ESE 5310: Digital Signal Processing

Lecture 6: January 31, 2023

Inverse z-Transform

Lecture Outline

- Inverse z-transform
  - Inspection
  - Partial fraction
  - Power series expansion
- z-transform of difference equations

z-Transform

z-Transform

- Define the **forward z-transform** of x[n] as

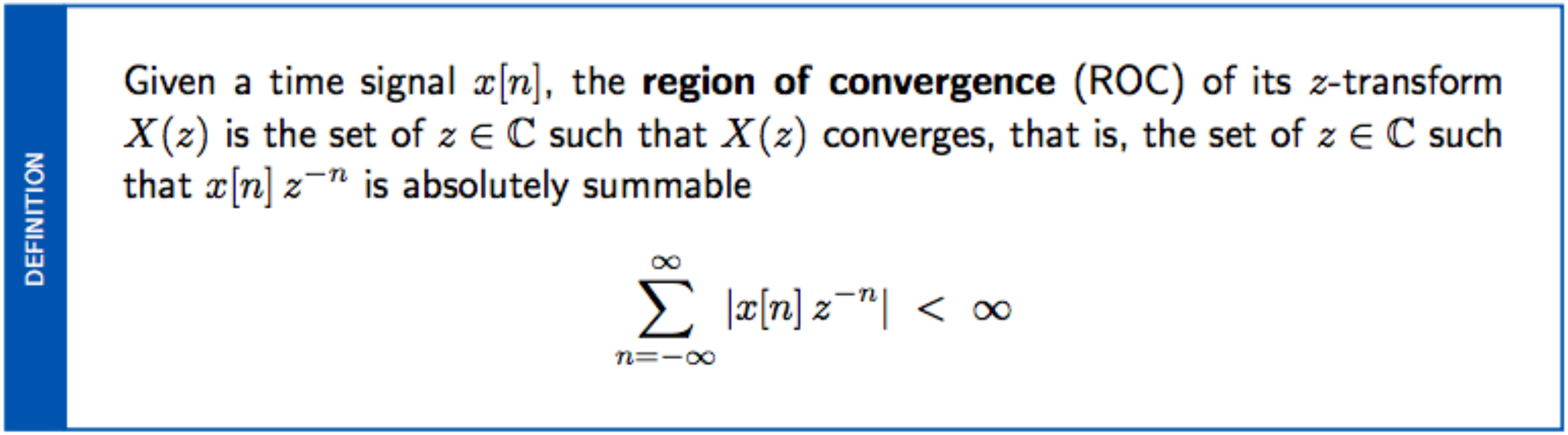
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

- The core “basis functions” of the z-transform are the complex exponentials z^n with arbitrary z ∈ C; these are the eigenfunctions of LTI systems for infinite-length signals

- **Notation abuse alert:** We use X(•) to represent both the DTFT X(ω) and the z-transform X(z); they are, in fact, intimately related

$$X_{\mathrm{DTFT}}(\omega) = X_z(z)|_{z=e^{j\omega}} = X_z(e^{j\omega})$$

Region of Convergence (ROC)



