ESE 5310: Digital Signal Processing

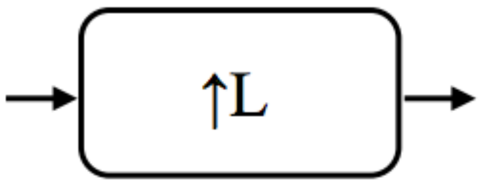
Lecture 11: February 22, 2024

Polyphase Decomposition and Multi-rate Filter Banks

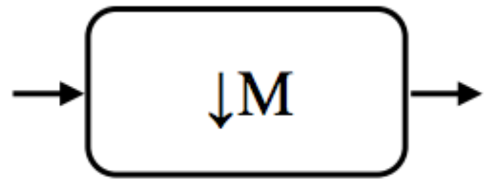
Lecture Outline

- Multi-Rate Filter Banks
- Polyphase Decomposition
- Haar Filter Example

Expander and Compressor



"expander"



"compressor"

<u>not LTI!</u>

Upsampling
-expanding in time
-compressing in frequency

Downsampling
-compressing in time
-expanding in frequency

Interchanging Operations - Summary

Filter and expander ≡ **Expander and expanded filter***

$$x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

Compressor and filter ≡ **Expanded filter* and compressor**

*Expanded filter = expanded impulse response, compressed freq response

Interchanging Operations - Summary

Compressed filter* and expander

Expander and filter

$$x[n] \rightarrow \boxed{H(z^{-L})} \rightarrow \boxed{\uparrow L} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z)} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z^{-M})} \rightarrow y[n] \quad \equiv \quad x[n] \rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow M} \rightarrow y[n]$$

Compressor and Compressed filter*

Filter and compressor

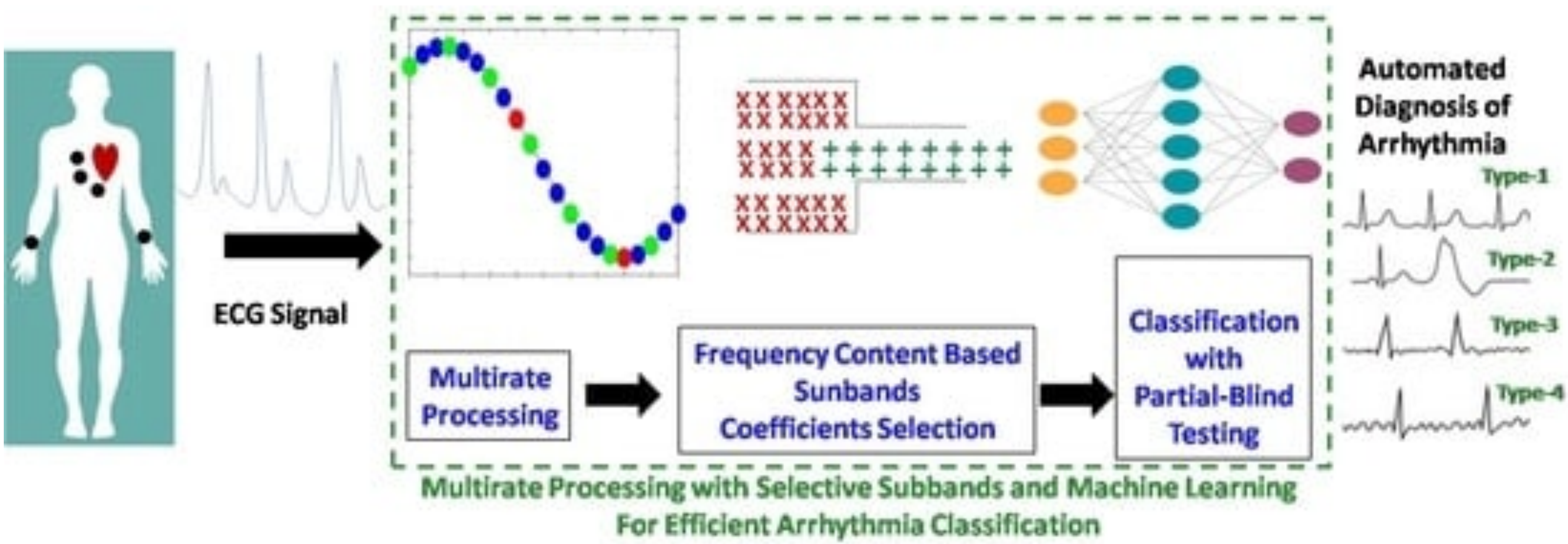
*Compressed filter = compressed impulse response, expanded freq response

Motivation

- Multirate DSP finds application in communications, speech processing, image compression, antenna systems, analog voice privacy systems, and in the digital audio industry to enable efficient processing
  - subband coding of waveforms
  - voice privacy systems
  - integral and fractional sampling rate conversion (such as in digital audio)
  - digital crossover networks
  - multirate coding of narrow-band filter coefficients.

P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial," in Proceedings of the IEEE, vol. 78, no. 1, pp. 56-93, Jan. 1990, doi: 10.1109/5.52200.

Example: Arrhythmia Detection



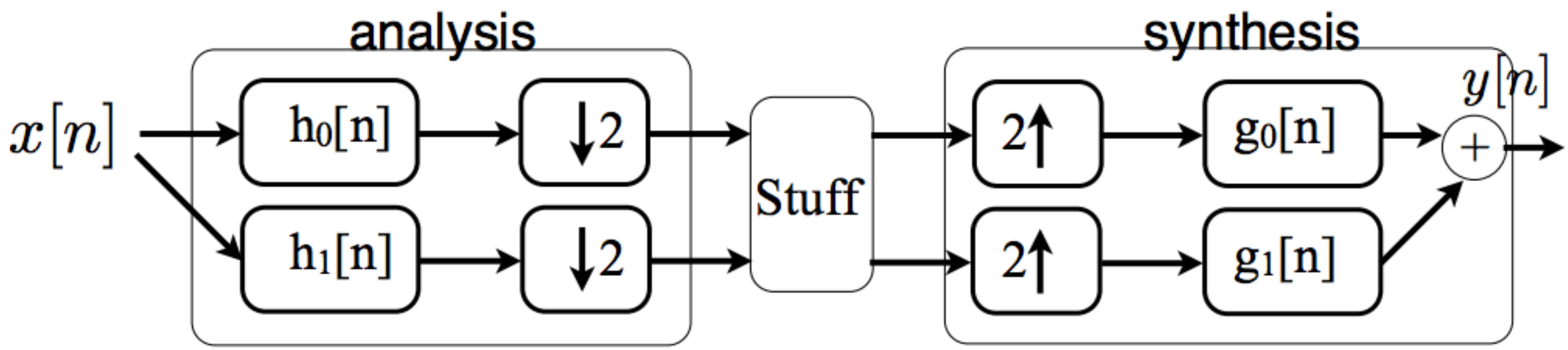
Multi-Rate Filter Banks

- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

Multi-Rate Filter Banks

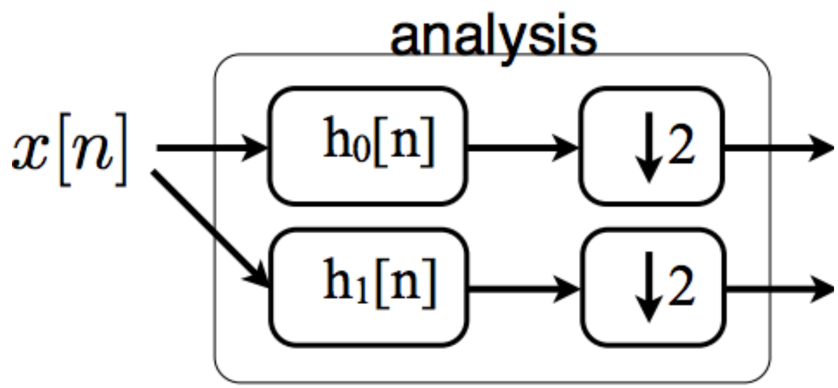
- Use filter banks to operate on a signal differently in different frequency bands
  - To save computation, reduce the rate after filtering

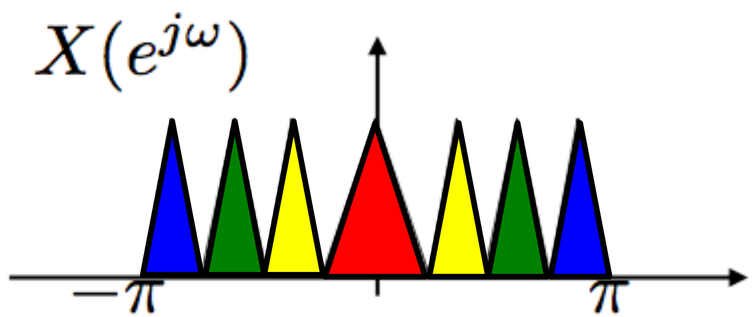
- $h_0[n]$ is low-pass, $h_1[n]$ is high-pass
  - Often $h_1[n]=e^{j\pi n}h_0[n]$ ← shift freq resp by π



Multi-Rate Filter Banks: Analysis

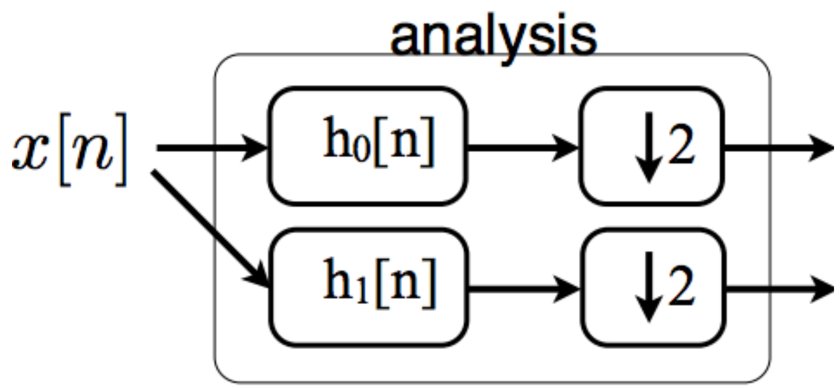
- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$

