

ESE 5310: Digital Signal Processing

Lecture 4: January 24, 2023
Discrete Time Fourier Transform



Lecture Outline

- ❑ Difference Equations
- ❑ Eigenfunctions
- ❑ Discrete Time Fourier Transform
 - Definition
 - Properties
- ❑ Frequency Response of LTI Systems

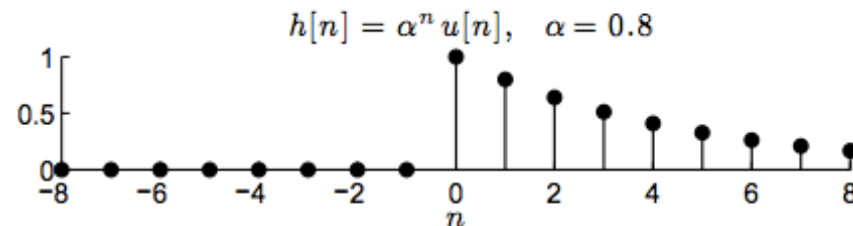
Causal System Revisited

DEFINITION

A system $\mathcal{H}$ is **causal** if the output $y[n]$ at time n depends only the input $x[m]$ for times $m \leq n$. In words, causal systems do not look into the future

- An LTI system is causal if its impulse response is causal:

$$h[n] = 0 \text{ for } n < 0$$



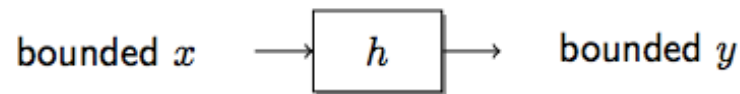
- To prove, note that the convolution does not look into the future if the impulse response is causal

$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m] \qquad h[n-m] = 0 \text{ when } m > n;$$

BIBO Stability Revisited

DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y

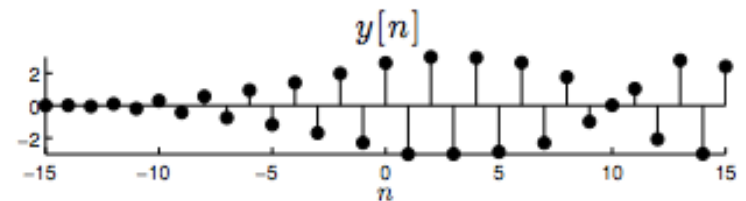
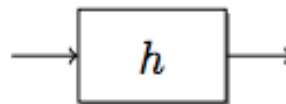
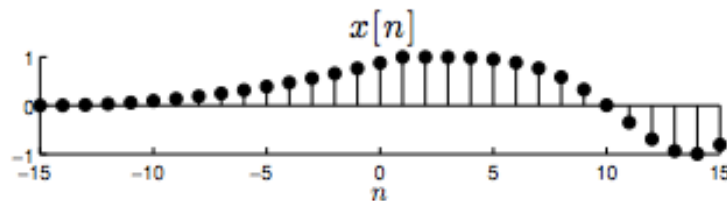


- Bounded input and output:

$$\|x\|_{\infty} < \infty \quad \text{and} \quad \|y\|_{\infty} < \infty$$

- Where

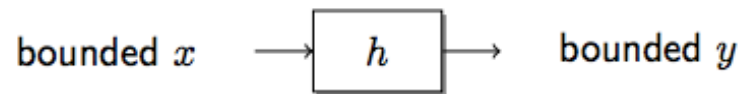
$$\|x\|_{\infty} = \max |x[n]|$$



BIBO Stability Revisited

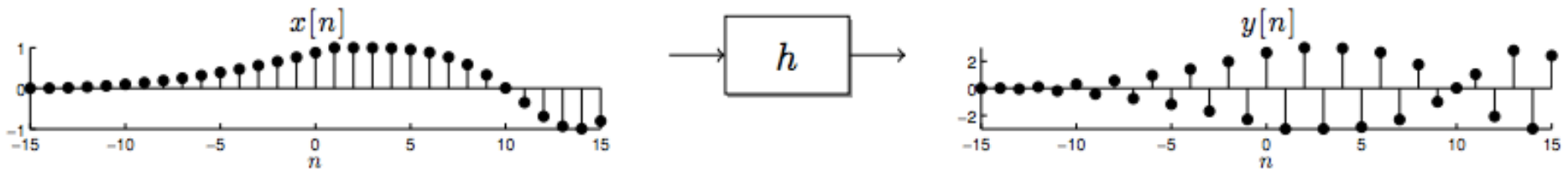
DEFINITION

An LTI system is **bounded-input bounded-output (BIBO) stable** if a bounded input x always produces a bounded output y



- Bounded input and output:

$$\|x\|_{\infty} < \infty \quad \text{and} \quad \|y\|_{\infty} < \infty$$



- An LTI system is BIBO stable **if and only if**

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Difference Equations

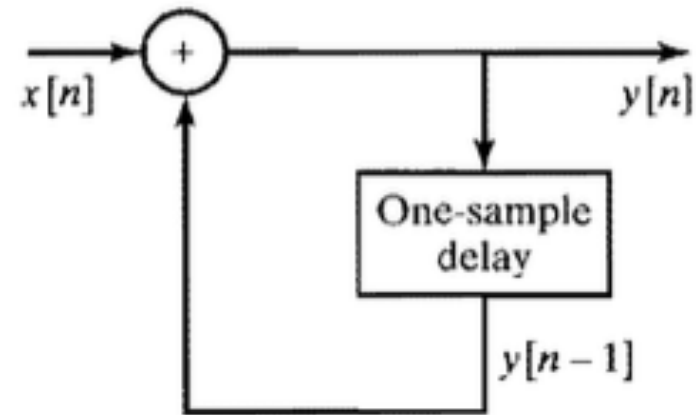
❑ Accumulator example

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$



Example: Difference Equation

□ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

□ Causal?



Example: Difference Equation

□ Moving Average System

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

□ Let $M_1=0$ (i.e. system is causal)

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k]$$

Eigenfunctions



Eigenfunction



- $x[n] = e^{j\omega n}$

- Eigenfunction: function when acted upon by a linear operator is scaled by an eigenvalue

- $Df = \lambda f$

Eigenvalue (frequency response)

□ $x[n] = e^{j\omega n}$

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega(n-k)}h[k] \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\ &= H(e^{j\omega})e^{j\omega n} \end{aligned}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

- Describes the change in amplitude and phase of signal at frequency ω
- Frequency response
- Complex value
 - Re and Im
 - Mag and Phase

DT Frequency Response

□ $H(e^{j(\omega+2\pi)})?$

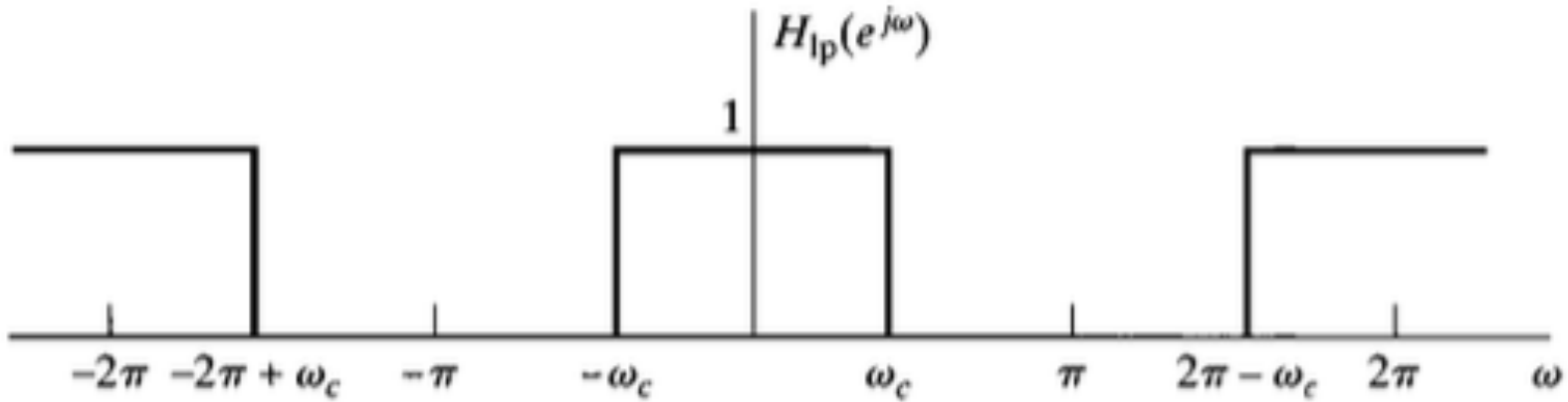
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