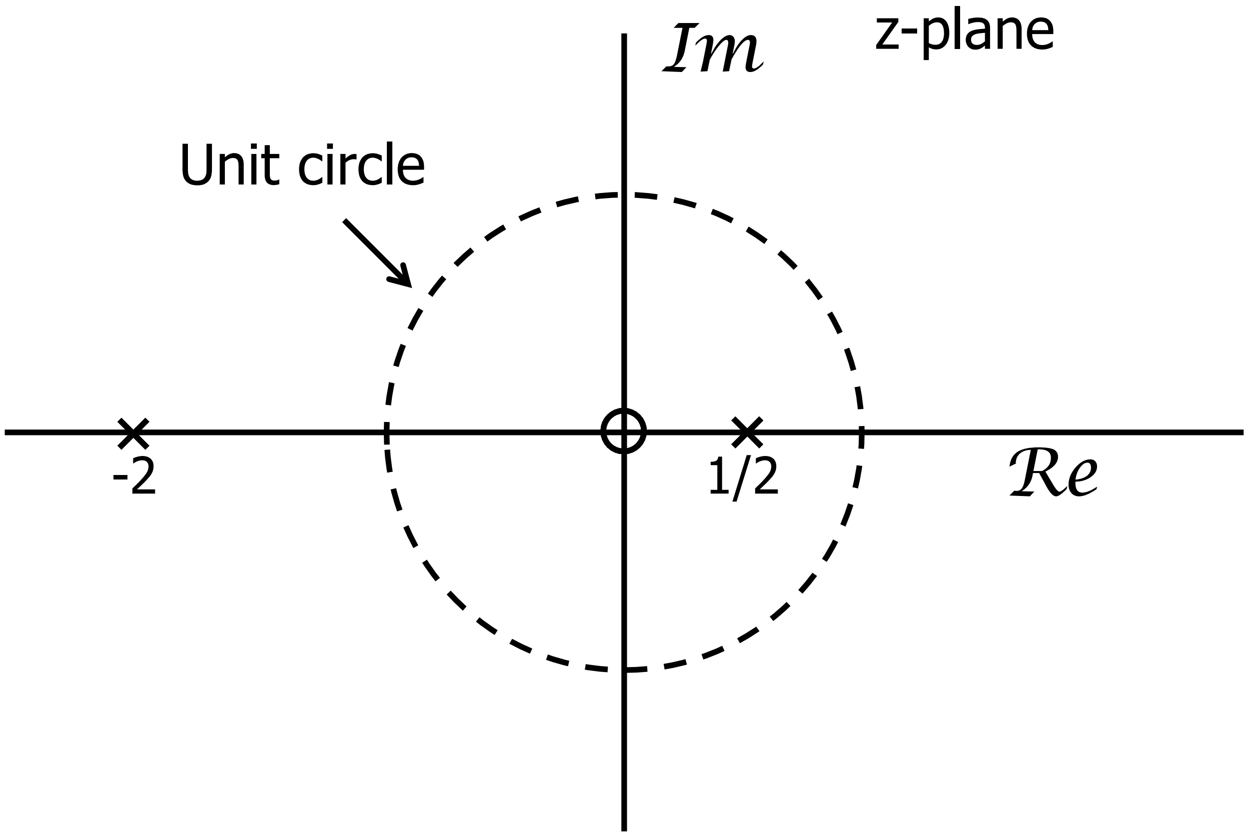
DEFINITION

Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| < \infty$$

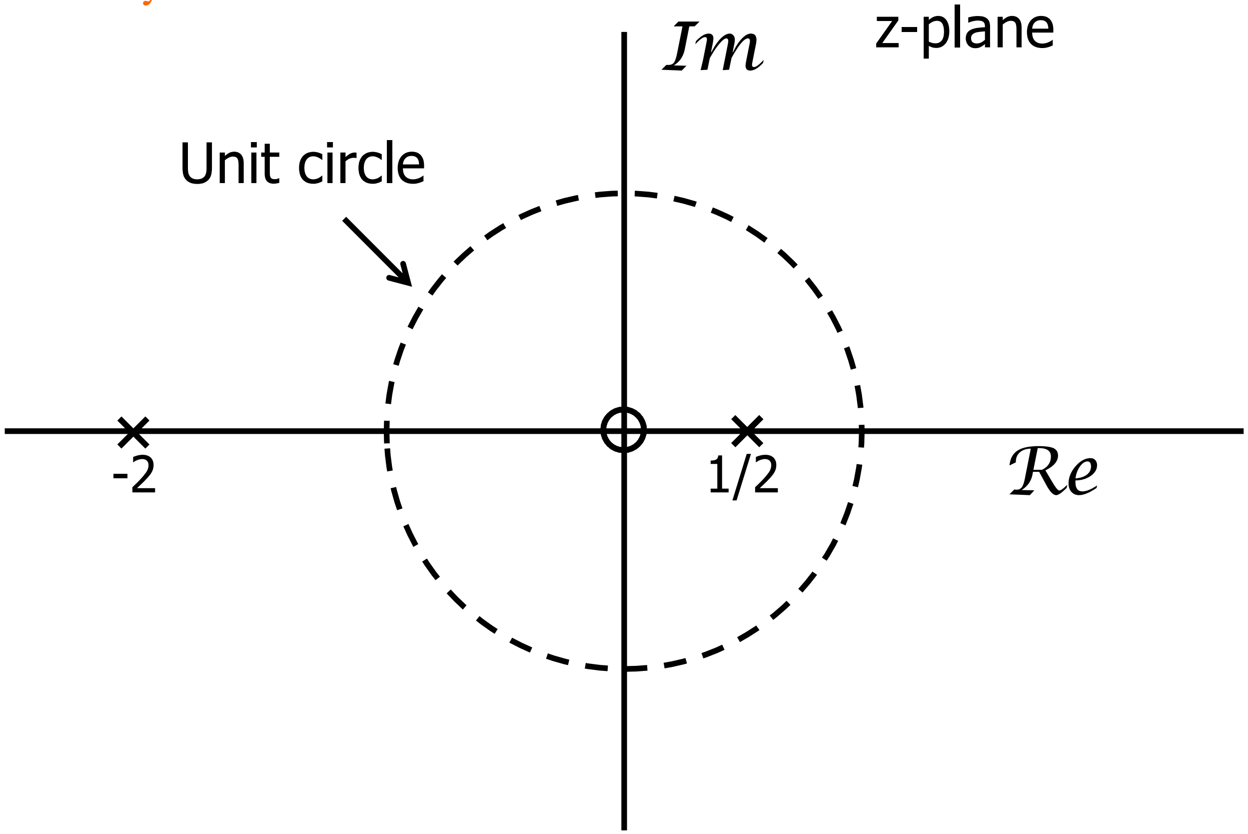
Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?



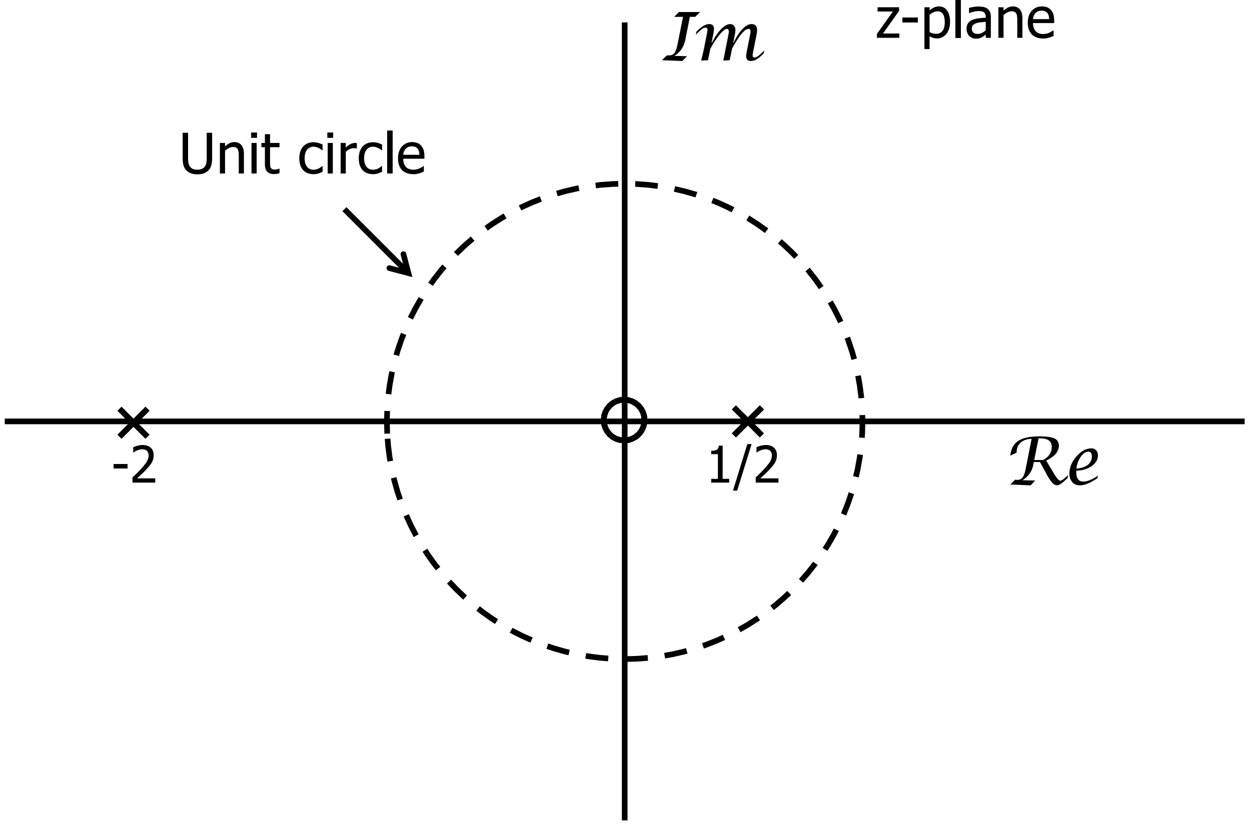
Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is causal?

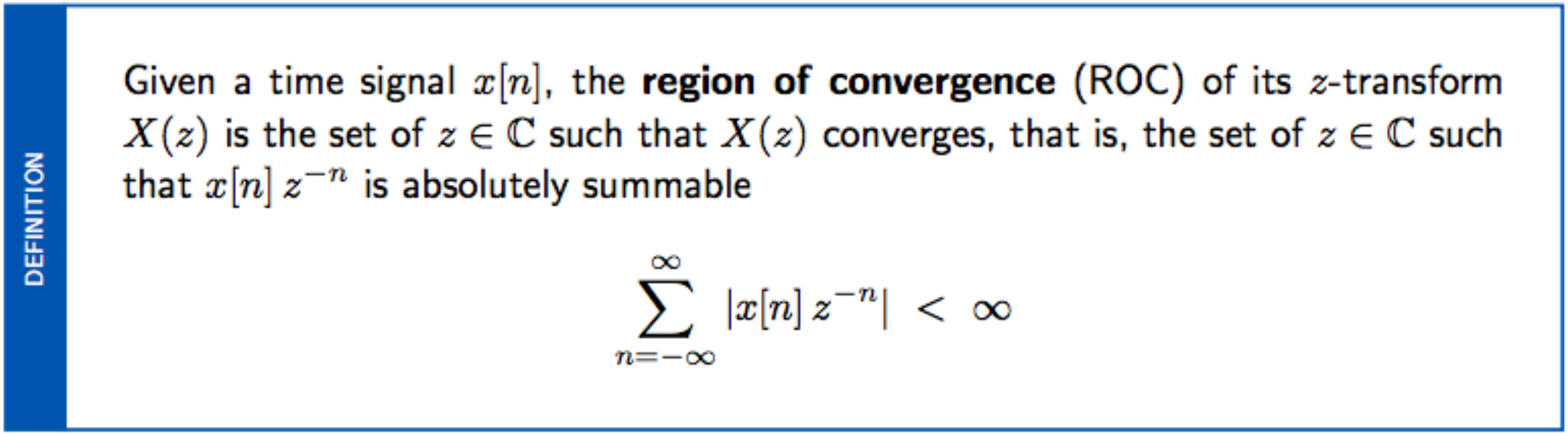


Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - What if system is stable?



