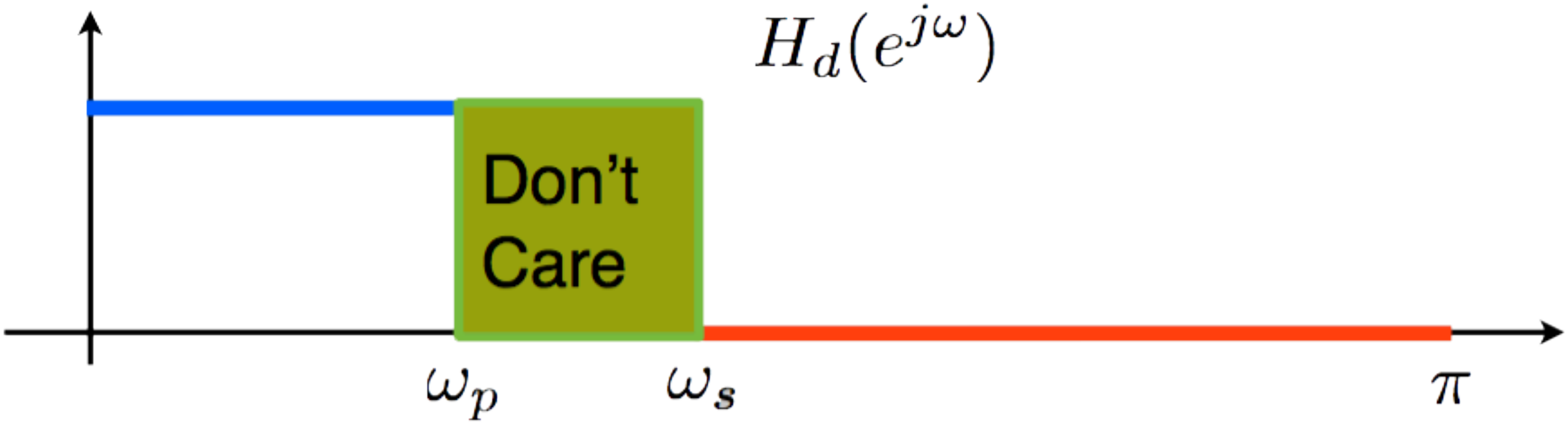
Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Tradeoff – Ripple vs. Transition Width



Optimality



- Least Squares:

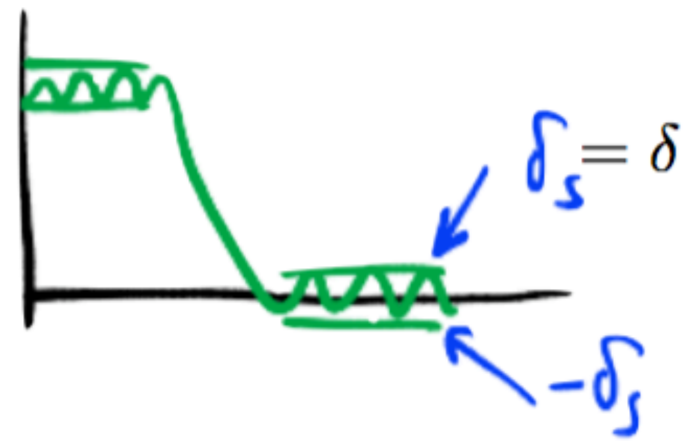
$$\text{minimize} \quad \int_{\omega \in \text{care}} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

- Variation: Weighted Least Squares:

$$\text{minimize} \quad \int_{-\pi}^{\pi} W(\omega)|H(e^{j\omega}) - H_d(e^{j\omega})|^2 d\omega$$

Min-Max Ripple Design

- Recall, $\tilde{H}(e^{j\omega})$ is symmetric and real
- Given ω_p, ω_s, M, find δ, $\tilde{h}_+$



$$
\begin{aligned}
&\text{minimize} \quad \delta \\
&\text{Subject to :} \\
&1-\delta \leq \tilde{H}(e^{j\omega_k}) \leq 1+\delta \qquad 0 \leq \omega_k \leq \omega_p \\
&-\delta \leq \tilde{H}(e^{j\omega_k}) \leq \delta \qquad \omega_s \leq \omega_k \leq \pi \\
&\delta > 0
\end{aligned}
$$

- Formulation is a linear program with solution δ, $\tilde{h}_+$
- A well studied class of problems

IIR Filter Design

IIR Filter Design

- Transform continuous-time filter into a discrete-time filter meeting specs
  - Pick suitable transformation from s (Laplace variable) to z (or t to n)
  - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$

