University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Midterm Thursday, March 16

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators allowed. (non cell phone)
- Closed book = No text allowed. One two-sided 8.5x11 cheat sheet allowed.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 58.2, Stdev: 21

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Trigonometric Identity:

$$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$$

Geometric Series:

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r\rvert < 1$$

DTFT Equations:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Z-Transform Equations:

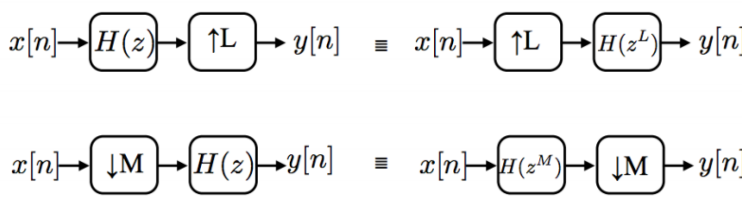
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Upsampling/Downsampling:

Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$

Downsampling by M (↓M): $X_{down} = \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M} i)})$

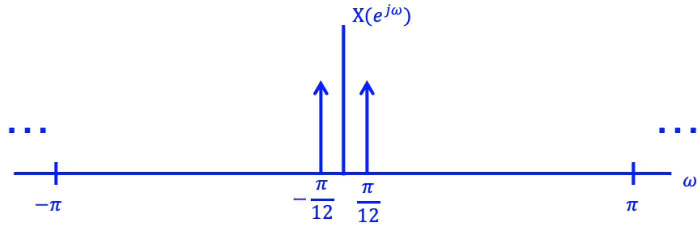
Interchange Identities:



1. (18 points) A frequency of 46 kHz is higher than the normal audible range of 20 Hz to 20 kHz for a human being. Consider a continuous-time signal $x(t) = cos(2\pi f_0 t)$ where $f_0 = 46$ kHz. Sample the signal using a sampling rate of $f_s = 48$ kHz. You can assume ideal sampling.

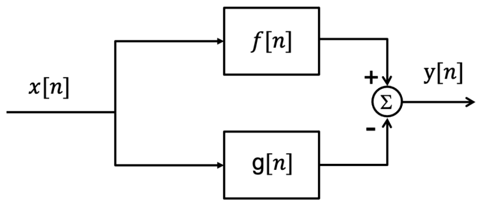
   (a) Derive a formula for the discrete-time signal $x[n]$ that results from sampling $x(t)$.
   $x[n] = cos(2\pi \frac{f_0}{f_s} n) = cos(2\pi \frac{46}{48} n)$

   (b) Draw the frequency response, $X(e^{j\omega})$, of the discrete-time signal $x[n]$.



   (c) The discrete-time signal is then put through an ideal reconstruction block. Would the resulting signal be audible? Explain your reasoning.
   Yes. The ideal reconstruction would place our DT signal at 2kHz in the continuous time reconstruction so we would hear it.

2. (30 points) An input, $x[n]$, is filtered by a system comprising two stable subsystems, with impulse responses $f[n]$ and $g[n]$, whose outputs are subtracted to form the output, $y[n]$, as shown in the figure below.



Given the above system with $F(z) = \frac{5z-4}{z-\frac{1}{2}}$, and $g[n] = 3 \cdot \left(-\frac{1}{2}\right)^n u[n]$,

(a) find the difference equation relating input $x[n]$ and output $y[n]$.

(b) find the impulse response of the entire system.

(c) find the output, $y[n]$, if the input $x[n] = e^{j\frac{\pi}{2}n} + e^{j\frac{\pi}{4}n}$.

a) We can write $y[n] = x[n] * f[n] - x[n] * g[n]$ which we can take the z-transforn of both sides:

$$\begin{aligned}
Y(z) &= X(z) \cdot F(z) - X(z) \cdot G(z) \\
Y(z) &= X(z) \cdot \frac{5z-4}{z-\frac{1}{2}} - X(z) \cdot \frac{3}{1+\frac{1}{2}z^{-1}} \\
Y(z)(z-\frac{1}{2})(1+\frac{1}{2}z^{-1}) &= X(z)(5z-4)(1+\frac{1}{2}z^{-1}) - X(z)(3)(z-\frac{1}{2}) \\
Y(z)(1-\frac{1}{4}z^{-2}) &= X(z)(2-2z^{-2})
\end{aligned}$$

From this we can take the inverse z-transform using the time-delay property:

$$y[n] - \frac{1}{4}y[n-2] = 2x[n] - 2x[n-2]$$

b)

