ESE 5310: Digital Signal Processing

Lecture 3: January 19, 2023

Discrete Time Signals and Systems, Pt 2

Lecture Outline

- Discrete Time Systems
- System Properties
- LTI Systems
- Difference Equations

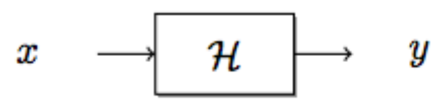
Discrete-Time Systems

Discrete Time Systems

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$



- Systems manipulate the information in signals
- Examples
  - Speech recognition system that converts acoustic waves into text
  - Radar system transforms radar pulse into position and velocity
  - fMRI system transform frequency into images of brain activity
  - Moving average system smooths out the day-to-day variability in a stock price

System Properties

- Causality
  - y[n] only depends on x[m] for m<=n
- Linearity
  - Scaled sum of arbitrary inputs results in output that is a scaled sum of corresponding outputs
    - $Ax_1[n]+Bx_2[n] \rightarrow Ay_1[n]+By_2[n]$
- Memoryless
  - y[n] depends only on x[n]
- Time Invariance
  - Shifted input results in shifted output
    - $x[n-q] \rightarrow y[n-q]$
- BIBO Stability
  - A bounded input results in a bounded output (ie. max signal value exists for output if max)

Proving Linearity

- A system that is not linear is called **nonlinear**
- To prove that a system is linear, you must prove rigorously that it has **both** the scaling and additive properties for **arbitrary** input signals
- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**

Linearity Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

- **Scaling:** (Strategy to prove – Scale input x by α, compute output y via the formula at top and verify that is scaled as well)
  - Let

$$x'[n] \;=\; \alpha x[n], \quad \alpha \in \mathbb{C}$$

Linearity Example: Moving Average

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- **Scaling:** (Strategy to prove – Scale input x by α, compute output y via the formula at top and verify that is scaled as well)
  - Let

$$x'[n] = \alpha x[n], \quad \alpha \in \mathbb{C}$$

  - Let y' denote the output when x' is input
  - Then

$$y'[n] = \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(\alpha x[n] + \alpha x[n-1]) = \alpha\left(\frac{1}{2}(x[n] + x[n-1])\right) = \alpha y[n] \quad \checkmark$$

Linearity Example: Moving Average

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- **Additive:** (Strategy to prove – Input two signals into the system and verify the output equals the sum of the respective outputs
  - Let

$$x'[n] = x_1[n] + x_2[n]$$

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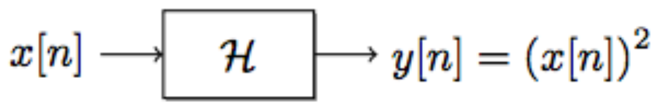
- **Additive:** (Strategy to prove – Input two signals into the system and verify the output equals the sum of the respective outputs
  - Let

$$x'[n] \;=\; x_1[n] + x_2[n]$$

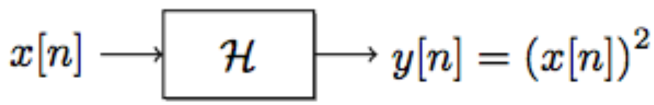
  - Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input
  - Then

$$\begin{aligned} y'[n] &= \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(\{x_1[n] + x_2[n]\} + \{x_1[n-1] + x_2[n-1]\}) \\ &= \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1]) \;=\; y_1[n] + y_2[n] \quad \checkmark \end{aligned}$$

Example: Squaring



Example: Squaring is Nonlinear



- **Additive:** Input two signals into the system and see what happens

  - Let

$$y_1[n] = (x_1[n])^2, \qquad y_2[n] = (x_2[n])^2$$

  - Set

$$x'[n] \ = \ x_1[n] + x_2[n]$$

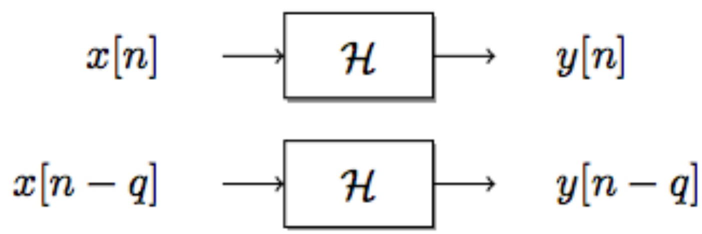
  - Then

$$y'[n] \ = \ (x'[n])^2 \ = \ (x_1[n] + x_2[n])^2 \ = \ (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \ \neq \ y_1[n] + y_2[n]$$

Time-Invariant Systems

DEFINITION

A system $\mathcal{H}$ processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal



- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called time-varying

Example: Moving Average

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \frac{1}{2}(x[n] + x[n-1])$$

- Let

$$x'[n] = x[n-q], \quad q \in \mathbb{Z}$$

- Let y' denote the output when x' is input

Example: Moving Average

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- Let

$$x'[n] = x[n-q], \quad q \in \mathbb{Z}$$

- Let y' denote the output when x' is input

- Then

$$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(x[n-q] + x[n-q-1]) \;=\; y[n-q] \;\checkmark$$

Causal Systems

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

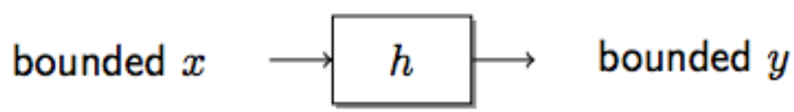
- Forward difference system:
  - y[n]=x[n+1]-x[n] causal?
- Backward difference system:
  - y[n]=x[n]-x[n-1] causal?

Stability

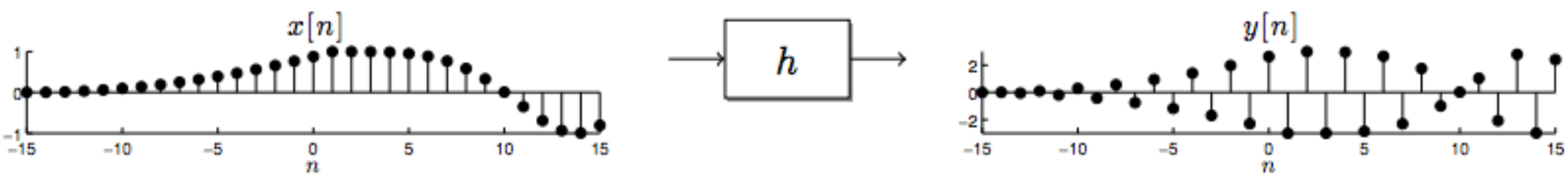
- BIBO Stability
  - Bounded-input bounded-output Stability

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$, or that there exist constants $A, C < \infty$ such that $|x[n]| < A$ and $|y[n]| < C$ for all n



Examples

- Causal? Linear? Time-invariant? Memoryless? BIBO Stable?
- Time Shift:
  - $$y[n] = x[n-m]$$
- Accumulator:
  - $$y[n] = \sum_{k=-\infty}^{n} x[k]$$
- Compressor (M>1):