- We've seen this before… impulse invariance

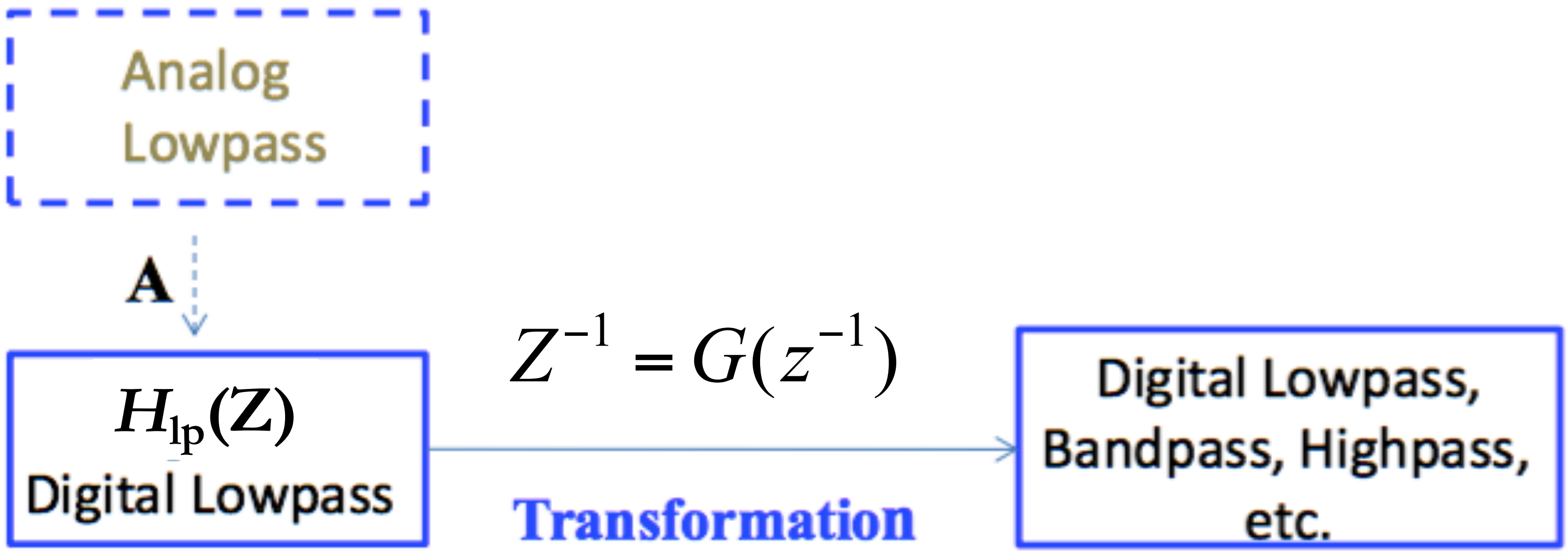
Bilinear Transformation

- The technique uses an algebraic transformation between the variables s and z that maps the entire jΩ-axis in the s-plane to one revolution of the unit circle in the z-plane.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right).$$

Transformation of DT Filters



- Map Z-plane→z-plane with transformation G

$$H(z) = H_{\text{lp}}(Z)\Big|_{Z^{-1}=G(z^{-1})}$$

General Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \frac{z^{-1} - \alpha}{1 - az^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Highpass	$Z^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ $\omega_p =$ desired cutoff frequency
Bandpass	$Z^{-1} = -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency
Bandstop	$Z^{-1} = \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right) \tan\left(\frac{\theta_p}{2}\right)$ $\omega_{p1} =$ desired lower cutoff frequency $\omega_{p2} =$ desired upper cutoff frequency

Discrete Fourier Transform

DFT

Discrete Fourier Transform

- The DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$ Inverse DFT, synthesis

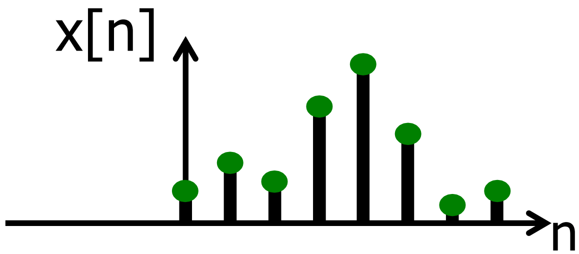
$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$ DFT, analysis

- It is understood that,

$$x[n] = 0 \quad \text{outside } 0 \le n \le N-1$$
$$X[k] = 0 \quad \text{outside } 0 \le k \le N-1$$

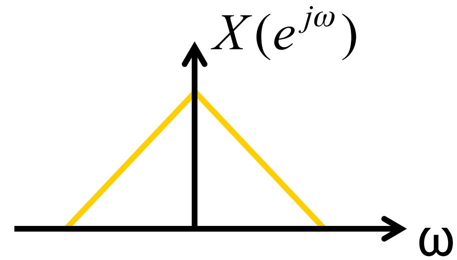
DFT Intuition

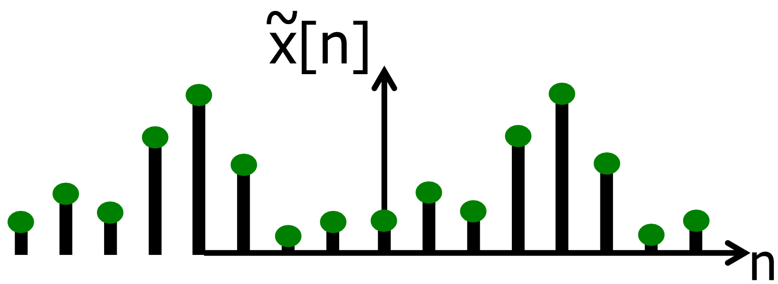
Time	Transform	Frequency



DTFT →

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

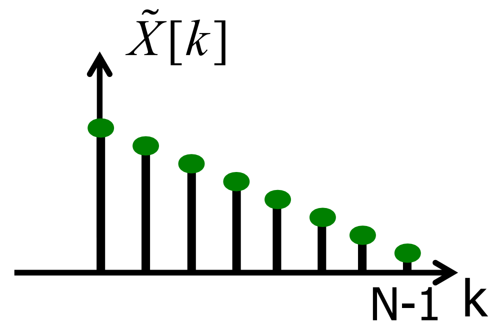


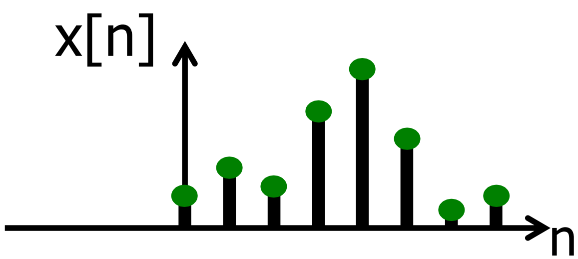


Periodic in N

DFS →

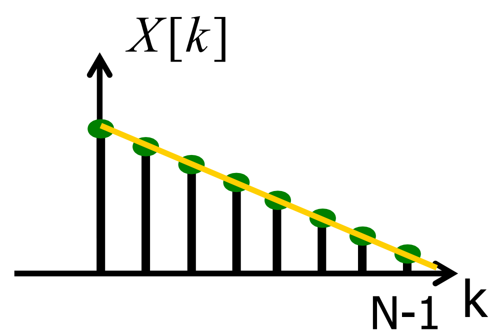
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}$$