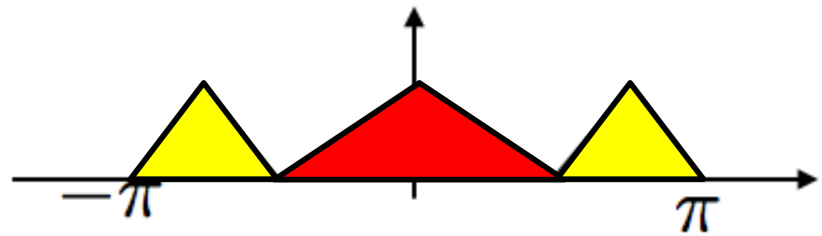


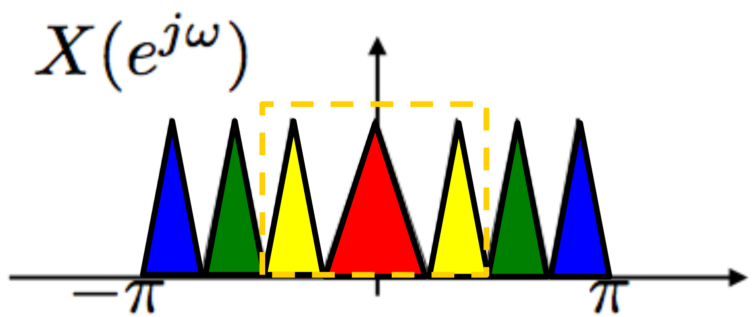


Multi-Rate Filter Banks: Analysis

- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$

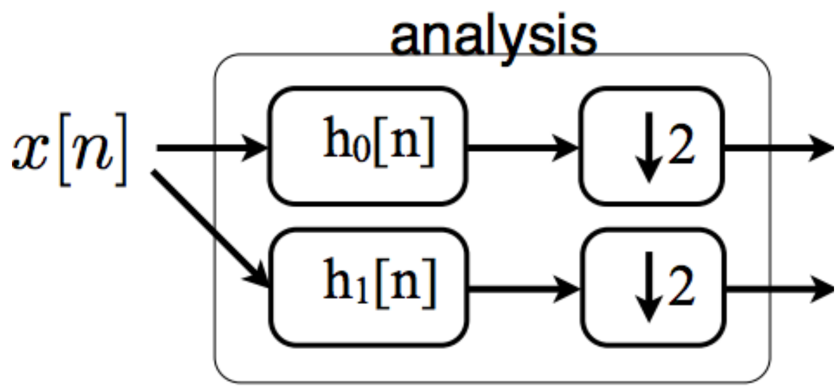


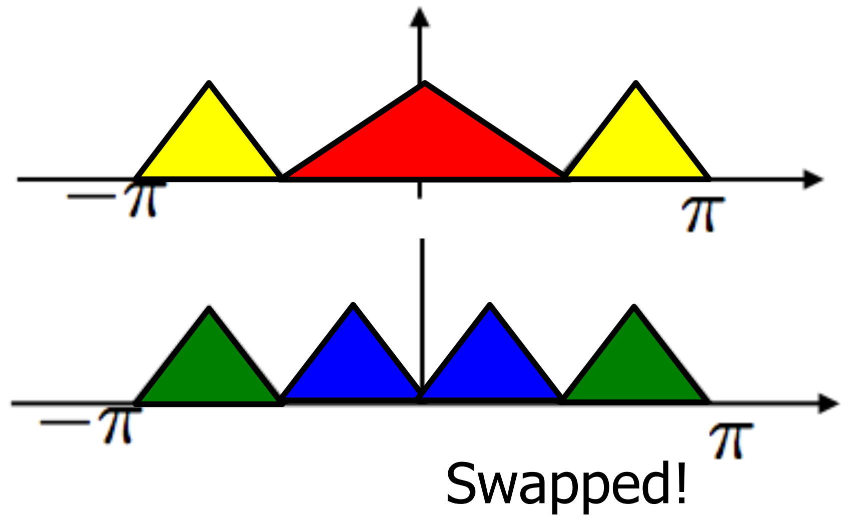


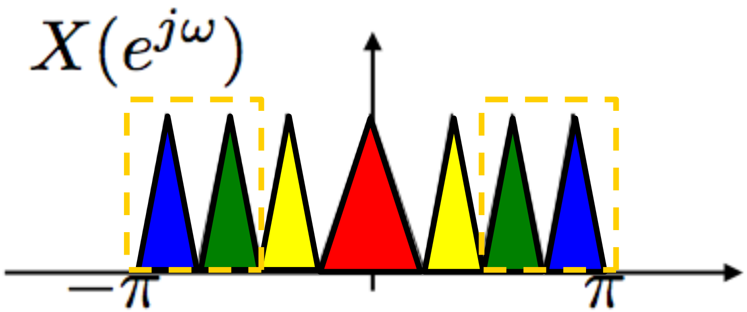


Multi-Rate Filter Banks: Analysis

- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$

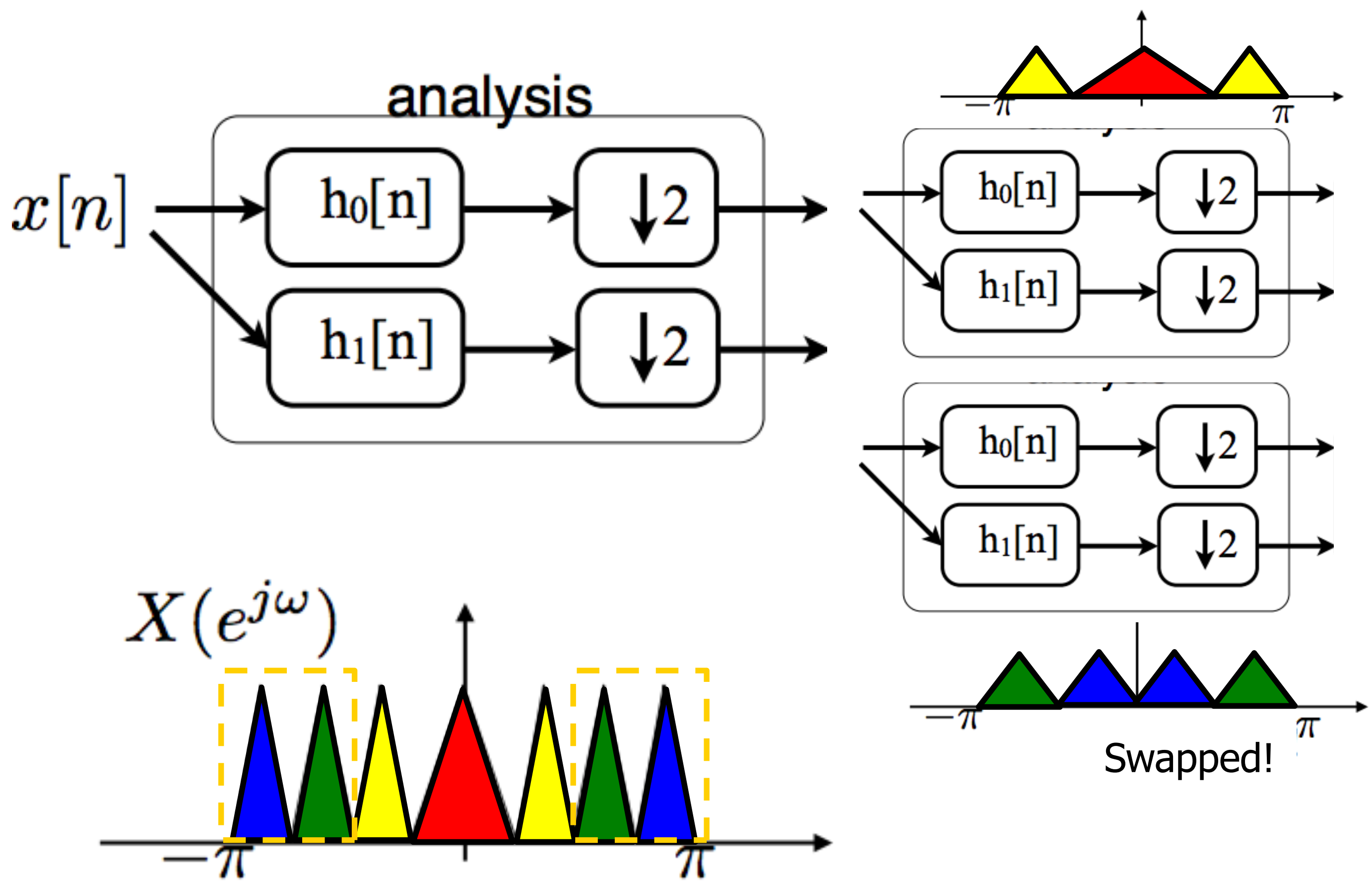






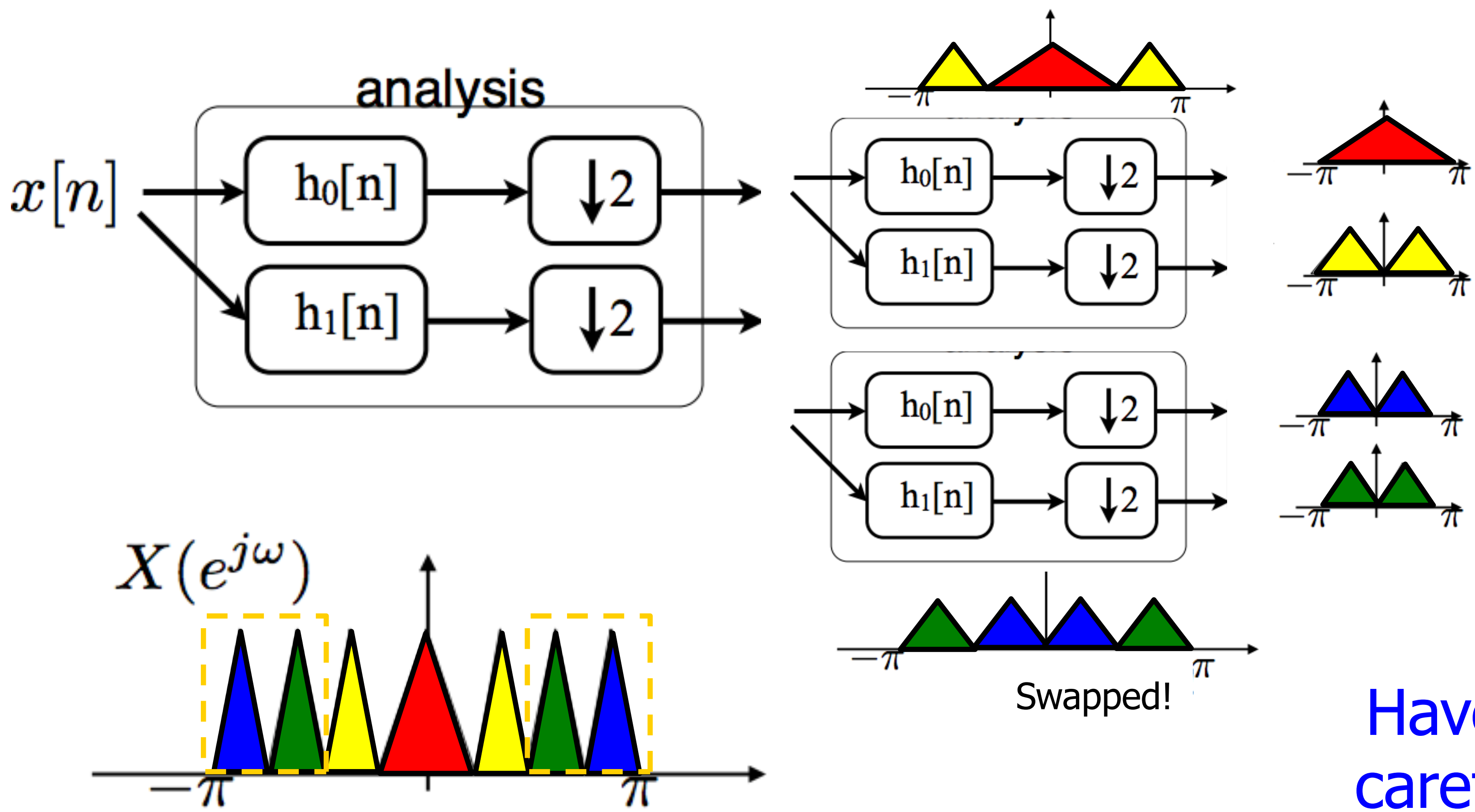
Multi-Rate Filter Banks: Analysis

- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$



