

ESE 5310: Digital Signal Processing

Lecture 9: February 9, 2023

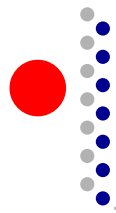
Downsampling/Upsampling and Practical
Interpolation



Lecture Outline

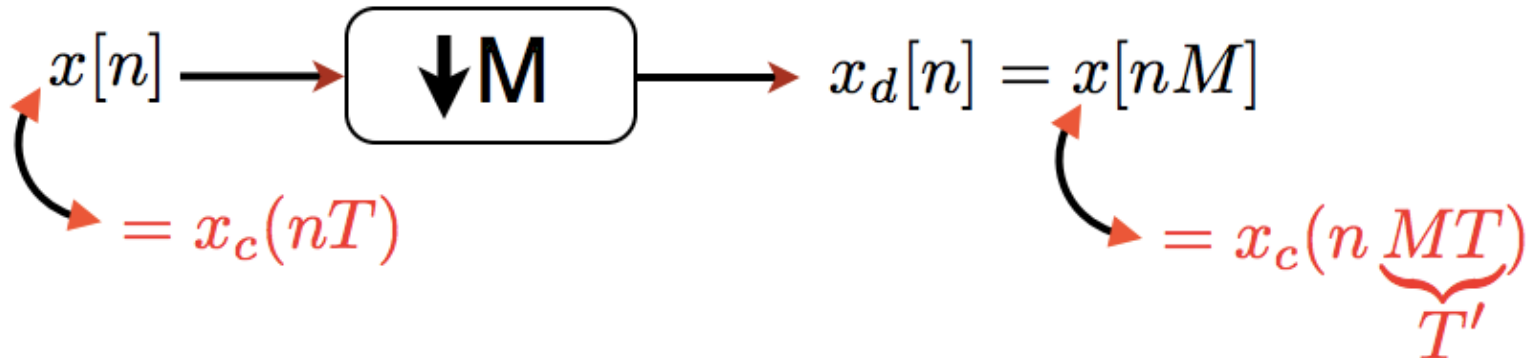
- ❑ Downsampling
- ❑ Upsampling
- ❑ Practical Interpolation (time permitting)

Downsampling



Downsampling

- Definition: Reducing the sampling rate by an integer number ($M > 1$)





Downsampling

- ❑ Similar to C/D conversion
 - Need to worry about aliasing
 - Use anti-aliasing filter to mitigate aliasing effects

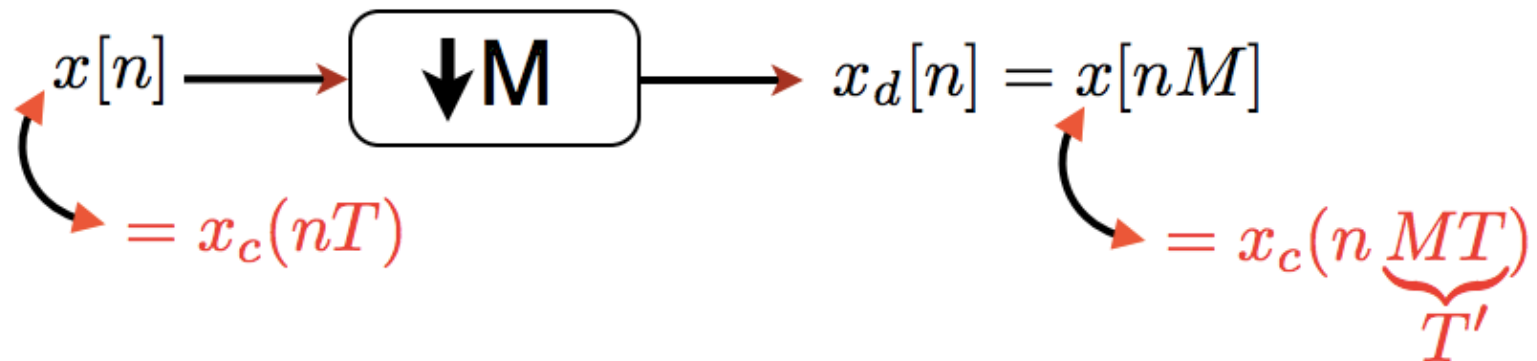


Downsampling

- ❑ Similar to C/D conversion
 - Need to worry about aliasing
 - Use anti-aliasing filter to mitigate aliasing effects

- ❑ If your discrete time signal is finely sampled (i.e. oversampled) almost like a CT signal
 - Downsampling is just like sampling (C/D conversion)

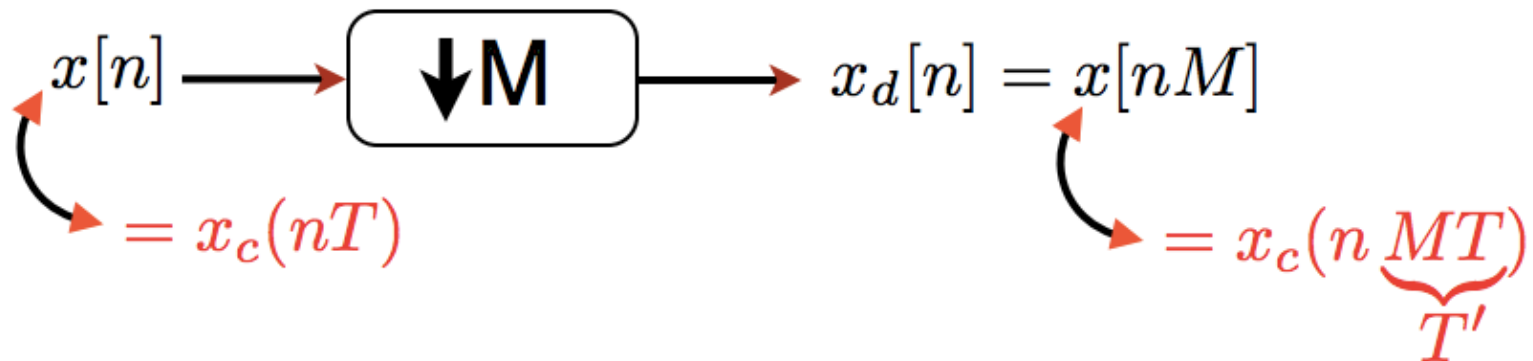
Downsampling



The DTFT:

$$X(e^{j\omega}) = \frac{1}{T} \sum_k X_c \left(j \left(\underbrace{\frac{\omega}{T}}_{\Omega} - \underbrace{\frac{2\pi}{T}}_{\Omega_s} k \right) \right)$$

Downsampling



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$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right)$$



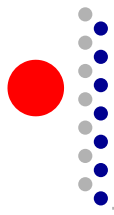
Downsampling

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$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right)$$

- ❑ Want to relate $X_d(e^{j\omega})$ to $X(e^{j\omega})$ not $X_c(j\Omega)$
- ❑ Separate sum into two sums—fine sum and coarse sum (i.e like counting minutes within hours)



Downsampling

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$$X_d(e^{j\omega}) = \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right)$$

□ $k = rM + i$

■ $i = 0, 1, \dots, M-1$

■ $r = -\infty, \dots, \infty$



Downsampling

$$\begin{aligned} X_d(e^{j\omega}) &= \frac{1}{MT} \sum_k X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} k \right) \right) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} \underbrace{\frac{1}{T} \sum_{r=-\infty}^{\infty} X_c \left(j \left(\frac{\omega}{MT} - \frac{2\pi}{MT} i - \frac{2\pi}{T} r \right) \right)} \end{aligned}$$

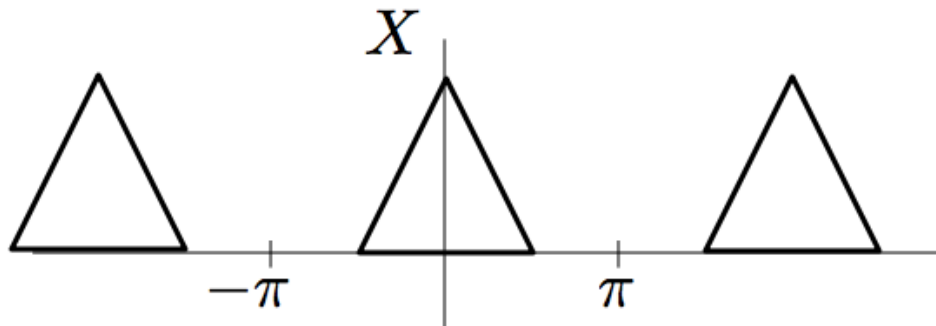
$$X(e^{j\omega}) = \frac{1}{T} \sum_k X_c \left(j \left(\underbrace{\frac{\omega}{T}} - \underbrace{\frac{2\pi}{T} k} \right) \right) \quad X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$$

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$$

scale by 1/M stretch by M replicate

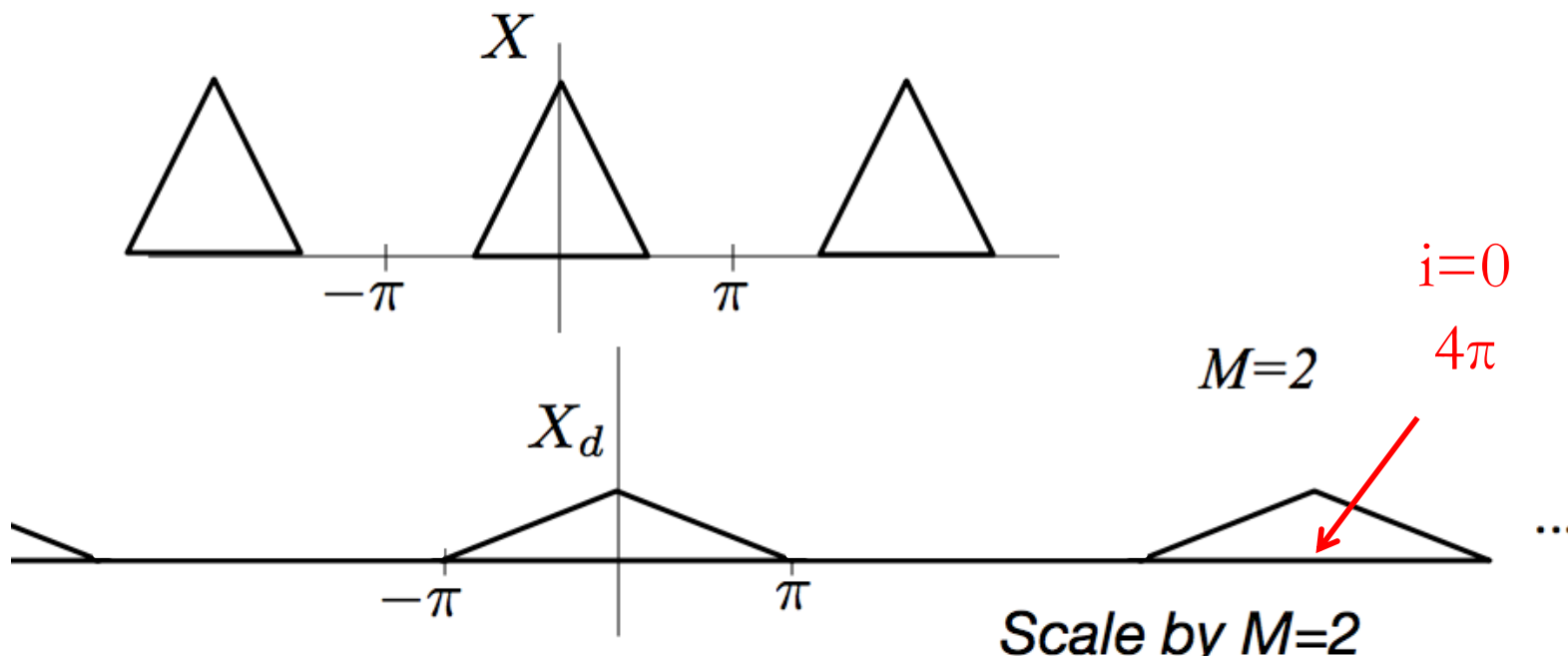
Example: M=2

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{i=0}^{M-1} X \left(e^{j \left(\frac{\omega}{M} - \frac{2\pi}{M} i \right)} \right)$$