DFT →

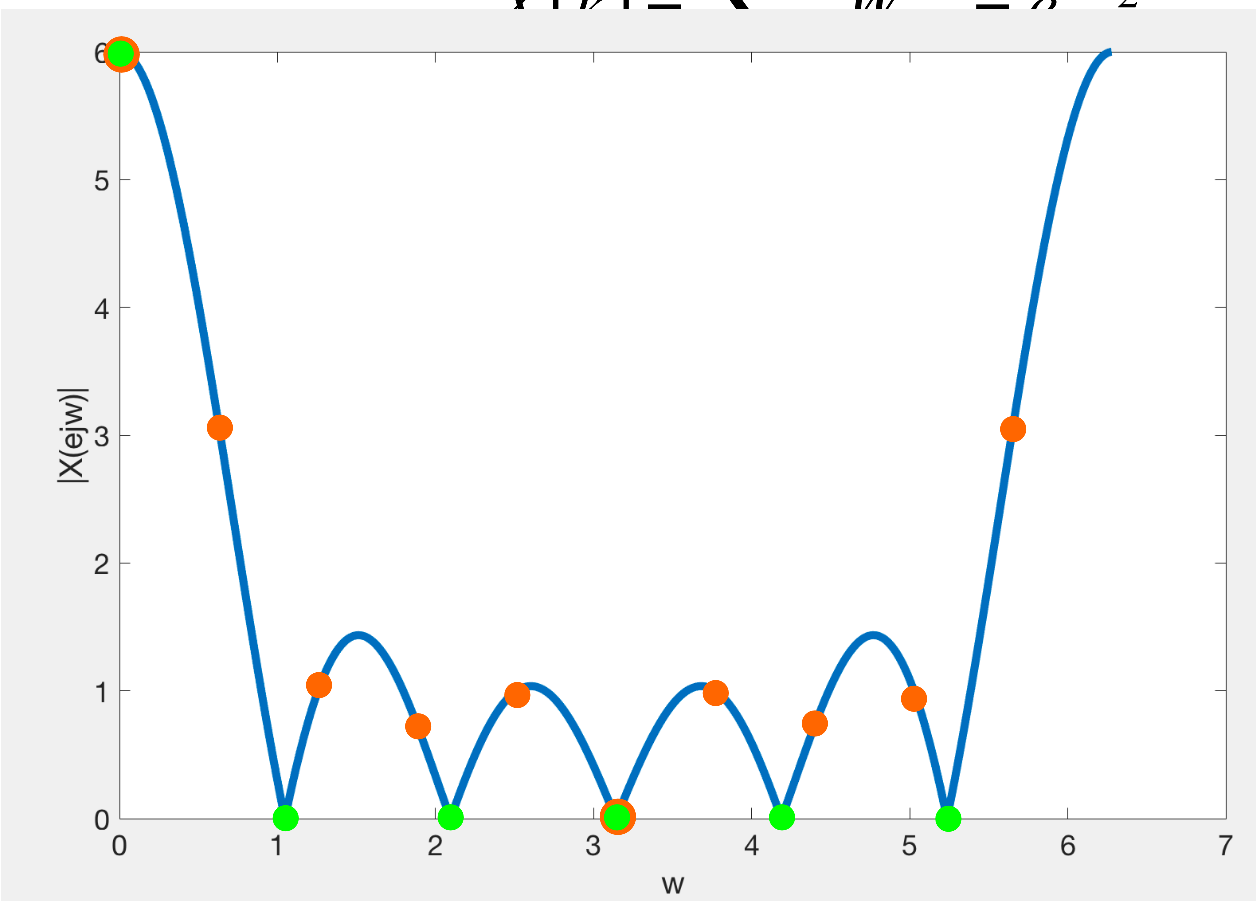
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$



DFT vs DTFT

- Back to example

$$X[k]=\sum^{5} W^{nk} = e^{-j\frac{\pi}{2}k}\frac{\sin\left(\frac{3\pi}{5}k\right)}{\sin\left(\frac{\pi}{10}k\right)}$$



"6-point" DFT

"10-point" DFT

Use `fftshift` to center around dc

Circular Convolution

- For $x_1[n]$ and $x_2[n]$ with length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  - Very useful!! (for linear convolutions with DFT)

Linear Convolution via Circular Convolution

- Zero-pad x[n] by P-1 zeros

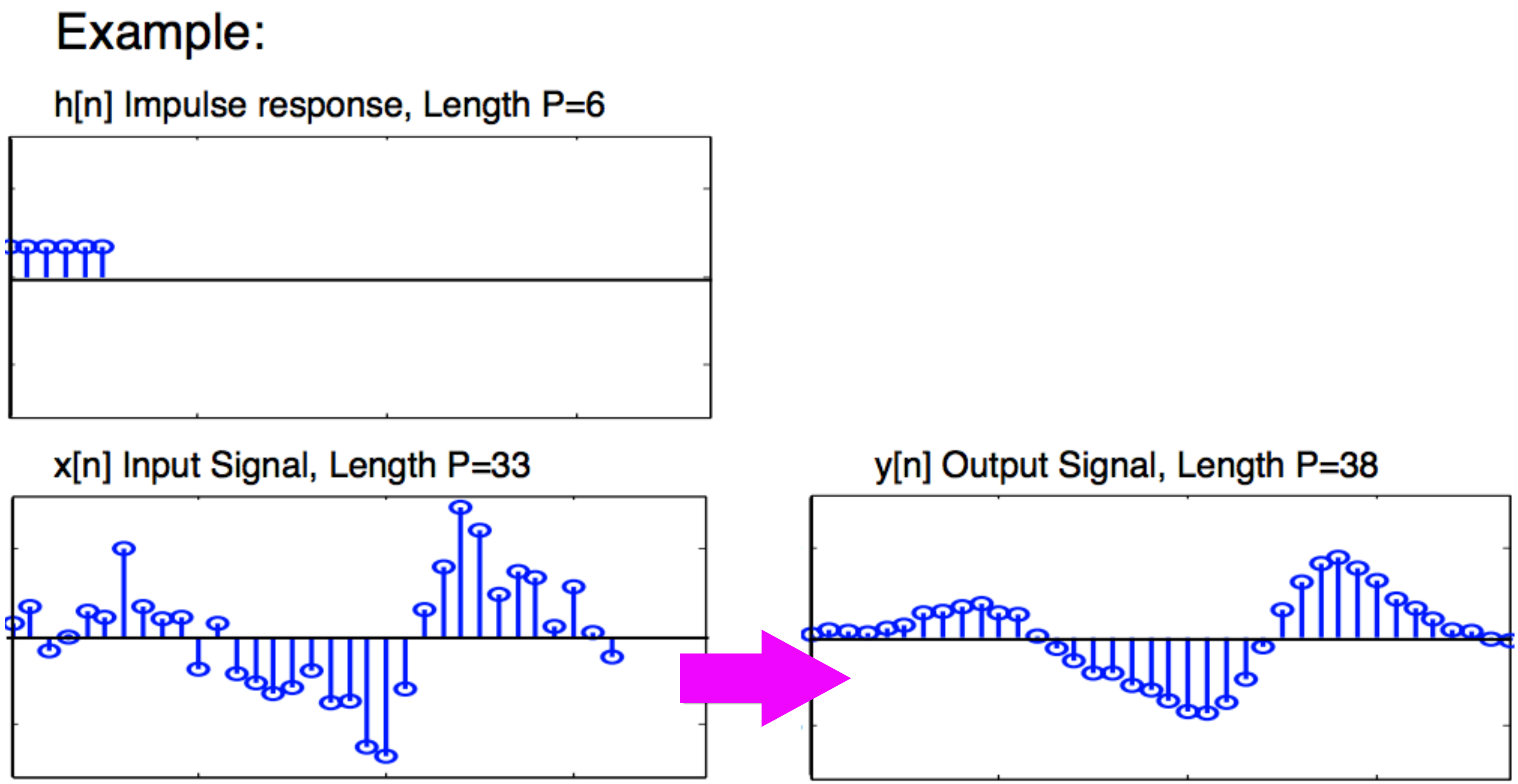
$$x_{\mathrm{zp}}[n] = \begin{cases} x[n] & 0 \leq n \leq L-1 \\ 0 & L \leq n \leq L+P-2 \end{cases}$$

