University of Pennsylvania
Department of Electrical and System Engineering
Digital Signal Processing

Final

Friday, May 10

- 4 Problems with point weightings shown. All 4 problems must be completed.
- Calculators (non-cellphone) allowed.
- Closed book = No text allowed.
- Two two-sided 8.5x11 cheat sheet allowed.
- All answers and work here.

Name: Answers

Grade:

Q1	
Q2	
Q3	
Q4	
Total	Mean: 66.1, Stdev: 14.4

DTFT Equations:	Z-Transform Equations:
$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ $x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d] \quad$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|X(e^{j\omega})\|^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\|z\| > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

DFT Equations:

N-point DFT of $\{x[n], n = 0,1,...,N-1\}$ is $X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$, for $k = 0,1,...,N-1$

N-point IDFT of $\{X[k], k = 0,1,...,N-1\}$ is $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$, for $n = 0,1,...,N-1$

Trigonometric Identities:	**Geometric Series:**
$e^{j\Theta} = cos(\Theta) + jsin(\Theta)$ $cos(\Theta) = \frac{1}{2}(e^{j\Theta} + e^{-j\Theta})$ $sin(\Theta) = \frac{1}{2j}(e^{j\Theta} - e^{-j\Theta})$	$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$ $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \lvert r \rvert < 1$

Upsampling/Downsampling:

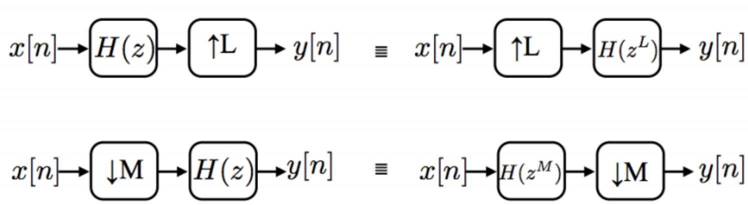
Upsampling by L (↑L): $X_{up} = X(e^{j\omega L})$
Downsampling by M (↓M): $X_{down} = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi}{M}i)})$

Generalized Linear Phase Systems:

	Type I	Type II
Symmetry	Even, $h[n] = h[M-n]$	Even, $h[n] = h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$A(e^{j\omega})e^{-j\omega M/2}$	$A(e^{j\omega})e^{-j\omega M/2}$
$A(e^{j\omega})$	$\sum_{k=0}^{M/2} a[k]cos(\omega k)$ $a[0] = h[M/2]$ $a[k] = 2h[(M/2) - k]$ for $k = 1, 2, ..., M/2$	$\sum_{k=1}^{(M+1)/2} b[k]cos(\omega(k - \frac{1}{2}))$ $b[k] = 2h[(M+1)/2 - k]$ for $k = 1, 2, ..., (M+1)/2$

	Type III	Type IV
Symmetry	Odd, $h[n] = -h[M-n]$	Odd, $h[n] = -h[M-n]$
M	Even	Odd
$H(e^{j\omega})$	$A(e^{j\omega})je^{-j\omega M/2}$	$A(e^{j\omega})je^{-j\omega M/2}$
$A(e^{j\omega})$	$\sum_{k=1}^{M/2} c[k]sin(\omega k)$ $c[k] = 2h[(M/2) - k]$ for $k = 1, 2, ..., M/2$	$\sum_{k=1}^{(M+1)/2} d[k]sin(\omega(k - \frac{1}{2}))$ $d[k] = 2h[(M+1)/2 - k]$ for $k = 1, 2, ..., (M+1)/2$

Interchange Identities:



1. (18 points) Window-based filter design. We want to design a lowpass filter (LPF) with a cutoff frequency of $\omega_c = 0.6$, transition band, $\Delta\omega$, and peak stopband attenuation, δ_s.

   (a) Write the impulse response, $h_{ideal}[n]$, of the ideal LPF with a cutoff frequency of $\omega_c = 0.6$.

   From table 2.3 we can write:

$$h_{ideal}[n] = \frac{sin(0.6n)}{\pi n} \tag{1}$$

   When using windows to design low-pass filters, there is a tradeoff between transition width and peak stopband attenuation. However for increasing filter length, N, the transition band improves while the peak stopband attenuation does not change much and can be fit as a constant value. The table below shows this best fit characteristic for various window types.