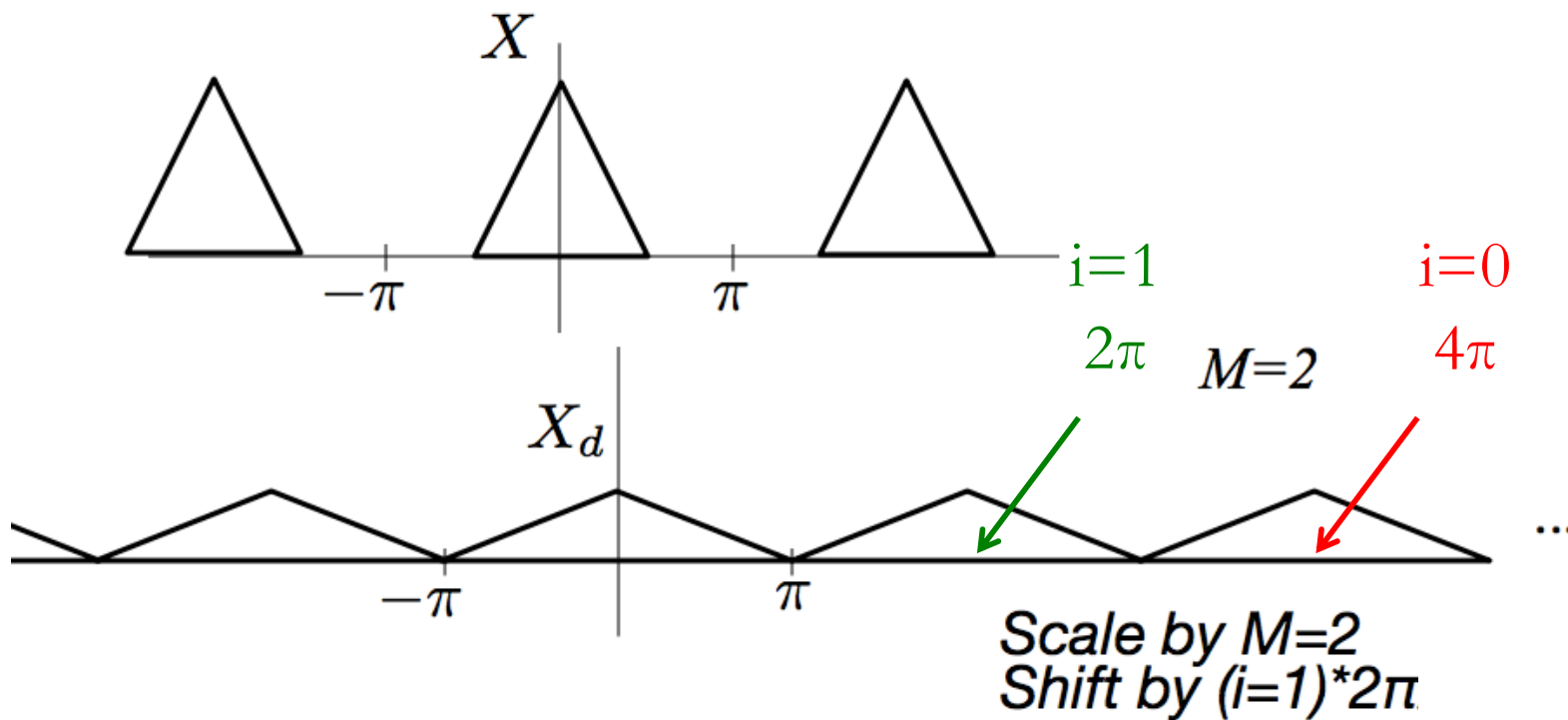
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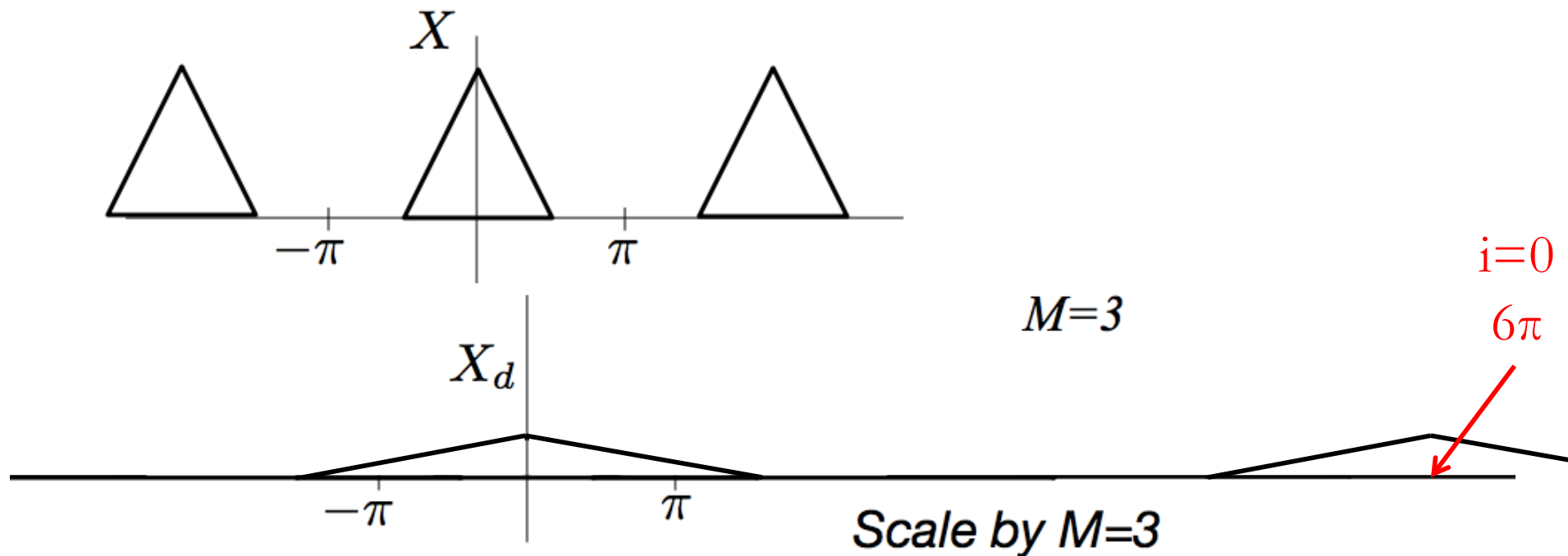
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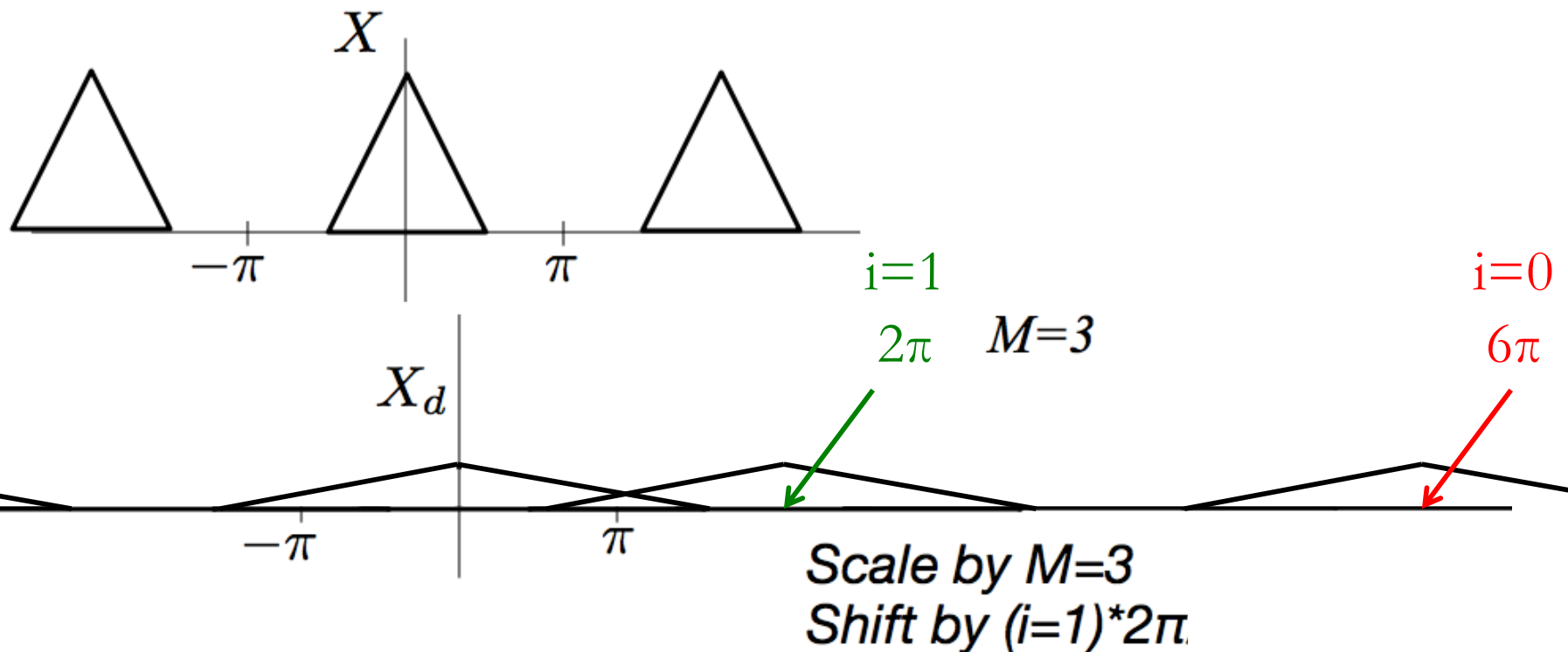
Example: $M=3$

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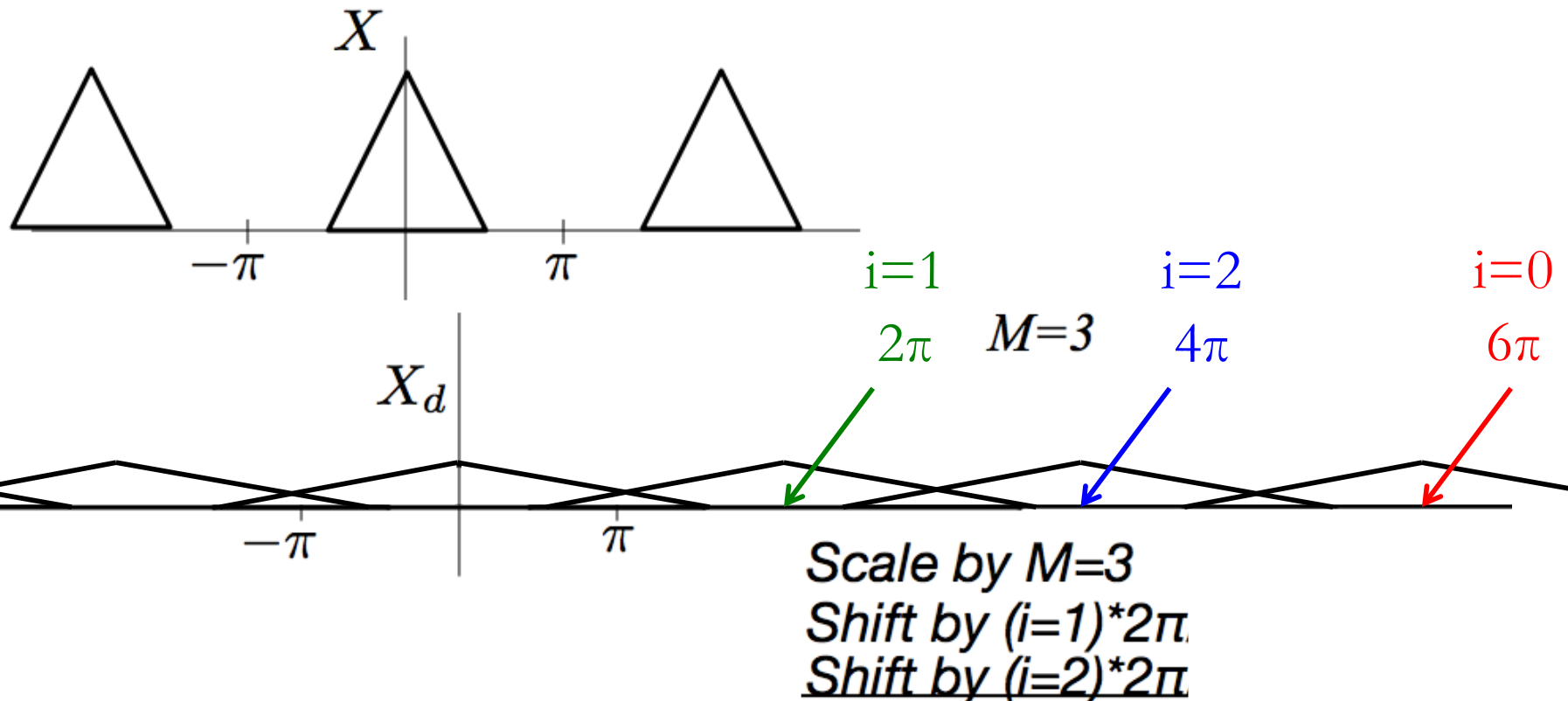
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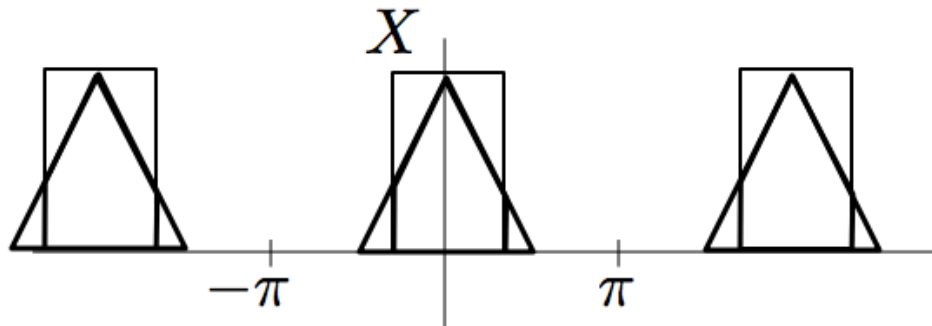
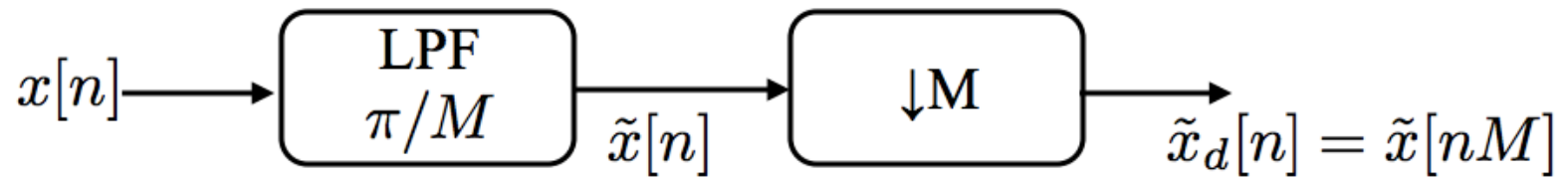