Reminder: Region of Convergence (ROC)



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Given a time signal $x[n]$, the **region of convergence** (ROC) of its z-transform $X(z)$ is the set of $z \in \mathbb{C}$ such that $X(z)$ converges, that is, the set of $z \in \mathbb{C}$ such that $x[n]\, z^{-n}$ is absolutely summable

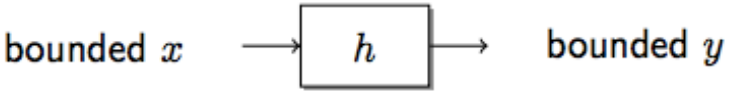
$$\sum_{n=-\infty}^{\infty} |x[n]\, z^{-n}| \; < \; \infty$$

Reminder: BIBO Stability Revisited

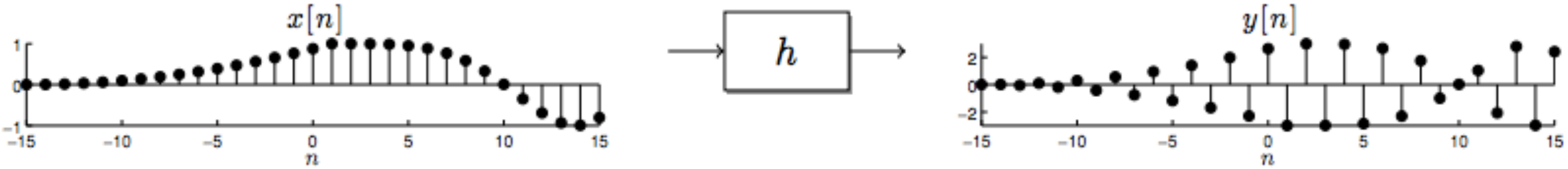
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

DEFINITION

An LTI system is **bounded-input bounded-output** (
input x always produces a bounded output y



- Bounded input and output means $\|x\|_\infty < \infty$ and $\|y\|_\infty < \infty$

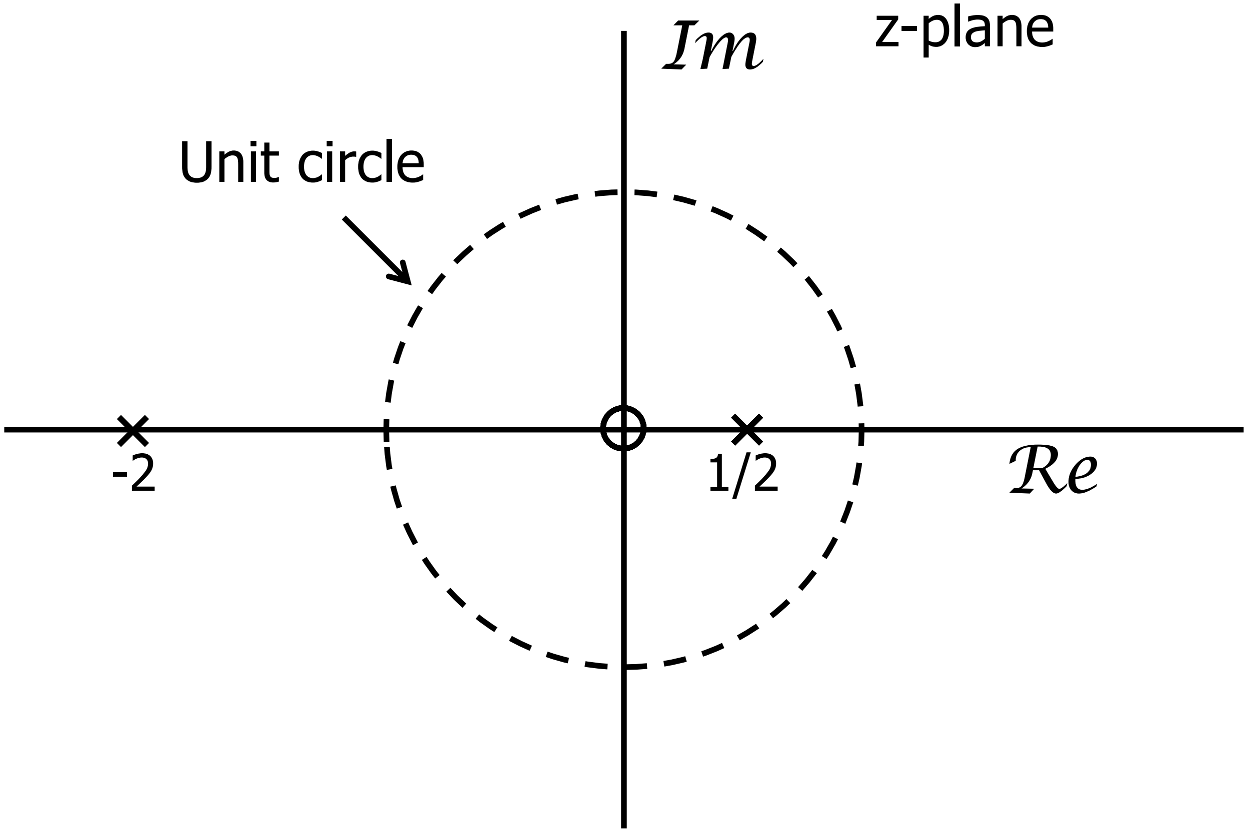


- **Fact:** An LTI system with impulse response h is BIBO stable if and only if

$$\|h\|_1 = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Pole-Zero Plot

- H(z) for an LTI System
  - How many possible ROCs?
  - Stable and causal?