Multi-Rate Filter Banks: Analysis

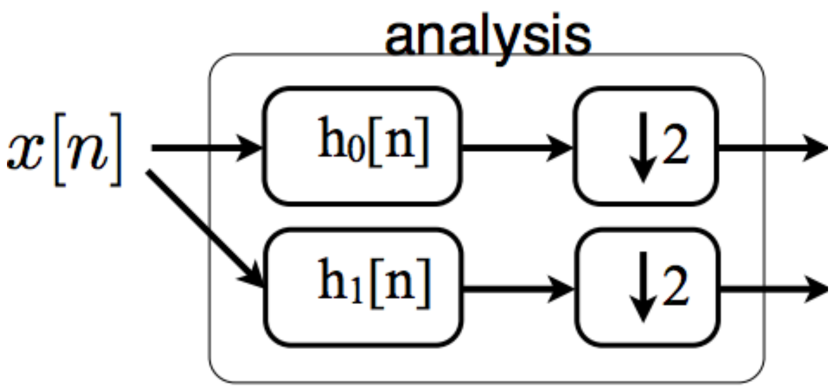
- Assume h_0, h_1 are ideal low/high pass with $\omega_C=\pi/2$

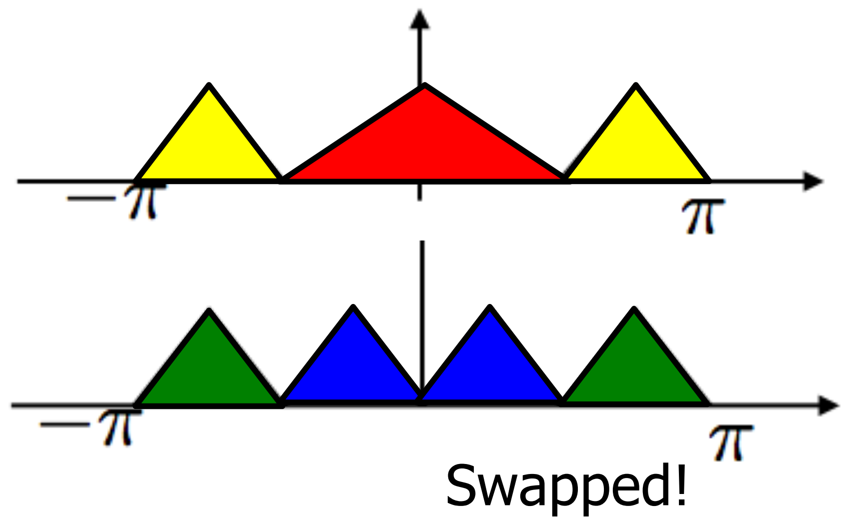


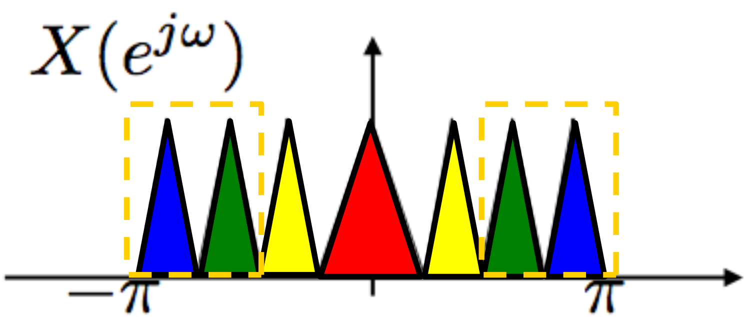
Have to be careful with order!

Multi-Rate Filter Banks: Analysis

- Assume h_0, h_1 are ideal low/high pass with $\omega_C = \pi/2$

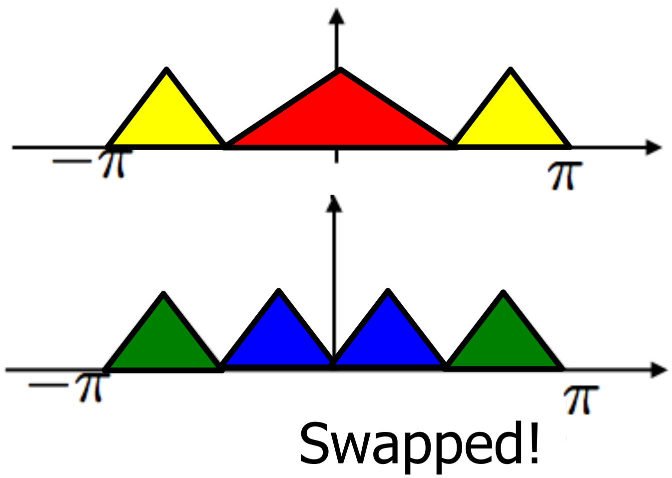


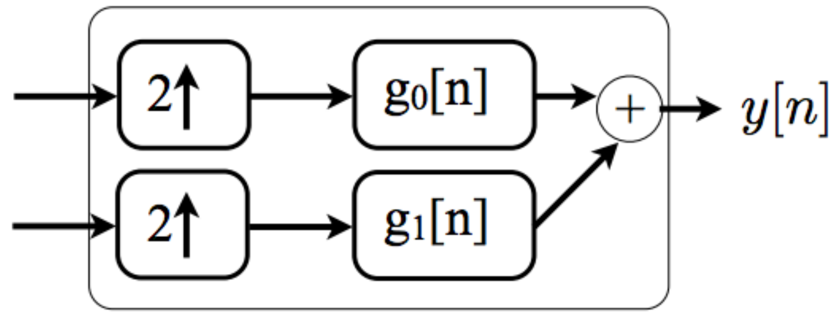




Multi-Rate Filter Banks: Synthesis

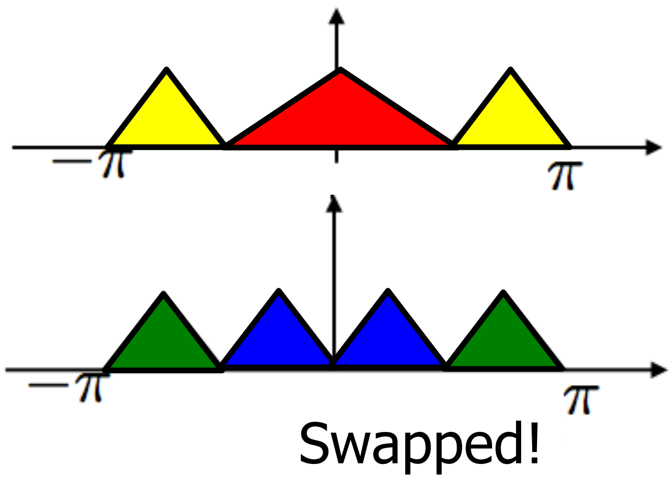
- Assume g_0, g_1 are ideal low/high pass with $\omega_C=\pi/2$