Example: M=3

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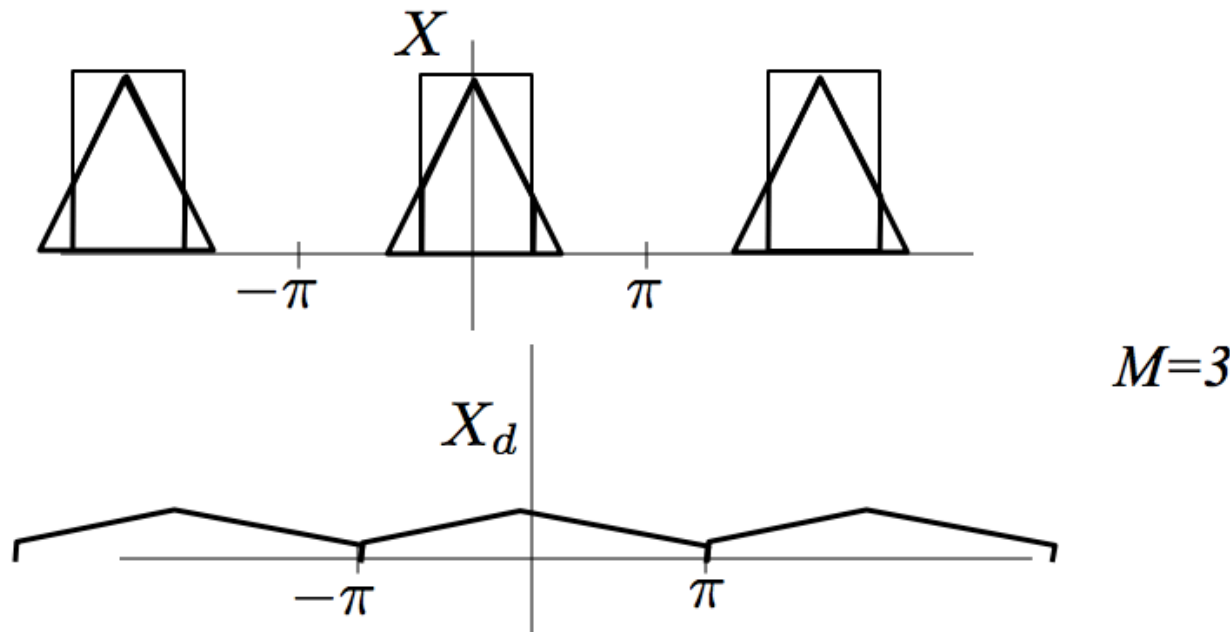
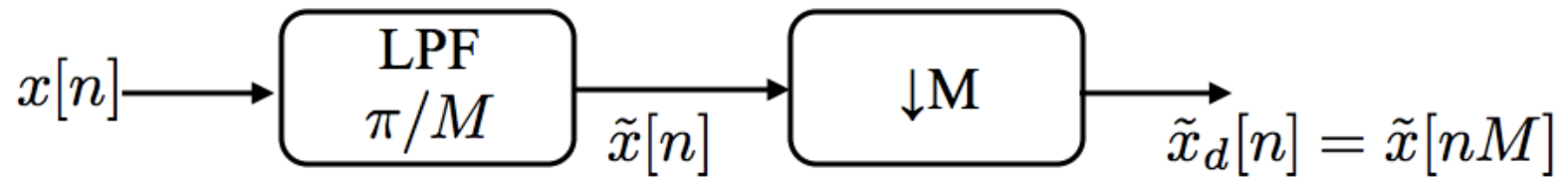


Example: $M=3$



$M=3$

Example: $M=3$



Upsampling



Upsampling

- Definition: Increasing the sampling rate by an integer number

$$x[n] = x_c(nT)$$

$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \quad L \text{ integer}$$



Upsampling

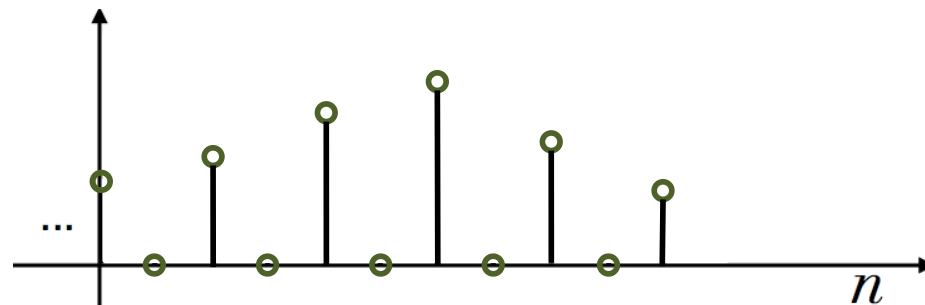
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$$x[n] = x_c(nT)$$

$$x_i[n] = x_c(nT') \quad \text{where } T' = \frac{T}{L} \quad L \text{ integer}$$

Obtain $x_i[n]$ from $x[n]$ in two steps:

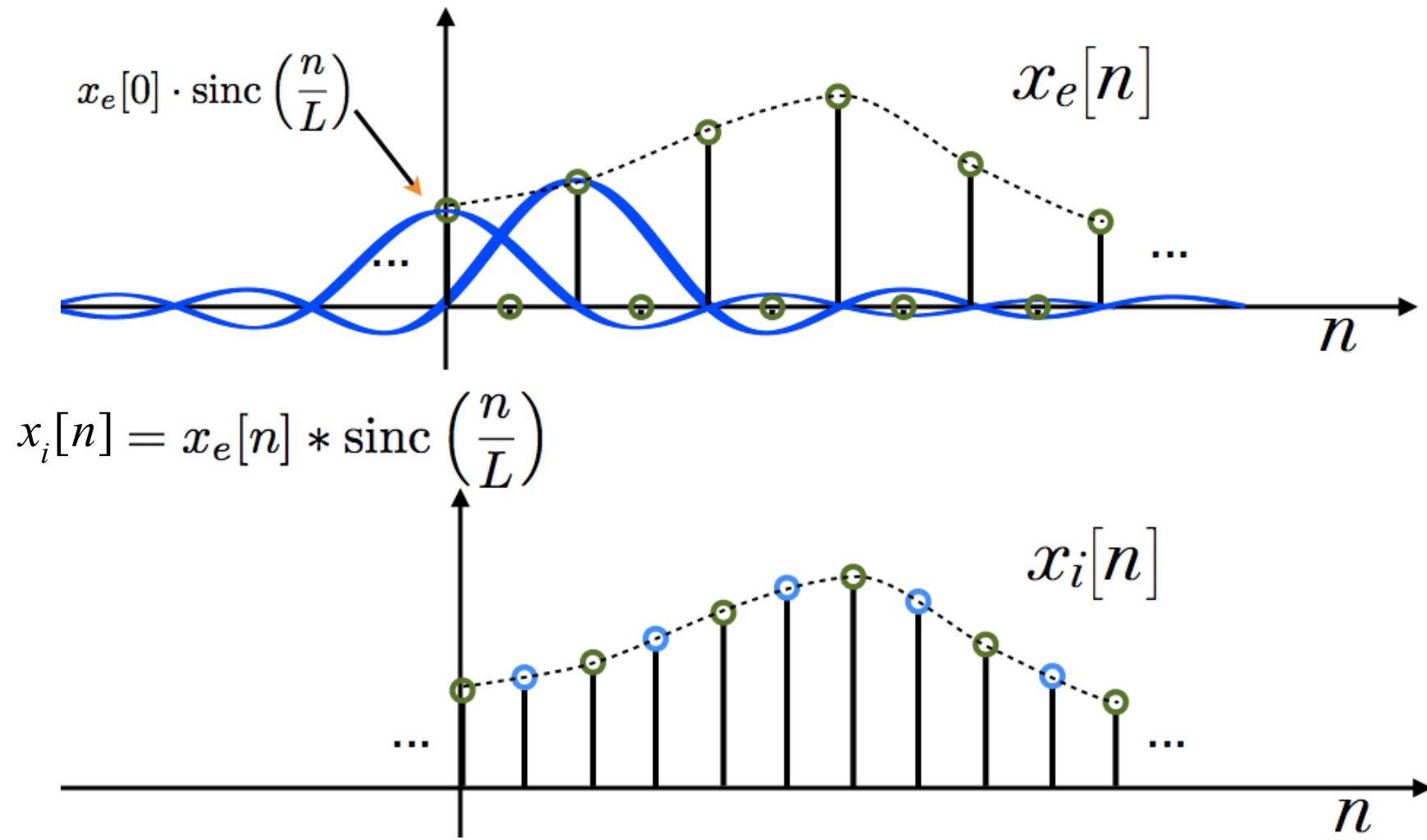
(1) Generate:
$$x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$





Upsampling

(2) Obtain $x_i[n]$ from $x_e[n]$ by bandlimited interpolation:





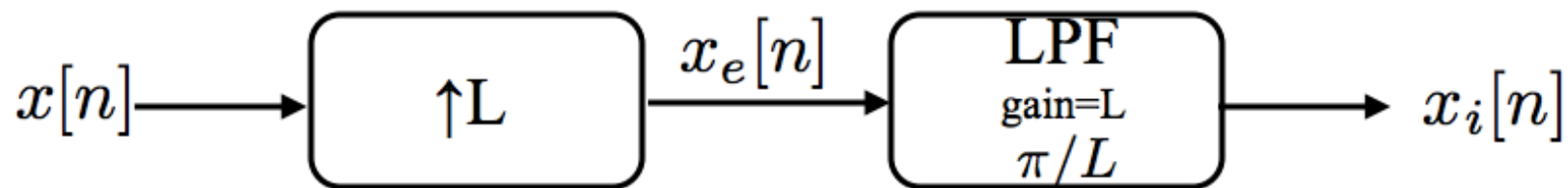
Upsampling

- ❑ Much like D/C converter
- ❑ Upsample by A LOT $\rightarrow$ almost continuous

- ❑ Intuition:
 - Recall our D/C model: $x[n] \rightarrow x_s(t) \rightarrow x_c(t)$
 - Approximate “ $x_s(t)$ ” by placing zeros between samples
 - Convolve with a sinc to obtain “ $x_c(t)$ ”