- Zero-pad h[n] by L-1 zeros

$$h_{\mathrm{zp}}[n] = \begin{cases} h[n] & 0 \leq n \leq P-1 \\ 0 & P \leq n \leq L+P-2 \end{cases}$$

- Now, both sequences are length M=L+P-1

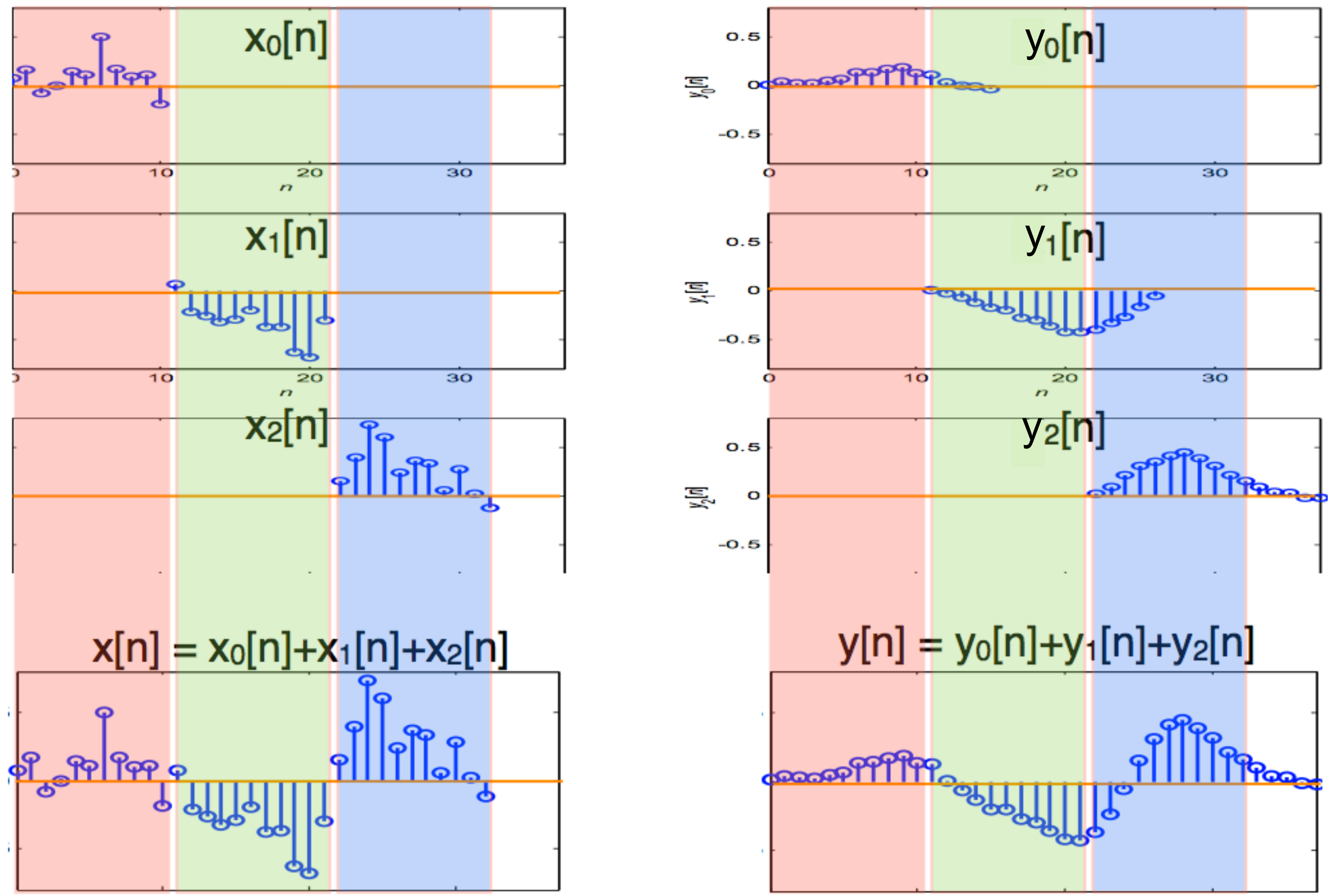
Block Convolution



Example of Overlap-Add

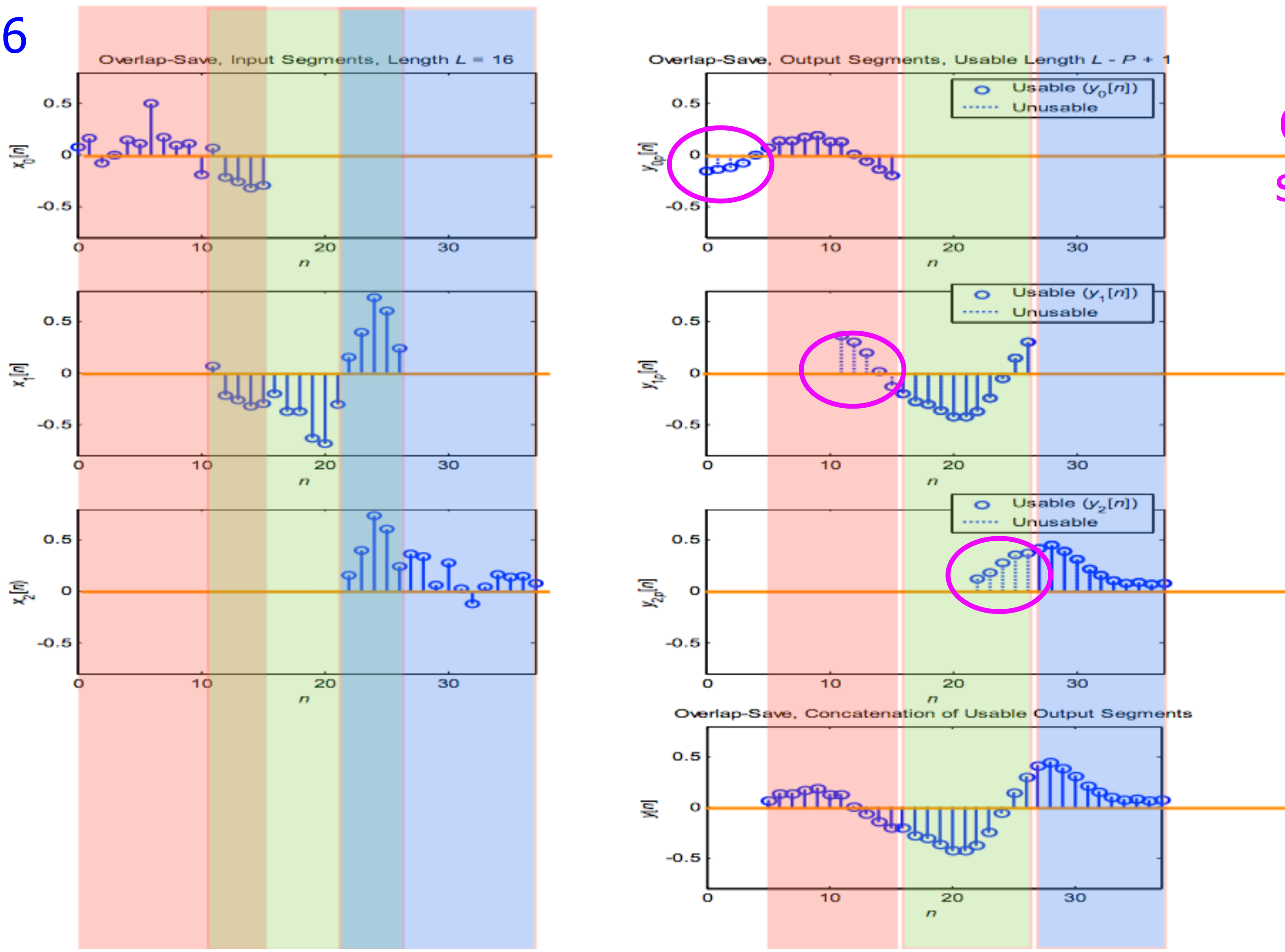
L+P-1=16

L=11



Example of Overlap-Save

L+P-1=16



P-1=5
Overlap