DT Frequency Response

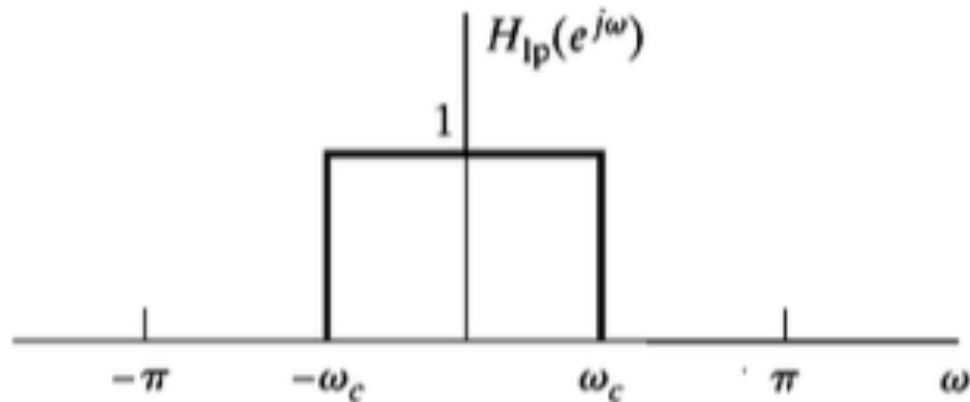
□ $H(e^{j(\omega+2\pi)})?$

$$\begin{aligned} H(e^{j(\omega+2\pi)}) &= \sum_{k=-\infty}^{\infty} h[k] e^{-j(\omega+2\pi)k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} e^{-2\pi k} \\ &= \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \\ &= H(e^{j\omega}) \end{aligned}$$

Periodicity of Low Pass Freq Response



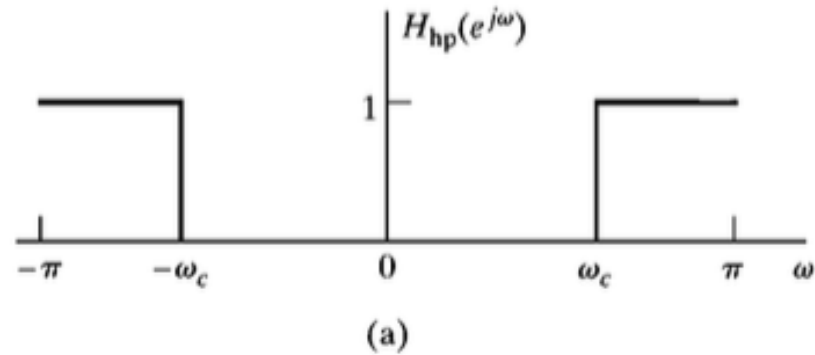
(a)



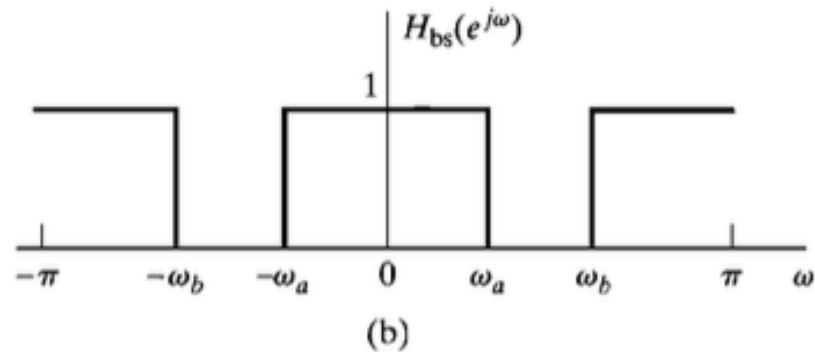
(b)

Other Filters

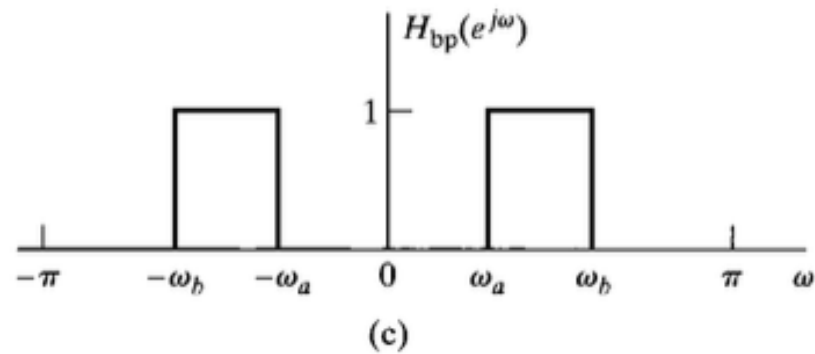
High-pass



Band-stop



Band-pass



Discrete-Time Fourier Transform (DTFT)



DTFT Definition

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

DTFT Definition

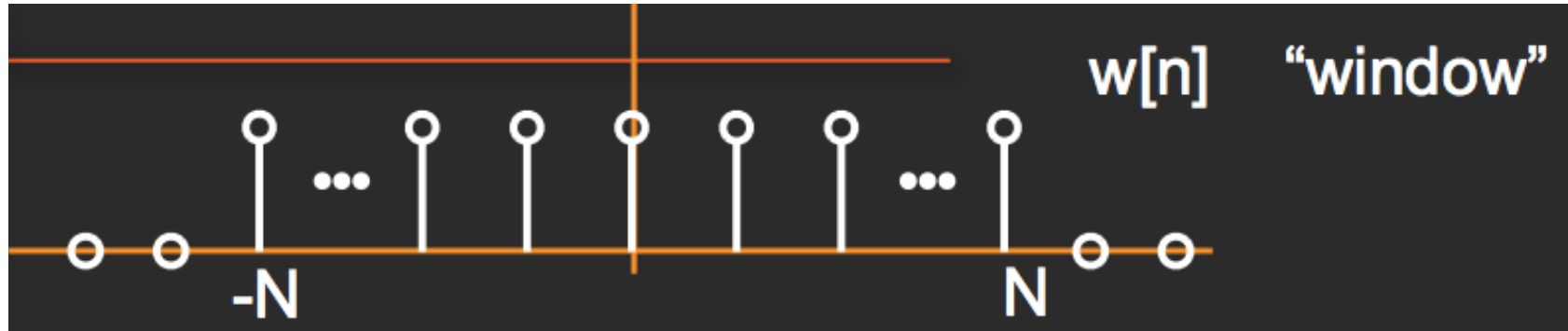
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

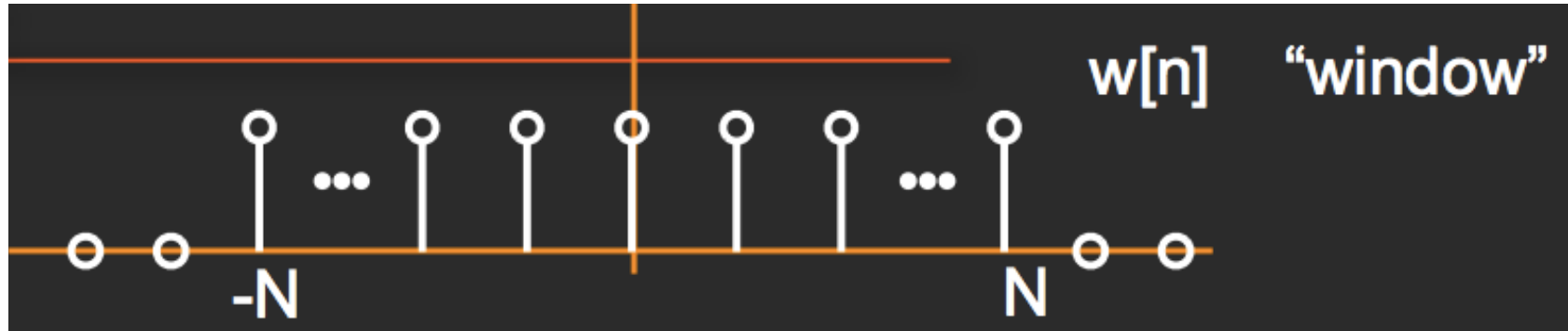
Alternate

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi f k}$$
$$x[n] = \int_{-0.5f_s}^{0.5f_s} X(f) e^{j2\pi f n} df$$

Example: Window DTFT



Example: Window DTFT



$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} w[k] e^{-j\omega k} \\ &= \sum_{k=-N}^N e^{-j\omega k} \end{aligned}$$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^N e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \dots + e^{j\omega 0} + \dots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Useful sum: $1 + p + p^2 + \dots + p^M = \frac{1 - p^{M+1}}{1 - p}$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^N e^{-j\omega k} \\ &= e^{j\omega N} + e^{j\omega(N-1)} + \dots + e^{j\omega 0} + \dots e^{-j\omega(N-1)} + e^{-j\omega N} \\ &= e^{-j\omega N} (1 + e^{j\omega} + \dots + e^{j\omega N} + \dots + e^{j\omega(2N-1)} + e^{j\omega 2N}) \end{aligned}$$