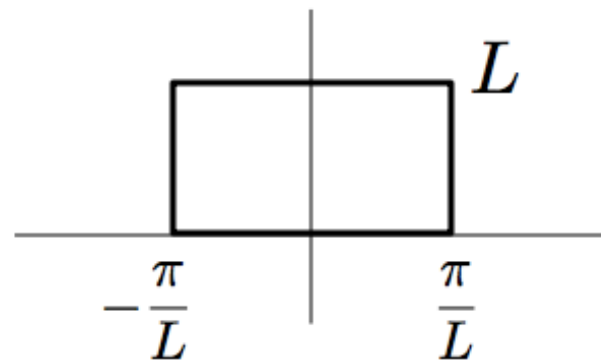
Frequency Domain Interpretation

$$x_i[n] = x_e[n] * \text{sinc}(n/L)$$

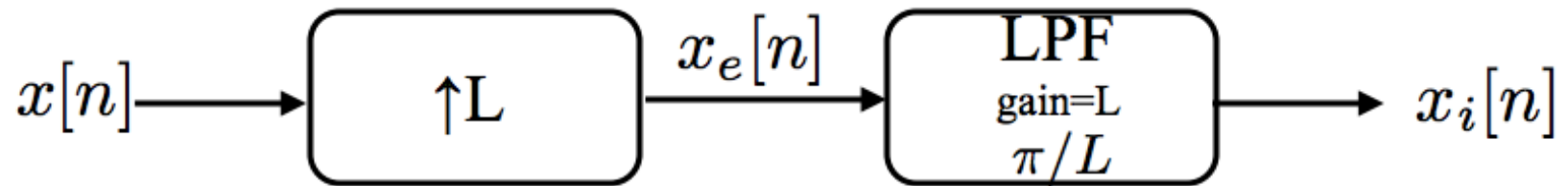


$$\text{sinc}(n/L)$$

DTFT $\Rightarrow$

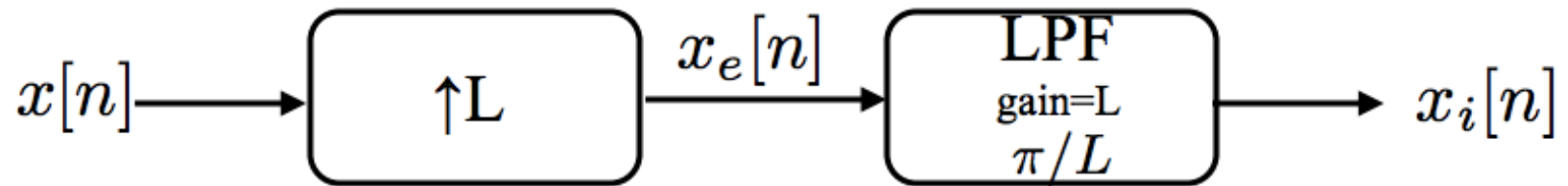


Frequency Domain Interpretation



$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]} e^{-j\omega n}$$

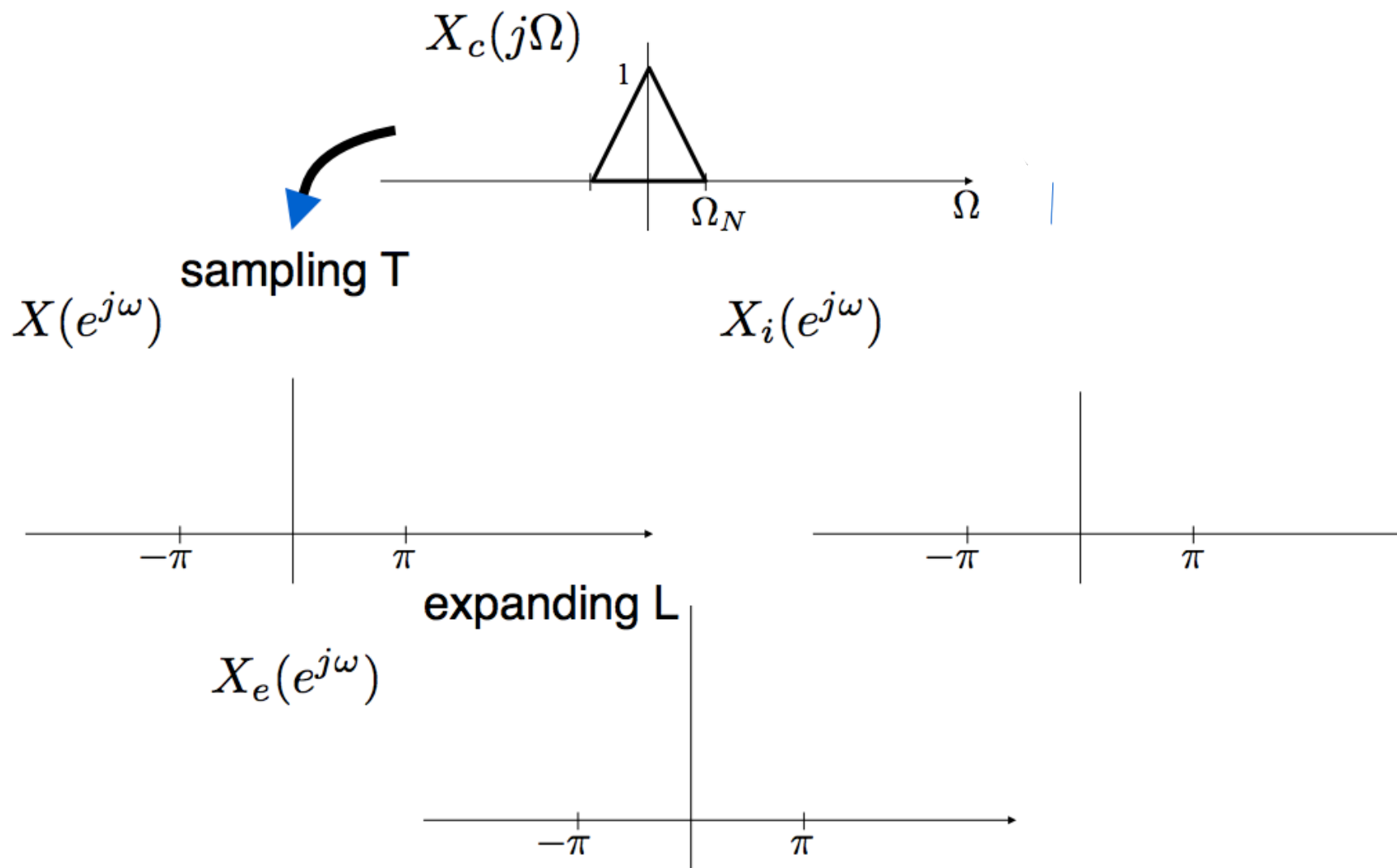
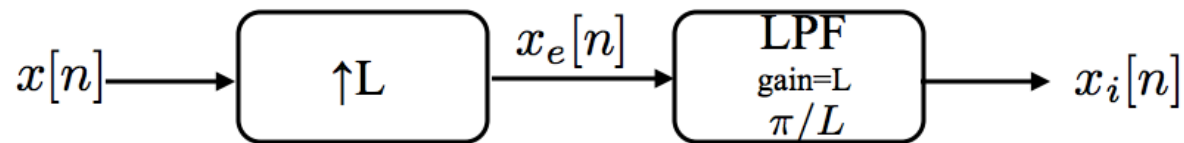
Frequency Domain Interpretation



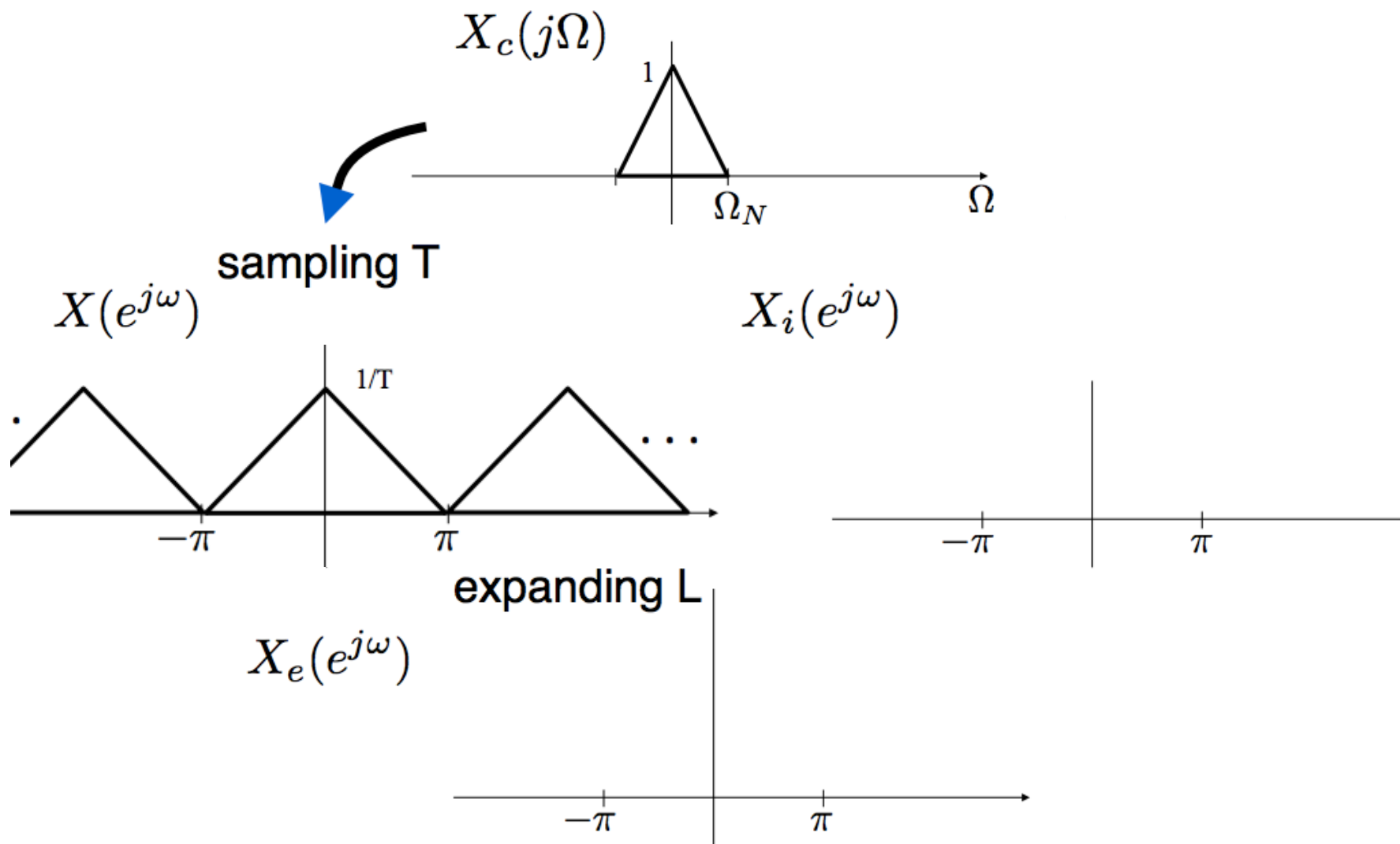
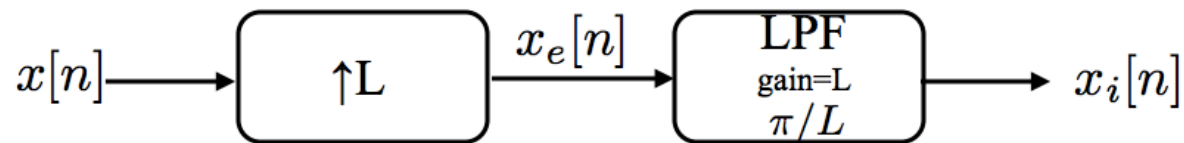
$$\begin{aligned}
 X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \underbrace{x_e[n]}_{\substack{\neq 0 \text{ only for } n=mL \\ \text{(integer } m)}} e^{-j\omega n} \\
 &= \sum_{m=-\infty}^{\infty} \underbrace{x_e[mL]}_{=x[m]} e^{-j\omega mL} = X(e^{j\omega L})
 \end{aligned}$$

Shrink DTFT by a factor of L!