samples

Circular Conv. as Linear Conv. w/ Aliasing

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n - rN], & 0 \le n \le N-1, \\ 0, & \text{otherwise}, \end{cases}$$

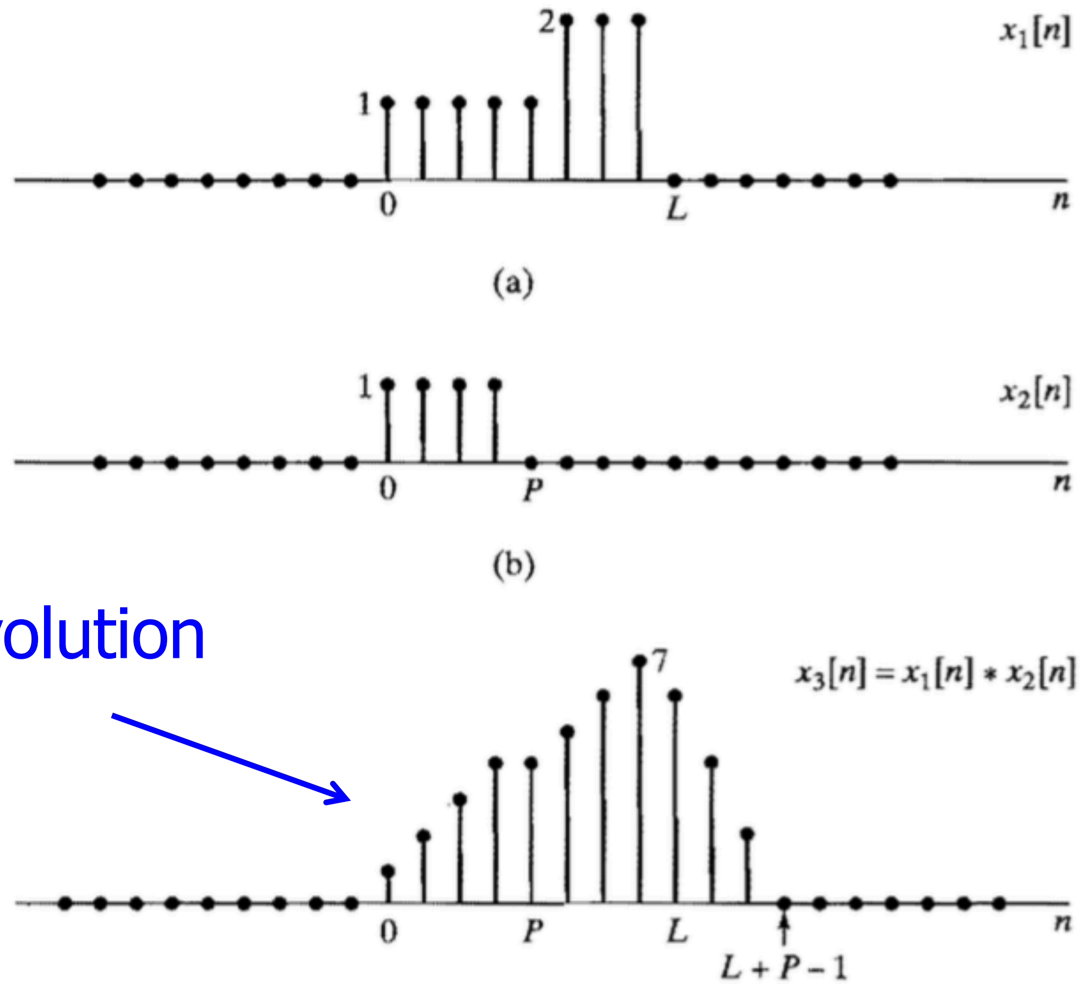
- Therefore

$$x_{3p}[n] = x_1[n] \, \text{Ⓝ} \, x_2[n]$$

- The N-point circular convolution is the sum of linear convolutions shifted in time by N