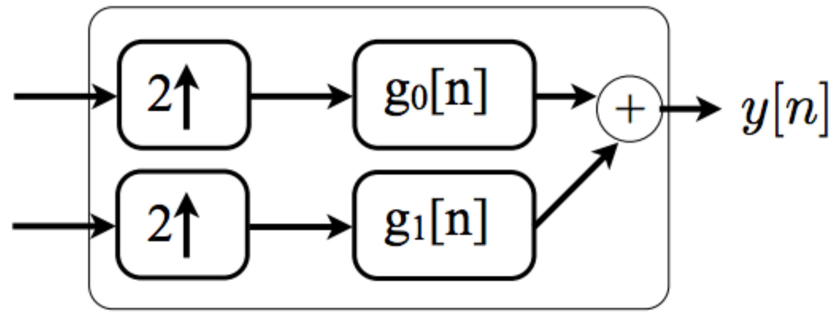


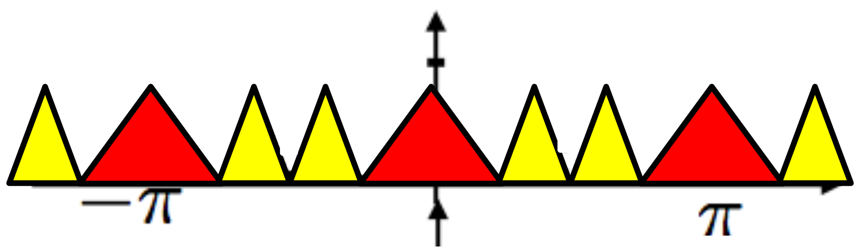


Multi-Rate Filter Banks: Synthesis

- Assume g_0, g_1 are ideal low/high pass with $\omega_C=\pi/2$

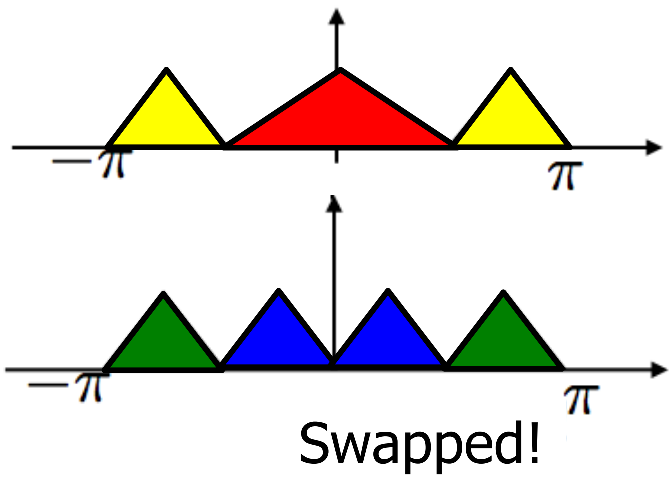


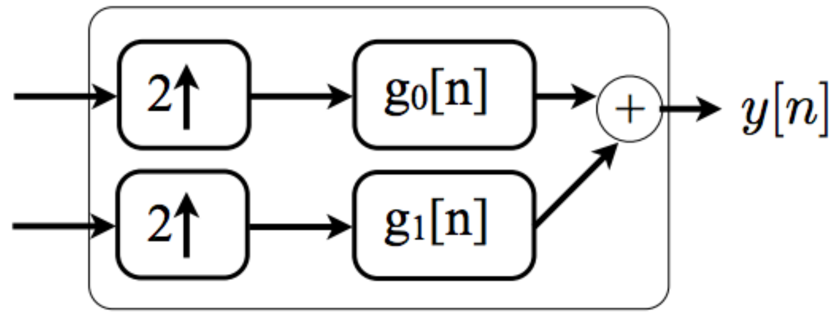


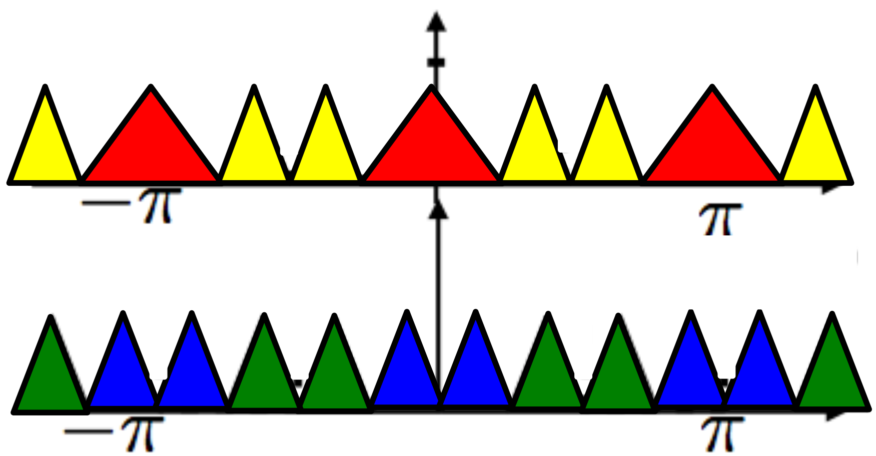


Multi-Rate Filter Banks: Synthesis

- Assume g_0, g_1 are ideal low/high pass with $\omega_C = \pi/2$

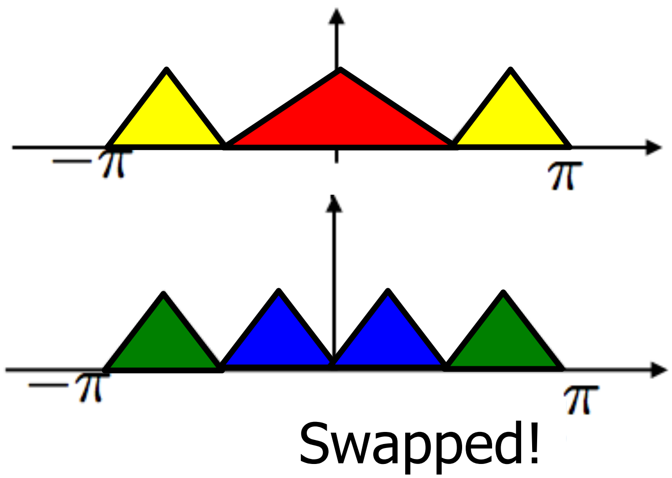


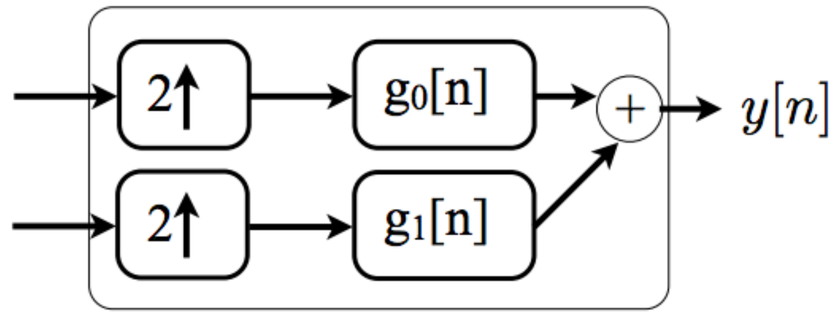


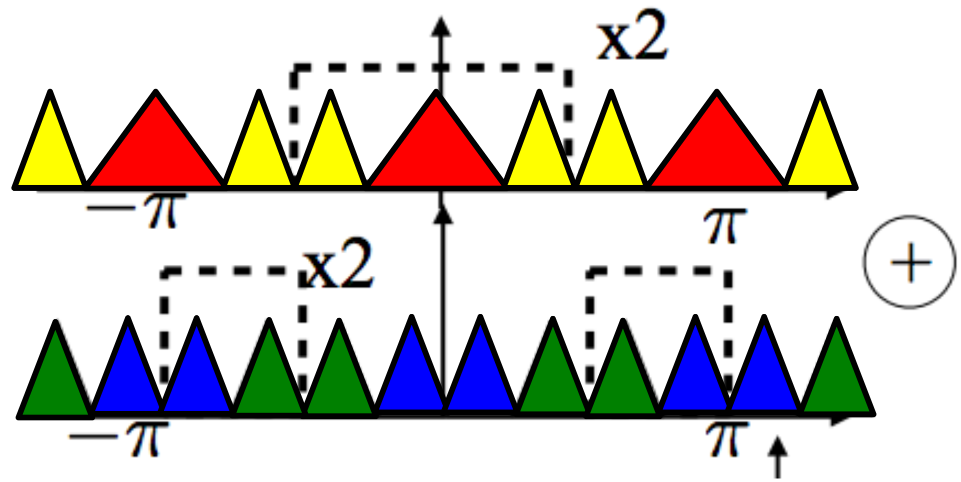


Multi-Rate Filter Banks: Synthesis

- Assume g_0, g_1 are ideal low/high pass with $\omega_C=\pi/2$

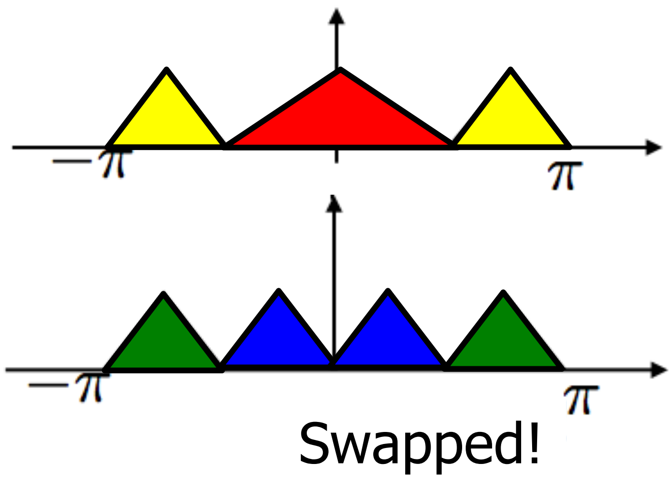


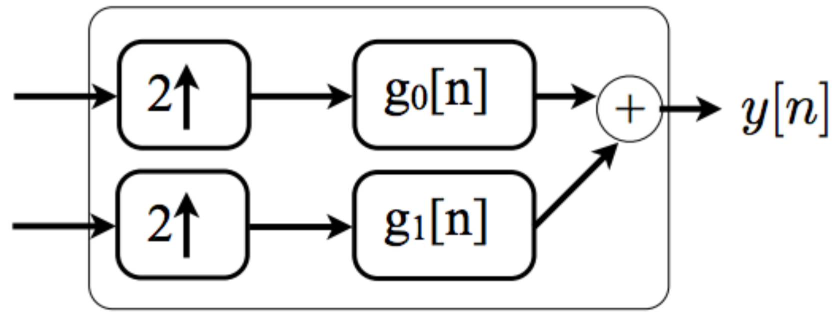


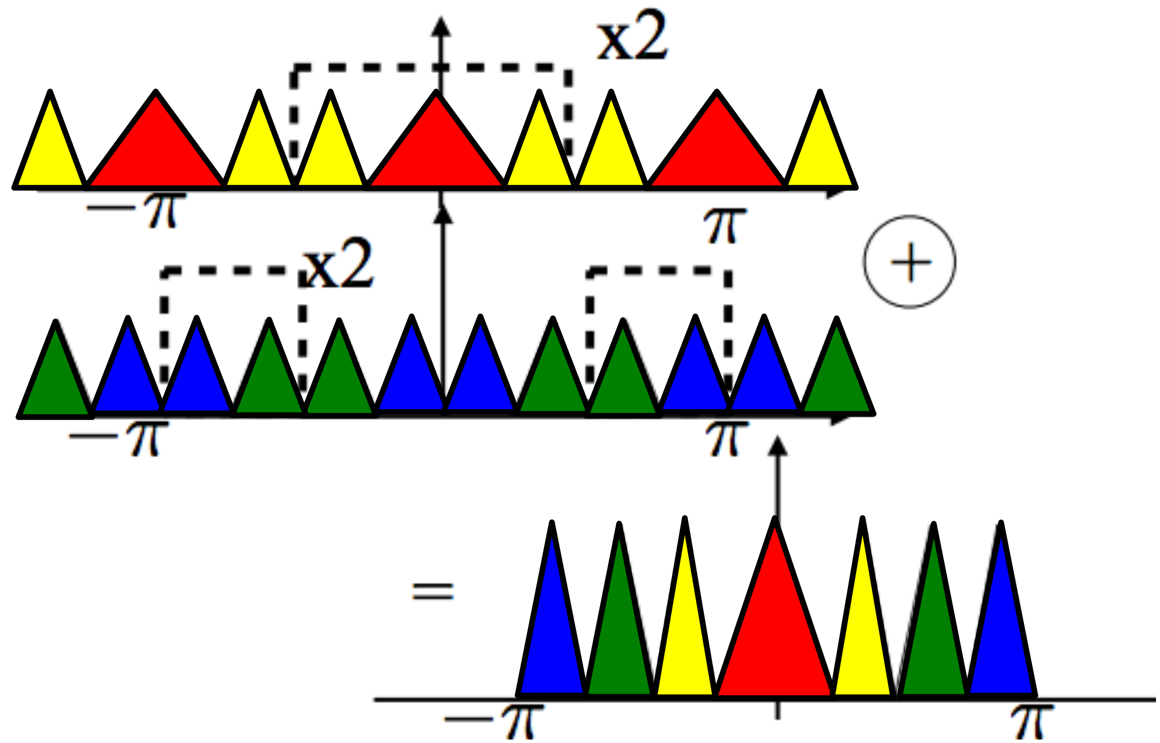


Multi-Rate Filter Banks: Synthesis

