**University of Pennsylvania**
**Department of Electrical and System Engineering**
**Digital Signal Processing**

Midterm Thursday, March 16

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- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed. (non cell phone)
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

**Name:**

Grade:

| | |
|---|---|
| Q1 | |
| Q2 | |
| Q3 | |
| Q4 | |
| Total | |

**TABLE 2.3** FOURIER TRANSFORM PAIRS

| Sequence | Fourier Transform |
|---|---|
| 1. $\delta[n]$ | $1$ |
| 2. $\delta[n-n_0]$ | $e^{-j\omega n_0}$ |
| 3. $1 \quad (-\infty < n < \infty)$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$ |
| 4. $a^n u[n] \quad (\|a\|<1)$ | $\frac{1}{1-ae^{-j\omega}}$ |
| 5. $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$ |
| 6. $(n+1)a^n u[n] \quad (\|a\|<1)$ | $\frac{1}{(1-ae^{-j\omega})^2}$ |
| 7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\|<1)$ | $\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$ |
| 8. $\frac{\sin\omega_c n}{\pi n}$ | $X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$ |
| 9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$ | $\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$ |
| 10. $e^{j\omega_0 n}$ | $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$ |
| 11. $\cos(\omega_0 n+\phi)$ | $\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$ |

**TABLE 2.2** FOURIER TRANSFORM THEOREMS

| Sequence | Fourier Transform |
|---|---|
| $x[n]$ | $X(e^{j\omega})$ |
| $y[n]$ | $Y(e^{j\omega})$ |
| 1. $ax[n]+by[n]$ | $aX(e^{j\omega})+bY(e^{j\omega})$ |
| 2. $x[n-n_d] \quad (n_d$ an integer$)$ | $e^{-j\omega n_d}X(e^{j\omega})$ |
| 3. $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 4. $x[-n]$ | $X(e^{-j\omega})$ <br> $X^*(e^{j\omega}) \quad$ if $x[n]$ real. |
| 5. $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| 6. $x[n]*y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 7. $x[n]y[n]$ | $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |

Parseval's theorem:

8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

**TABLE 3.1** SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\|z\| < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m>0$) or $\infty$ (if $m<0$) |
| 5. $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| > \|a\|$ |
| 6. $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\|z\| < \|a\|$ |
| 7. $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| > \|a\|$ |
| 8. $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| < \|a\|$ |
| 9. $\cos(\omega_0 n)u[n]$ | $\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| 10. $\sin(\omega_0 n)u[n]$ | $\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| 11. $r^n\cos(\omega_0 n)u[n]$ | $\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |
| 12. $r^n\sin(\omega_0 n)u[n]$ | $\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$ | $\|z\| > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\frac{1-a^Nz^{-N}}{1-az^{-1}}$ | $\|z\| > 0$ |

**TABLE 3.2** SOME $z$-TRANSFORM PROPERTIES

| Property Number | Section Reference | Sequence | Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R_x$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| 1 | 3.4.1 | $ax_1[n]+bx_2[n]$ | $aX_1(z)+bX_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| 2 | 3.4.2 | $x[n-n_0]$ | $z^{-n_0}X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\|z_0\|R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\frac{dX(z)}{dz}$ | $R_x$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z)+X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{I}m\{x[n]\}$ | $\frac{1}{2j}[X(z)-X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| 9 | 3.4.7 | $x_1[n]*x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |

## Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

## Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1$$

## DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

**Z-Transform Equations:**

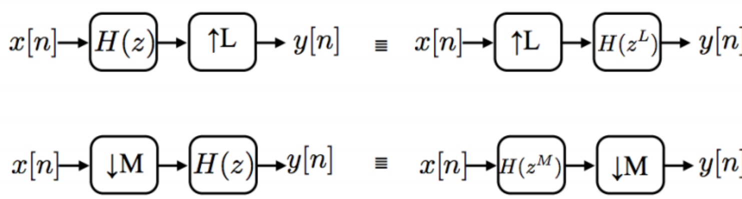
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

**Upsampling/Downsampling:**

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

**Interchange Identities:**



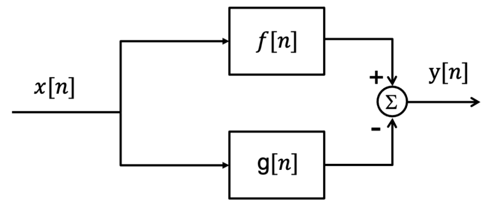
1. (18 points) A frequency of 46 kHz is higher than the normal audible range of 20 Hz to 20 kHz for a human being. Consider a continuous-time signal $x(t) = cos(2\pi f_0 t)$ where $f_0 = 46$ kHz. Sample the signal using a sampling rate of $f_s = 48$ kHz. You can assume ideal sampling.

   (a) Derive a formula for the discrete-time signal $x[n]$ that results from sampling $x(t)$.

   (b) Draw the frequency response, $X(e^{j\omega})$, of the discrete-time signal $x[n]$.

   (c) The discrete-time signal is then put through an ideal reconstruction block. Would the resulting signal be audible? Explain your reasoning.