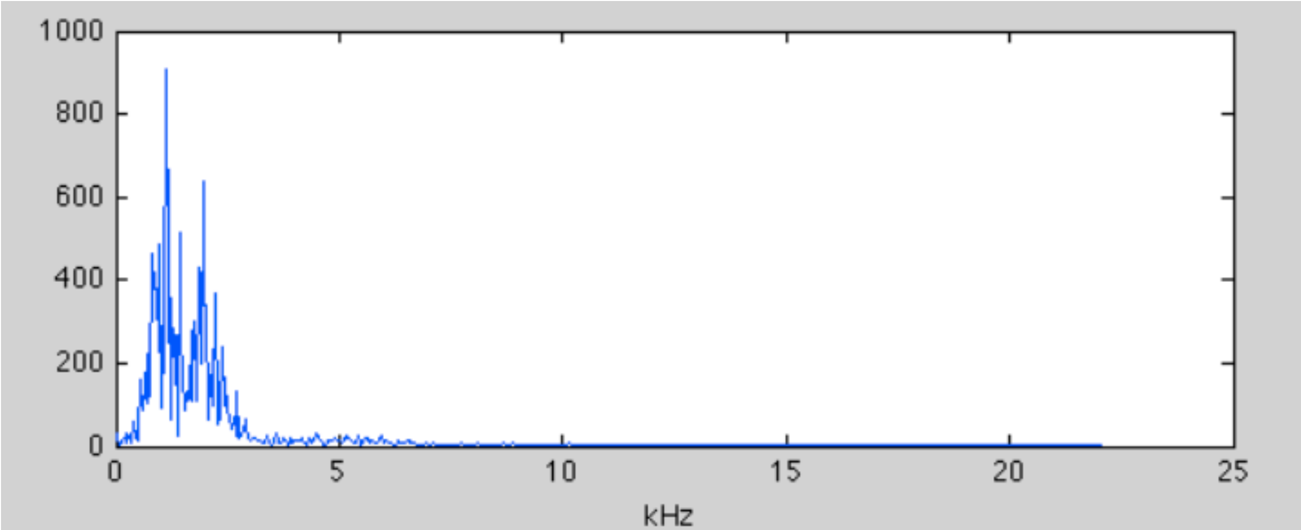
$$y[n] = x[Mn]$$

Non-Linear System Example

- Median Filter
  - $y[n]=MED\{x[n-k], \ldots x[n+k]\}$
  - Let $k=1$
  - $y[n]=MED\{x[n-1], x[n], x[n+1]\}$

Spectrum of Speech

Speech



Spectrum of Speech

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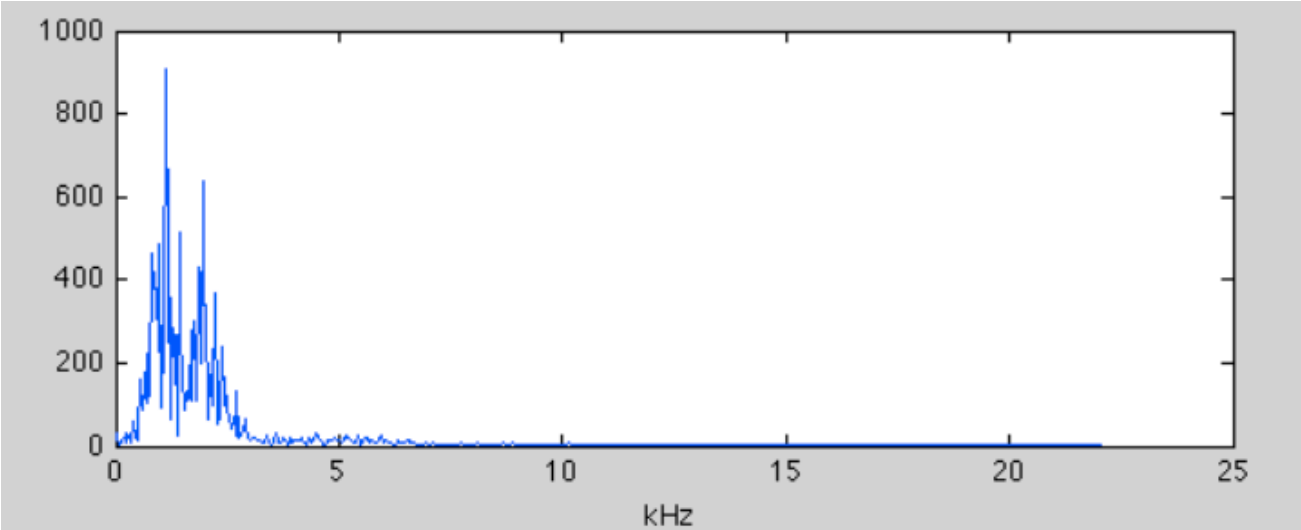
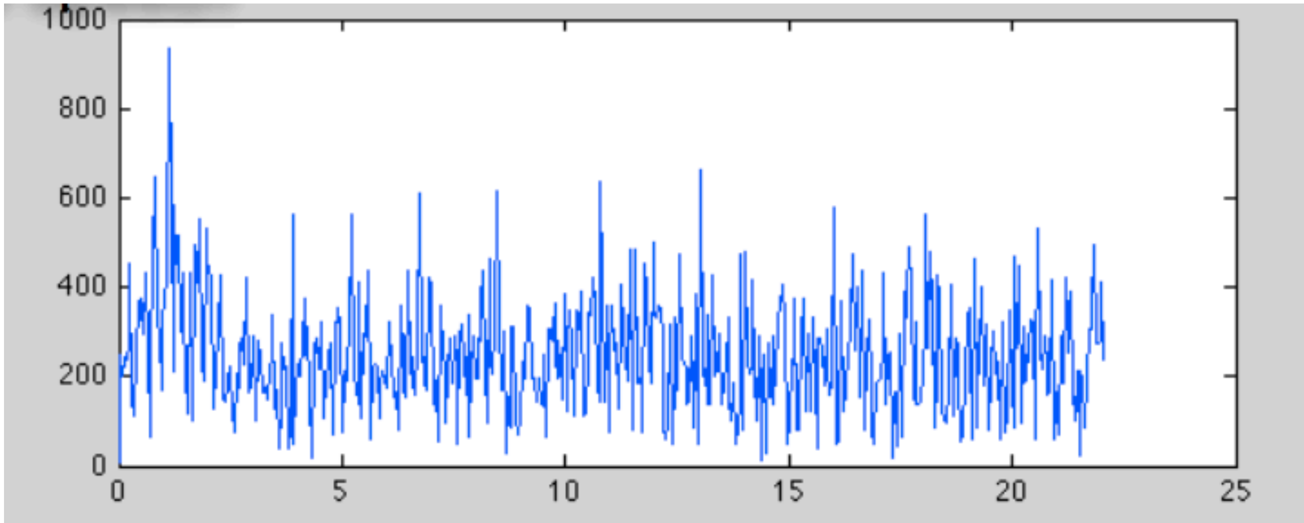
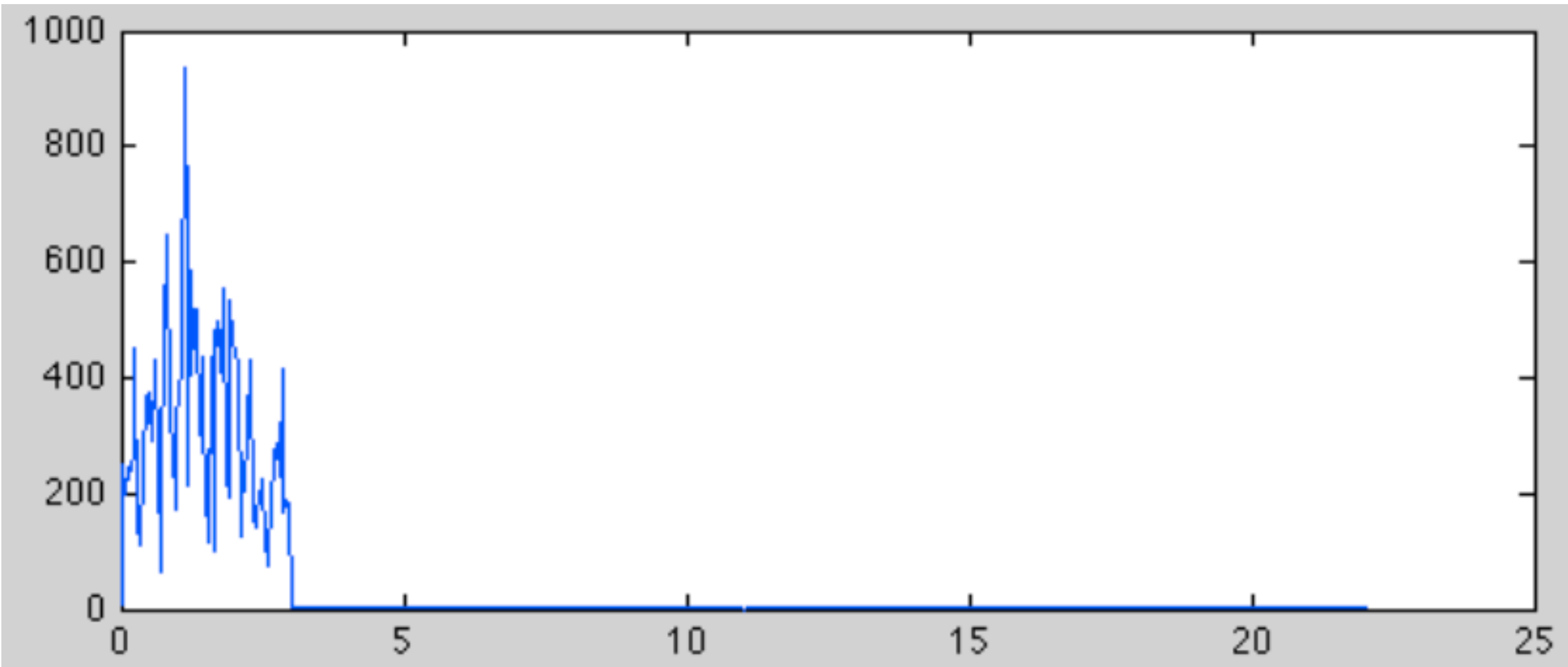