- Assume g_0, g_1 are ideal low/high pass with $\omega_C=\pi/2$

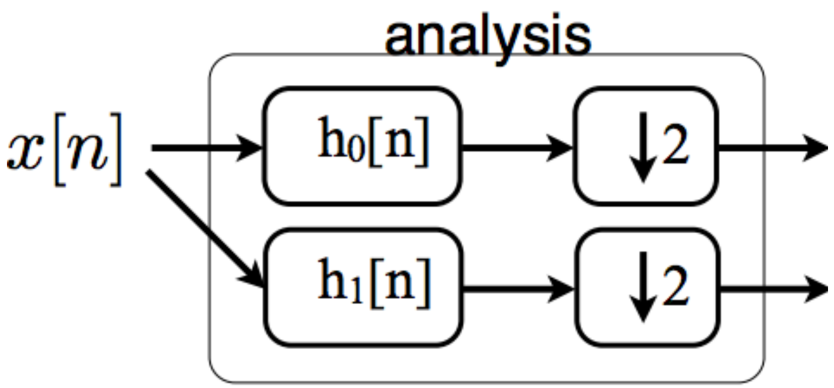


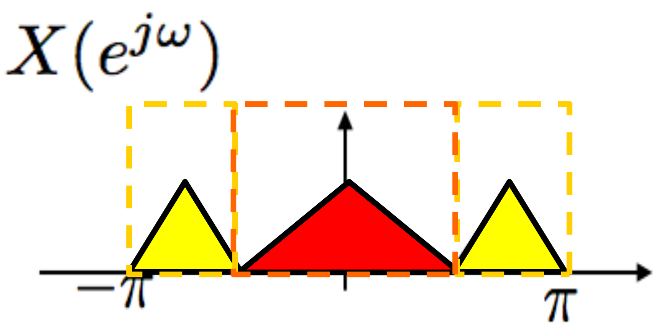


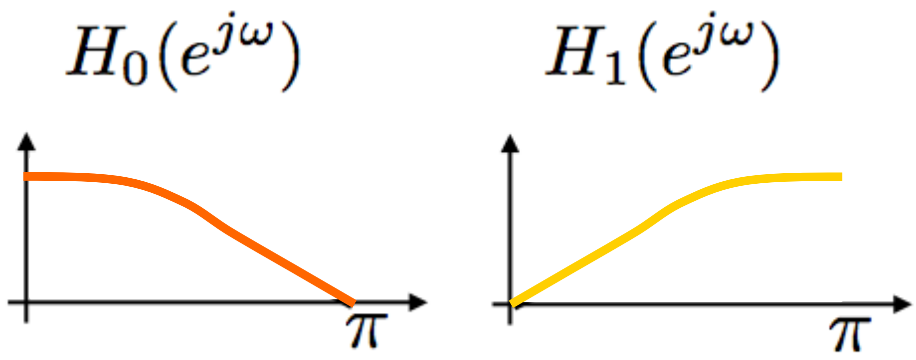


Multi-Rate Filter Banks

- h_0, h_1 are NOT ideal low/high pass

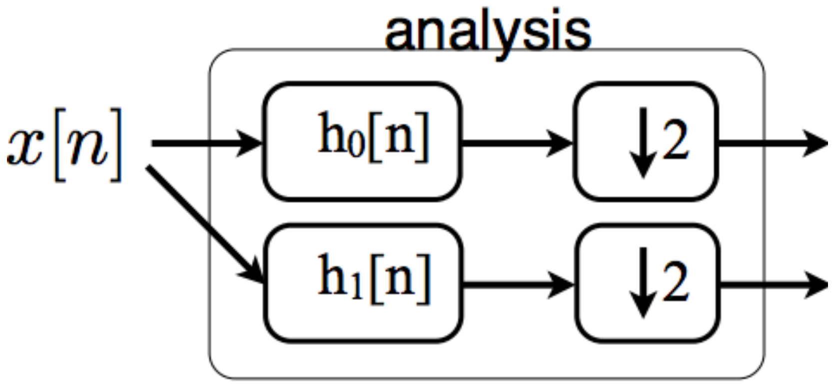


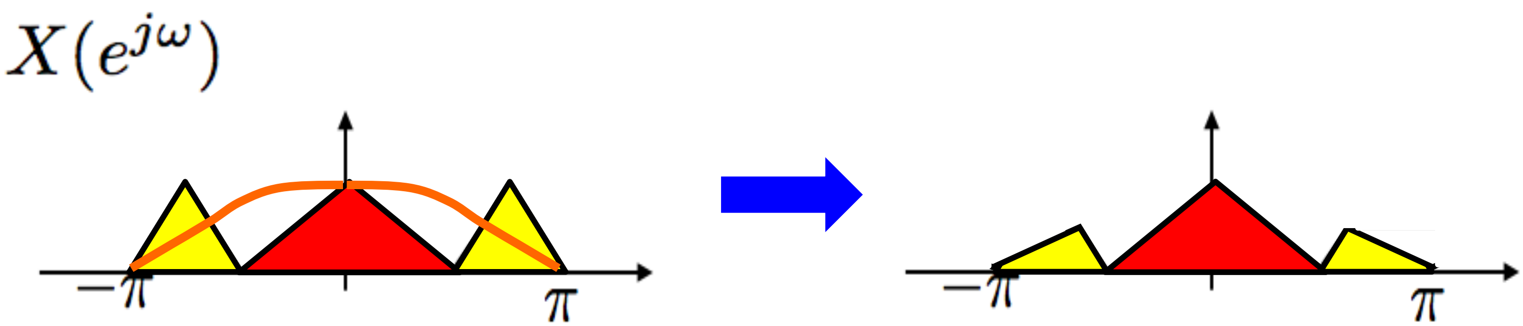




Non Ideal Filters

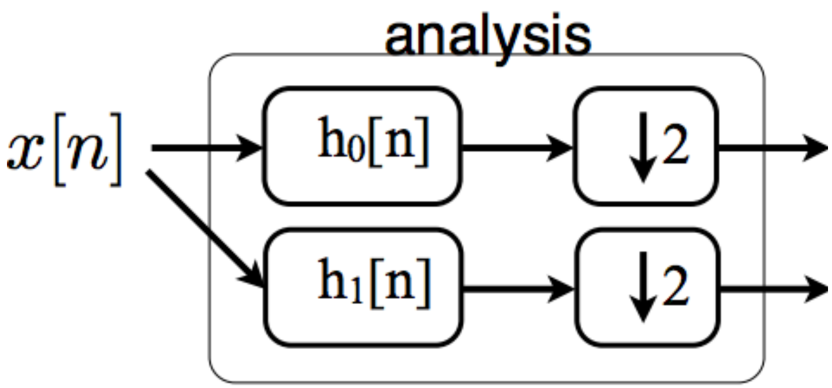
- h_0, h_1 are NOT ideal low/high pass

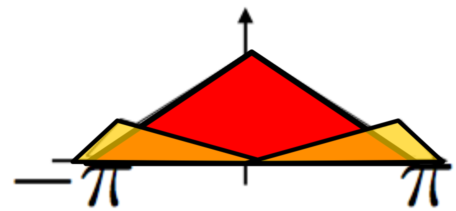


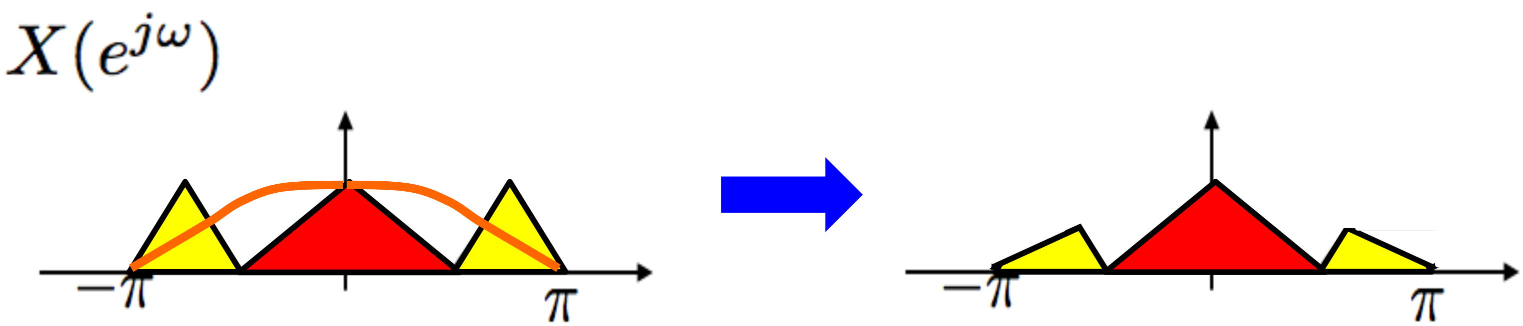


Non Ideal Filters

- h_0, h_1 are NOT ideal low/high pass

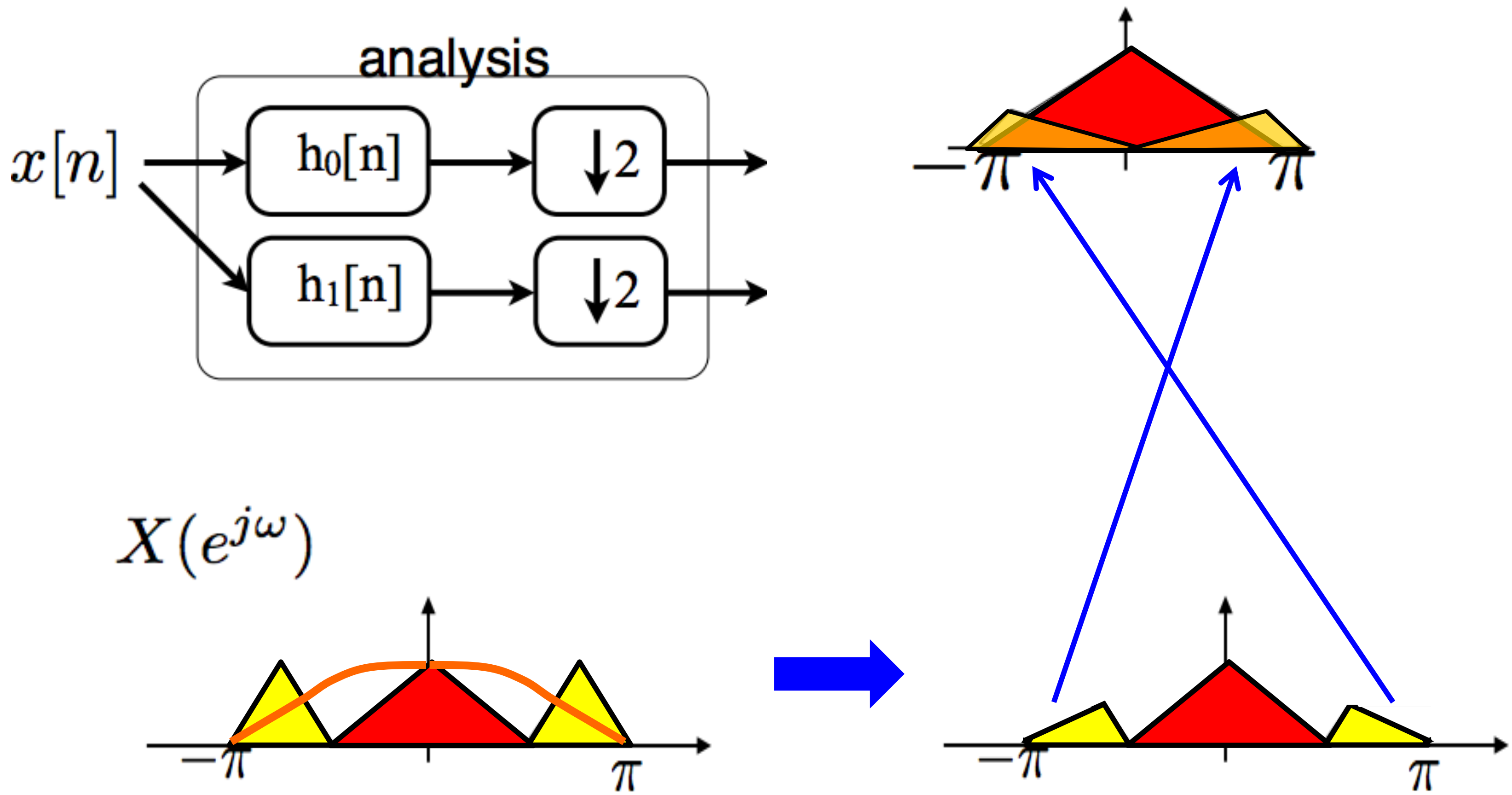






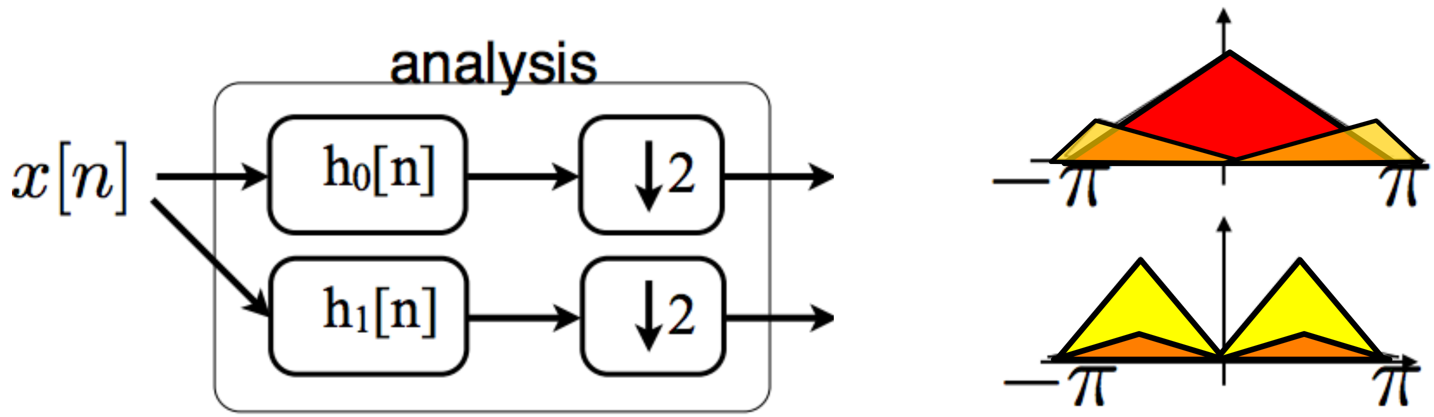
Non Ideal Filters

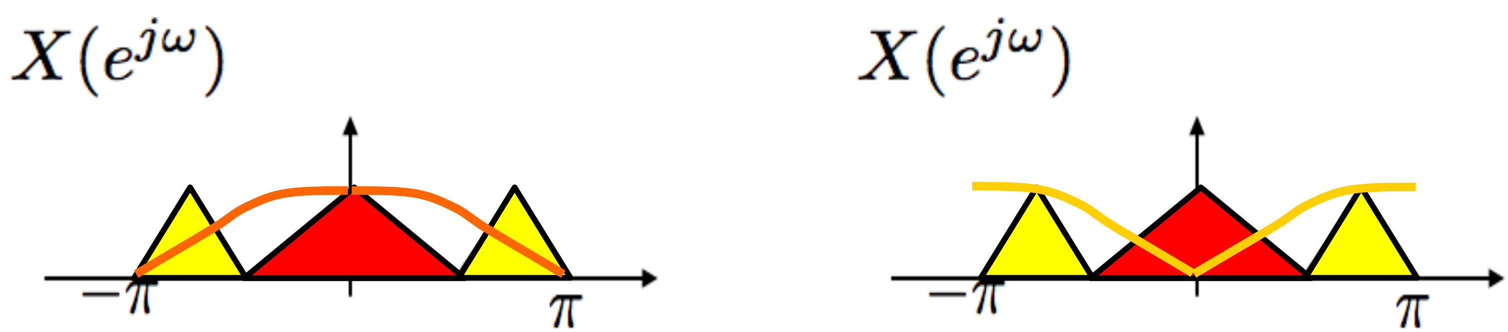
- h_0, h_1 are NOT ideal low/high pass



Non Ideal Filters