Useful sum: $1 + p + p^2 + \dots + p^M = \frac{1 - p^{M+1}}{1 - p}$

$$p = e^{j\omega} \quad M = 2N$$

$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$



Example: Window DTFT



$$W(e^{j\omega}) = e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

Example: Window DTFT

$$\begin{aligned} W(e^{j\omega}) &= e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}} \\ &= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \\ &= \frac{e^{-j\omega N} - e^{j\omega(N+1)}}{1 - e^{j\omega}} \times \frac{e^{-j\omega/2}}{e^{-j\omega/2}} \\ &= \frac{e^{-j\omega(N+1/2)} - e^{j\omega(N+1/2)}}{e^{-j\omega/2} - e^{j\omega/2}} = \frac{\sin((N+1/2)\omega)}{\sin(\omega/2)} \end{aligned}$$

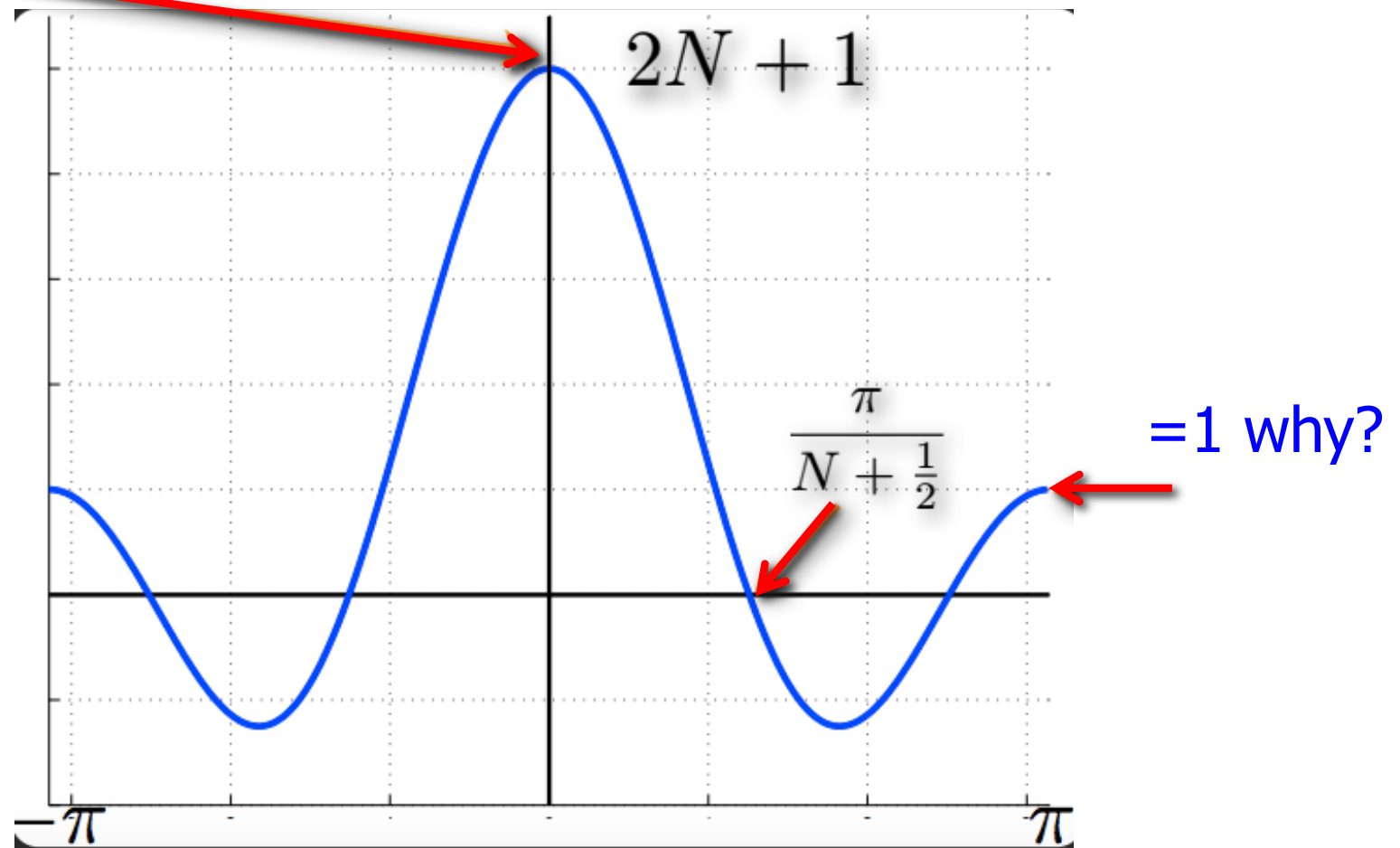
Periodic sinc

Example: Window DTFT

Also, $\sum w[n]$

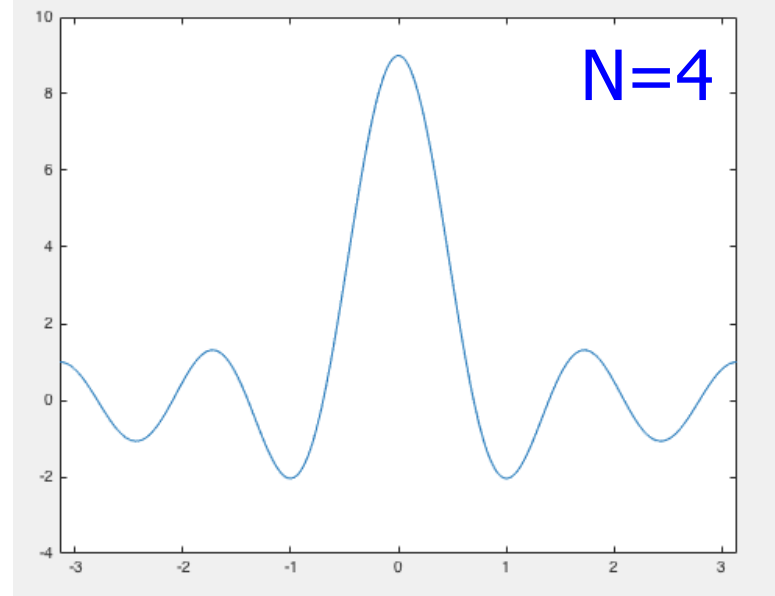
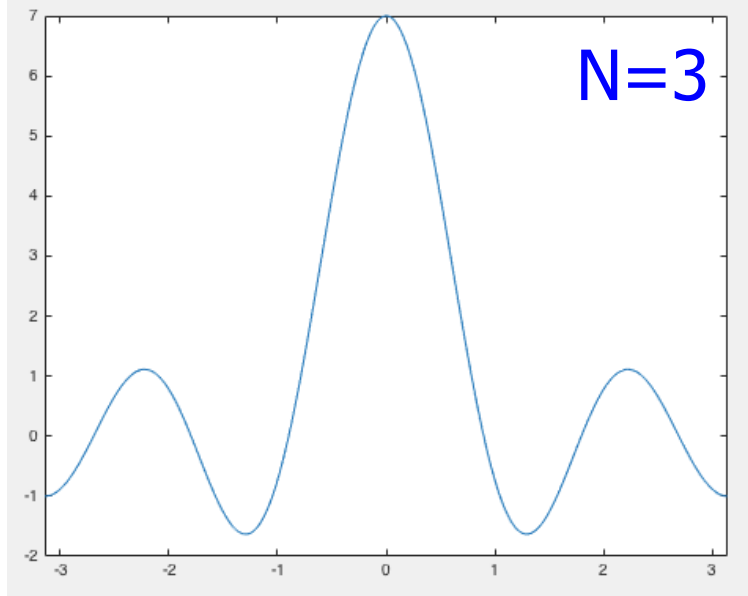
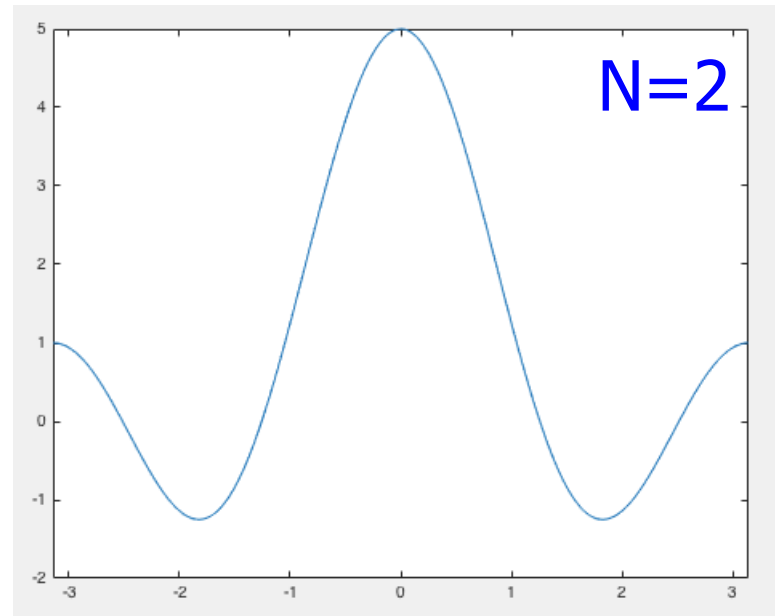
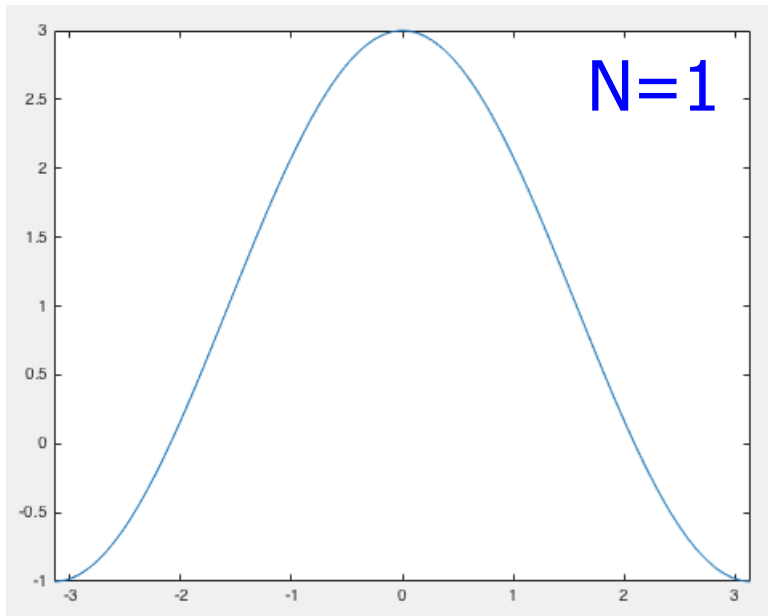
$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Plot for $N=2$