Window	$\pi/\boldsymbol{\Delta}\omega$	$\mathbf{max}(\delta_\mathbf{s})$
Rectangular	$0.4069N$	$-21dB$
Hann	$0.1470N$	$-44dB$
Hamming	$0.1398N$	$-54dB$
Gaussian	$0.0891N$	$-60dB$
Blackman	$0.0704N$	$-75dB$

   (b) Which windows can be used to design a window-based LPF with a transition band of 0.15π and peak stopband attenuation less than -50dB?

   Any window with the $max(\delta_s) \leq -50dB$ will work. So we can say Hamming, Gaussian and Blackman.

   (c) Which window would you pick to optimize your filter design? Explain your reasoning.

   Choosing a Hamming window will enable us to meet the transition band requirement with the shortest filter. So we choose a Hamming window of length $N = 48$ when rounding:

$$\frac{\pi}{\Delta\omega} = \frac{1}{0.15} = 0.1398 \cdot N \rightarrow N = 47.69$$

2. (24 points) *Impulse invariance* and the *bilinear transformation* are two methods for designing discrete-time filters. Both methods transform a continuous-time system function $H_c(s)$ into a discrete-time system function $H(z)$. Answer the following questions by indicating which method(s) will yield the desired result. Show justification only for partial credit. For your reference Impulse Invariance and Bilinear Transformation equations are given below:

Impulse Invariance:	**Bilinear Transformation:**
$h[n] = Th_c(nT)$	$H(z) = H_c\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$
$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} H_c\left[j\left(\frac{\omega}{T} + \frac{2\pi}{T}k\right)\right]$	$\omega = 2arctan(\Omega T/2)$

(a) A minimum-phase continuous-time system has all its poles and zeros in the left-half s-plane. If a minimum-phase continuous-time system is transformed into a discrete-time system, which method(s) will result in a minimum-phase discrete-time system?

Bilinear Transformation

Impulse Invariance maps left-half s-plane poles inside the unit circle in the z-plane, however zeros won't necessarily be mapped inside the unit circle. A counter example can be found, for e.g. $H_c(s) = \frac{s+10}{(s+1)(s+2)}$ with T=1.

The bilinear transform maps a pole or zero at $s = s_0$ the same to $z_0 = \frac{1+\frac{T}{2}s_0}{1-\frac{T}{2}s_0}$:

$$|z_0| = \left|\frac{1+\frac{T}{2}s_0}{1-\frac{T}{2}s_0}\right| \tag{2}$$

$$|z_0| = \sqrt{\frac{(1+\frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}{(1-\frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} \tag{3}$$

Since $H_c(s)$ is minimum phase $s_0 = \sigma + j\Omega$ and $\sigma < 0$. Therefore $|z_0| < 1$ and all zeros and poles will be inside the unit circle, so the DT filter will be minimum phase.

(b) If the continuous-time system is an all-pass system, its poles will be at locations s_k in the left-half s-plane, and its zeros will be at corresponding locations $-s_k$ in the right-half s-plane. Which design method(s) will result in an all-pass discrete-time system?

Bilinear Transformation

Impulse invariance can result in aliasing and destroy the all pass nature of the filter. However, bilinear transform only warps the frequency axis and the magnitude is not affected.

(c) Which design method(s) will guarantee that

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

Bilinear Transformation

Again because of aliasing, impulse invariance does not guarantee this in the general case, however bilinear transform maps $\Omega = 0$ to $\omega = 0$ and there is no aliasing so it does guarantee this.