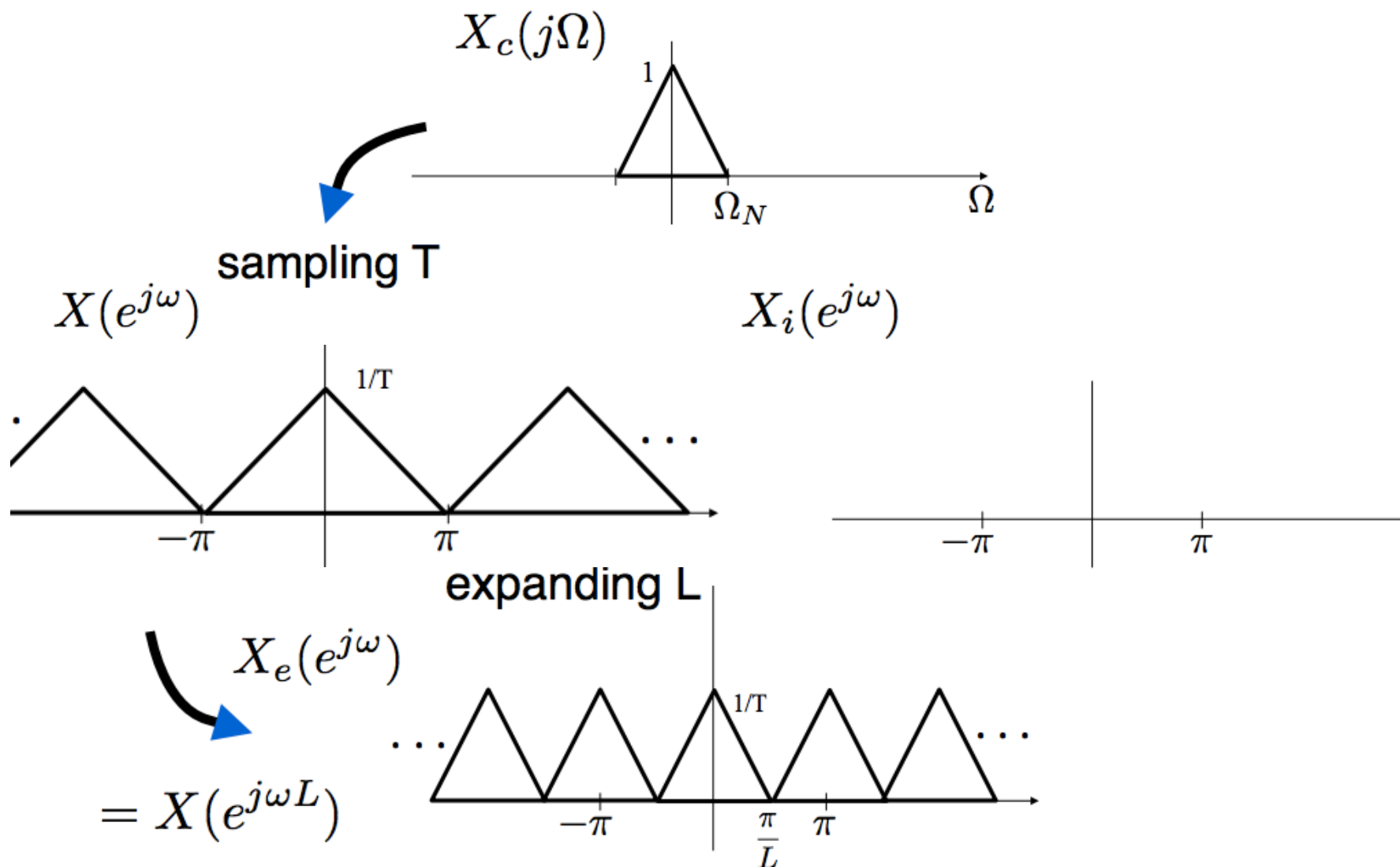
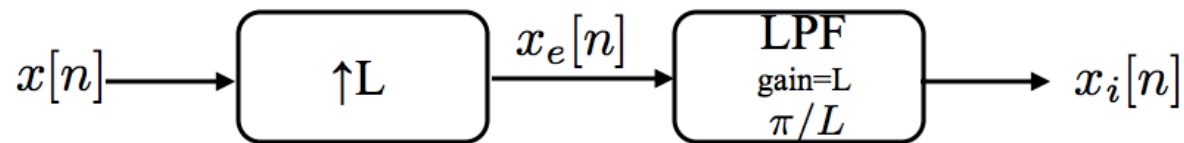
Example



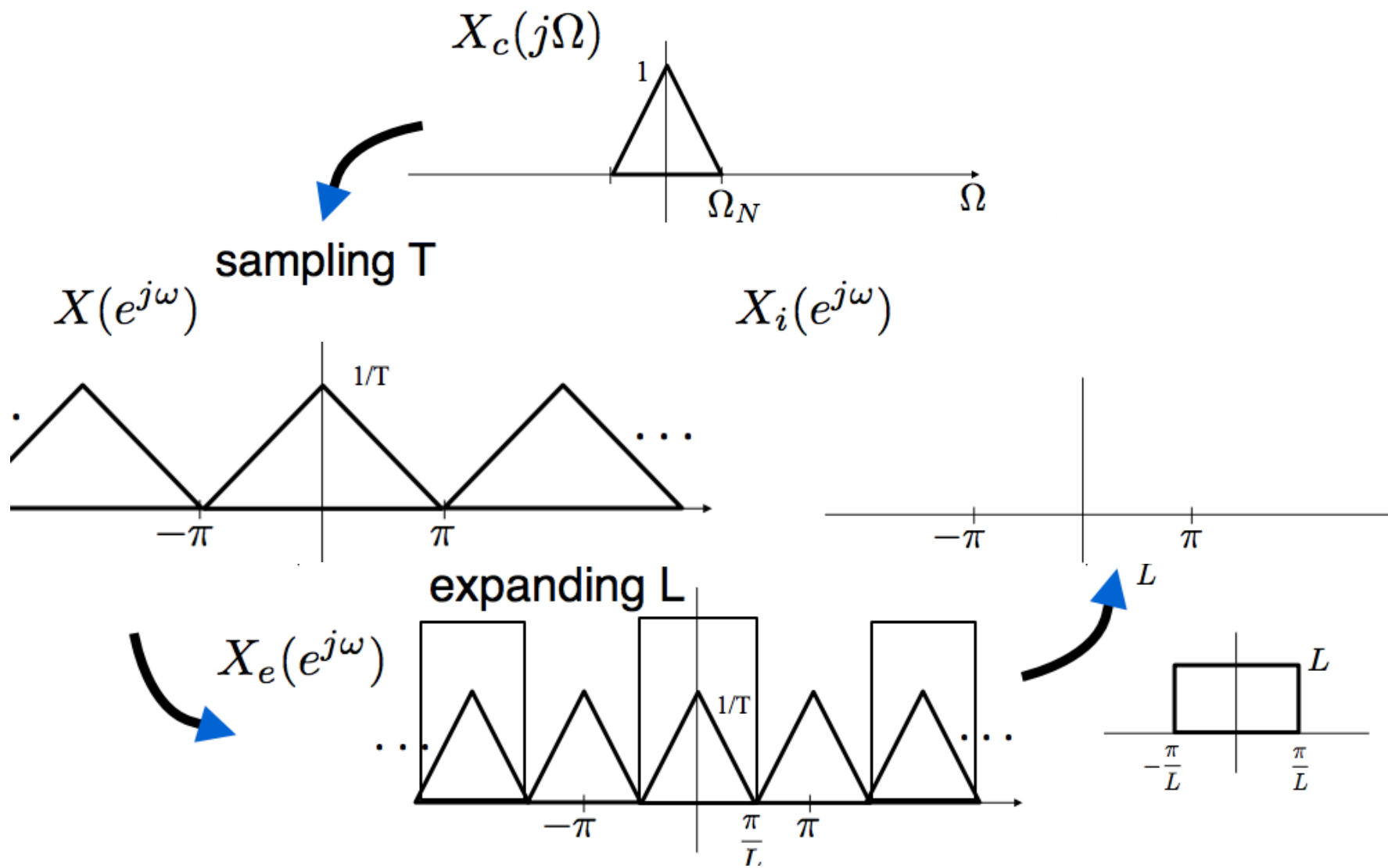
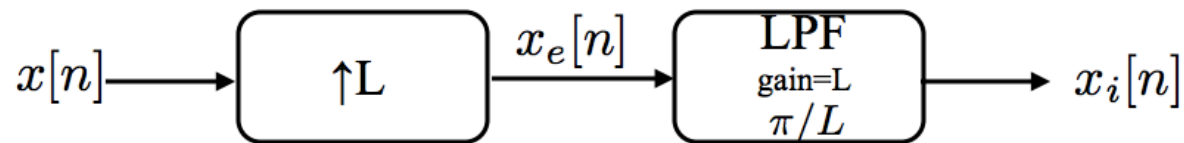
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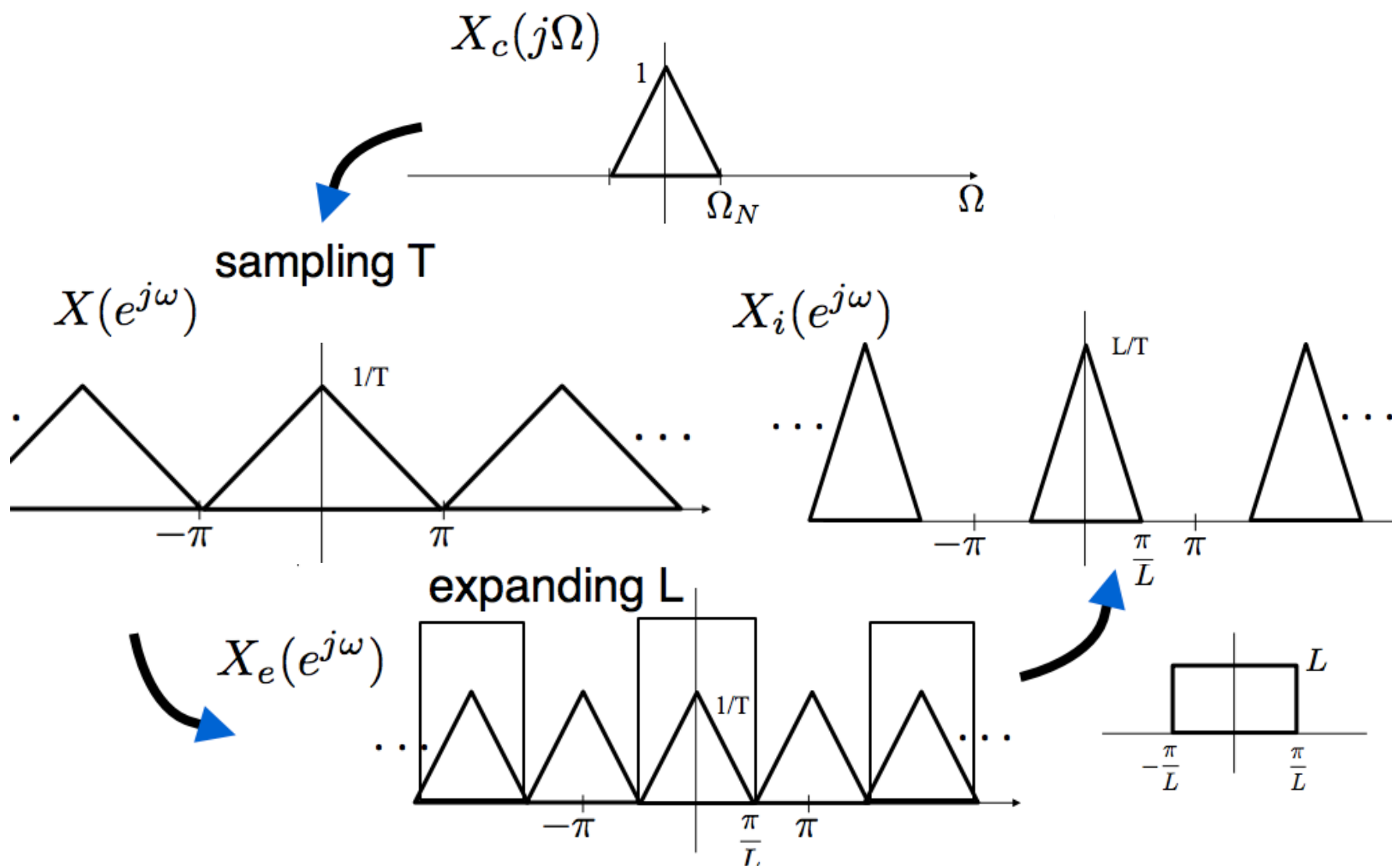
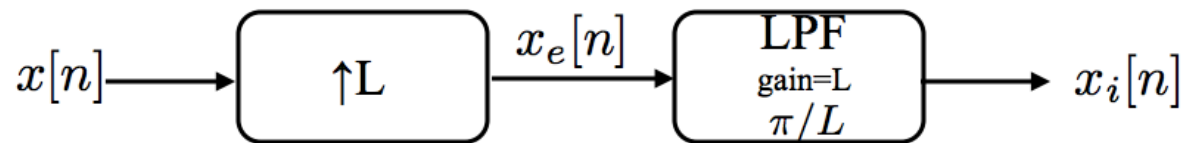
Example