Periodic Sinc



Fourier Transform Pairs

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. 1 $(-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi k)$
4. $a^n u[n]$ $(a < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n]$ $(a < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin \omega_p (n+1)}{\sin \omega_p} u[n]$ $(r < 1)$	$\frac{1}{1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \omega < \omega_c, \\ 0, & \omega_c < \omega \leq \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k)]$

Properties of the DTFT

□ Linearity:

$$ax_1[n] + bx_2[n] \Leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

□ Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

□ Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega}) \quad \text{If } x[n] \text{ real}$$

$$\operatorname{Re}\{X(e^{-j\omega})\} = \operatorname{Re}\{X(e^{j\omega})\}$$

$$\operatorname{Im}\{X(e^{-j\omega})\} = -\operatorname{Im}\{X(e^{j\omega})\}$$

Properties of the DTFT

□ Time Reversal:

$$x[n] \leftrightarrow X(e^{j\omega})$$

If $x[n]$ real

$$x[-n] \leftrightarrow X(e^{-j\omega})$$

$$x[-n] \leftrightarrow X^*(e^{j\omega})$$

□ Time/Freq Shifting:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$$



Properties of the DTFT

□ Differentiation in Frequency:

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$$

□ Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

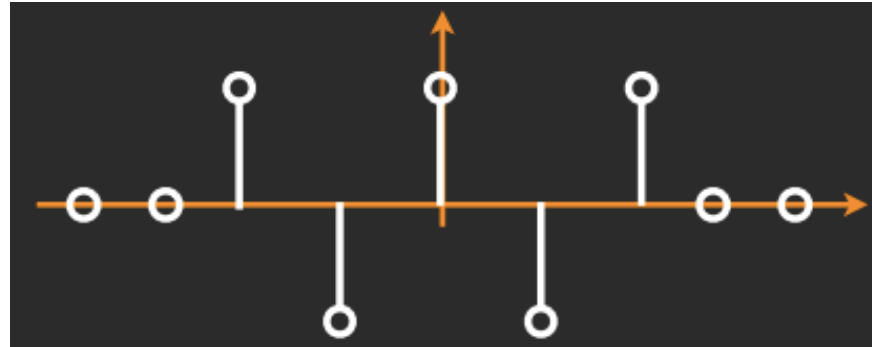
Fourier Transform Theorems

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} x[n] ^2$	$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) ^2 d\omega$
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

Example:

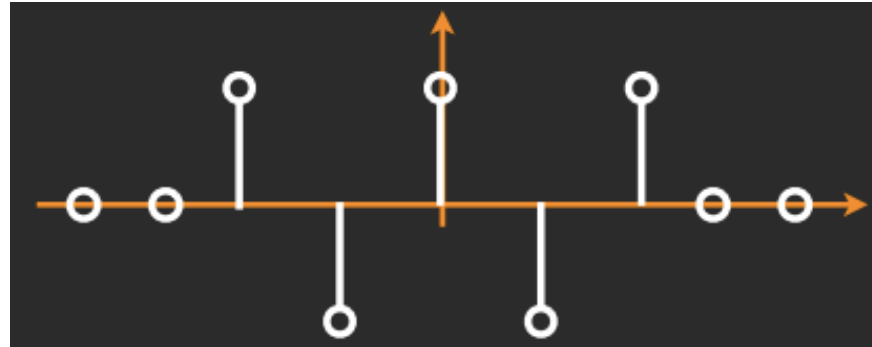
□ What is DTFT of:



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

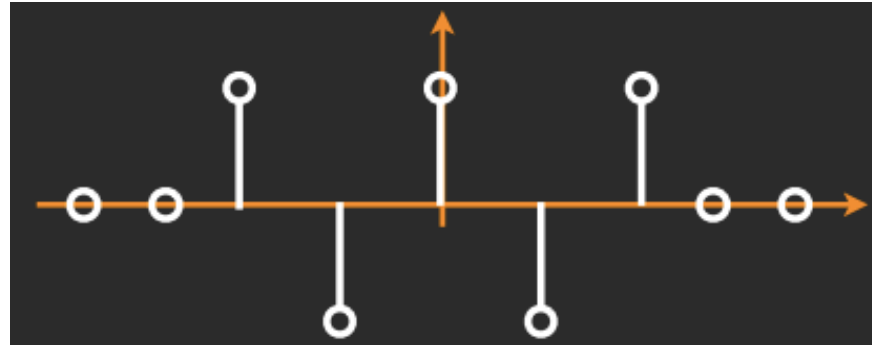
Example: Windowed $\cos(\pi n)$

□ What is DTFT of:



Example: Windowed $\cos(\pi n)$

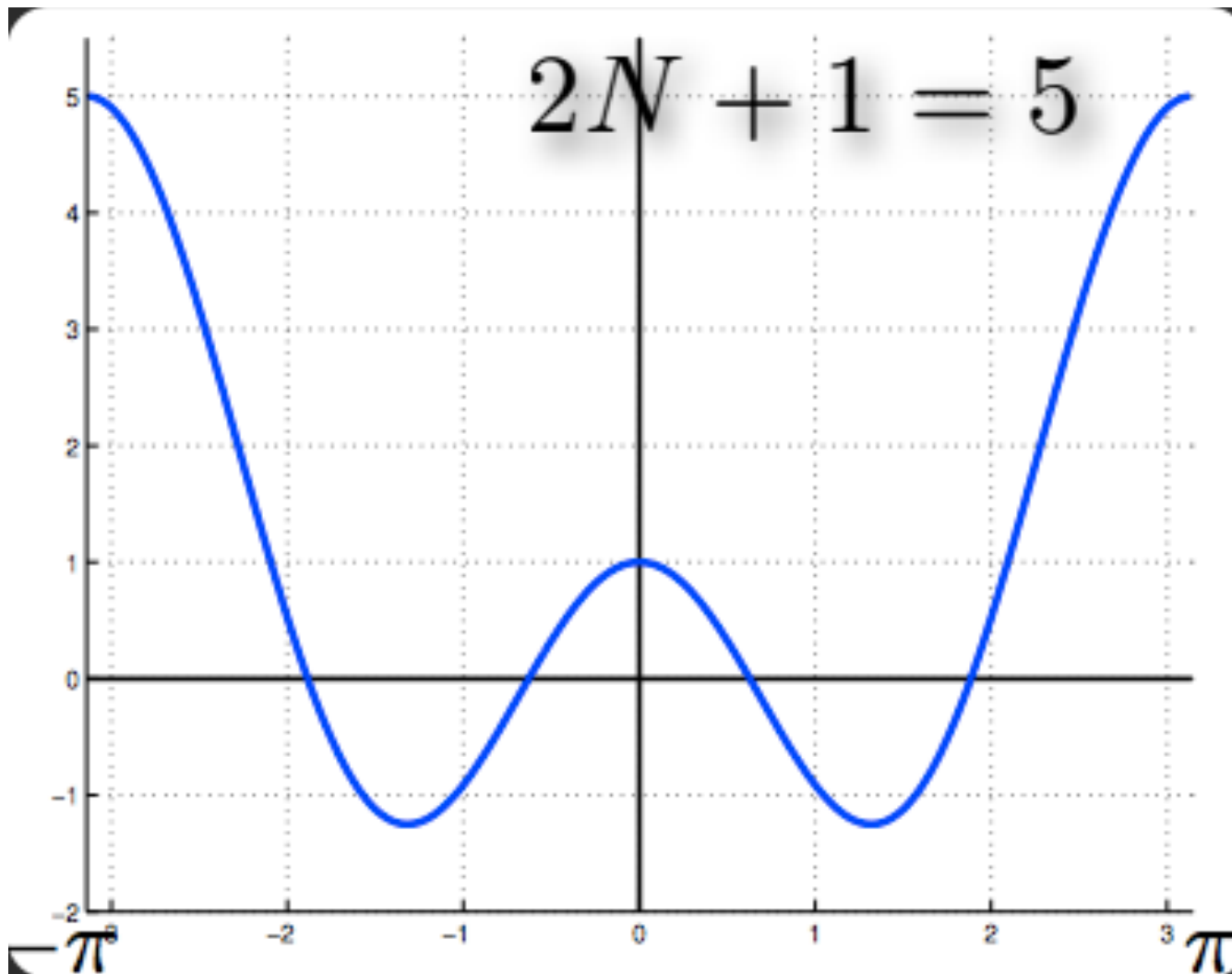
□ What is DTFT of:



$$e^{j\omega_0 n} x[n] \Leftrightarrow X(e^{j(\omega - \omega_0)})$$



Example: Windowed $\cos(\pi n)$

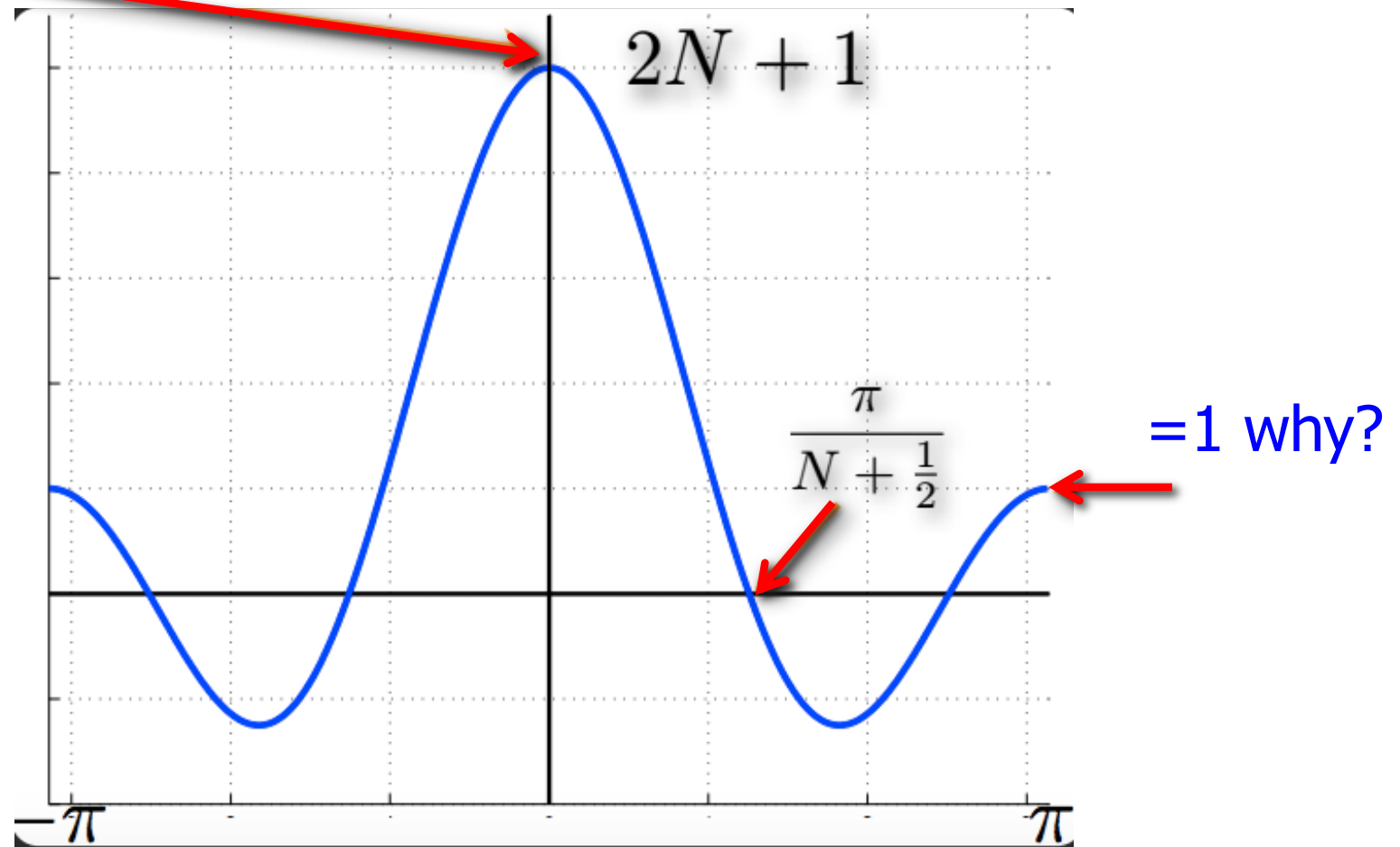


Example: Window DTFT

Also, $\sum w[n]$

$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Plot for $N=2$



Frequency Response of LTI Systems

LTI Systems

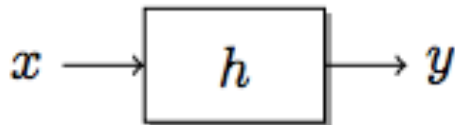
DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI system can be characterized by its impulse response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

- Then the output for an arbitrary input is a sum of weighted, delay impulse responses

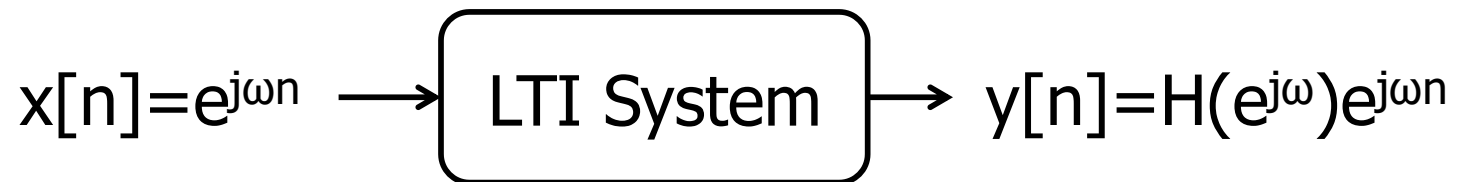


$$y[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m]$$

$$y[n] = x[n] * h[n]$$

LTI System Frequency Response

- (DT)Fourier Transform of impulse response



$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$



Example: Moving Average



- Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Example: Moving Average

□ Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response



Freq response?