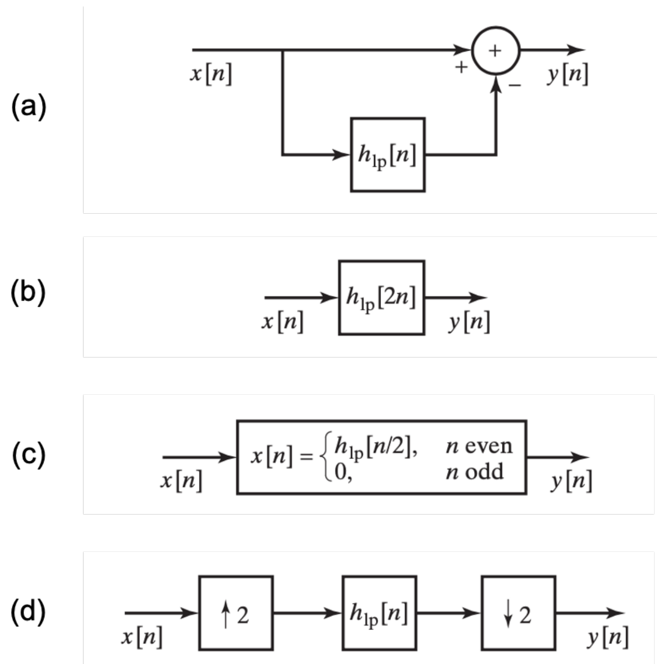
$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} = F(z) - G(z) \\
H(z) &= \frac{5z-4}{z-\frac{1}{2}} - \frac{3}{1+\frac{1}{2}z^{-1}} \\
H(z) &= \frac{5}{1-\frac{1}{2}z^{-1}} - \frac{4z^{-1}}{1-\frac{1}{2}z^{-1}} - \frac{3}{1+\frac{1}{2}z^{-1}}
\end{aligned}$$

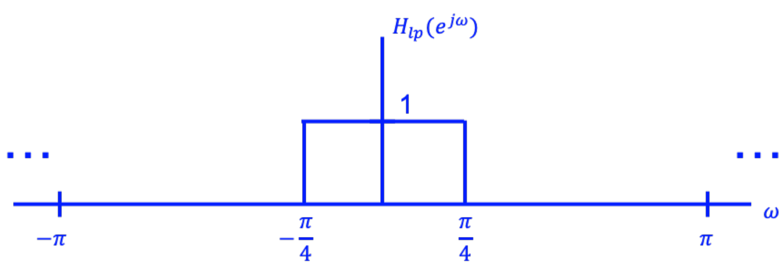
Therefore $h[n] = 5\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{2}\right)^{n-1} u[n-1] - 3\left(-\frac{1}{2}\right)^n u[n]$.

c) $H(e^{j\omega}) = \frac{2-2e^{-2j\omega}}{1-\frac{1}{4}e^{-2j\omega}}$

$$\begin{aligned} y[n] &= H(e^{j\omega})|_{\frac{\pi}{2}} e^{j\frac{\pi}{2}n} + H(e^{j\omega})|_{\frac{\pi}{4}} e^{j\frac{\pi}{4}n} \\ y[n] &= \frac{16}{5} e^{j\frac{\pi}{2}n} + \left(\frac{40}{17} + \frac{24}{17}j\right) e^{j\frac{\pi}{4}n} \\ y[n] &= 3.2 e^{j\frac{\pi}{2}n} + 2.744 e^{j(\frac{\pi}{4}n + 0.54)} \end{aligned}$$

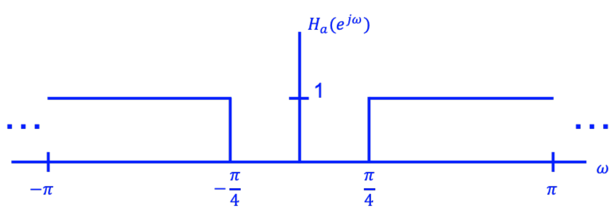
3. (30 points) Let $h_{lp}[n]$ denote the impulse response of an ideal lowpass filter with unity passband gain and cutoff frequency $\omega_c = \frac{\pi}{4}$. The figure below shows four systems, each of which is equivalent to an ideal LTI frequency-selective filter. For each system shown, sketch the equivalent frequency response, $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$, indicating explicitly the band-edge frequencies in terms of ω_c and labeling all axes. In each case, specify whether the system is a lowpass, highpass, bandpass, or bandstop filter.





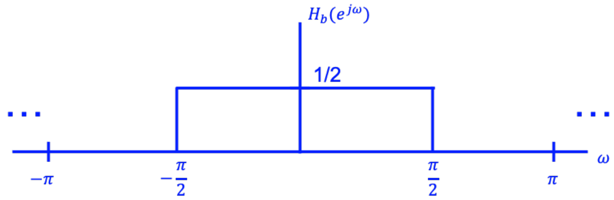
a) $y[n] = x[n] - x[n] * h_{lp}[n]$ and taking the DTFT of both sides:

$$Y(e^{j\omega}) = X(e^{j\omega}) - X(e^{j\omega}) \cdot H_{lp}(e^{j\omega})$$
$$H_a(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$$



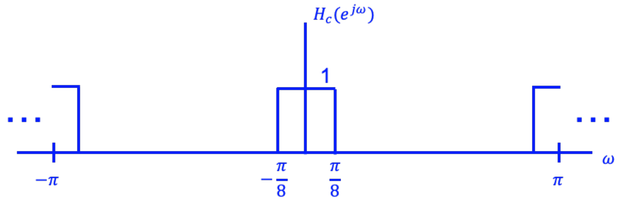
This is a highpass filter.

b) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be stretched by a factor of two and scaled by a factor of $\frac{1}{2}$.