Corrupted Speech

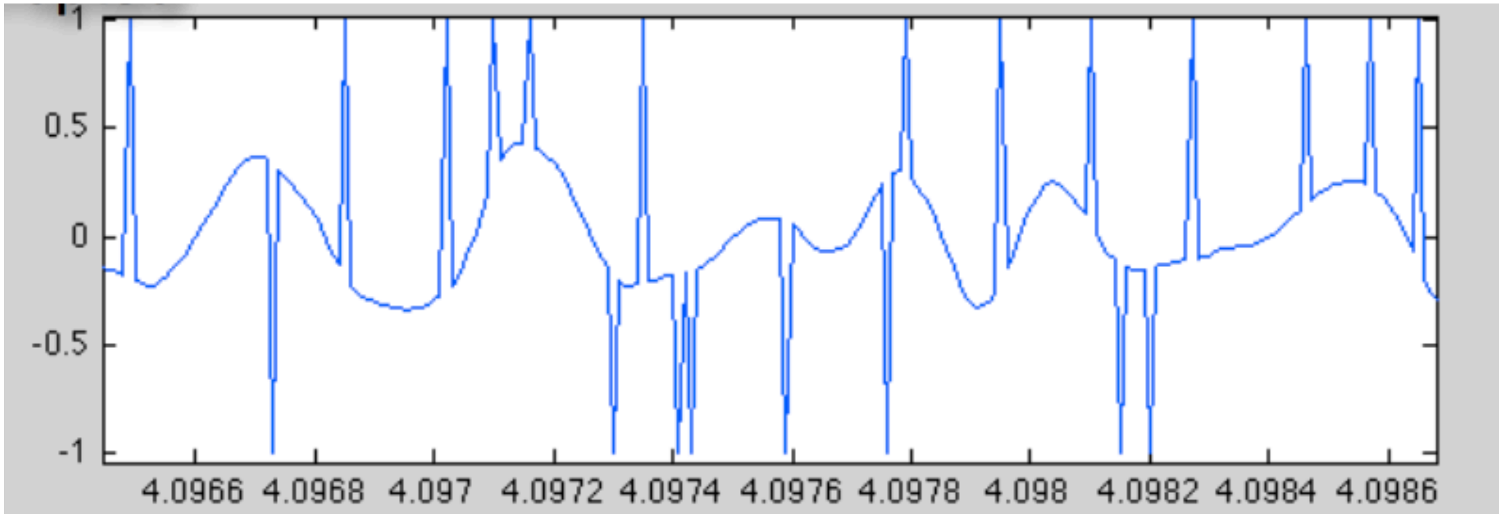


Low Pass Filtering



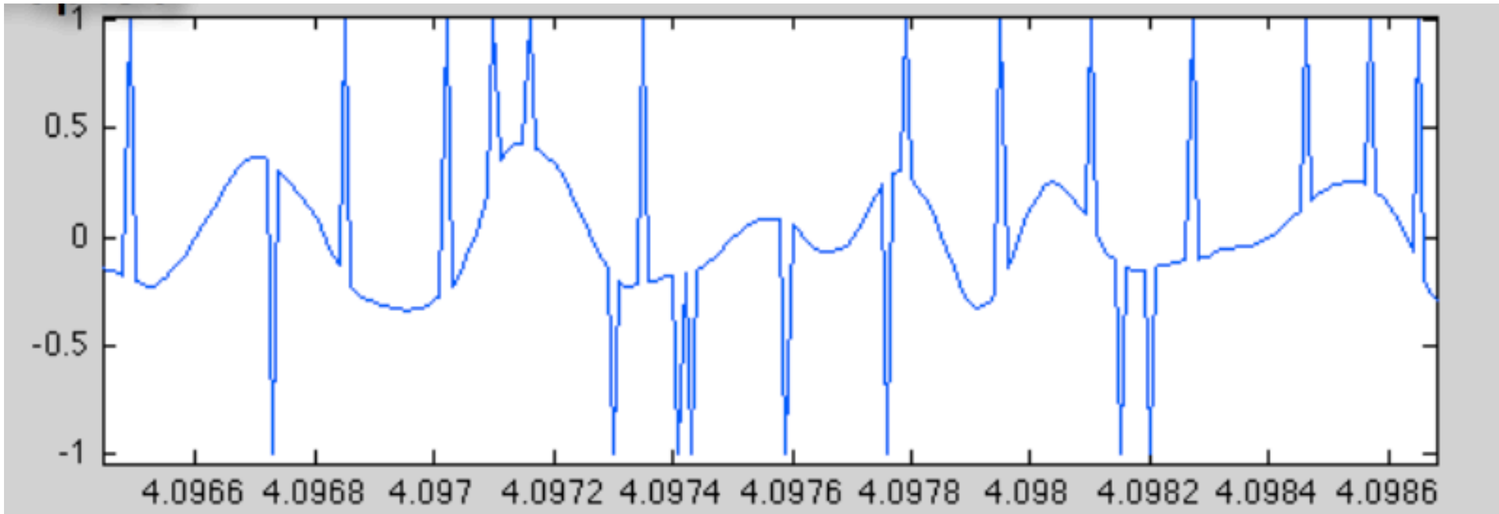
Speech in Time

Corrupted Speech

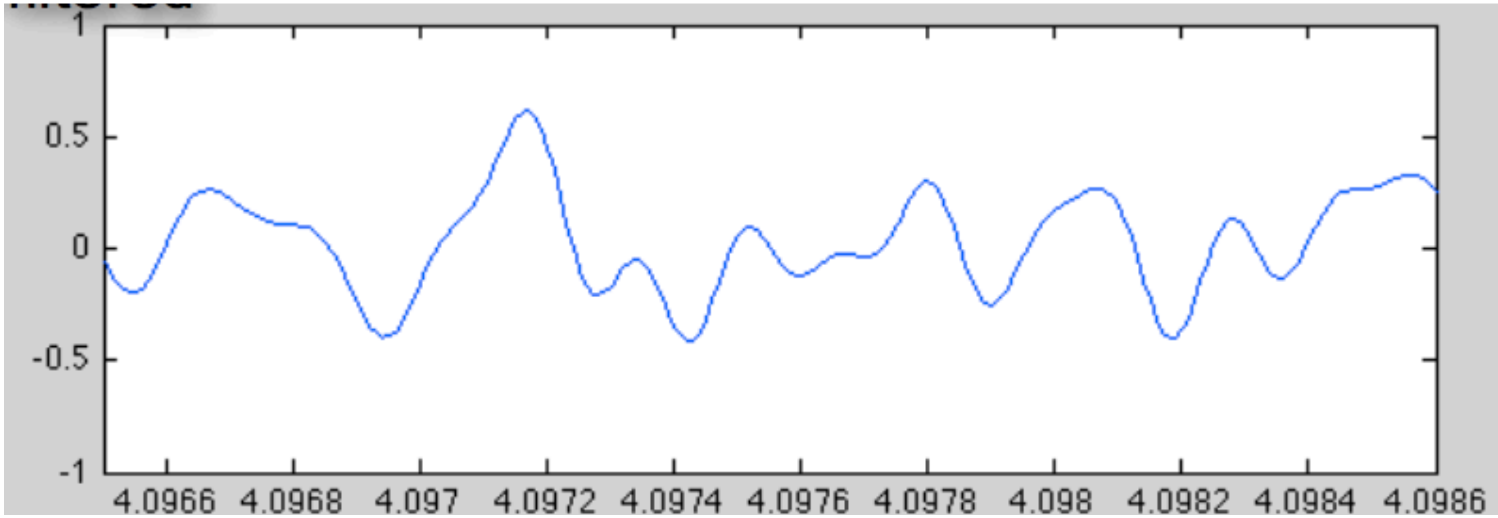


Low Pass Filtering

Corrupted Speech

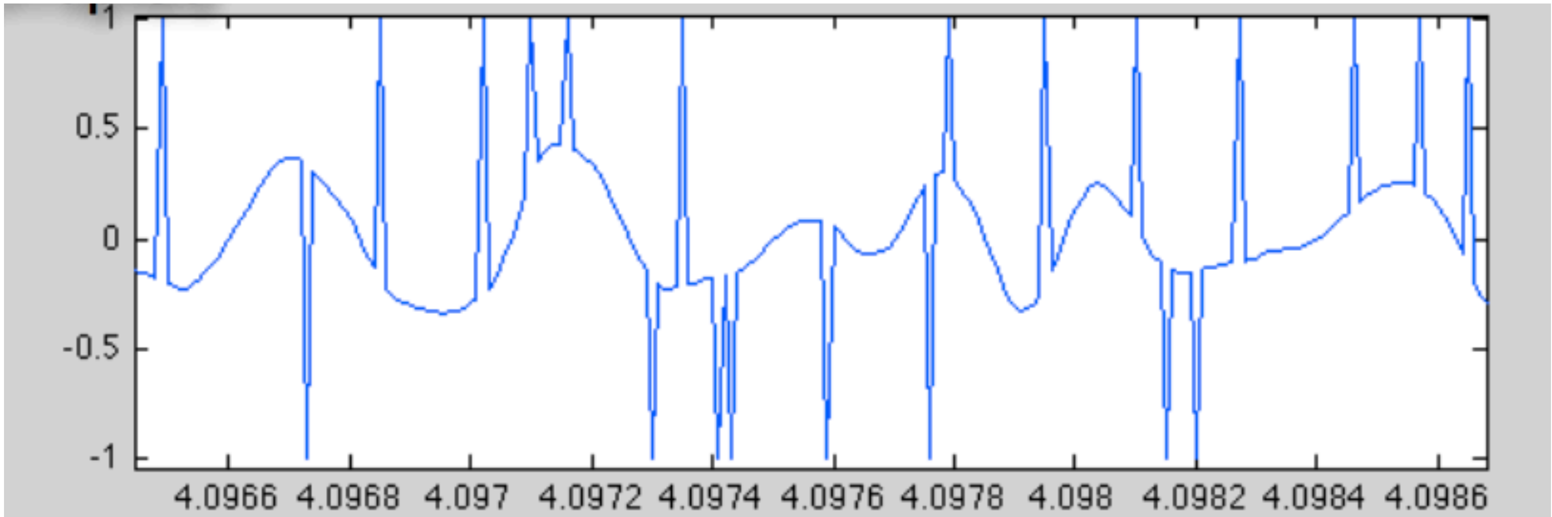


LP-Filtered Speech

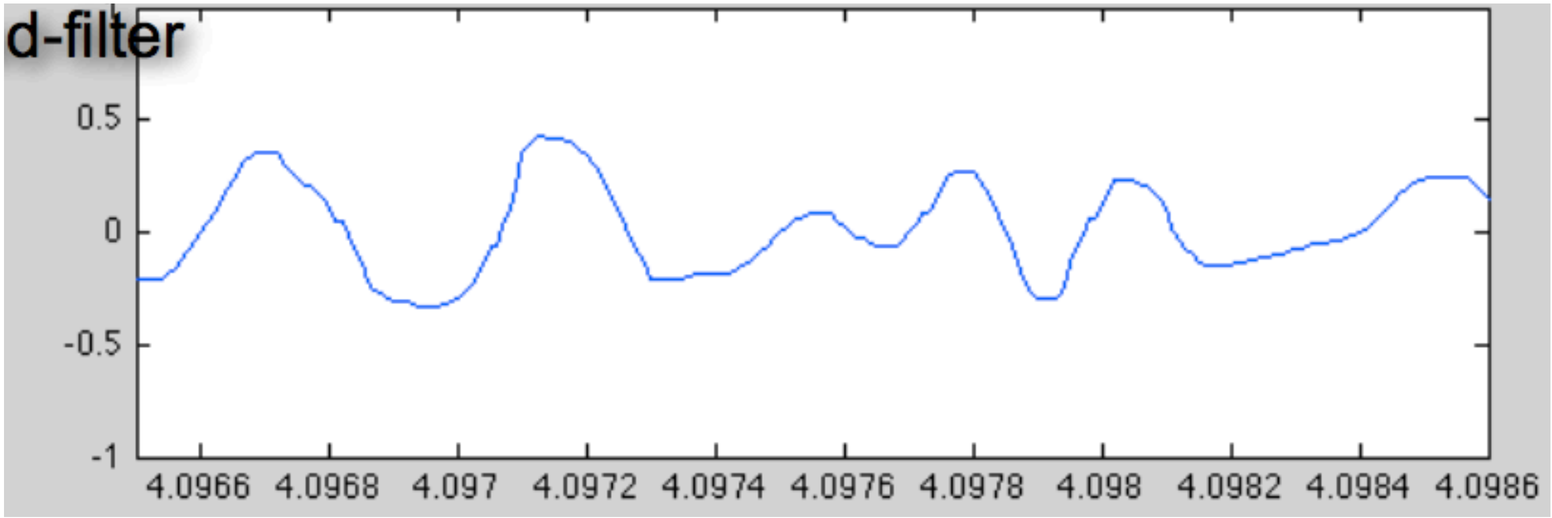


Median Filtering

Corrupted Speech



Med-Filter Speech

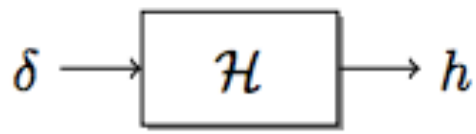


LTI Systems

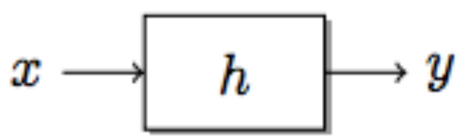
LTI Systems

> **DEFINITION**
> A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be completely characterized by its impulse response