z-transform Pairs

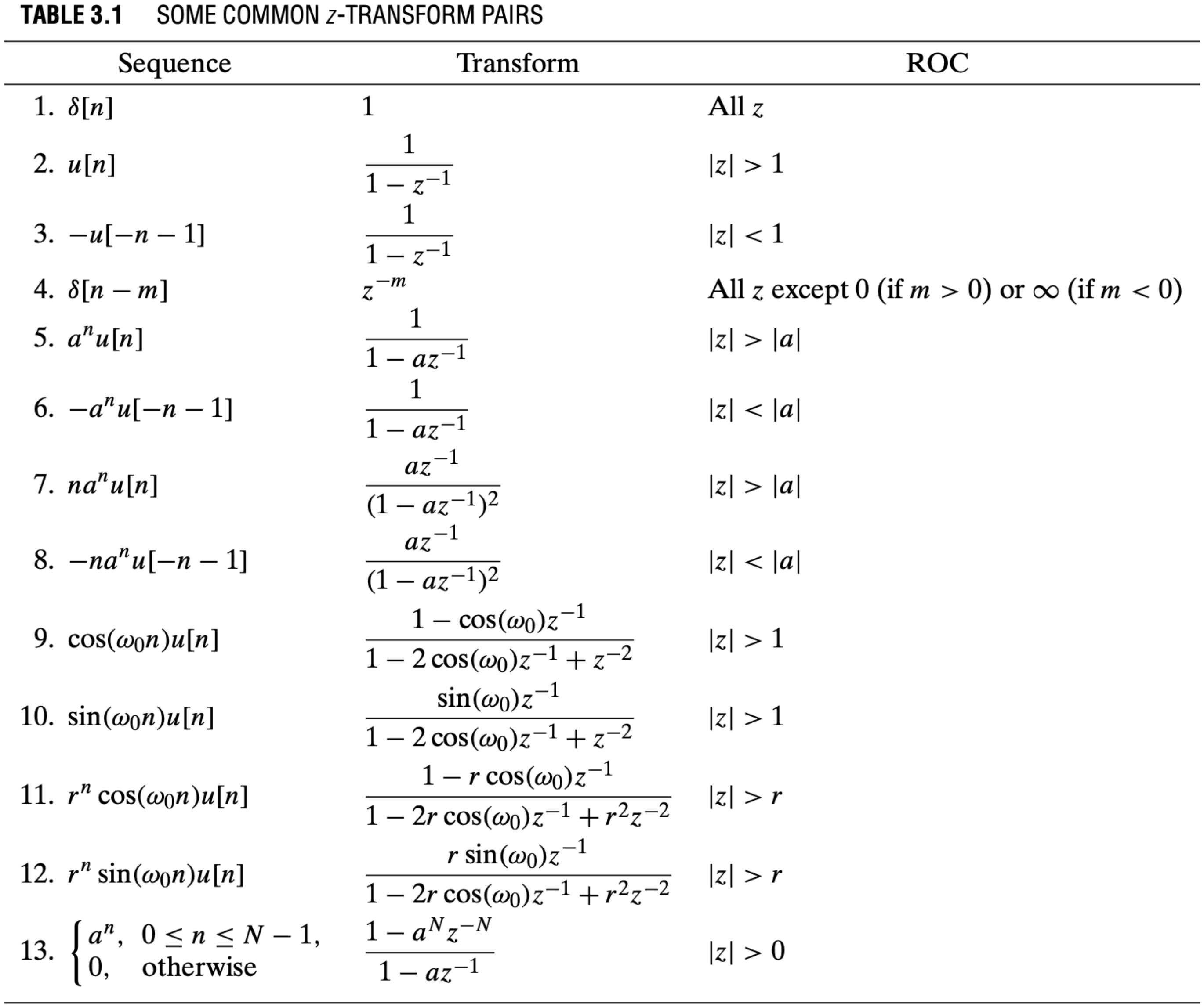


TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\dfrac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n \cos(\omega_0 n)u[n]$	$\dfrac{1 - r\cos(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
12. $r^n \sin(\omega_0 n)u[n]$	$\dfrac{r\sin(\omega_0)z^{-1}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1 - a^N z^{-N}}{1 - az^{-1}}$	$\lvert z\rvert > 0$

Properties of z-Transform

- Linearity:

$$ax_1[n]+bx_2[n]\leftrightarrow aX_1(z)+bX_2(z)$$

- Time shifting:

$$x[n]\leftrightarrow X(z)$$

$$x[n-n_d]\leftrightarrow z^{-n_d}X(z)$$

- Multiplication by exponential sequence

$$x[n]\leftrightarrow X(z)$$

$$z_0^n x[n]\leftrightarrow X\left(\frac{z}{z_0}\right)$$

Properties of z-Transform

- Time Reversal:

$$x[n] \leftrightarrow X(z)$$

$$x[-n] \leftrightarrow X(z^{-1})$$

- Differentiation of transform:

$$x[n] \leftrightarrow X(z)$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$$

- Convolution in Time:

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z)$$

ROC_Y at least $\mathrm{ROC}_x \wedge \mathrm{ROC}_H$

z-transform Properties

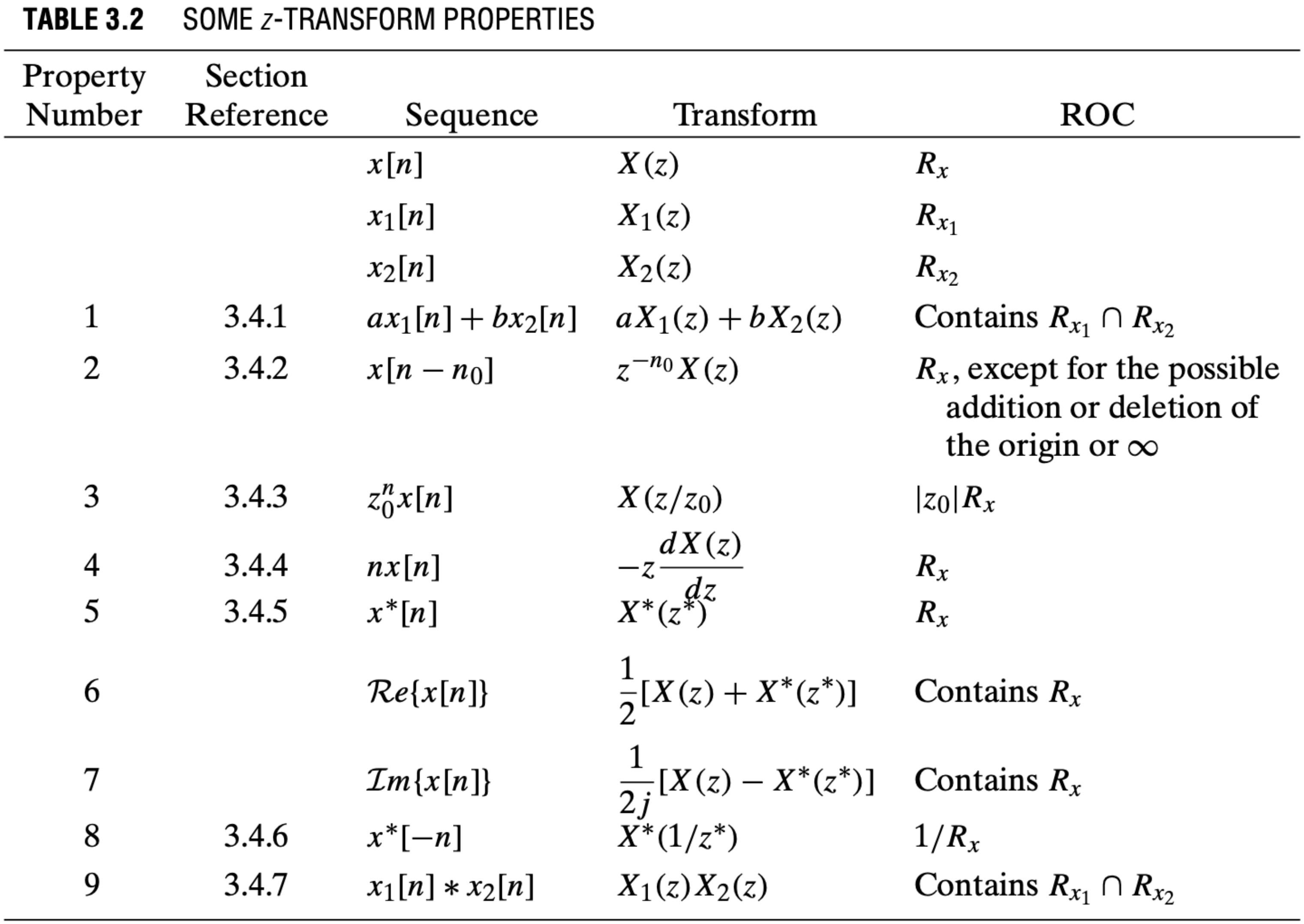


TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Inverse z-Transform

Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

Inverse z-Transform

- Recall the inverse DTFT

$$x[n] = \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, \frac{d\omega}{2\pi}$$