2. (30 points) An input, $x[n]$, is filtered by a system comprising two stable subsystems, with impulse responses $f[n]$ and $g[n]$, whose outputs are subtracted to form the output, $y[n]$, as shown in the figure below.




Given the above system with $F(z) = \frac{5z-4}{z-\frac{1}{2}}$, and $g[n] = 3 \cdot \left(-\frac{1}{2}\right)^n u[n]$,

(a) find the difference equation relating input $x[n]$ and output $y[n]$.

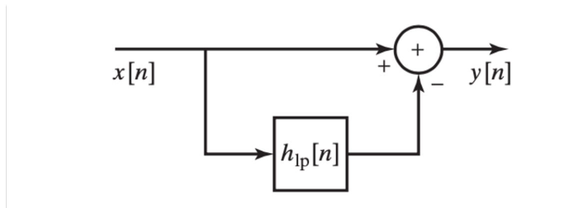
(b) find the impulse response of the entire system.

(c) find the output, $y[n]$, if the input $x[n] = e^{j\frac{\pi}{2}n} + e^{j\frac{\pi}{4}n}$.

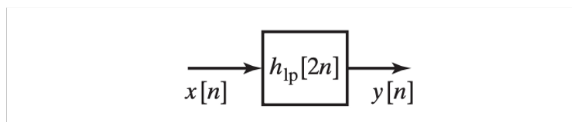
(You may continue the problem on this almost blank page.)

3. (30 points) Let $h_{lp}[n]$ denote the impulse response of an ideal lowpass filter with unity passband gain and cutoff frequency $\omega_c = \frac{\pi}{4}$. The figure below shows four systems, each of which is equivalent to an ideal LTI frequency-selective filter. For each system shown, sketch the equivalent frequency response, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$, indicating explicitly the band-edge frequencies in terms of $\omega_c$ and labeling all axes. In each case, specify whether the system is a lowpass, highpass, bandpass, or bandstop filter.

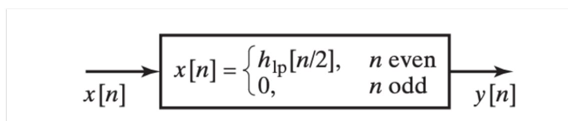
(a)



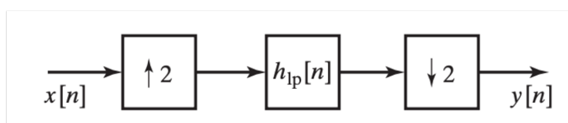

(b)




(c)

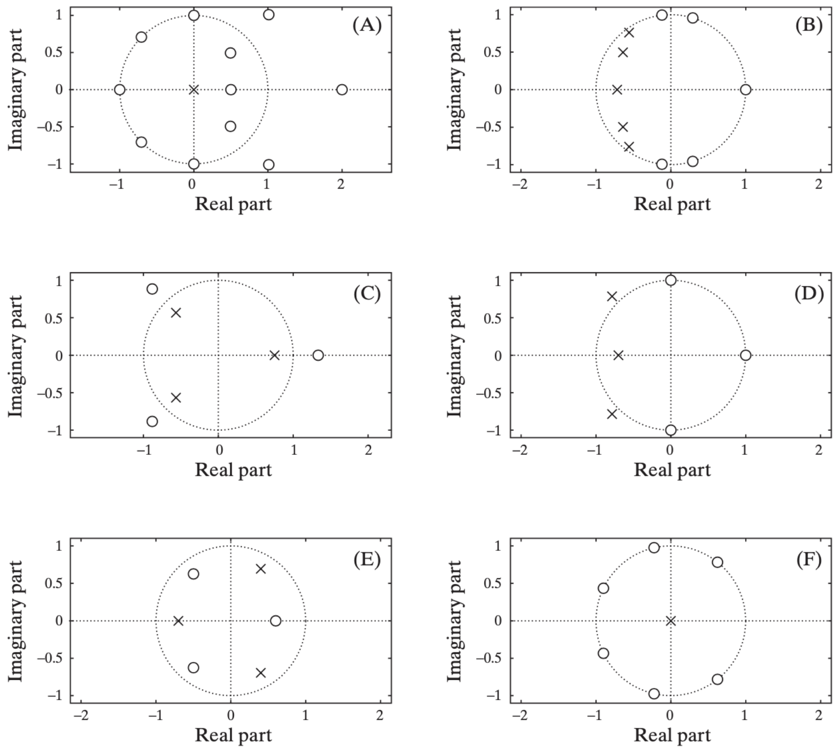



(d)



(You may continue the problem on this almost blank page.)

4. (22 points) The polezero plots below describe six different causal LTI systems.




(a) Answer the following questions about the systems having the above polezero plots. In each case, an acceptable answer could be none or all.

i. Which systems are stable systems?

ii. Which systems are minimum-phase systems?

iii. Which systems have $|H(e^{j\omega})|=$ constant for all $\omega$?

iv. Which systems have corresponding stable and causal inverse systems?

(b) Below is a zoomed in and annotated version of system C. Write the transfer function for this system, $H(z)$, if $H(0) = 1$.