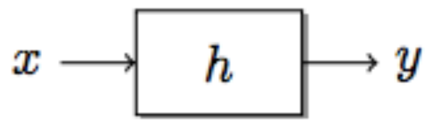
- Then the output for an arbitrary input is a sum of weighted, delay impulse responses



$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

$$y[n] = x[n] * h[n]$$

Convolution



- Convolution formula:

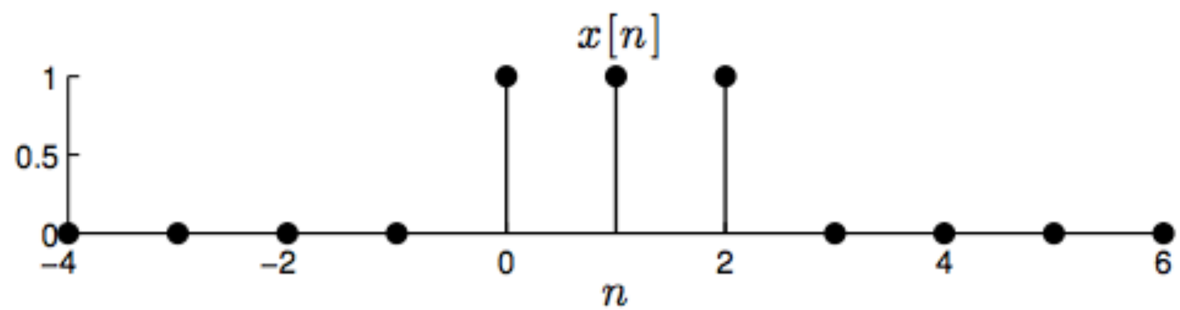
$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolution method:
  - 1) Time reverse the impulse response and shift it *n* time steps to the right
  - 2) Compute the inner product between the shifted impulse response and the input vector
  - Repeat for evey *n*