- There is a similar formula for the inverse z-transform using a contour integral

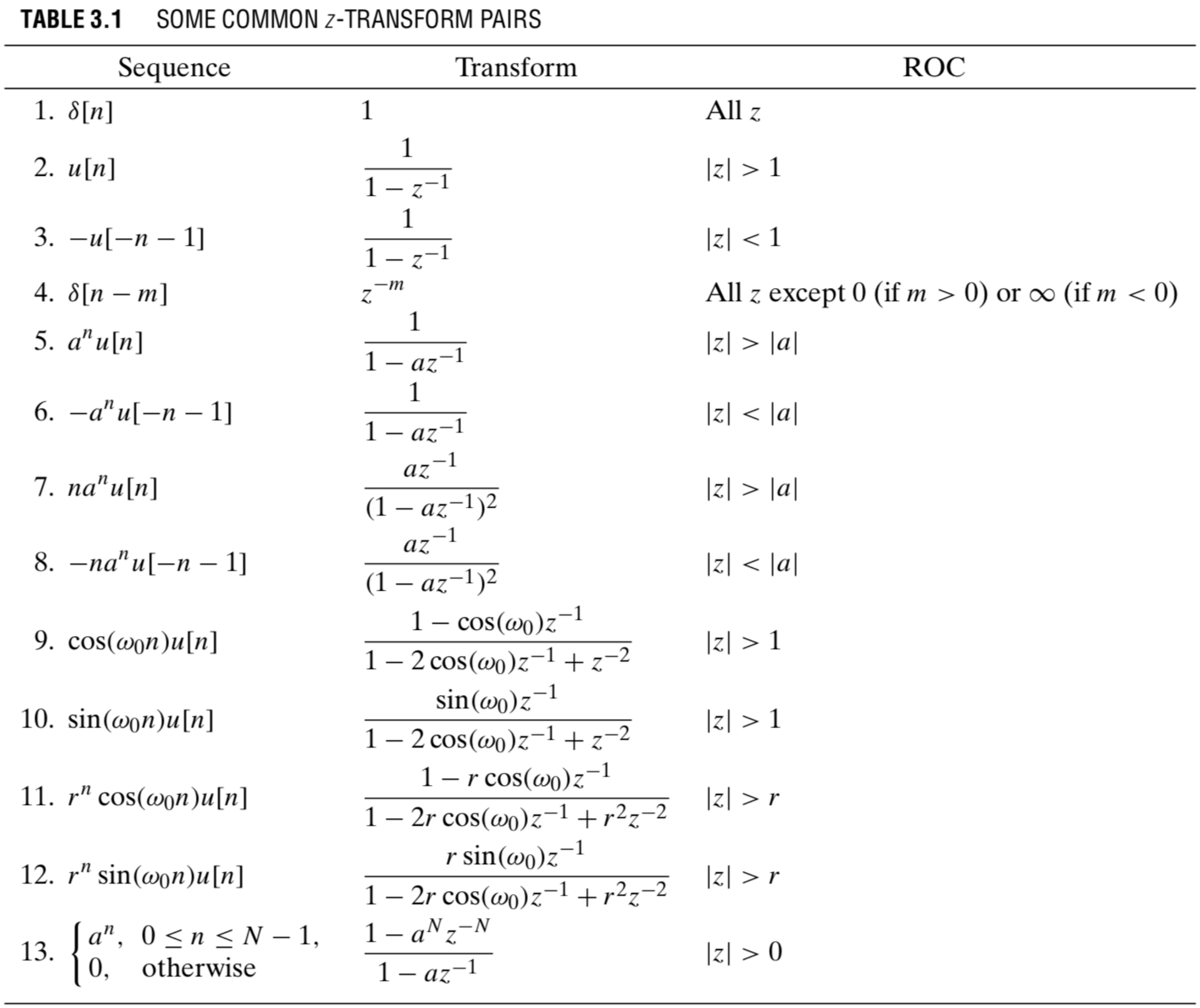
$$x[n] = \oint_C X(z)\, z^n\, \frac{dz}{j2\pi z}$$

- Contour integrals are fun but beyond the scope of this course!

Inverse z-Transform

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion

Z-Transform Pairs



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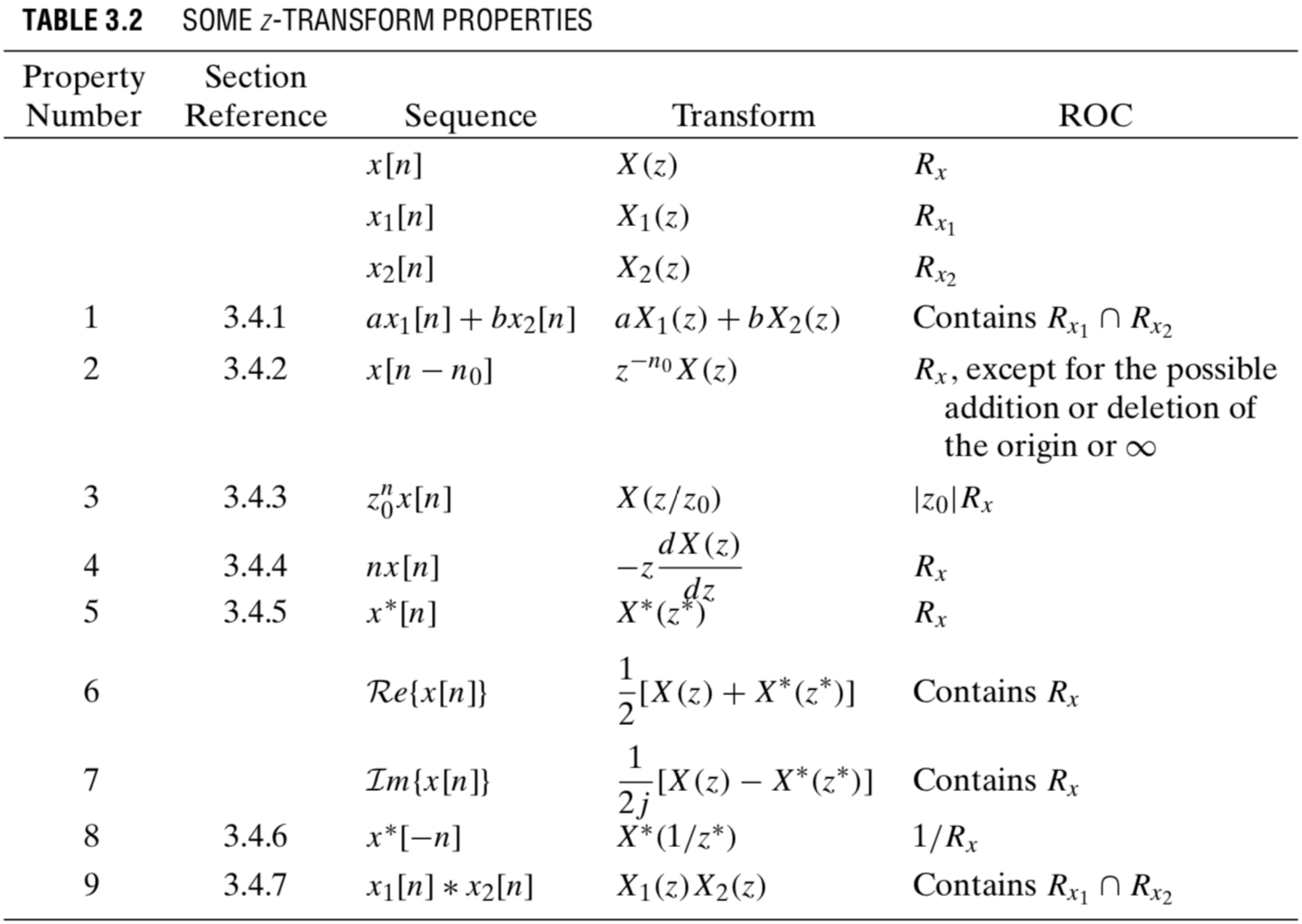
Z-Transform Properties