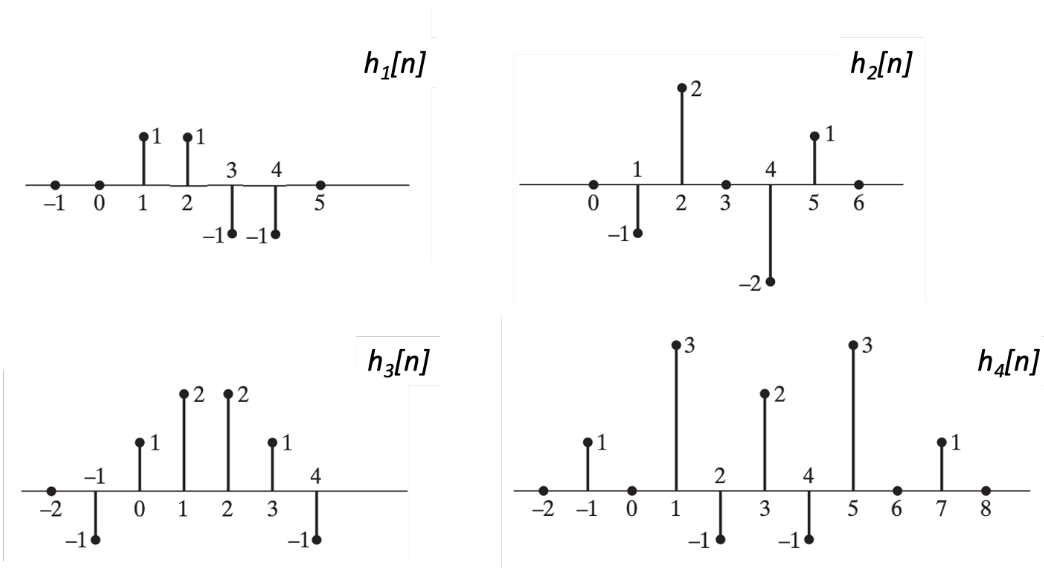
(d) Suppose that $H_1(z)$, $H_2(z)$, and $H(z)$ are transformed versions of $H_{c1}(s)$, $H_{c2}(s)$, and $H_c(s)$, respectively. Which design method(s) will guarantee that $H(z) = H_1(z)H_2(z)$ whenever $H_c(s) = H_{c1}(s)H_{c2}(s)$?

Bilinear Transformation

Impulse invariance may result in aliasing. Since the order of aliasing and multiplication are not interchangeable, the desired identity does not hold. Consider $H_1(s) = H_2(2) = e^{-sT/2}$. By the bilinear transform:

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_{c1}\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)H_{c2}\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

3. (24 pts) Below are the impulse responses of four FIR filters. For each filter determine whether $H(e^{j\omega}) = 0$ for $\omega = 0$ and $\omega = \pi$.



Fill in each box with "yes" or "no".

Filter	$H(e^{j\omega})\|_{\omega=0} = 0?$	$H(e^{j\omega})\|_{\omega=\pi} = 0?$	GLP Type	M	$A(\omega)$
$h_1[n]$	yes	yes	IV	3	$2sin(0.5\omega) + 2sin(1.5\omega)$
$h_2[n]$	yes	yes	III	4	$4sin(\omega) - 2sin(2\omega)$
$h_3[n]$	no	yes	II	5	$4cos(0.5\omega) + 2cos(1.5\omega) - 2cos(2.5\omega)$
$h_4[n]$	no	no	I	8	$2 - 2cos(\omega) + 6cos(2\omega) + 2cos(4\omega)$

These are all GLP filters with a time shift. The time shift does not affect the magnitude, so we can just identify the type and order M of the filter and write $A(\omega)$ from the table given.

4. (34 pts) Spectral analysis via the DFT.

Consider the 32-point signal $x[n] = cos\left(\frac{2\pi(3)}{32}n\right), n = 0, ..., 31$.

(a) Determine an exact expression for the 32-point discrete Fourier transform (DFT) $X[k], k = 0, ..., 31$ of this signal.

(b) Determine the approximate values of k where you would find the peaks in the DFT.

(c) Are there any spectral leakage effects in this case? Why or why not?