Convolution Example

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m]$$

- Convolve a unit pulse with itself



Convolution is Commutative

- Convolution is commutative:

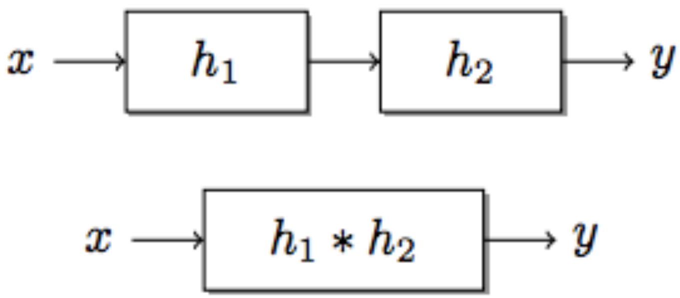
$$x * h = h * x$$

- These block diagrams are equivalent

- Implication: pick either *h* or *x* to flip and shift when convolving

LTI Systems in Series

- Impulse response of the cascade of two LTI systems:



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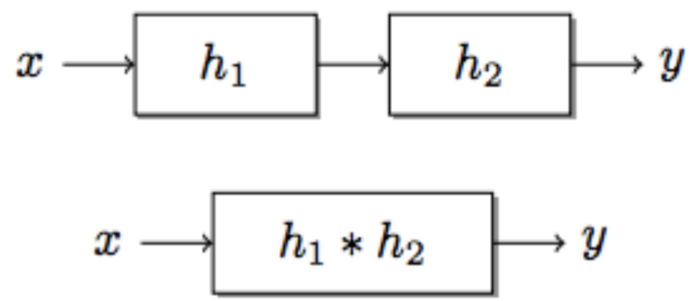
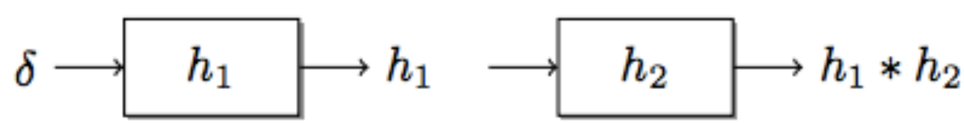
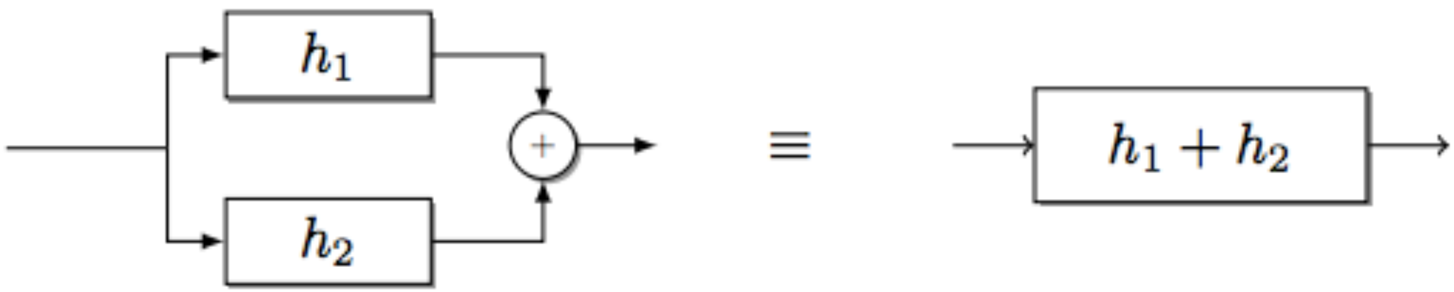