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Partial Fraction Expansion

- Let

$$X(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- M zeros and N poles at nonzero locations

Partial Fraction Expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- ☐ Factored numerator/denominator

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Partial Fraction Expansion

- If M<N and the poles are 1st order

$$X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_kz^{-1})}{\prod_{k=1}^{N}(1-d_kz^{-1})}=\sum_{k=1}^{N}\frac{A_k}{1-d_kz^{-1}}$$

- where

$$A_k=(1-d_kz^{-1})X(z)\Big|_{z=d_k}$$

Example: 2^{nd}-Order z-Transform

- 2^{nd}-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z : \frac{1}{2} < |z|\right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z : \ \frac{1}{2} < |z|\right\}$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

$$A_k = (1 - d_k z^{-1}) X(z)\Big|_{z=d_k}$$

$$X(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{2} z^{-1}\right)}, \qquad ROC = \left\{ z : \frac{1}{2} < |z| \right\}$$

$$A_1 = (1 - \frac{1}{4} z^{-1}) X(z)\Big|_{z=1/4} = \left.\frac{(1 - \frac{1}{4} z^{-1})}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}\right|_{z=1/4} = -1$$

$$A_2 = (1 - \frac{1}{2} z^{-1}) X(z)\Big|_{z=1/2} = \left.\frac{(1 - \frac{1}{2} z^{-1})}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}\right|_{z=1/2} = 2$$

Example: 2^{nd}-Order z-Transform

- 2^{nd}-order = two poles

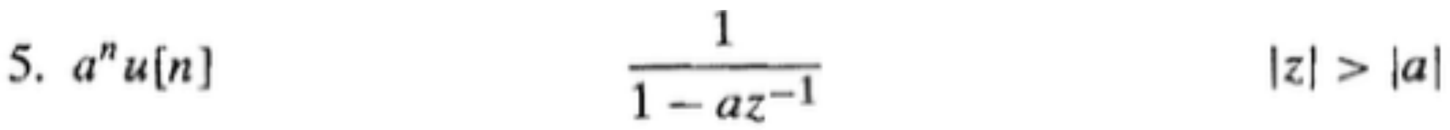
$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC=\left\{z:\ \frac{1}{2}<|z|\right\}$$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{ z : \; \frac{1}{2} < |z| \right\}$$



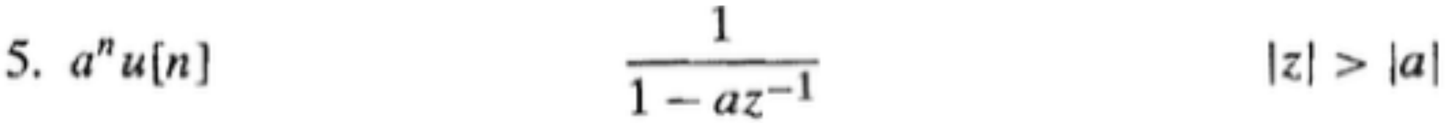
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

Example: 2nd-Order z-Transform

- 2nd-order = two poles

Right sided

$$X(z) = \frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1-\frac{1}{2}z^{-1}\right)}, \qquad ROC = \left\{z : \ \frac{1}{2} < |z|\right\}$$



5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$

$$x[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

Partial Fraction Expansion

- If M≥N and the poles are 1st order

$$X(z)=\sum_{r=0}^{M-N} B_r z^{-r}+\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

- Where B_k is found by long division and

$$A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \qquad ROC = \left\{ z : \ 1 < |z| \right\}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}, \qquad ROC=\left\{z:\ 1<\left|z\right|\right\}$$

$$=\frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}, \qquad ROC=\{z:\ 1<|z|\}$$

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\big)\overline{z^{-2}+2z^{-1}+1} \\ \underline{z^{-2}-3z^{-1}+2} \\ 5z^{-1}-1 \end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}, \qquad ROC = \left\{z:\ 1<\left|z\right|\right\}$$

$$\frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

Example: Partial Fractions

- M=N=2 and poles are first order

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}, \qquad ROC = \left\{ z : \; 1 < |z| \right\}$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Power Series Expansion

- Expansion of the z-transform definition

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$=\cdots+x[-2]z^{2}+x[-1]z+x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^{2}\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n = -2 \\ -1/2, & n = -1 \\ -1, & n = 0 \\ 1/2, & n = 1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Example: Finite-Length Sequence

- Poles and zeros?

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1})$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -1/2, & n=-1 \\ -1, & n=0 \\ 1/2, & n=1 \\ 0, & else \end{cases} = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

4. $\delta[n-m]$ $\quad$ z^{-m}

Reminder: Difference Equations

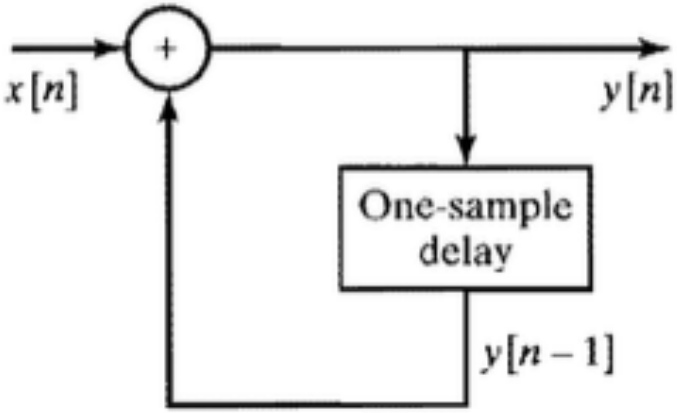
- Accumulator example

$$y[n]=\sum_{k=-\infty}^{n}x[k]$$

$$y[n]=x[n]+\sum_{k=-\infty}^{n-1}x[k]$$

$$y[n]=x[n]+y[n-1]$$

$$\boxed{y[n]-y[n-1]=x[n]}$$



$$\boxed{\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m x[n-m]}$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right) y[n-k] + \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right) x[n-m]$$

- Difference equations of this form behave as causal LTI systems
  - when the input is zero prior to n=0
  - Initial rest equations are imposed prior to the time when input becomes nonzero
    - i.e y[-N]=y[-N+1]=…=y[-1]=0

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N} \left(\frac{a_k}{a_0} \right) z^{-k} Y(z) = \sum_{m=0}^{M} \left(\frac{b_m}{a_0} \right) z^{-m} X(z)$$

Difference Equation to z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\sum_{k=0}^{N}\left(\frac{a_k}{a_0}\right) z^{-k} Y(z) = \sum_{m=0}^{M}\left(\frac{b_m}{a_0}\right) z^{-m} X(z)$$

$$\Rightarrow Y(z) = \frac{\sum_{m=0}^{M}\left(b_m\right) z^{-m}}{\sum_{k=0}^{N}\left(a_k\right) z^{-k}} X(z)$$

$$\boxed{H(z) = \frac{\sum_{m=0}^{M}\left(b_k\right) z^{-k}}{\sum_{k=0}^{N}\left(a_k\right) z^{-k}}}$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^{M} (b_k) z^{-k}}{\sum_{k=0}^{N} (a_k) z^{-k}}$$