Example: Moving Average

□ Moving Average Filter

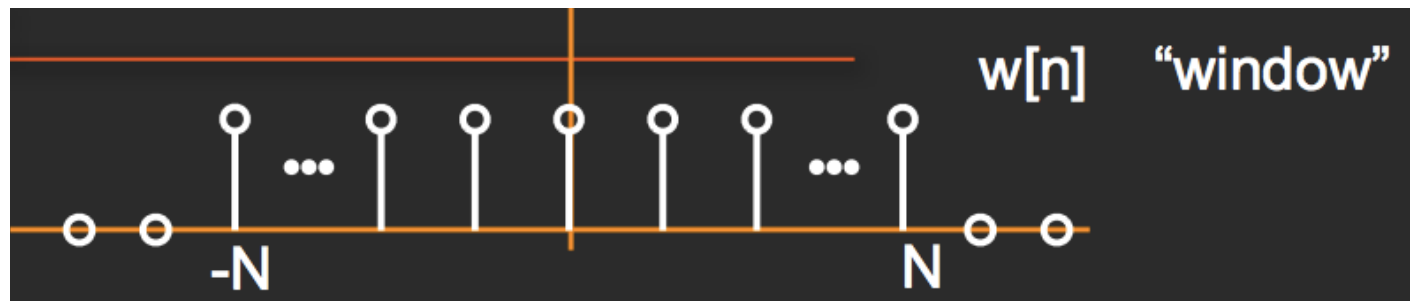
- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

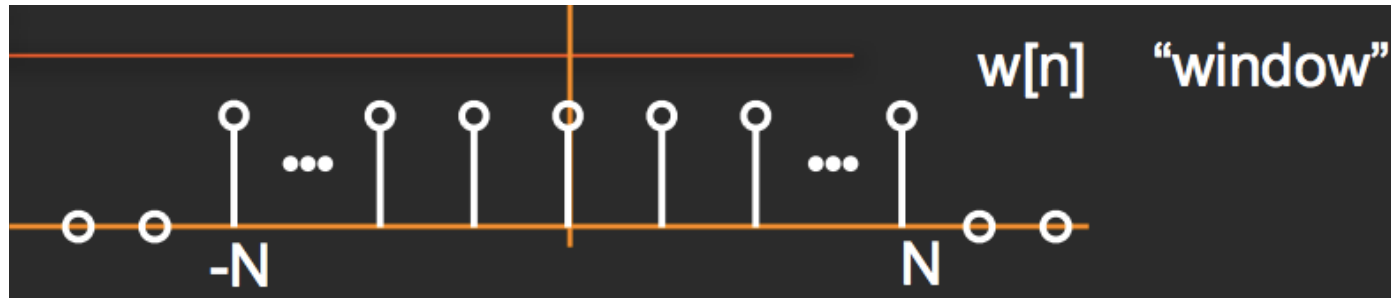
Impulse
response



Scaled & Time
Shifted window



Example: Moving Average



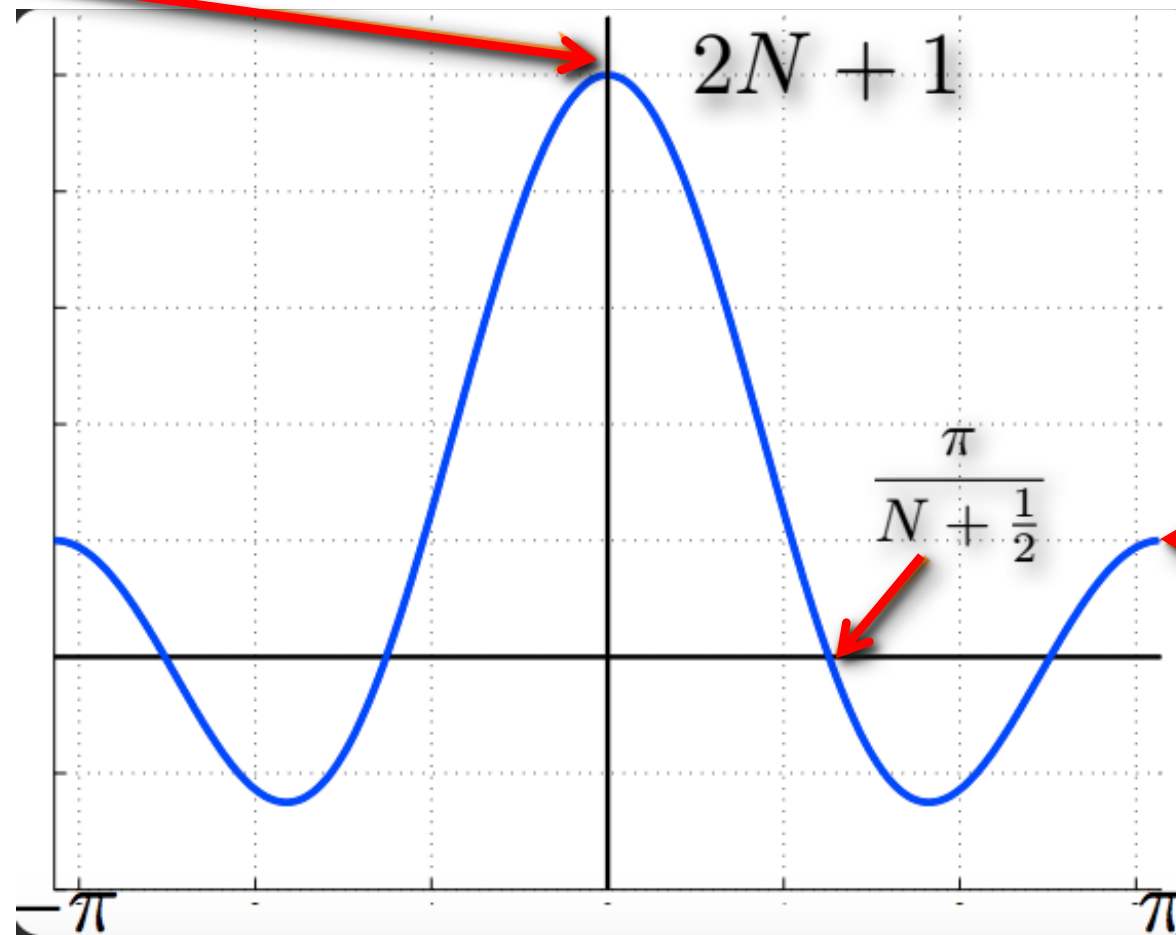
$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Window DTFT

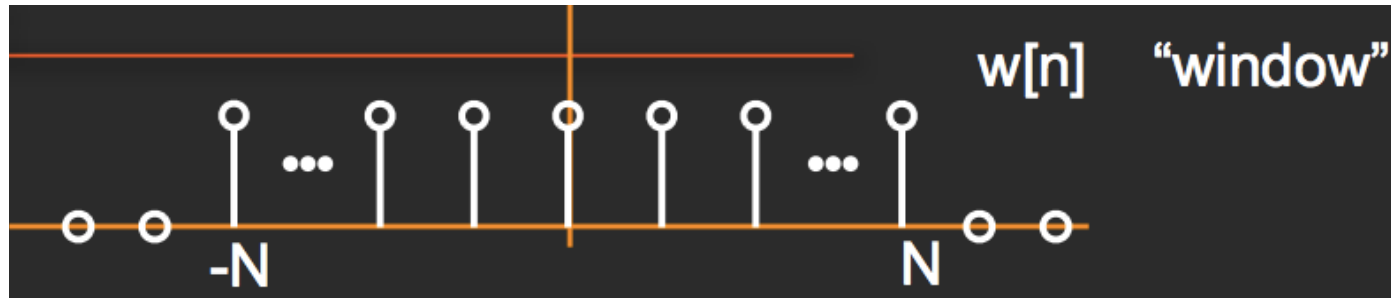
$$W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Also, $\sum x[n]$