- Proof by picture



LTI Systems in Parallel

- Impulse response of the parallel connection of two LTI systems:



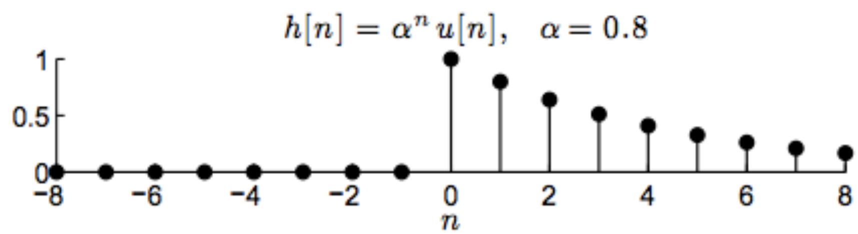
Causal System Revisited

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$



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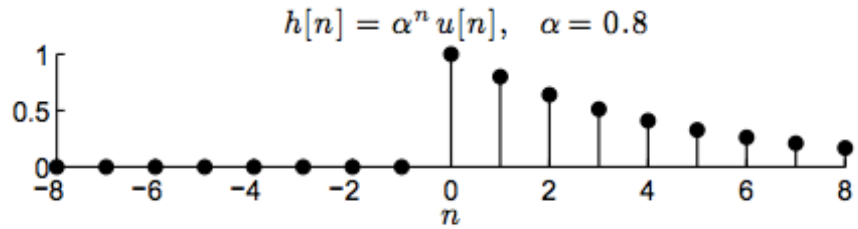

- To prove, note that the convolution does not look into the future if the impulse response is causal

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \qquad h[n-m] = 0 \text{ when } m > n;$$

Duration of Impulse

> **DEFINITION**
>
> An LTI system has a **finite impulse response** (FIR) if the duration of its impulse response h is finite

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- Example: Moving average

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \tfrac{1}{2}\left(x[n]+x[n-1]\right)$$

Duration of Impulse

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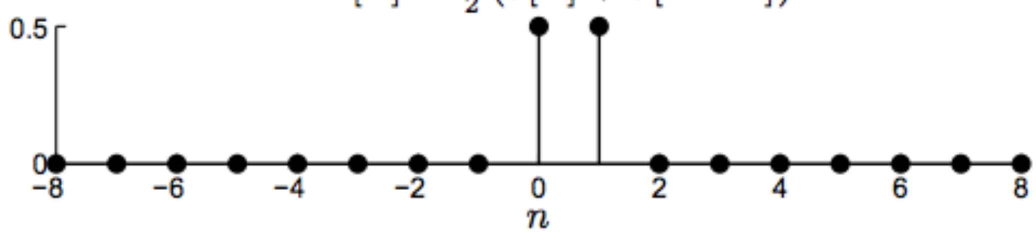
- Example: Moving average

$$y[n] = \mathcal{H}\{x[n]\} = \frac{1}{2}(x[n] + x[n-1])$$

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$



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Duration of Impulse

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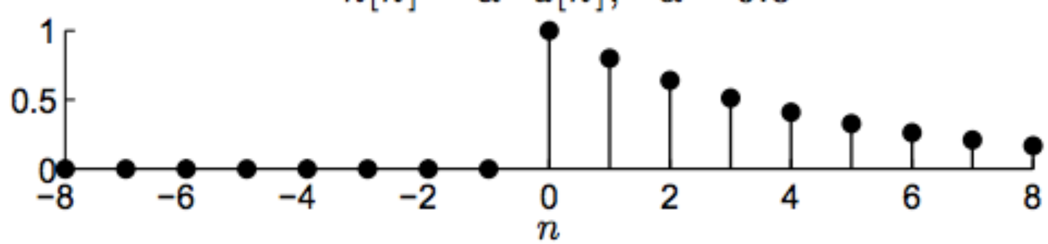
- Example: Recursive average

$$y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$$

$$h[n] = \alpha^n u[n], \quad \alpha = 0.8$$



BIBO Stability Revisited

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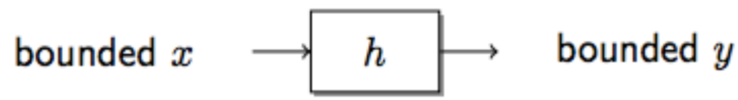
An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



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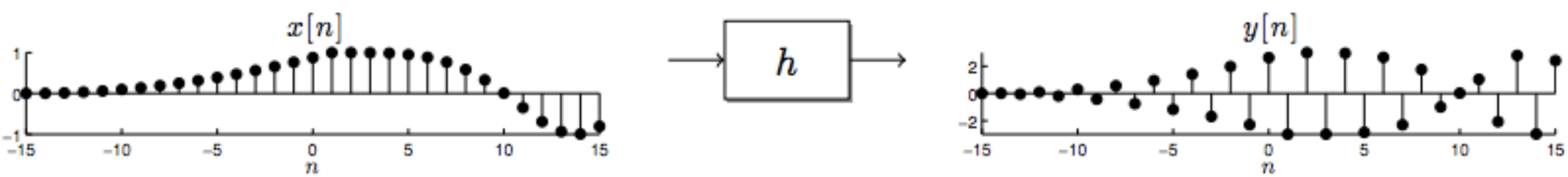
bounded x → h → bounded y

- Bounded input and output:

$$\|x\|_\infty < \infty \quad \text{and} \quad \|y\|_\infty < \infty$$

- Where

$$\|x\|_\infty = \max |x[n]|$$



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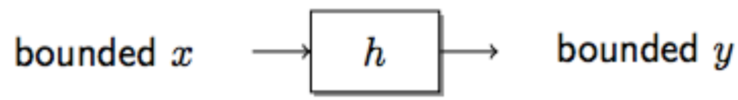

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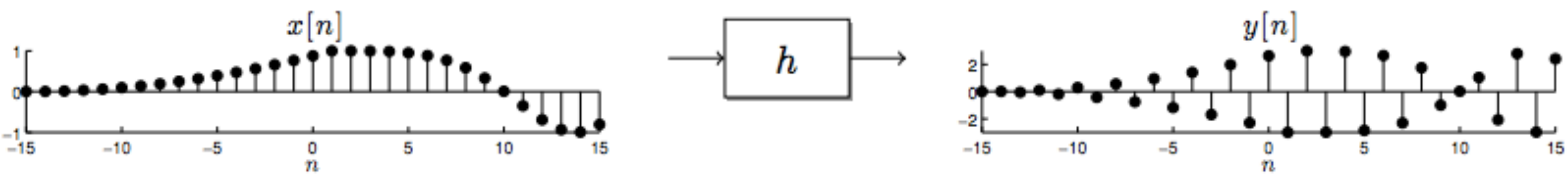