Example: 1st-Order System

$$y[n] = ay[n-1] + x[n]$$

$$H(z) = \frac{\overset{b_0\,\swarrow}{1}}{\underset{\uparrow\,a_0}{1} - \underset{\nwarrow\,a_1}{a}z^{-1}}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$H(z) = \frac{\sum_{m=0}^{M} \left(b_k\right) z^{-k}}{\sum_{k=0}^{N} \left(a_k\right) z^{-k}}$$

Example: 1st-Order System

$$y[n]=ay[n-1]+x[n]$$

$$H(z)=\frac{\overset{b_0}{1}}{\underset{a_0}{1}-\underset{a_1}{a}z^{-1}}$$

$$h[n]=a^n u[n]$$

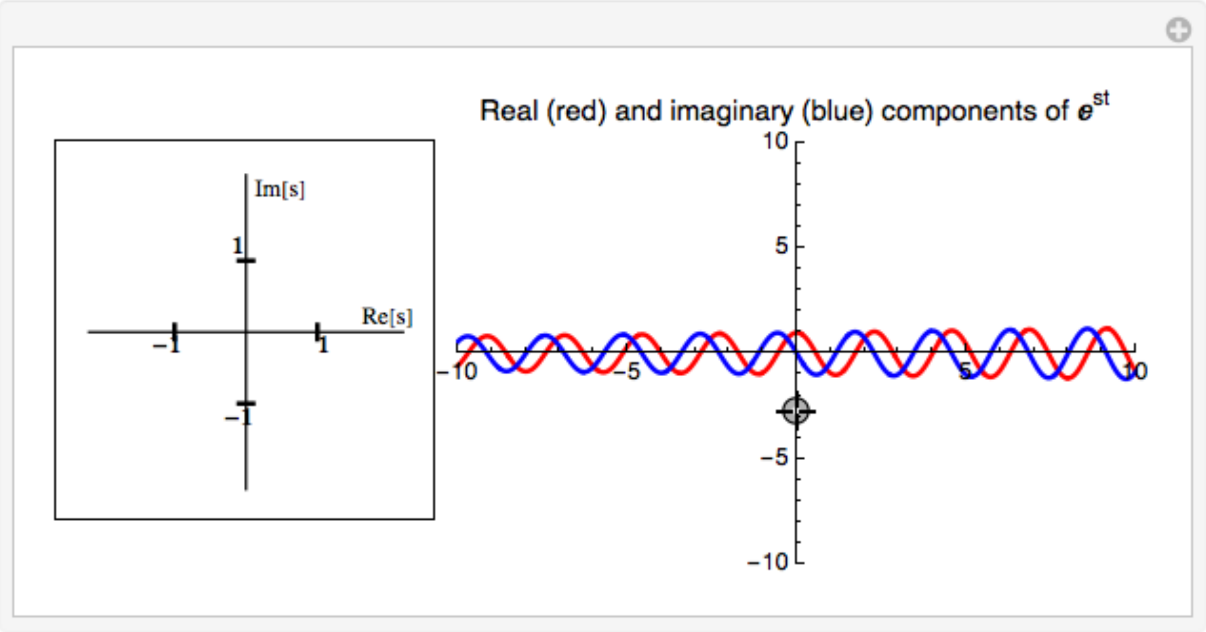
$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{m=0}^{M}b_m x[n-m]$$

$$H(z)=\frac{\sum_{m=0}^{M}\left(b_k\right)z^{-k}}{\sum_{k=0}^{N}\left(a_k\right)z^{-k}}$$

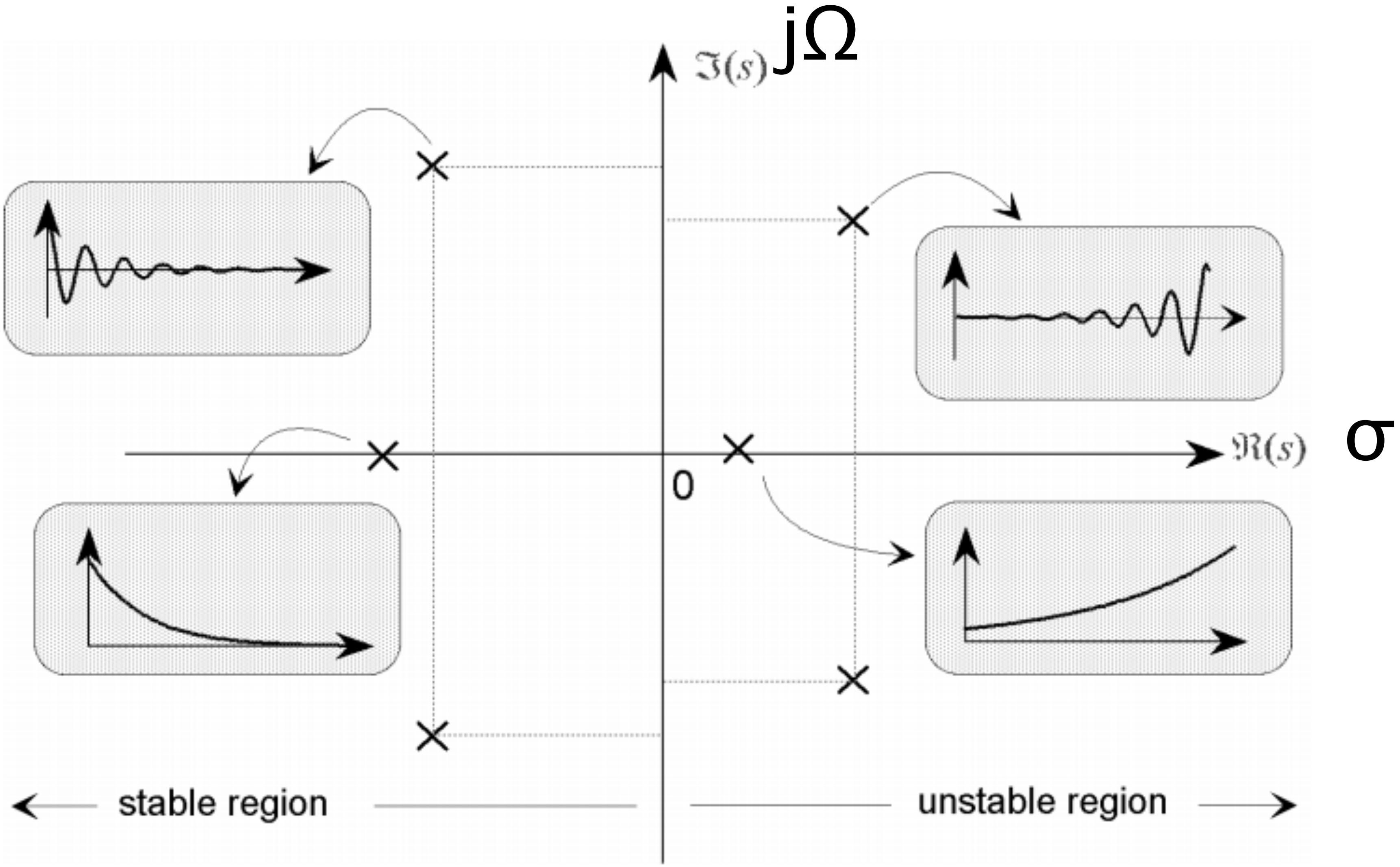
Why right sided?