Example:

- Let

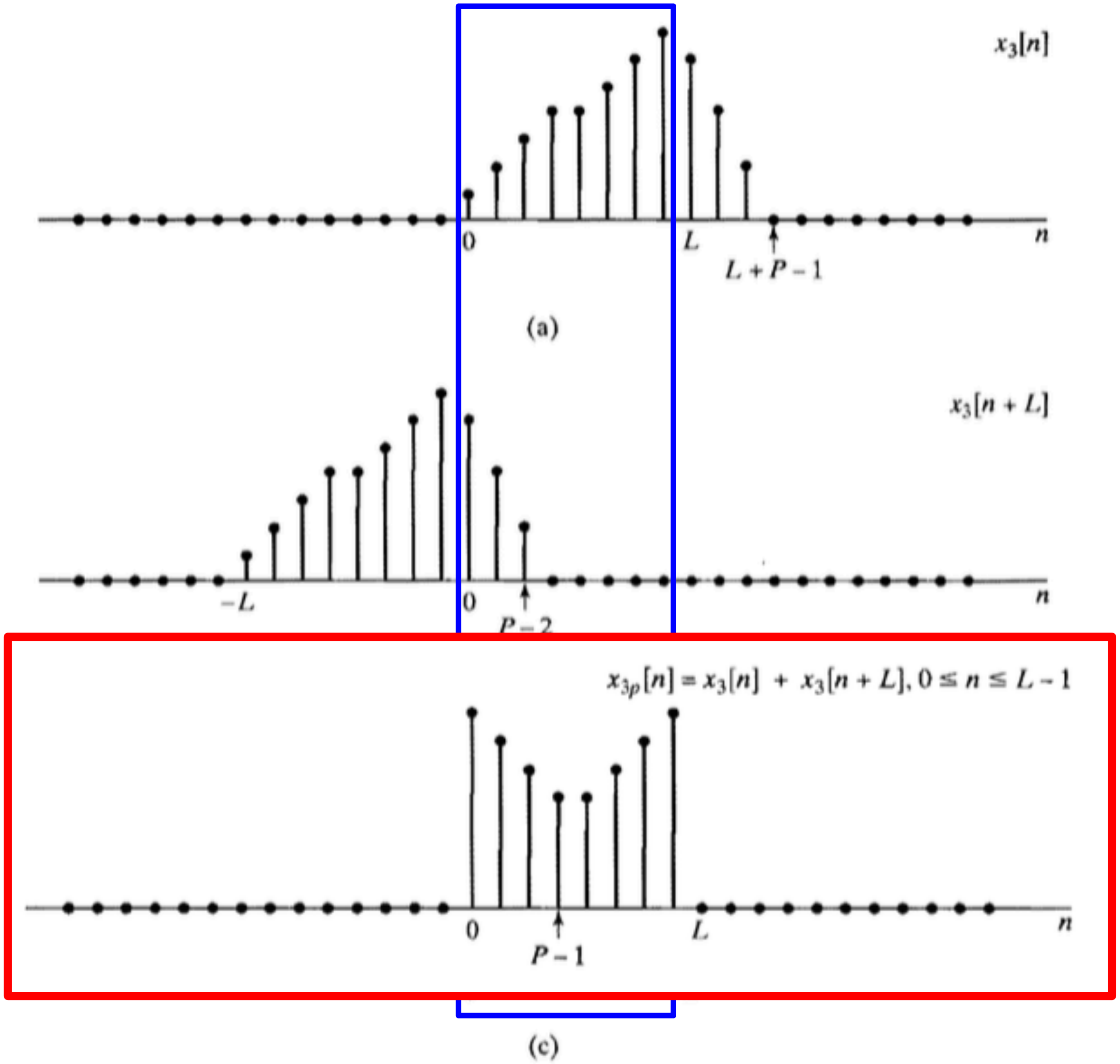


Linear convolution

- What does the L-point circular convolution look like?

Example:

- The L-shifted linear convolutions



Fast Fourier Transform

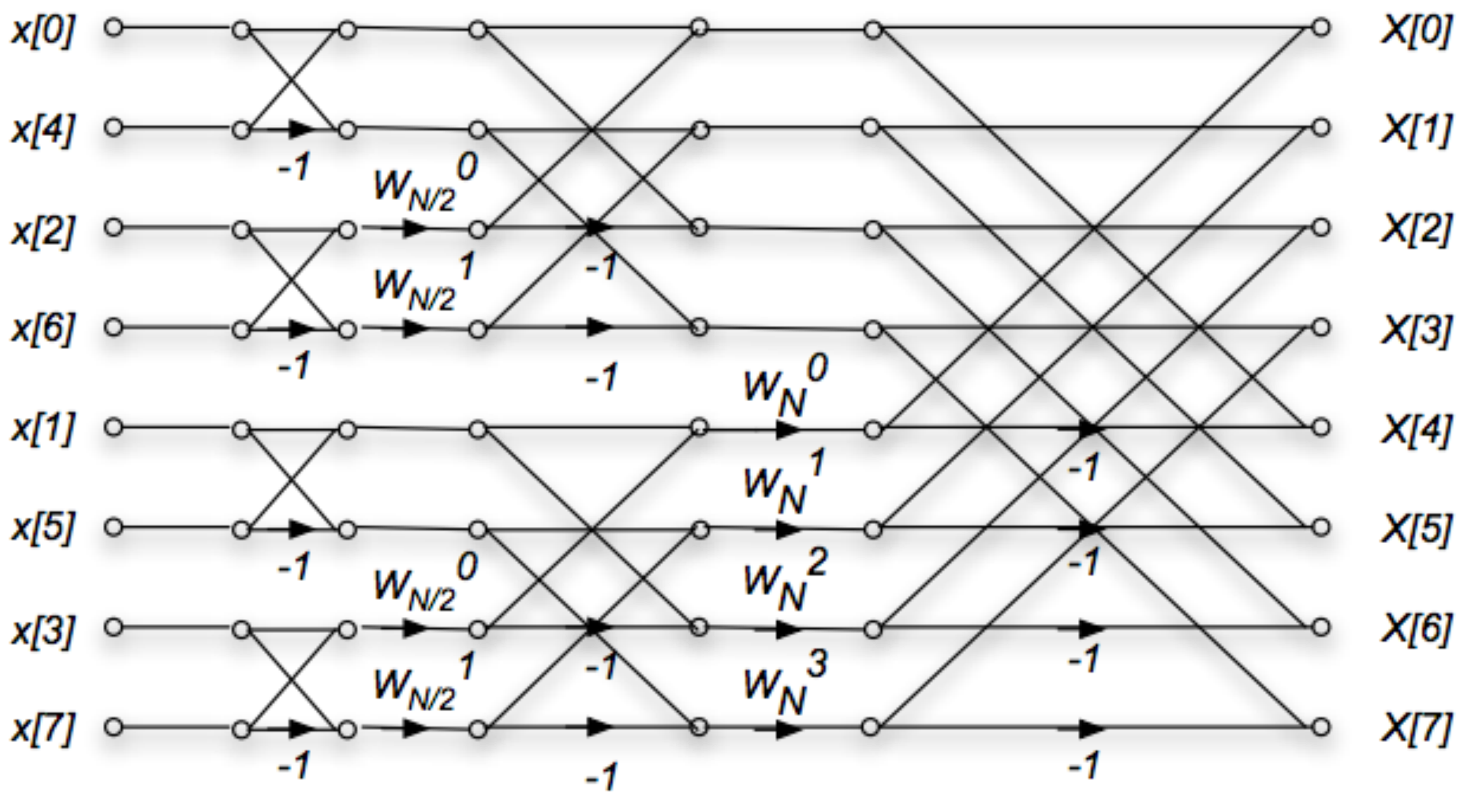
FFT

Fast Fourier Transform

- Enable computation of an N-point DFT (or DFT^{-1}) with the order of just $N \cdot \log_2 N$ complex multiplications.
- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - Decimation-in-time algorithms
  - Decimation-in-frequency
- Historically, power-of-2 DFTs had highest efficiency
- Modern computing has led to non-power-of-2 FFTs with high efficiency
- Sparsity leads to reduce computation on order $K \cdot \log N$

Decimation-in-Time FFT

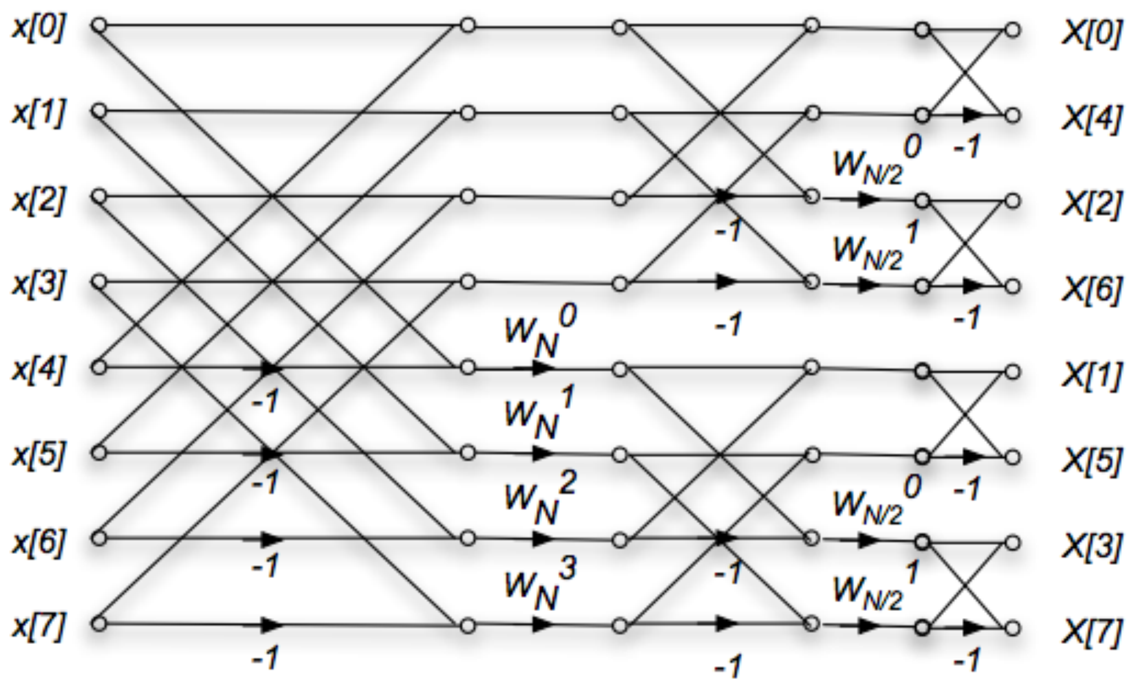
Combining all these stages, the diagram for the 8 sample DFT is:



- $3=\log_2(N)=\log_2(8)$ stages
- $4=N/2=8/2$ multiplications in each stage
  - 1st stage has trivial multiplication