- An LTI system is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

BIBO Stability – Sufficient Condition

- Prove that if $\|h\|_1 < \infty$ then the system is BIBO stable, then for any input $\|x\|_\infty < \infty$ the output $\|y\|_\infty < \infty$
- Recall that $\|x\|_\infty < \infty$ means there exist a constant A such that $|x[n]| < A < \infty$ for all n

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- Recall that $\|x\|_\infty < \infty$ means there exists a constant A such tha $|x[n]| < A < \infty$ for all n

- Let $\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| = B < \infty$
- Compute a bound on $|y[n]|$ using the convolution of x and h and the bounds A and B

$$
\begin{aligned}
|y[n]| &= \left| \sum_{m=-\infty}^{\infty} h[n-m]\, x[m] \right| \leq \sum_{m=-\infty}^{\infty} |h[n-m]|\, |x[m]| \\
&< \sum_{m=-\infty}^{\infty} |h[n-m]|\, A = A \sum_{k=-\infty}^{\infty} |h[k]| = AB = C < \infty
\end{aligned}
$$

- Since $|y[n]| < C < \infty$ for all n, $\|y\|_\infty < \infty$ ✓

BIBO Stability – Necessary Condition

- Prove that if $\|h\|_1 = \infty$ the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$
  - Assume that *x* and *h* are real-value; the proof for complex-valued signals is nearly identical

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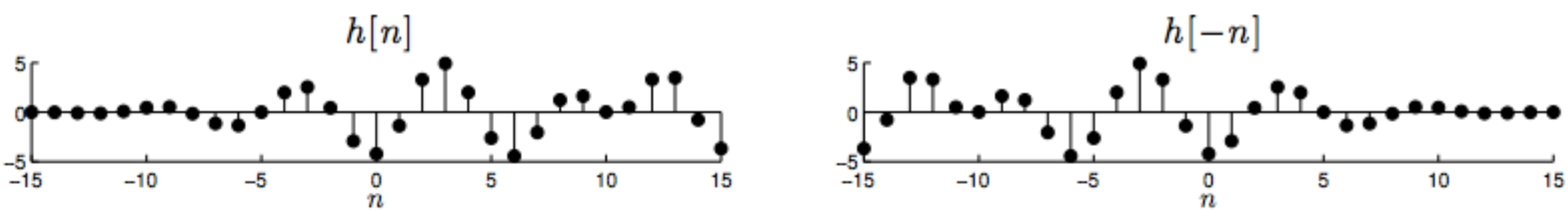
- Given an impulse response *h* with $\|h\|_1 = \infty$, form the tricky special signal

  $$x[n] = \text{sgn}(h[-n])$$
  - x[n] is the sign of the time-reversed impulse response h[-n]



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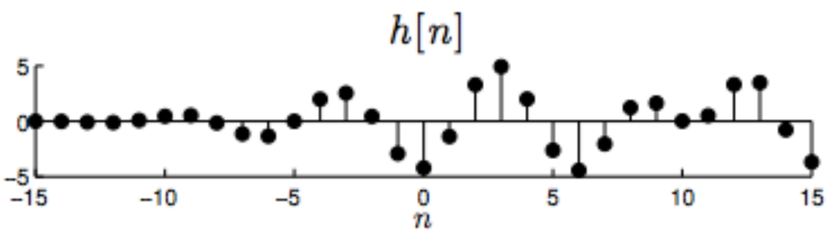
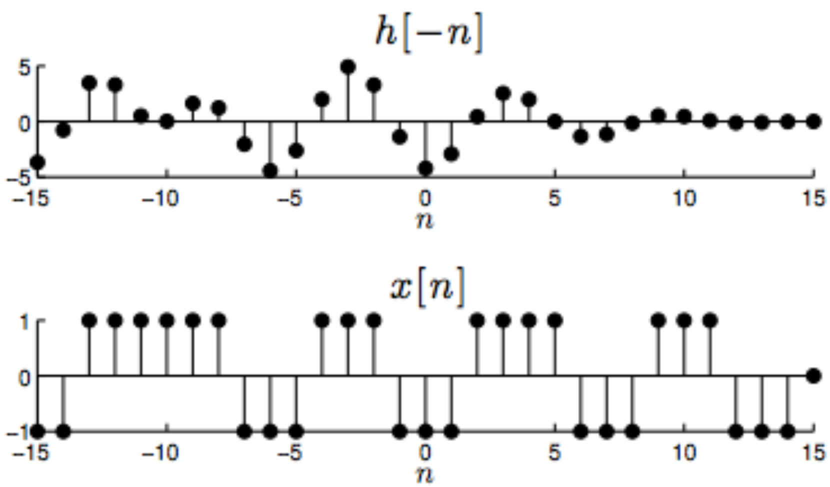
  - Assume that x and h are real-value; the proof for complex-valued signals is nearly identical
- Given an impulse response h with $\|h\|_1 = \infty$, form the tricky special signal

$$x[n] = \text{sgn}(h[-n])$$

  - x[n] is the sign of the time-reversed impulse response h[-n]
  - Note that x is bounded $|x[n]| \leq 1$ for all n





BIBO Stability – Necessary Condition

- We are proving that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$

- Armed with the tricky signal *x*, compute the output y[n] at n=0

BIBO Stability – Necessary Condition

- We are proving that if $\|h\|_1 = \infty$ then the system is not BIBO stable – there exists an input $\|x\|_\infty < \infty$ such that the output $\|y\|_\infty = \infty$

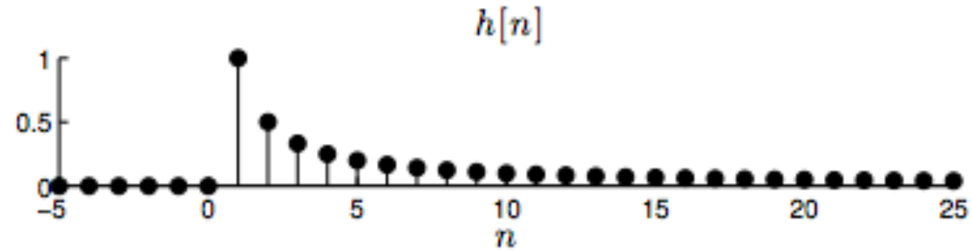
- Armed with the tricky signal *x*, compute the output y[n] at n=0

$$\begin{aligned} y[0] &= \sum_{m=-\infty}^{\infty} h[0-m]\, x[m] = \sum_{m=-\infty}^{\infty} h[-m]\, \mathrm{sgn}(h[-m]) \\ &= \sum_{m=-\infty}^{\infty} |h[-m]| = \sum_{k=-\infty}^{\infty} |h[k]| = \infty \end{aligned}$$