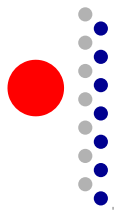


Example

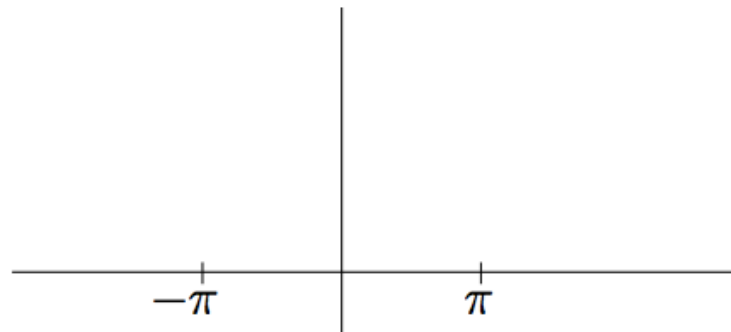
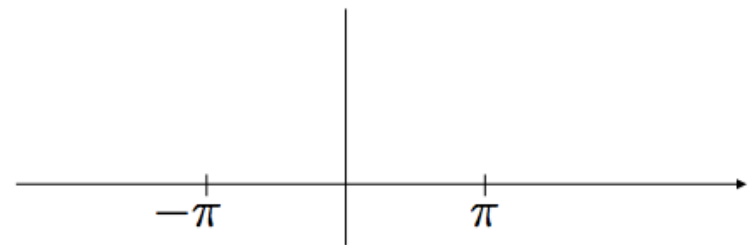
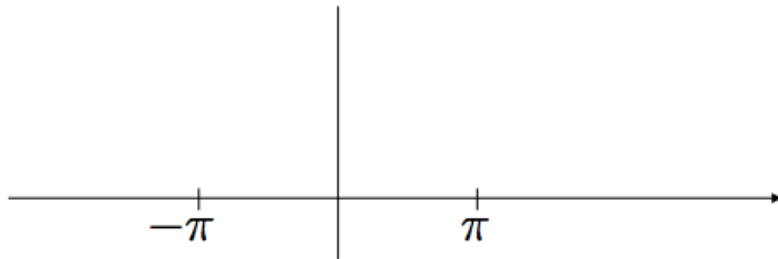
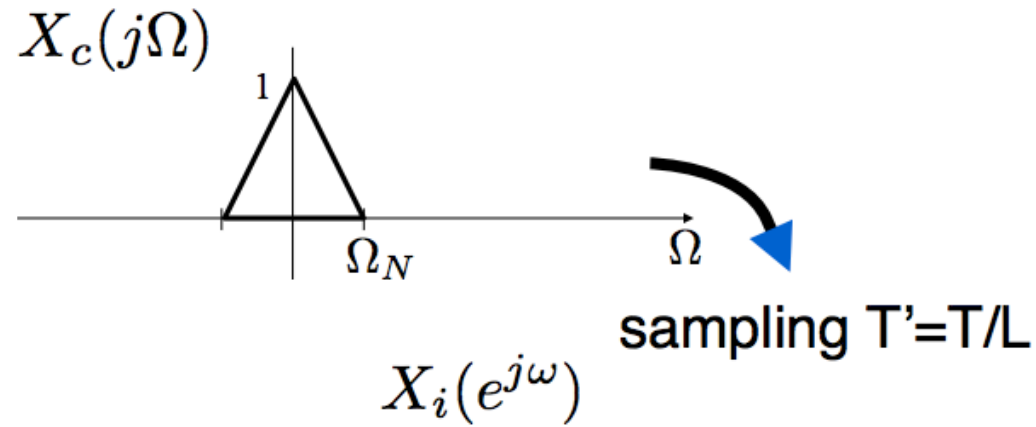
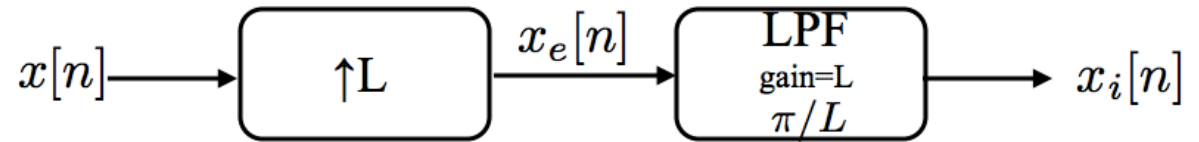


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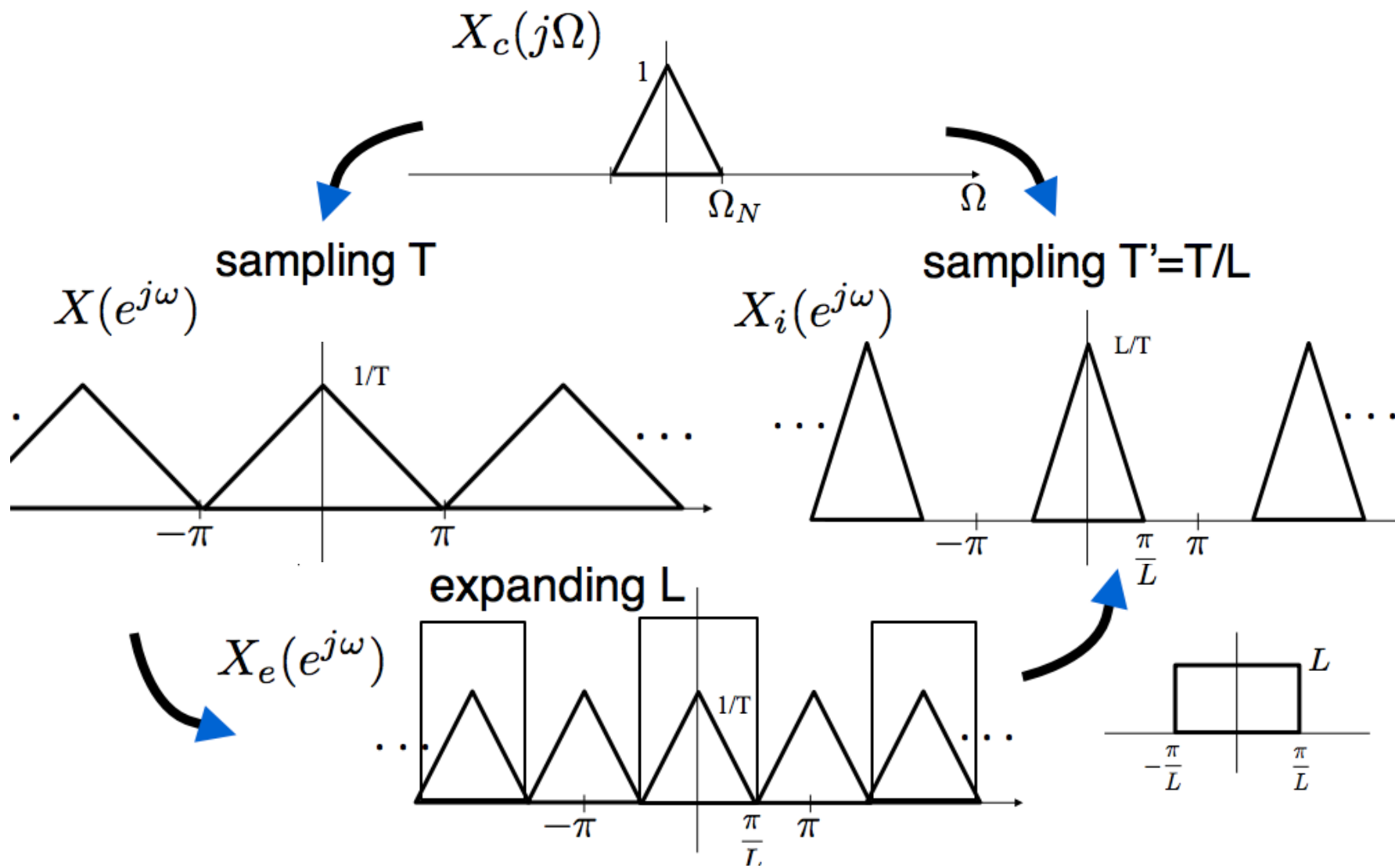
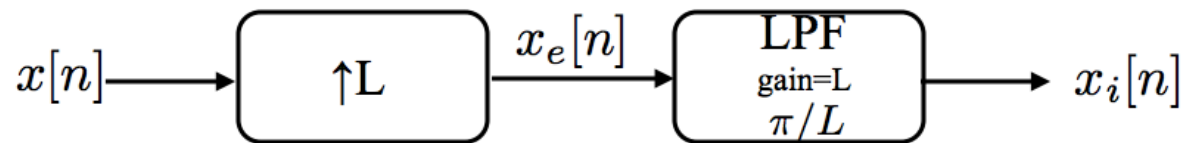




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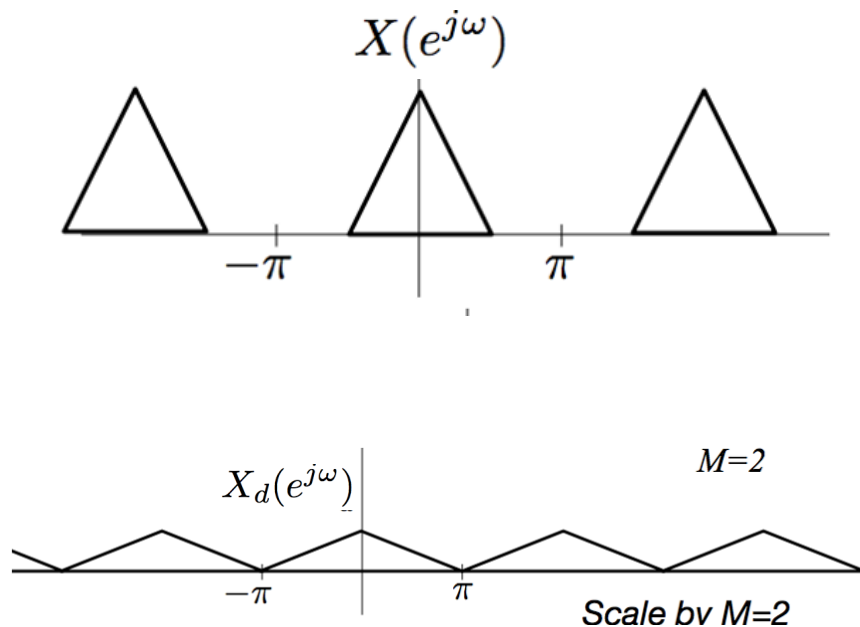


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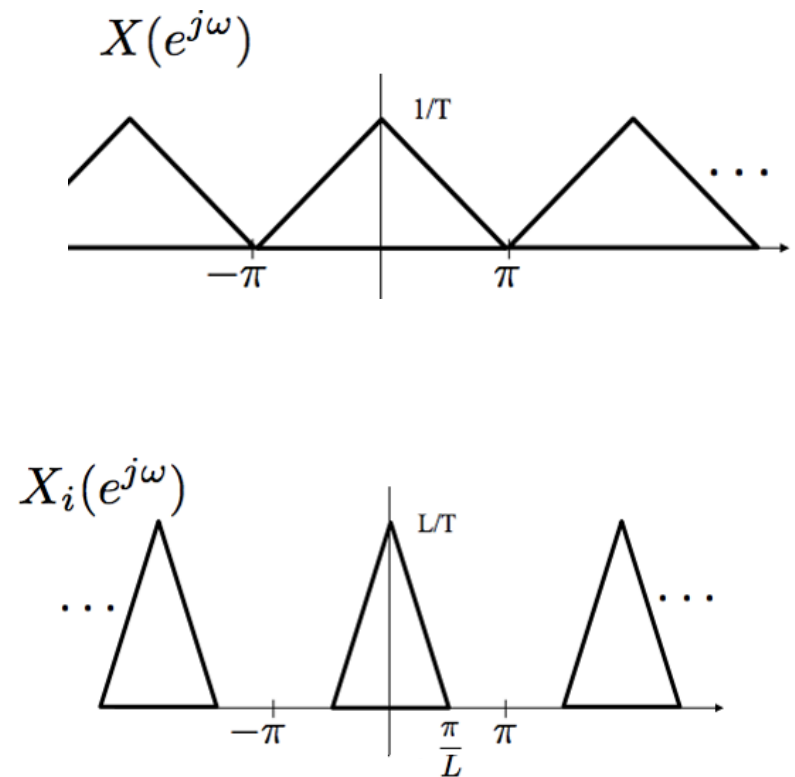


Downsampling vs. Upsampling

Downsampling



Upsampling



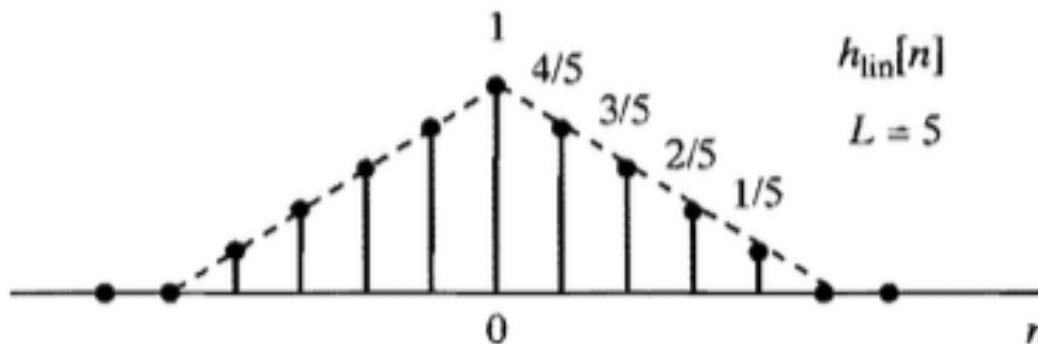
Practical Interpolation



Practical Interpolation

- ❑ Interpolate with simple, practical filters
 - Linear interpolation – samples between original samples fall on a straight line connecting the samples
 - Convolve with triangle instead of sinc