S-Plane

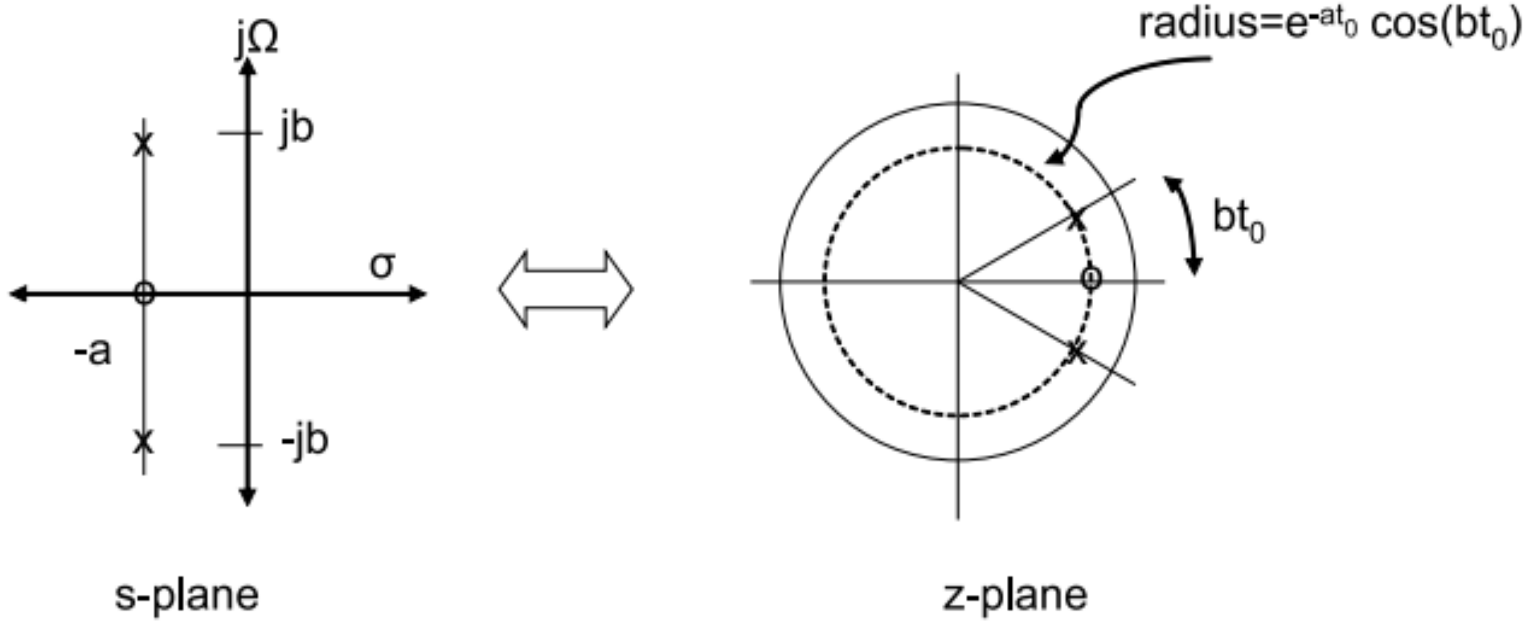
- s=σ+jΩ
- Wolfram Demo



S-plane and stability

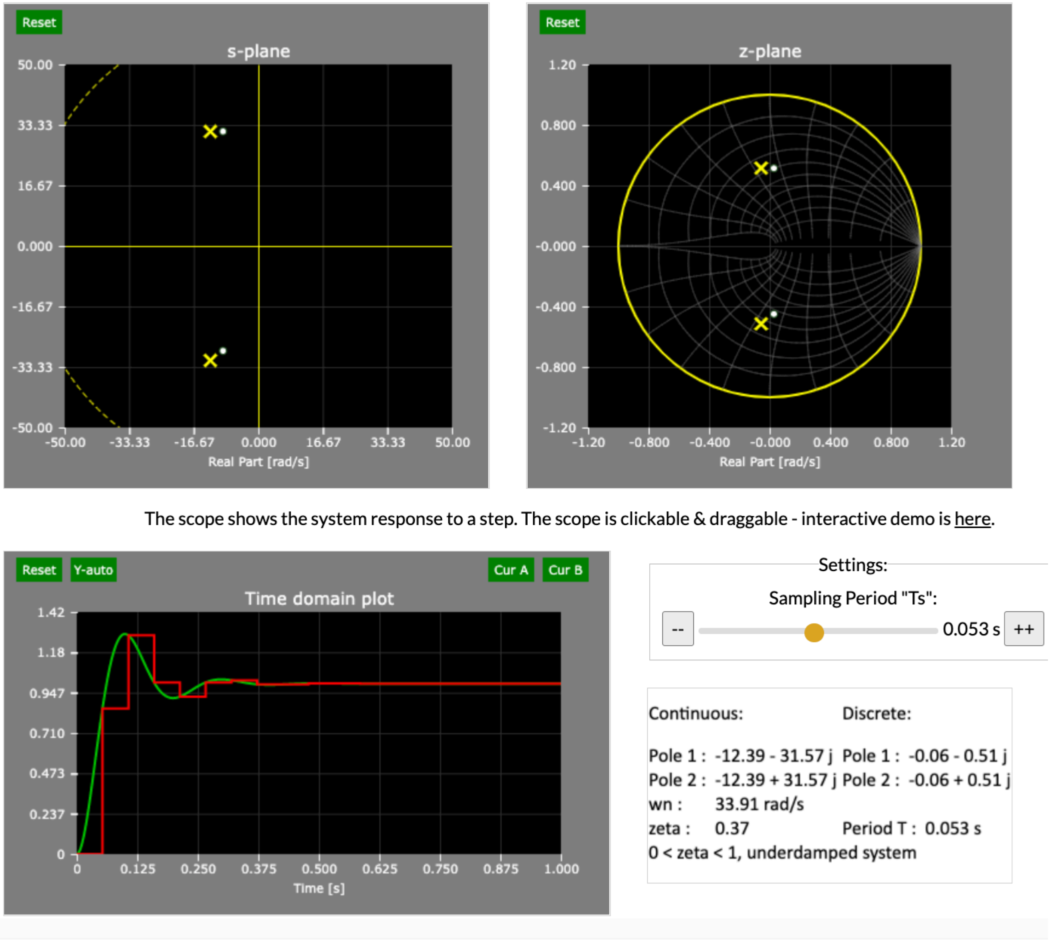


S-Plane Mapping to Z-Plane



S-Plane Mapping to Z-Plane

- Interactive Demo



The scope shows the system response to a step. The scope is clickable & draggable - interactive demo is here.

Settings:

Sampling Period "Ts": 0.053 s

Continuous:		Discrete:	
Pole 1 :	-12.39 - 31.57 j	Pole 1 :	-0.06 - 0.51 j
Pole 2 :	-12.39 + 31.57 j	Pole 2 :	-0.06 + 0.51 j
wn :	33.91 rad/s		
zeta :	0.37	Period T :	0.053 s

0 < zeta < 1, underdamped system

- https://controlsystemsacademy.com/0003/0003.html

Big Ideas

- z-Transform properties
  - Similar to DTFT
- Inverse z-transform
  - Avoid it!
  - Inspection, properties, partial fractions, power series
- Difference equations easy to transform
- S-plane to Z-plane mapping
  - Poles map directly. Zeros don't! More later…

Video Example

- https://www.youtube.com/watch?v=ByTsISFXUoY

Admin

- HW 2 out due Sunday 2/5 at midnight
  - Double check that your submission by viewing it!