- Thus y is not bounded while x is bounded, so the system is not BIBO stable

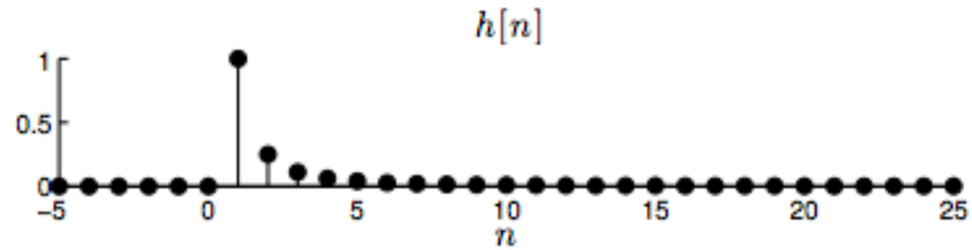
Examples

Example: $h[n] = \begin{cases} \frac{1}{n} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n}\right| = \infty \;\Rightarrow$ not BIBO

Example: $h[n] = \begin{cases} \frac{1}{n^2} & n \geq 1 \\ 0 & \text{otherwise} \end{cases}$



$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \;\Rightarrow$ BIBO

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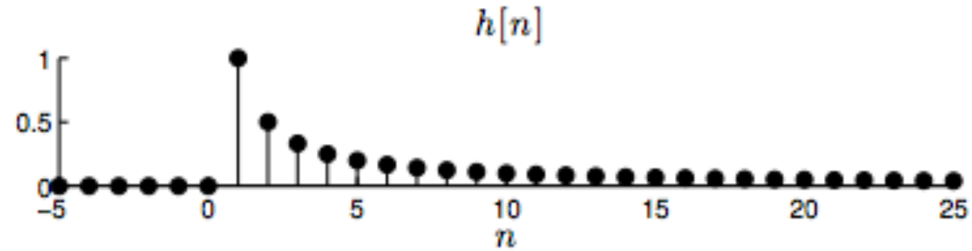

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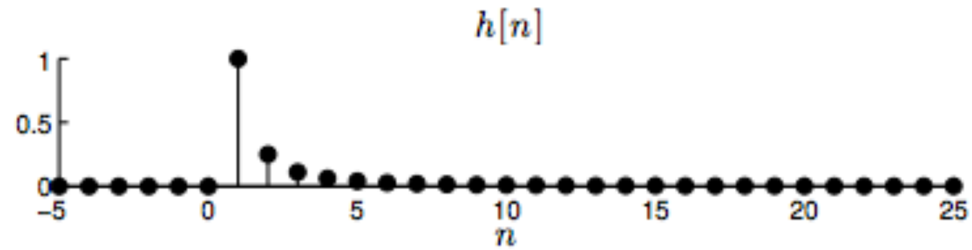
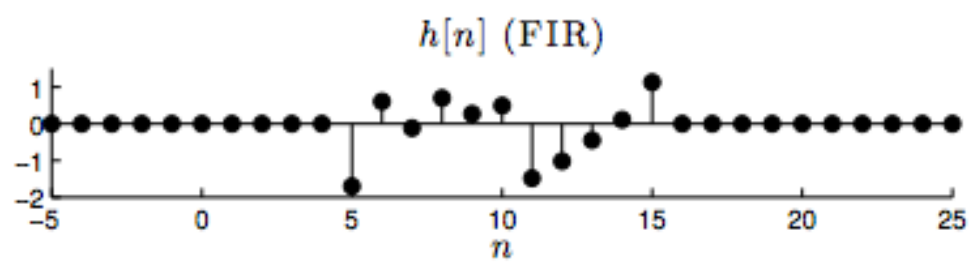


$\|h\|_1 = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \frac{\pi^2}{6} \;\Rightarrow$ BIBO

Example: h FIR $\;\Rightarrow$ BIBO



Example

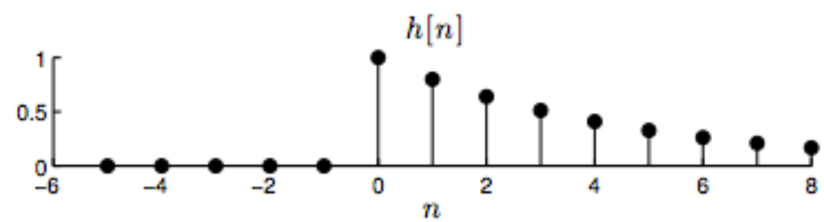
- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] = \alpha^n\, u[n]$

Example

- Example: Recall the recursive average system $y[n] = \mathcal{H}\{x[n]\} = x[n] + \alpha\, y[n-1]$
- Impulse response: $h[n] = \alpha^n\, u[n]$

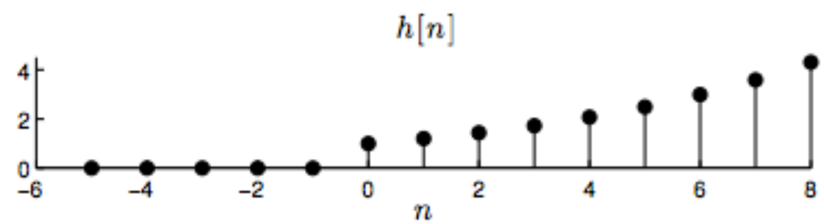
For $|\alpha| < 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} < \infty \;\Rightarrow$ BIBO



For $|\alpha| > 1$

$\|h\|_1 = \sum_{n=0}^{\infty} |\alpha|^n = \infty \;\Rightarrow$ not BIBO



Difference Equations

- Accumulator example

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$

Difference Equations

- Accumulator example

$$y[n]=\sum_{k=-\infty}^{n}x[k]$$

$$y[n]=x[n]+\sum_{k=-\infty}^{n-1}x[k]$$

$$y[n]=x[n]+y[n-1]$$

$$y[n]-y[n-1]=x[n]$$

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m x[n-m]$$

Difference Equations

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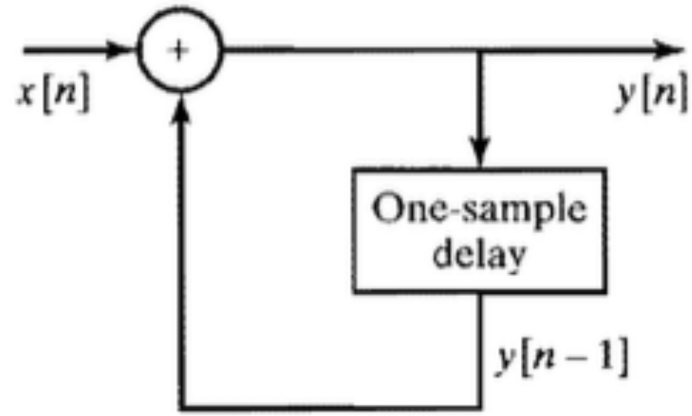

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Example: Difference Equation

- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Causal?

Example: Difference Equation

- Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Big Ideas

- LTI Systems are a special class of systems with significant signal processing applications
  - Can be characterized by the impulse response
- LTI System Properties
  - Causality and stability can be determined from impulse response
- Difference equations suggest implementation of systems
  - Give insight into complexity of system

Admin

- Complete Diagnostic Quiz by midnight tonight
  - Answers posted after due date
- HW 0: Brush up on background and Matlab tutorial
- HW 1 out now
- First recitation posted after lecture
  - Basic Matlab usage
  - (in general recitations will get posted Th or F every week)