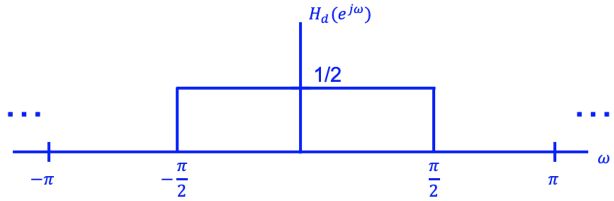
This is a lowpass filter.

c) This system upsamples $h_{lp}[n]$. Therefore, the frequency axis will be squished by a factor of two and the gain is not scaled.



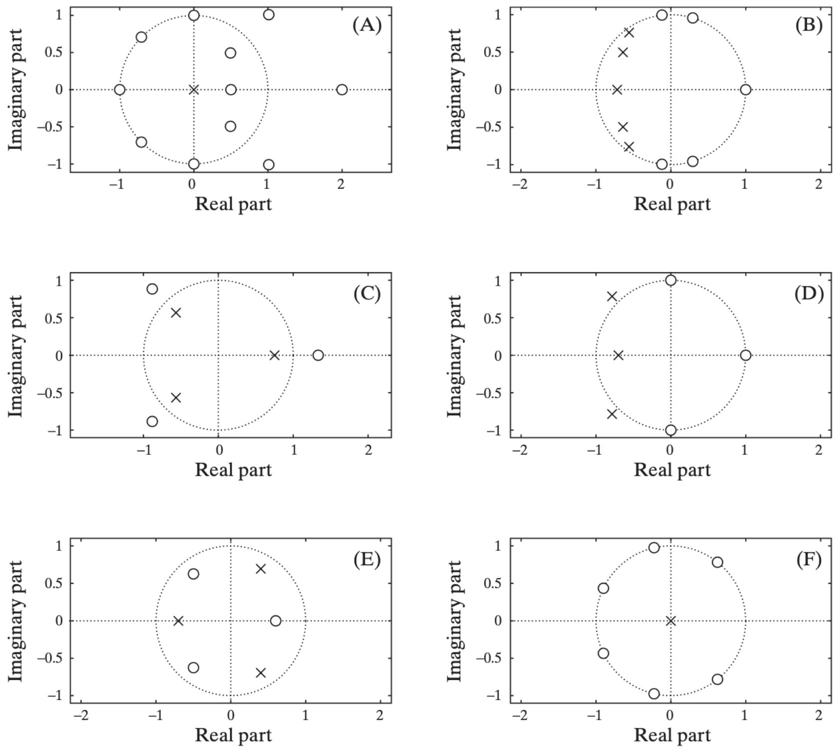
This is a bandstop filter.

d) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of H_{lp}. The downsampling then scales by a factor of two and stretches the frequency response of the filter output signal.



This is a lowpass filter.

4. (22 points) The polezero plots below describe six different causal LTI systems.



(a) Answer the following questions about the systems having the above polezero plots. In each case, an acceptable answer could be none or all.

i. Which systems are stable systems?

A, B, C, E, F

ii. Which systems are minimum-phase systems?

E

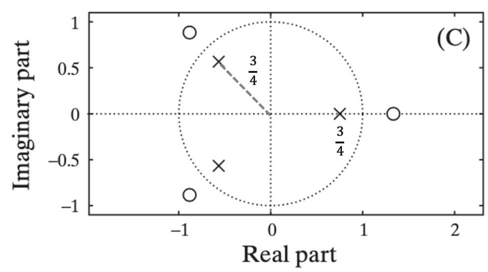
iii. Which systems have $|H(e^{j\omega})|=$ constant for all ω?

C

iv. Which systems have corresponding stable and causal inverse systems?

E

(b) Below is a zoomed in and annotated version of system C. Write the transfer function for this system, $H(z)$, if $H(0) = 1$.



This is an all-pass system, so the zeros are located at the reciprocal locations from the poles. Therefore we can write:

$$H(z) = A \cdot \frac{(z - \frac{4}{3})(z - \frac{4}{3}e^{j\frac{3\pi}{4}})(z - \frac{4}{3}e^{-j\frac{3\pi}{4}})}{(z - \frac{3}{4})(z - \frac{3}{4}e^{j\frac{3\pi}{4}})(z - \frac{3}{4}e^{-j\frac{3\pi}{4}})} \quad (1)$$

With $H(0) = 1$, we can solve $A = (\frac{3}{4})^6$.

Or using the general all-pass formula given in lecture 14:

$$H(z) = A \cdot \frac{(z^{-1} - \frac{3}{4})(z^{-1} - \frac{3}{4}e^{j\frac{3\pi}{4}})(z^{-1} - \frac{3}{4}e^{-j\frac{3\pi}{4}})}{(1 - \frac{3}{4}z^{-1})(1 - \frac{3}{4}e^{j\frac{3\pi}{4}}z^{-1})(1 - \frac{3}{4}e^{-j\frac{3\pi}{4}}z^{-1})} \quad (2)$$

With $H(0) = 1$, we can solve here $A = (-\frac{3}{4})^3$.