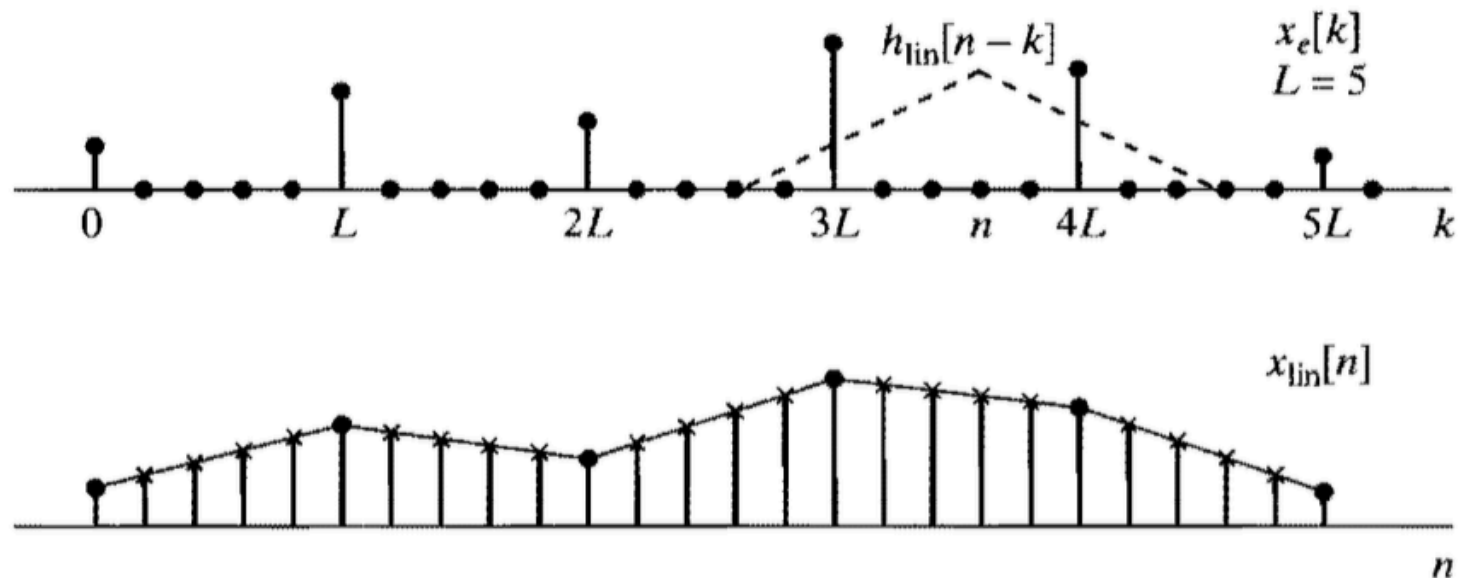
$$h_{\text{lin}}[n] = \begin{cases} 1 - |n|/L, & |n| \leq L, \\ 0, & \text{otherwise,} \end{cases}$$





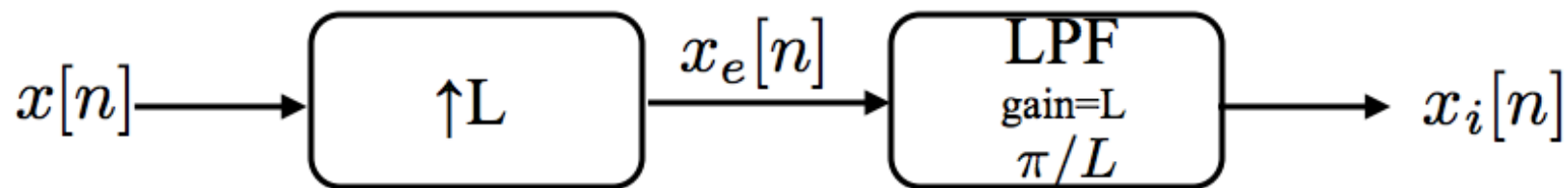
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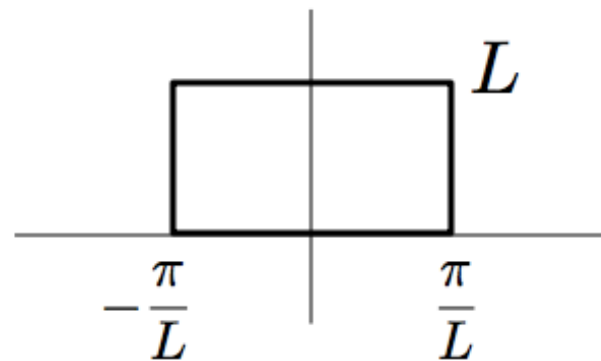
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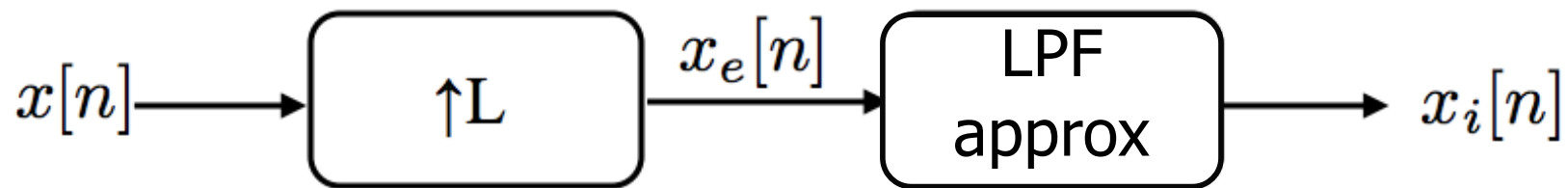
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DTFT $\Rightarrow$

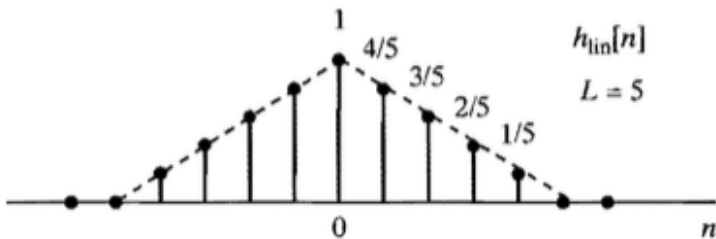


Linear Interpolation -- Frequency Domain

$$x_i[n] = x_e[n] * h_{lin}[n]$$

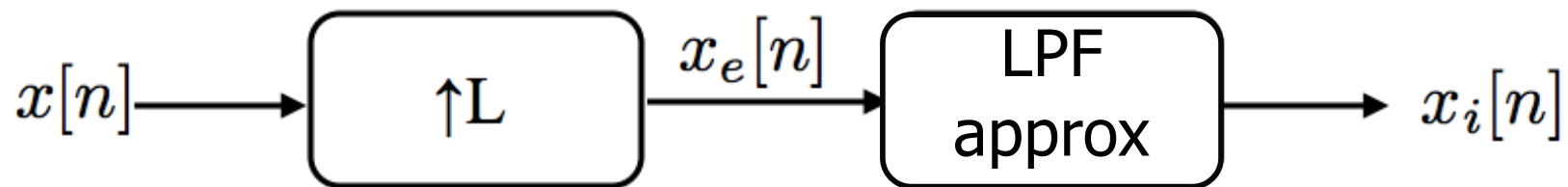


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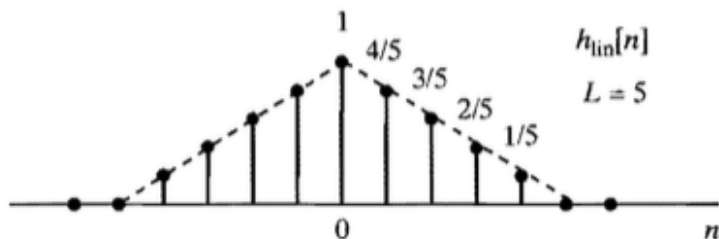


Linear Interpolation -- Frequency Domain

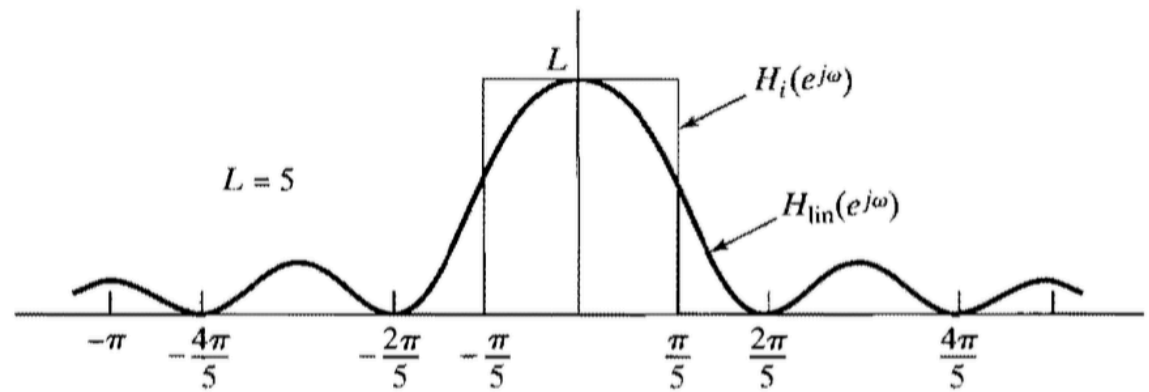
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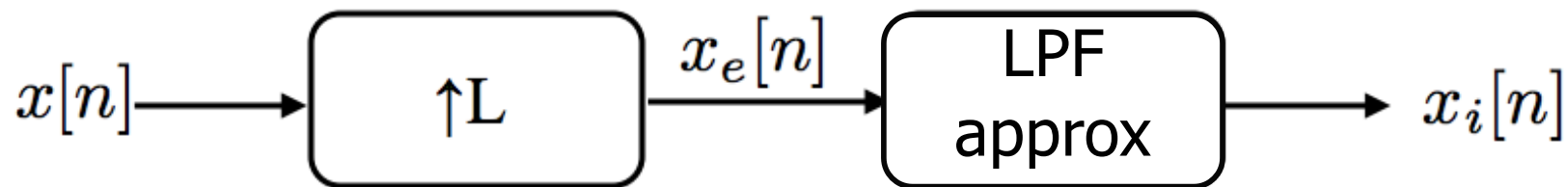


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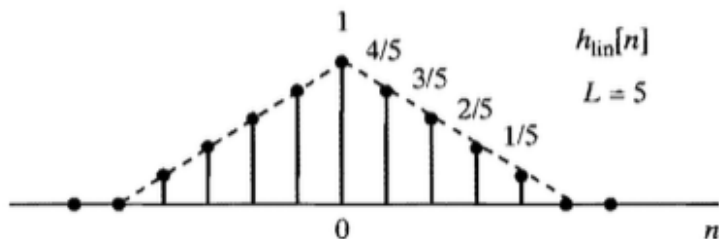


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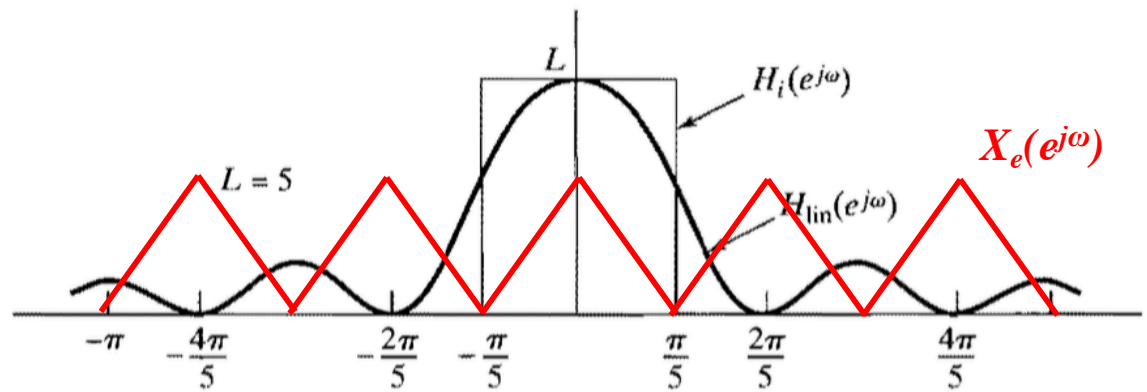
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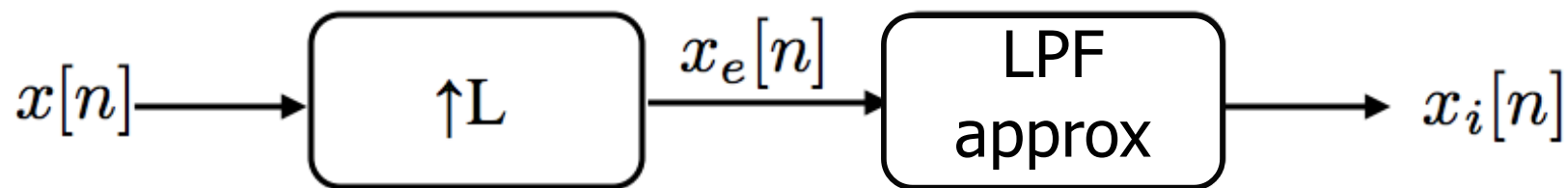