Decimation-in-Frequency FFT

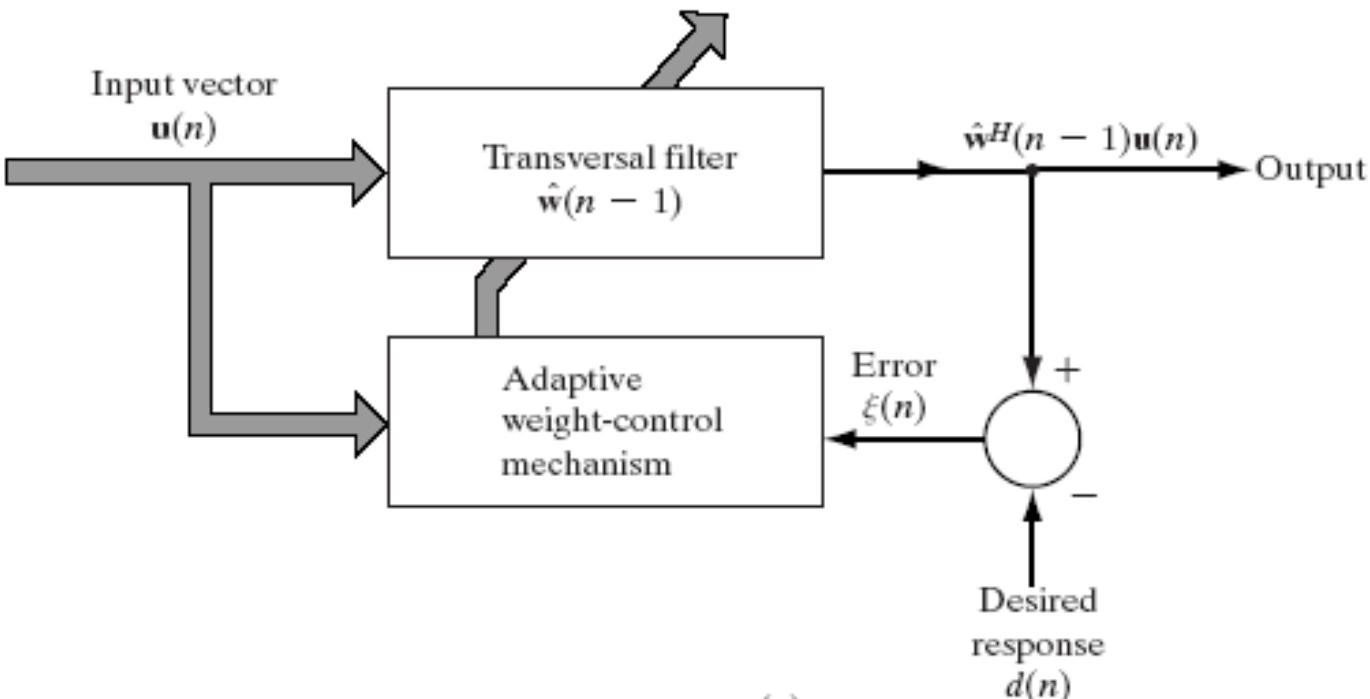
The diagram for and 8-point decimation-in-frequency DFT is as follows



This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Adaptive Filters

- Use LMS algorithm to update filter coefficients
- Applications like system ID, channel equalization, and signal prediction



(a)

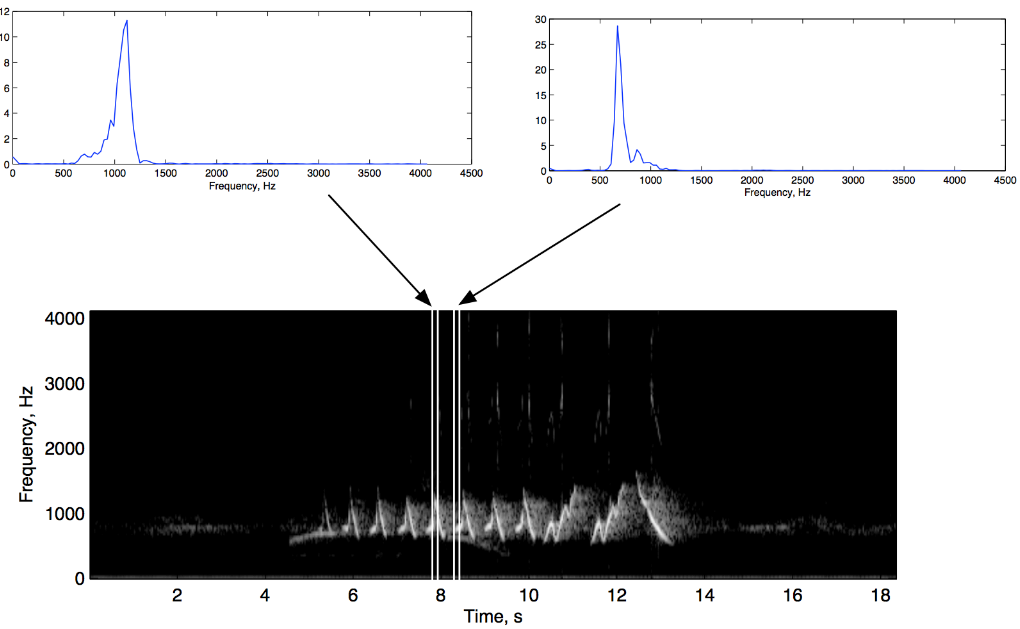
Spectral Analysis

- Frequency analysis with DFT
  - Nontrivial to choose sampling frequency, signal length, window type, DFT length (zero-padding)
  - Get accurate representation of DFT

Parameter	Symbol	Units
Sampling interval	T	s
Sampling frequency	$\Omega_s = \frac{2\pi}{T}$	rad/s
Window length	L	unitless
Window duration	$L \cdot T$	s
DFT length	$N \geq L$	unitless
DFT duration	$N \cdot T$	s
Spectral resolution	$\frac{\Omega_s}{L} = \frac{2\pi}{L \cdot T}$	rad/s
Spectral sampling interval	$\frac{\Omega_s}{N} = \frac{2\pi}{N \cdot T}$	rad/s

Spectral Analysis

- Time-dependent Fourier transform
  - Includes temporal information about signal
  - Useful for many applications
    - Analysis, Compression, Denoising, Detection, Recognition, Approximation (Sparse)



Admin

- Final Project due 4/26 – Tomorrow!
  - TA advice – "The report takes time. Leave time for it."
- Office hour schedule for rest of semester (See Ed for zoom links)
  - 25th T - Zhihan, 10-11:30 am and 7-8:30 pm
  - 26th W – Tania, 1-2:30 pm (Levine 262)
  - 27th Th – Tania 12-1pm, Shuang 5-7:30pm (LRSM 208)
  - 28th F - Shuang 3-4pm
  - 29th Sa - Shuang 2-4pm zoom
  - 1st M - Zhihan, 10-11:30 am
- Final Exam – May 1st

Admin - Final Exam Admin

- Final Exam – 5/1 (6-8pm)
  - Location DRLB A2
  - Cumulative – covers lec 1-23
    - Except data converters, noise shaping (lec 12)
    - Closed book
    - Data/Equation sheet provided by me (see old exams)
    - 2 8.5x11 two-sided cheat sheets allowed
    - Calculators allowed, no smart phones
      - Can't share. Bring your own.
  - Old exams posted
    - Disclaimers: old exams had different coverage for different years
  - TA Review session 4/28 (Fri) 1-2pm in Towne 307
    - Will not be recorded