(a) The signal is a windowed cosine function so $X[k]$ is the DTFT of the product of the rectangular window and cosine sampled at $\omega = \frac{2\pi}{N}k$, where $N = 32$:

$$\begin{aligned}
X[k] &= DTFT\{w[n] \times x[n]\}, \omega = \frac{2\pi}{N}k, k = 0, ..., 31 \\
X(e^{j\omega}) &= \pi\left[\delta\left(\omega - \frac{2\pi(3)}{32}\right) + \delta\left(\omega + \frac{2\pi(3)}{32}\right)\right], |\omega| \leq \pi \\
W(e^{j\omega}) &= e^{-j\omega(32-1)/2}\frac{sin(32\omega/2)}{sin(\omega/2)} \\
DTFT\{w[n] \times x[n]\} &= \frac{1}{2\pi}X(e^{j\omega}) * W(e^{j\omega}) \\
&= \frac{1}{2}\left[W\left(\omega - \frac{2\pi(3)}{32}\right) + W\left(\omega + \frac{2\pi(3)}{32}\right)\right], |\omega| \leq \pi \\
X[k] &= \frac{1}{2}\left[W\left(\frac{2\pi}{32}k - \frac{2\pi(3)}{32}\right) + W\left(\frac{2\pi}{32}k + \frac{2\pi(3)}{32}\right)\right] \\
&= \frac{1}{2}\left[W\left(\frac{2\pi}{32}(k-3)\right) + W\left(\frac{2\pi}{32}(k+3)\right)\right]
\end{aligned}$$

The first term is only non-zero for $k = 3$, and the second for $k = -3 + 32 = 29$:

$$X[k] = \frac{1}{2}\left(32\delta[k-3] + 32\delta[k-29]\right) = 16\delta[k-3] + 16\delta[k-29]$$

(b) It follows the peaks are at $k = 3$ and $k = 29$

(c) There is no spectral leakage because the frequency of the cosine belongs to the set of frequencies sampled by the DFT.

Now consider the 32-point signal $x[n] = cos\left(\frac{2\pi(7)}{64}n\right), n = 0, ..., 31$.

(d) Determine an exact expression for the 32-point discrete Fourier transform (DFT) $X[k], k = 0, ..., 31$ of this signal. You can use the function for a periodic sinc in your expression, $psinc_N(\omega) = \frac{sin(\omega N/2)}{sin(\omega/2)}$. Be sure to specify the value for N.

(e) Determine the approximate values of k where you would find the peaks in the DFT.

(f) Are there any spectral leakage effects in this case? Why or why not?

(a) The signal is a windowed cosine function so $X[k]$ is the DTFT of the product of the rectangular window and cosine sampled at $\omega = \frac{2\pi}{N}k$, where $N = 32$:

$$
\begin{aligned}
X[k] &= DTFT\{w[n] \times x[n]\}, \omega = \frac{2\pi}{N}k, k = 0, ..., 31 \\
X(e^{j\omega}) &= \pi\left[\delta\left(\omega - \frac{2\pi(7)}{64}\right) + \delta\left(\omega + \frac{2\pi(7)}{64}\right)\right], |\omega| \leq \pi \\
W(e^{j\omega}) &= e^{-j\omega(32-1)/2}\frac{sin(32\omega/2)}{sin(\omega/2)} = e^{-j\omega 31/2} psinc_{32}(\omega) \\
DTFT\{w[n] \times x[n]\} &= \frac{1}{2\pi}X(e^{j\omega}) * W(e^{j\omega}) \\
&= \frac{1}{2}\left[W\left(\omega - \frac{2\pi(7)}{64}\right) + W\left(\omega + \frac{2\pi(7)}{64}\right)\right], |\omega| \leq \pi \\
X[k] &= \frac{1}{2}\left[W\left(\frac{2\pi}{32}k - \frac{2\pi(7)}{64}\right) + W\left(\frac{2\pi}{32}k + \frac{2\pi(7)}{64}\right)\right] \\
&= \frac{1}{2}\left[W\left(\frac{2\pi}{32}(k - \frac{7}{2})\right) + W\left(\frac{2\pi}{32}(k + \frac{7}{2})\right)\right] \\
&= \frac{1}{2}e^{-j\omega 31/2}\left[psinc_{32}\left(\frac{2\pi}{32}(k - \frac{7}{2})\right) + psinc_{32}\left(\frac{2\pi}{32}(k + \frac{7}{2})\right)\right]
\end{aligned}
$$

(b) It follows the peaks are at $k = 7/2 = 3.5$ and $k = -7/2 + 32 = 28.5$

(c) There is now spectral leakage because the frequency of the cosine does not belong to the set of frequencies sampled by the DFT. This can be noted because we see that the peak frequencies occur at non-integer k values.