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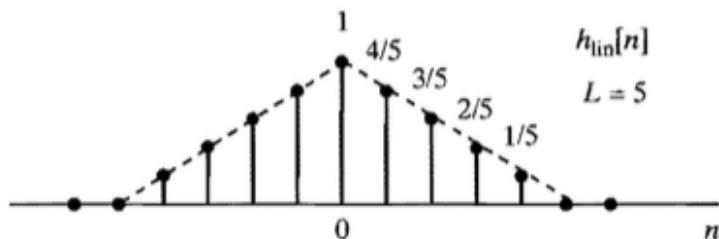


Linear Interpolation -- Frequency Domain

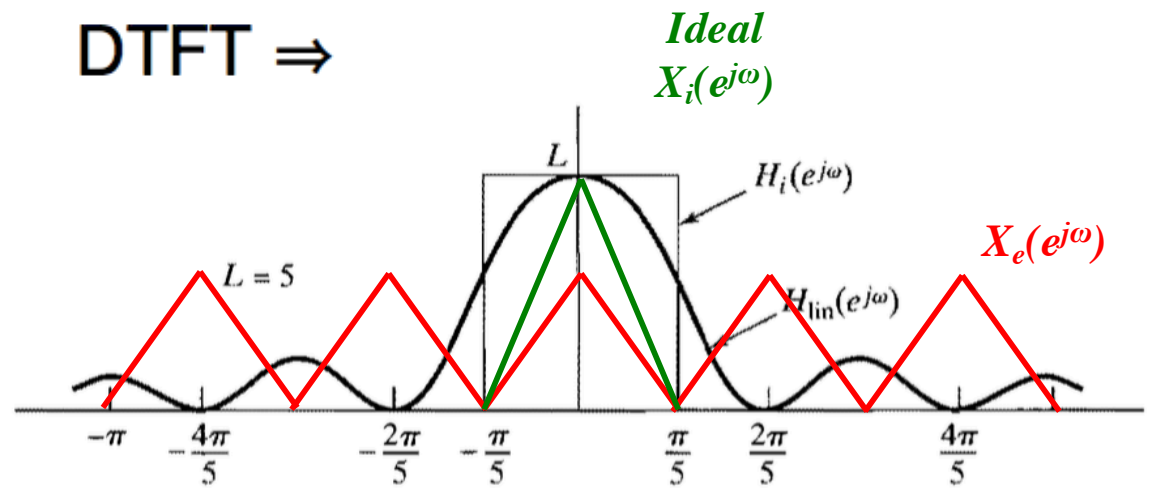
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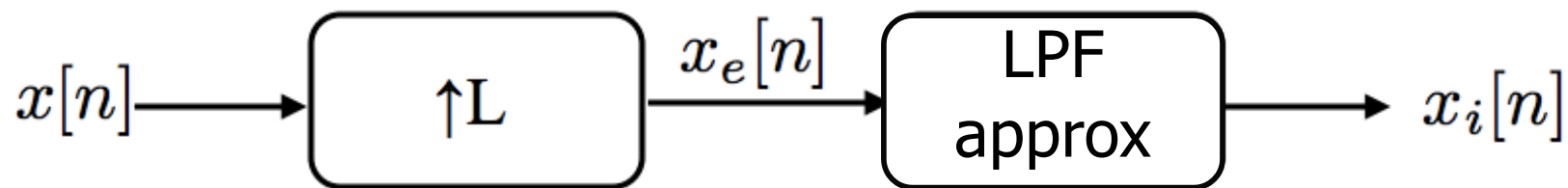


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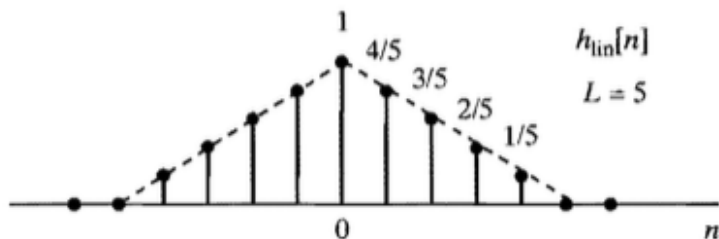


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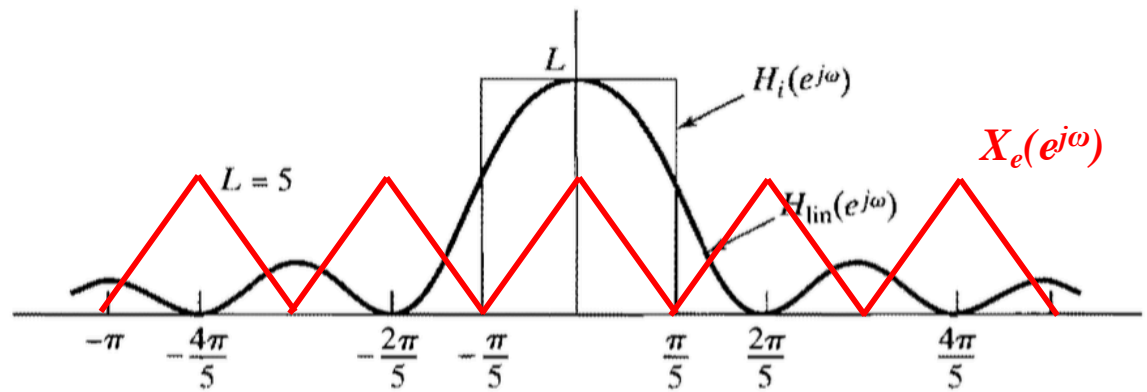
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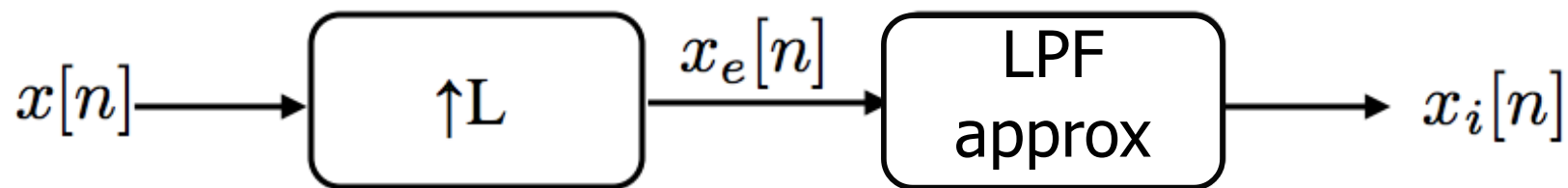


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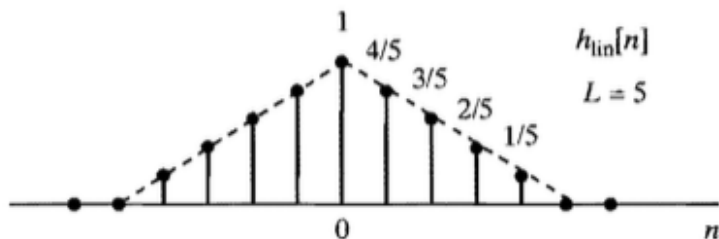


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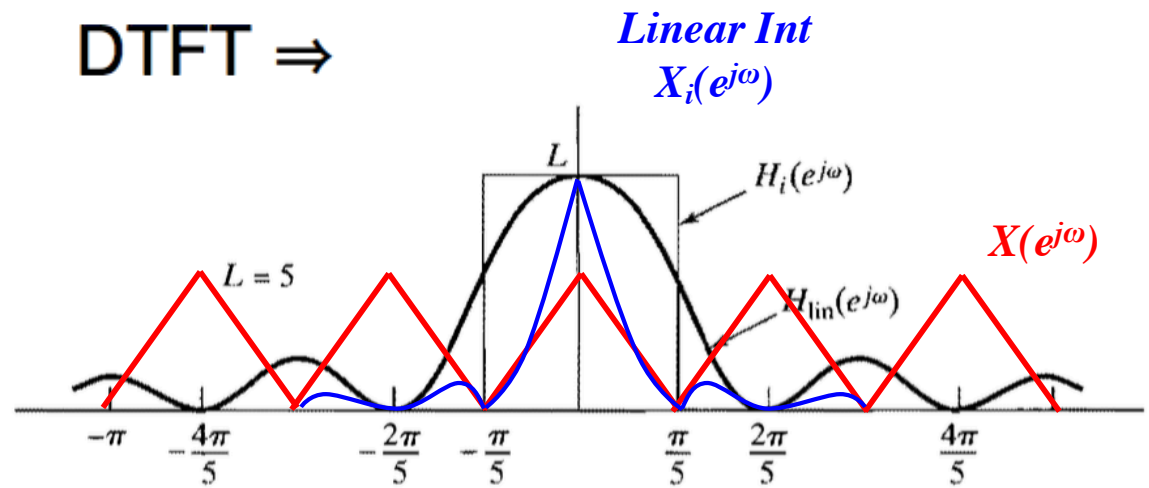
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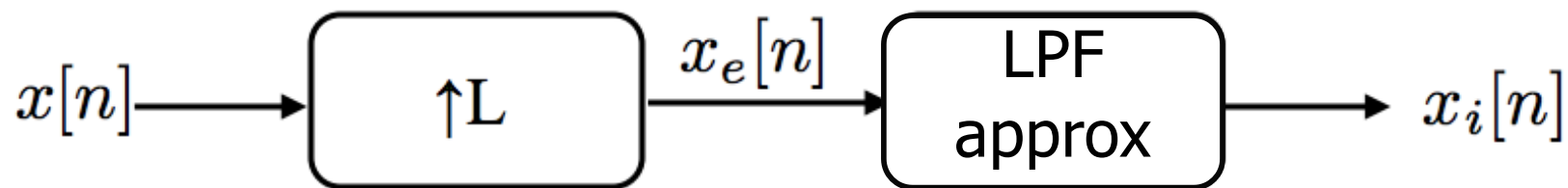


DTFT $\Rightarrow$

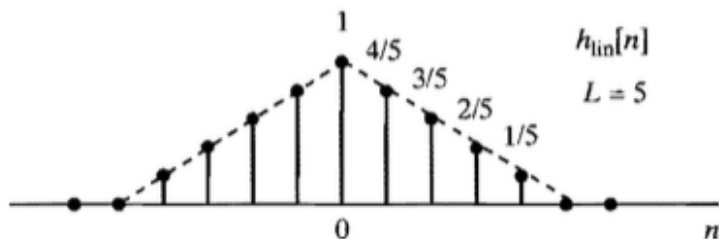


Linear Interpolation -- Frequency Domain

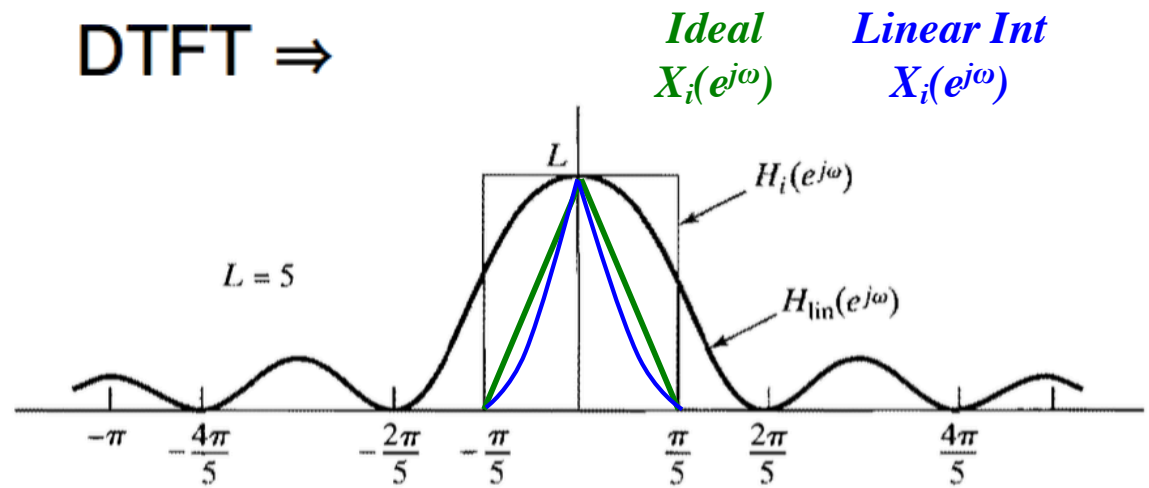
$$x_i[n] = x_e[n] * h_{lin}[n]$$



$$h_{lin}[n] = \begin{cases} 1 - |n|/L, & |n| \leq L, \\ 0, & \text{otherwise,} \end{cases}$$



DTFT $\Rightarrow$





Big Ideas

- ❑ Downsampling
 - Like a C/D converter
- ❑ Upsampling
 - Like a D/C converter
- ❑ Practical Interpolation
 - Linear interpolation
 - Approximate sinc function with triangle



Admin

- ❑ HW 3 due Sunday