Plot for $N=2$



Example: Moving Average

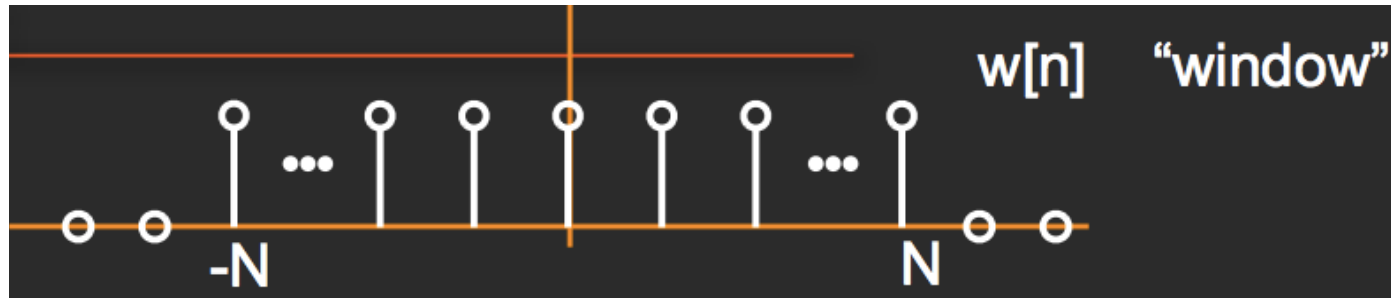


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] =$$

Example: Moving Average

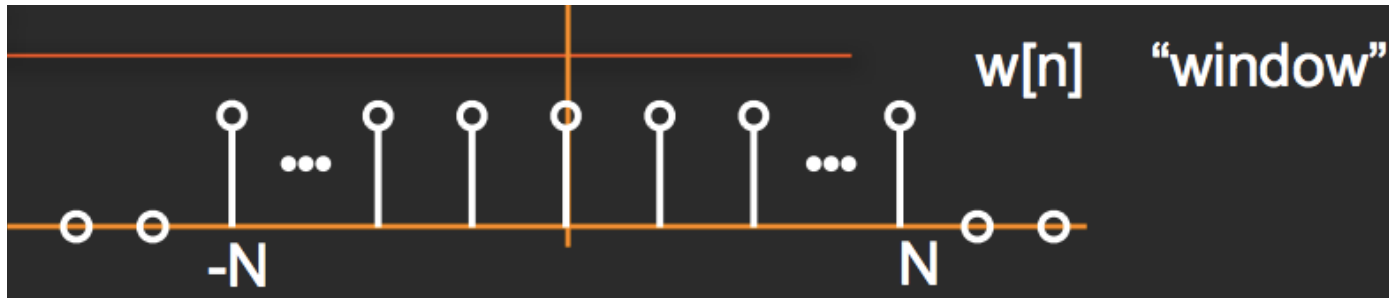


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

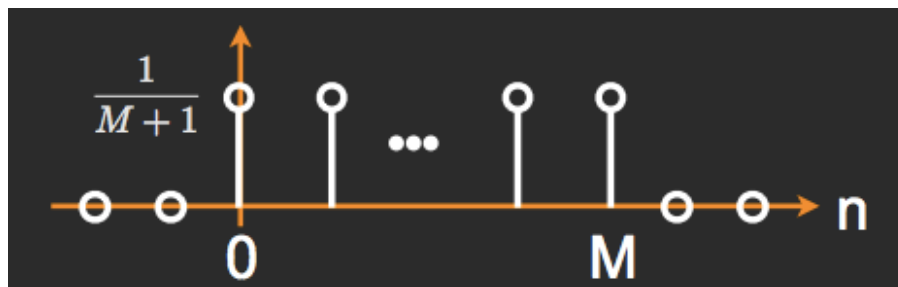


$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) =$$

Example: Moving Average



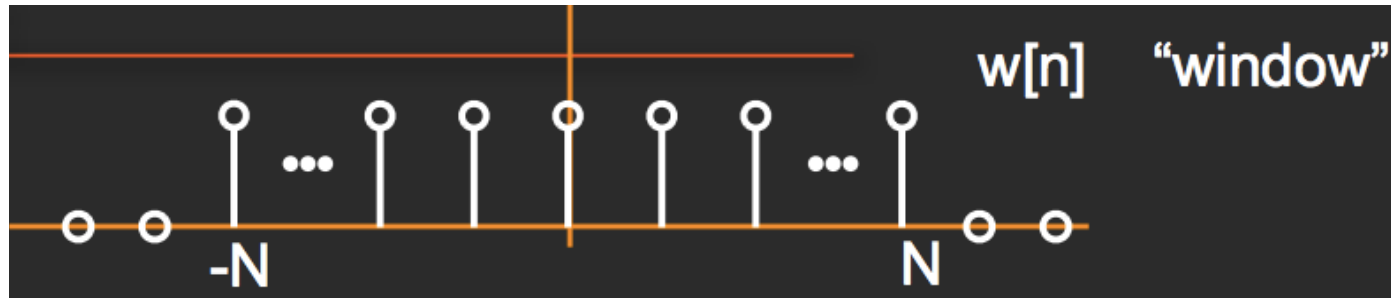
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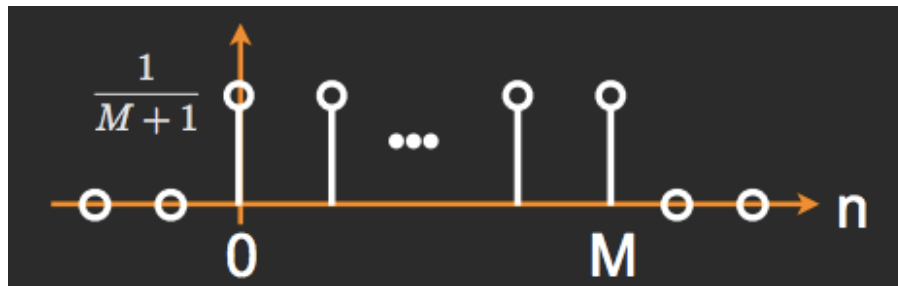
$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) =$$

$$x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$

Example: Moving Average

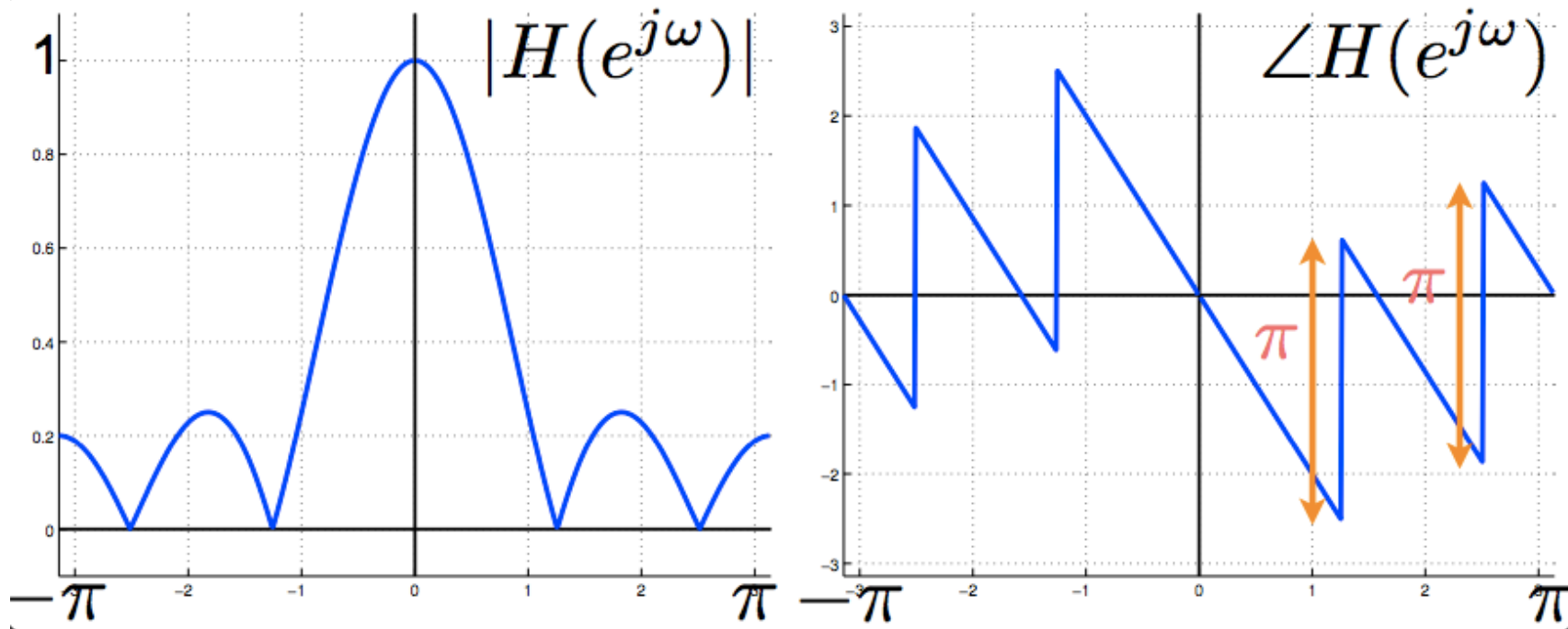


$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$



$$h[n] = \frac{1}{M+1} w[n - M/2] \leftrightarrow H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Moving Average



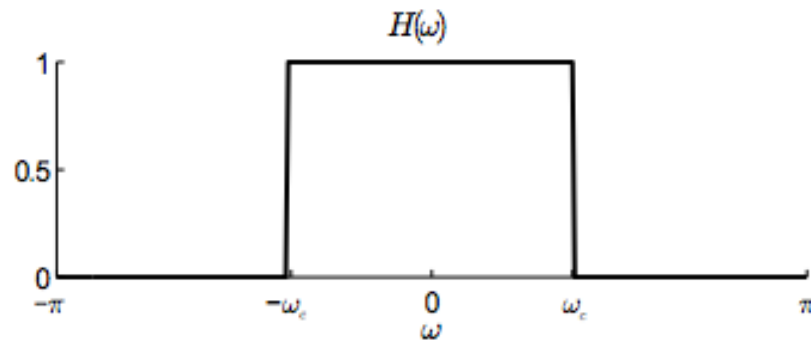
$M=4$
($N=2$)

$$H(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

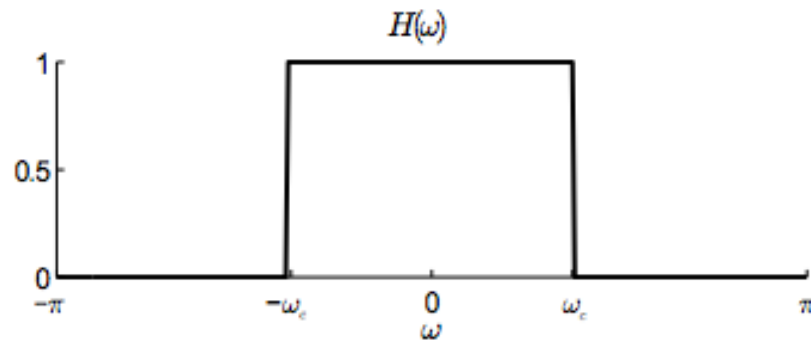
$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$