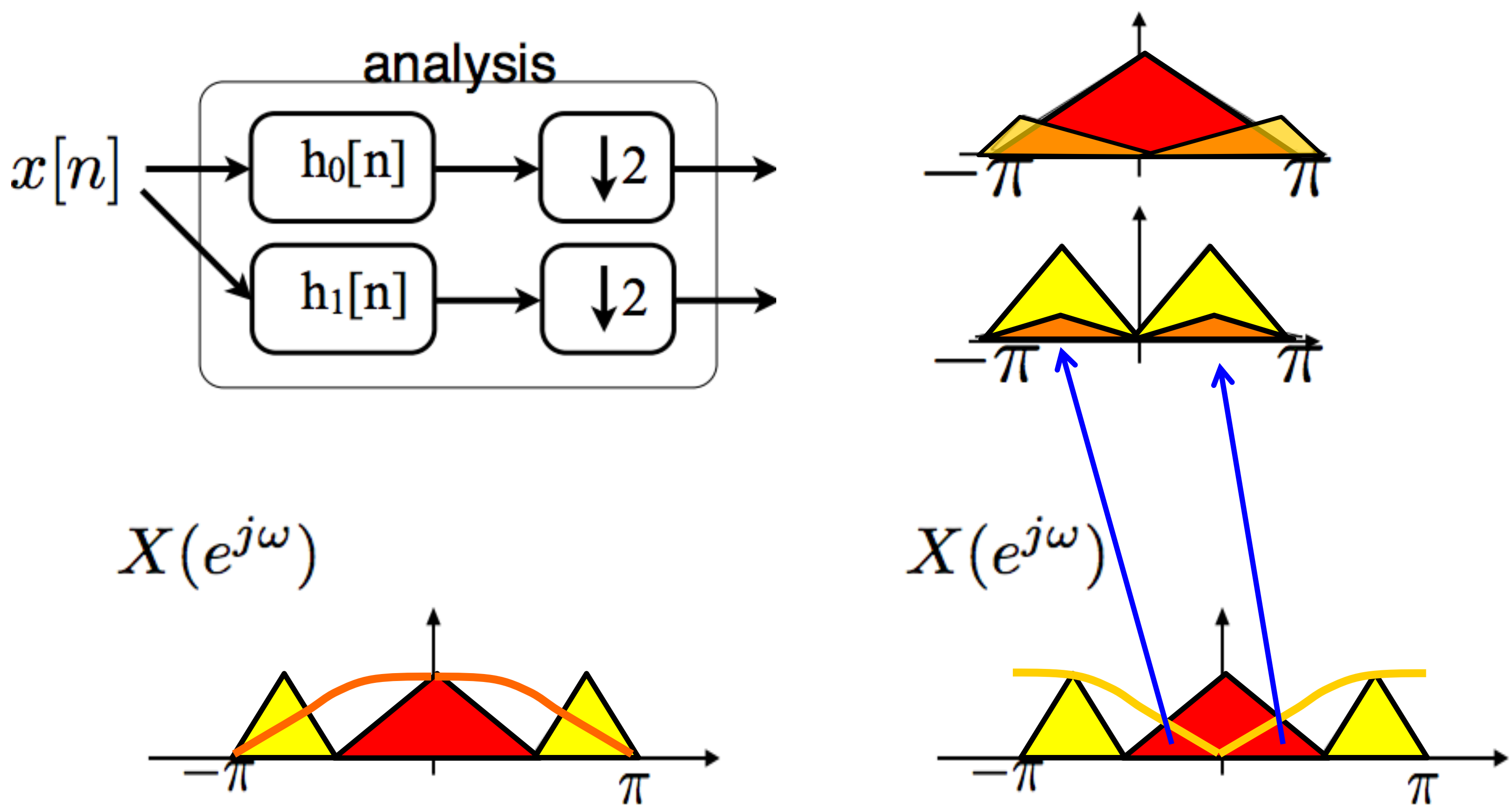
- h_0, h_1 are NOT ideal low/high pass



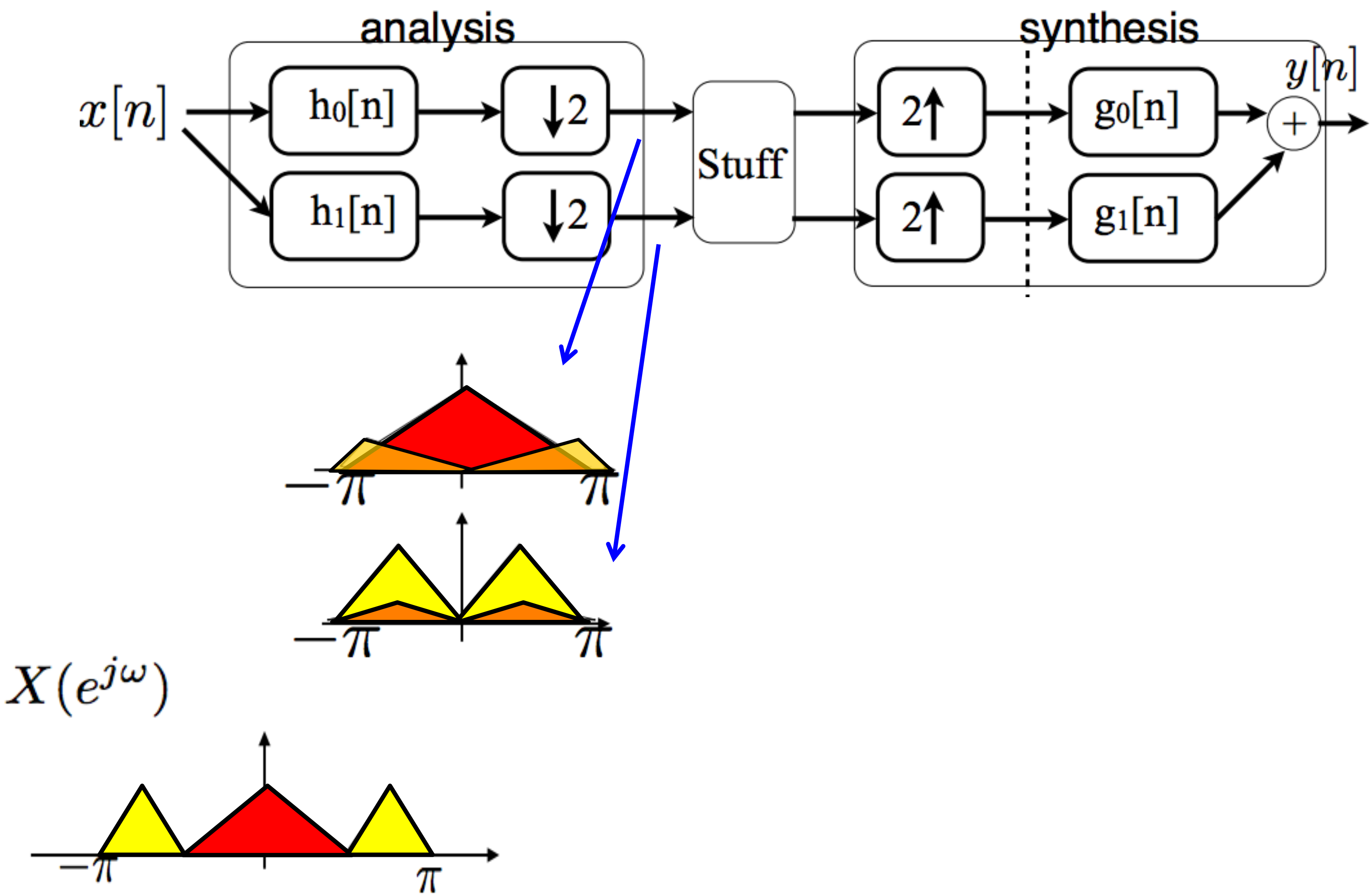


Non Ideal Filters

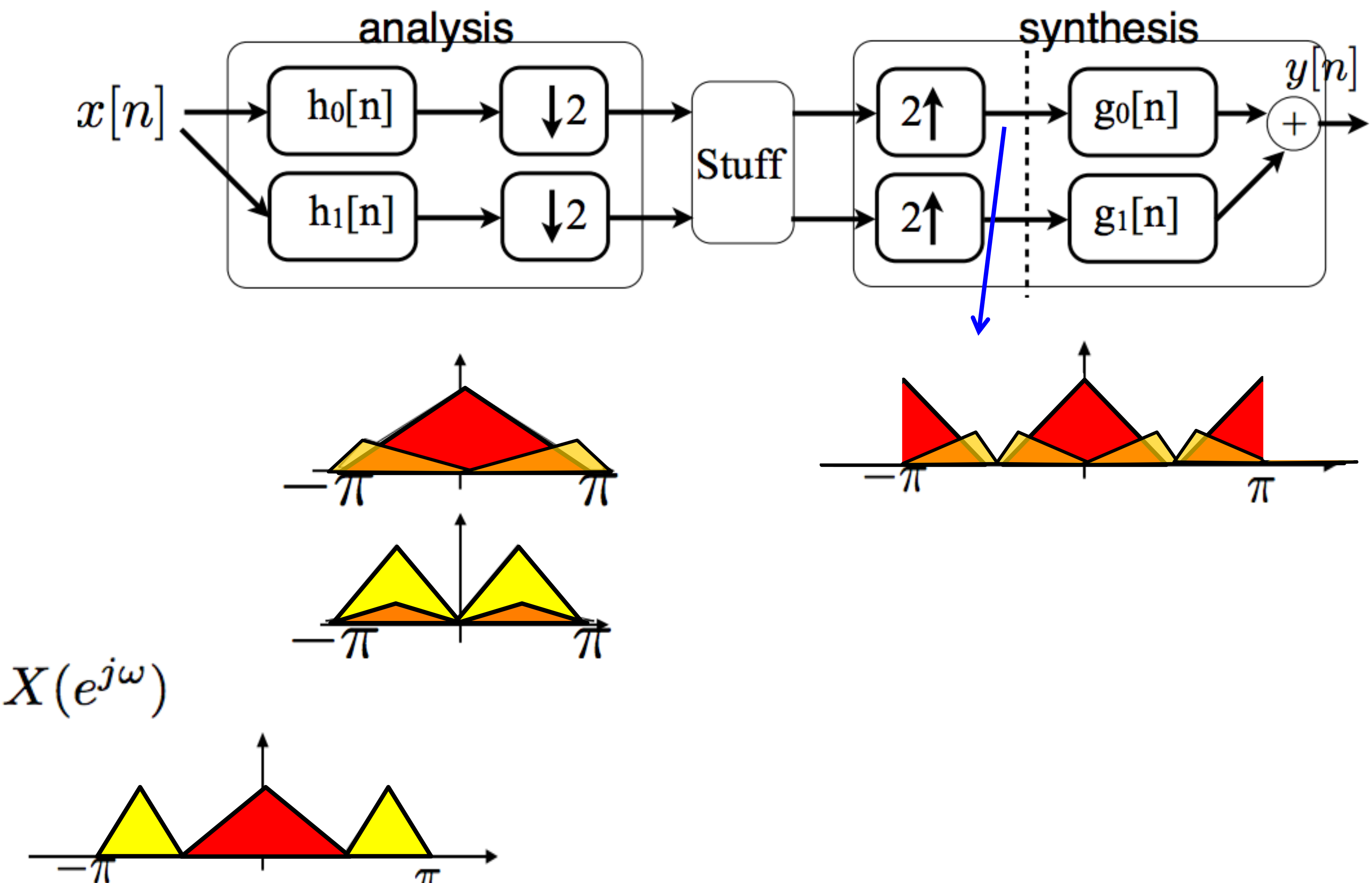
- h_0, h_1 are NOT ideal low/high pass



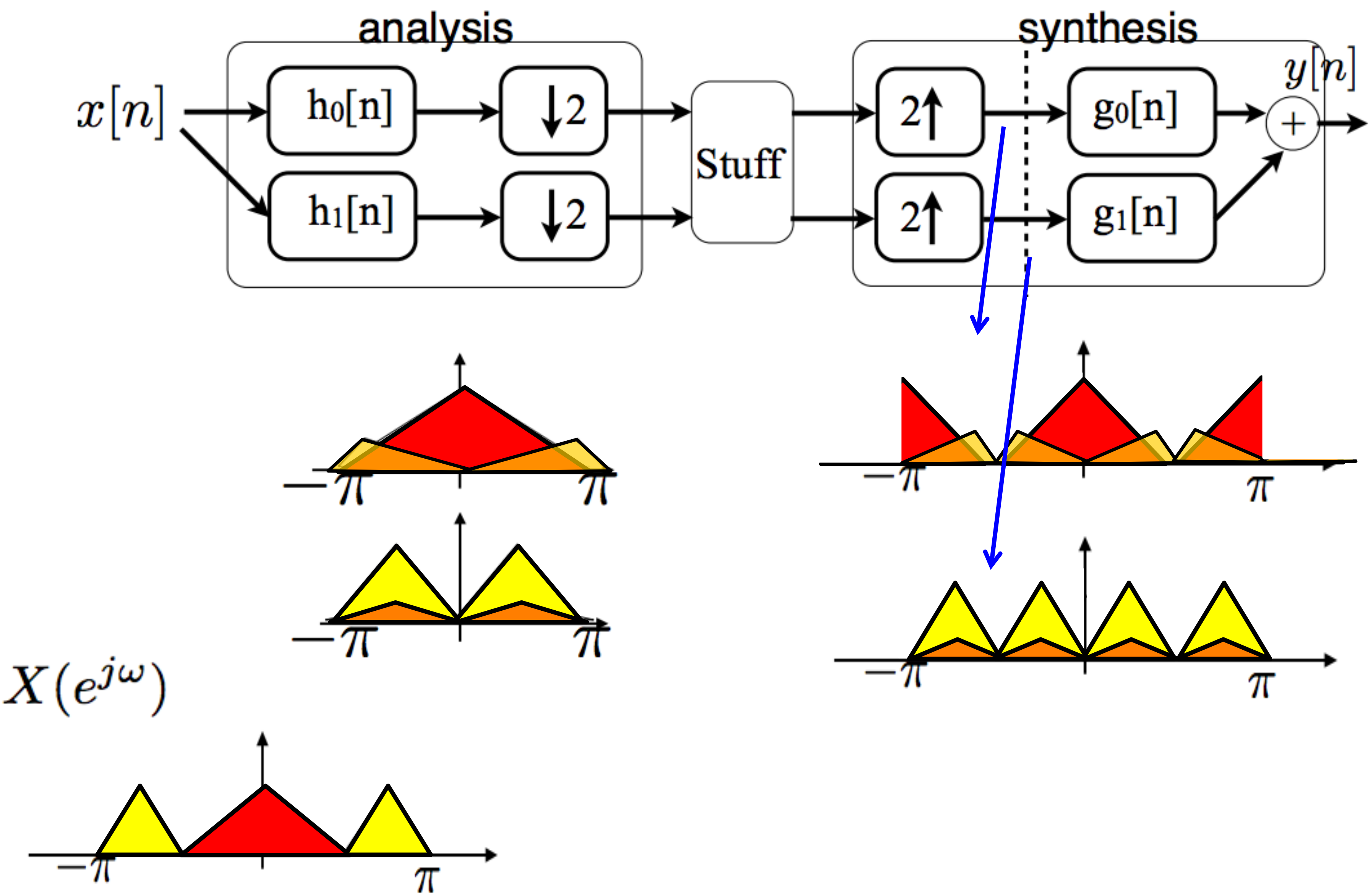
Non Ideal Filters



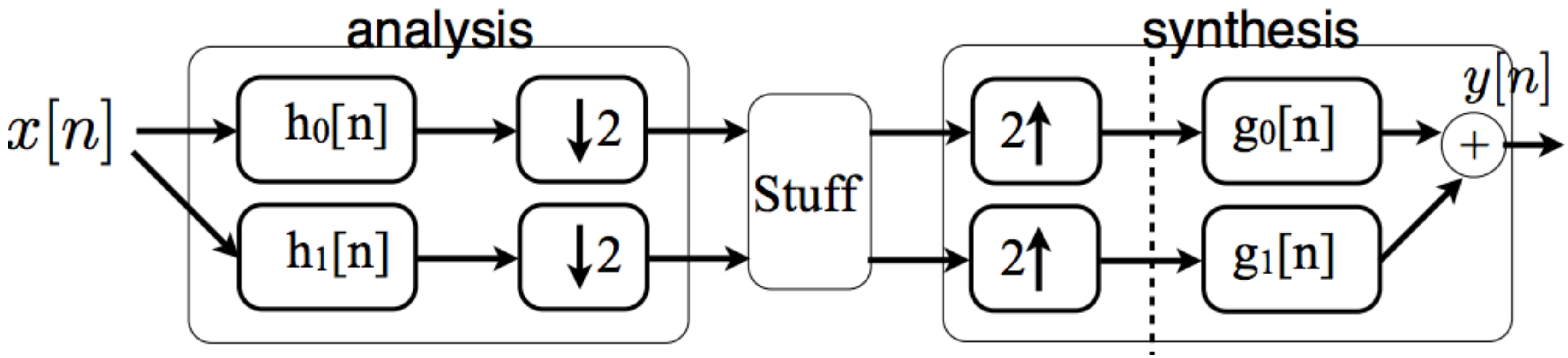
Non Ideal Filters



Non Ideal Filters



Perfect Reconstruction non-Ideal Filters

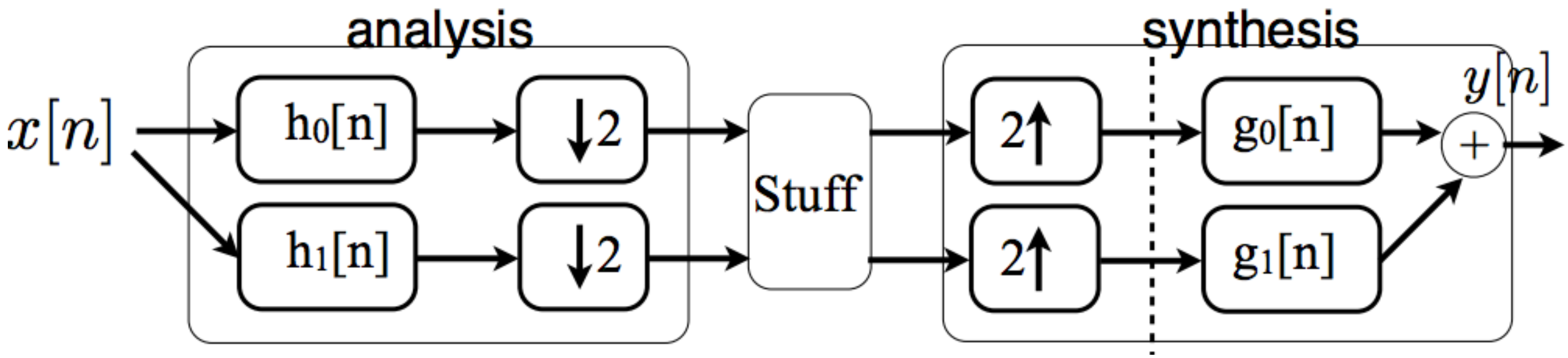


$$
\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega}) \\
&\quad + \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})\right] X(e^{j(\omega-\pi)})
\end{aligned}
$$

need to cancel!

aliasing

Quadrature Mirror Filters



Quadrature mirror filters

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

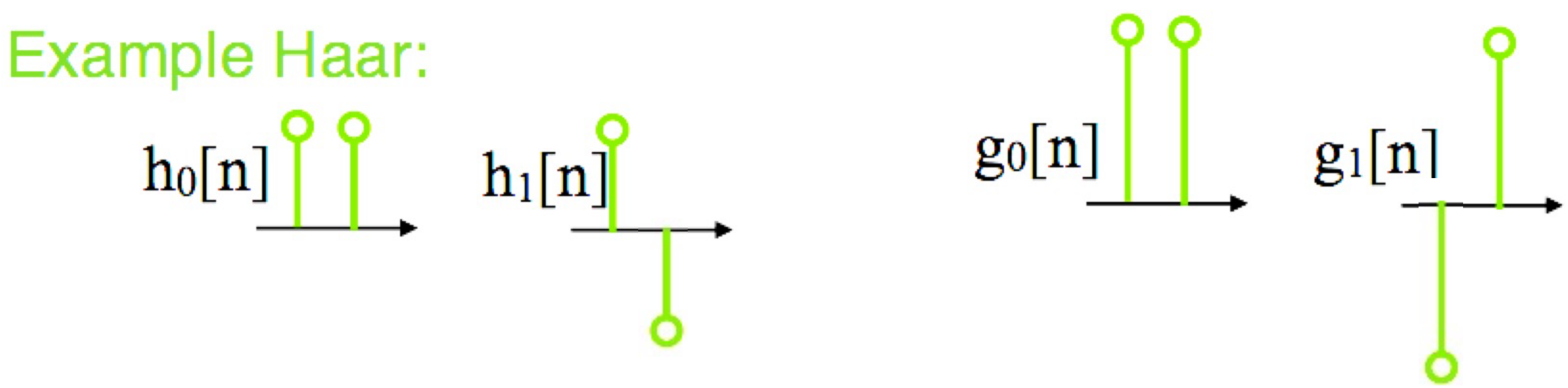