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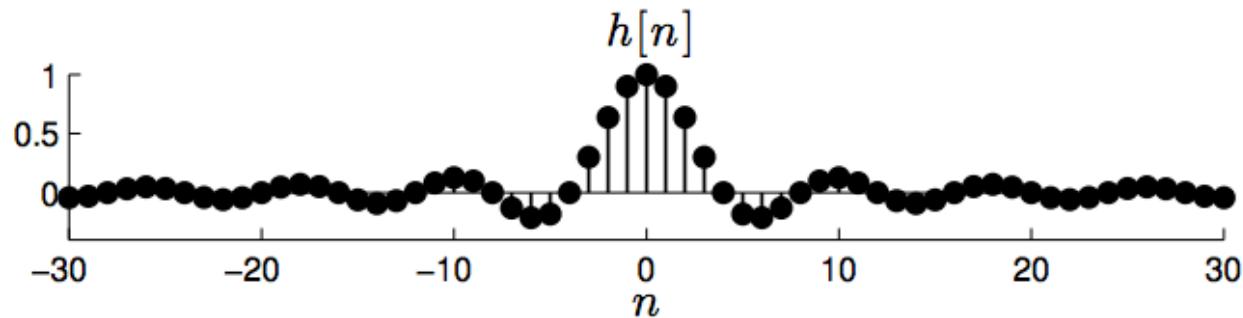
- Compute the impulse response $h[n]$ given this $H(\omega)$
- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \left. \frac{e^{j\omega n}}{2\pi j n} \right|_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

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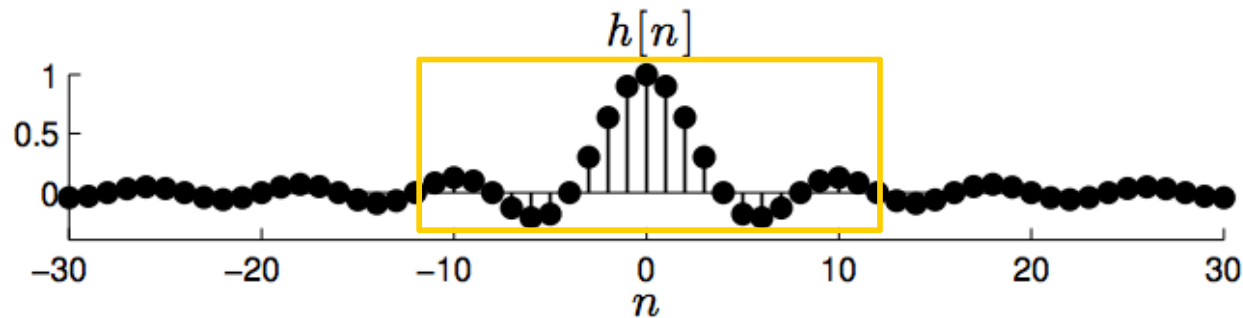


Example: Ideal Low-Pass Filter

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$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$

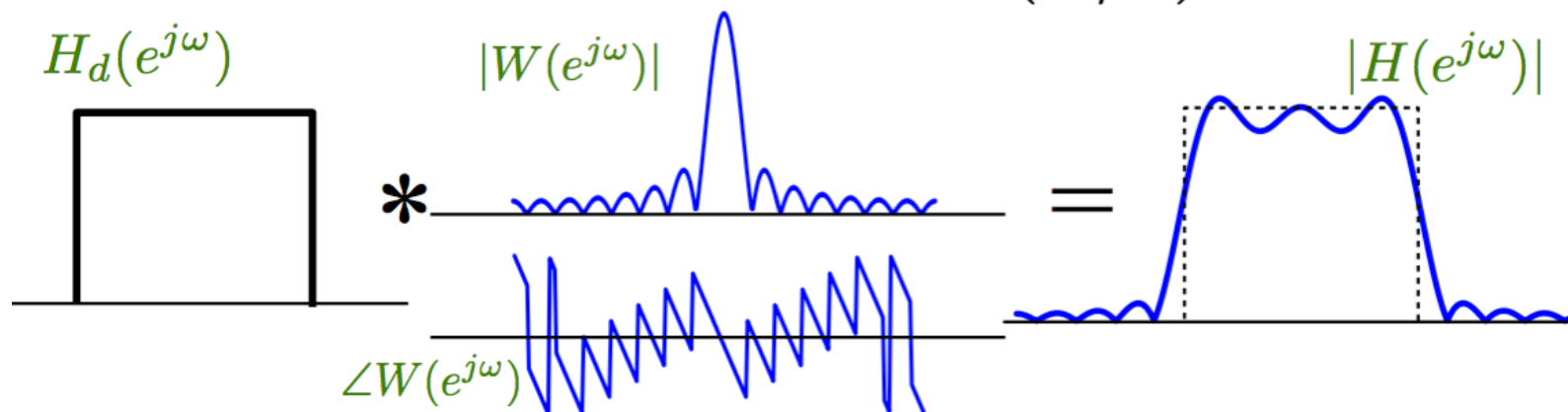
FIR Design by Windowing

- Desired filter,

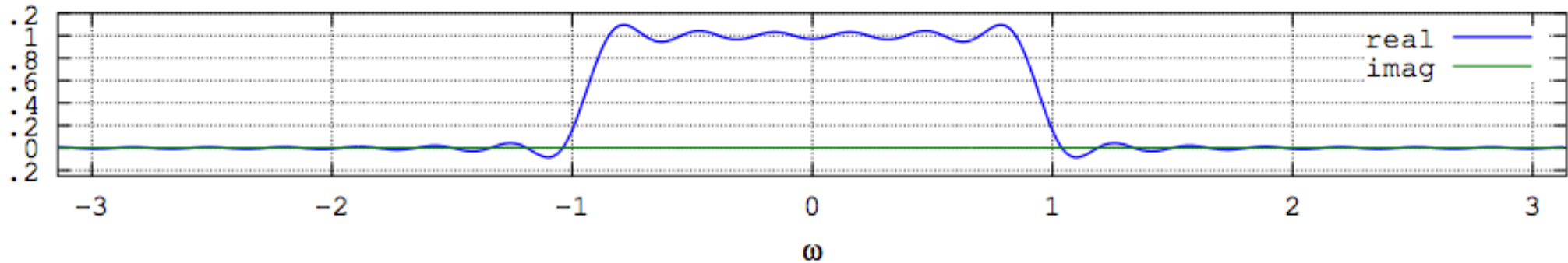
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

$$W(e^{j\omega}) = e^{-j\omega \frac{M}{2}} \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}$$



Example: Practical LP Filter



- ❑ Pass band smeared and rippled
 - Smearing determined by width of main lobe
 - Rippling determined by size of side lobes

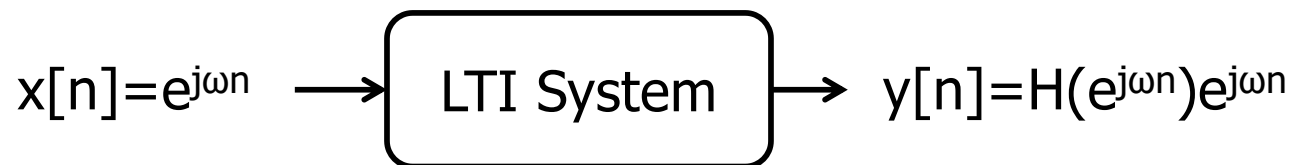


Big Ideas

- ❑ LTI Systems are a special class of systems with significant signal processing applications
 - Can be characterized by the impulse response
- ❑ LTI System Properties
 - Causality and stability can be determined from impulse response
- ❑ Difference equations suggest implementation of systems
 - Give insight into complexity of system

Big Ideas

- ❑ Discrete Time Fourier Transform
 - Represent signals as a sum of scaled and phase shifted complex sinusoids (eigenfunctions)
 - Continuous in frequency over 2π
- ❑ Frequency Response of LTI Systems
 - Frequency response of impulse response
 - Describes scaling and phase shifting of a pure frequency



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$



Admin

- ❑ HW 1 out since 1/19
 - Due **Sunday 1/29** at midnight
 - Submit in Gradescope (via Canvas)
 - Make sure you link the correct page to the correct question
- ❑ Diagnostic quiz grades
 - If you didn't submit, do it by midnight tomorrow and I'll give you credit