Perfect Reconstruction non-Ideal Filters

$$
\begin{aligned}
Y(e^{j\omega}) &= \frac{1}{2}\left[G_0(e^{j\omega})H_0(e^{j\omega}) + G_1(e^{j\omega})H_1(e^{j\omega})\right] X(e^{j\omega}) \\
&+ \frac{1}{2}\left[\underbrace{G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)})}_{\text{need to cancel!}}\right] \underbrace{X(e^{j(\omega-\pi)})}_{\text{aliasing}}
\end{aligned}
$$

$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

Haar Filter Example

$$\begin{aligned} H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\ G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\ G_1(e^{j\omega}) &= -2H_1(e^{j\omega}) \end{aligned}$$



Polyphase Decomposition

- The polyphase decomposition of a sequence is obtained by representing it as a superposition of M subsequences, each consisting of every M^{th} value of successively delayed versions of the sequence
- When this decomposition is applied to a filter impulse response, it can lead to efficient implementation structures for linear filters in several contexts

Polyphase Decomposition

- We can decompose an impulse response (of our filter) to M smaller impulse responses h_k for k=0...M-1

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

Polyphase Decomposition

- We can decompose an impulse response (of our filter) to M smaller impulse responses h_k for k=0…M-1

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

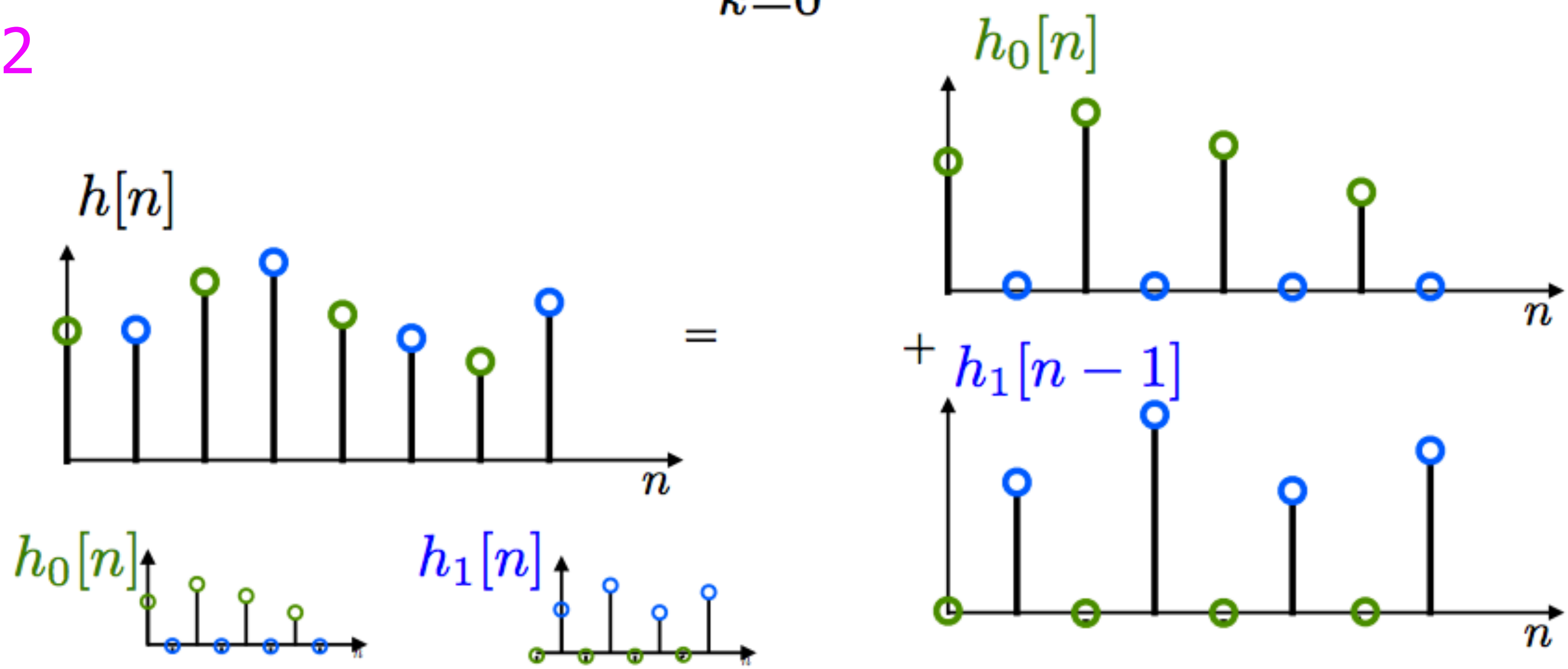
M=2

Polyphase Decomposition

- We can decompose an impulse response (of our filter) to M smaller impulse responses h_k for k=0...M-1

$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

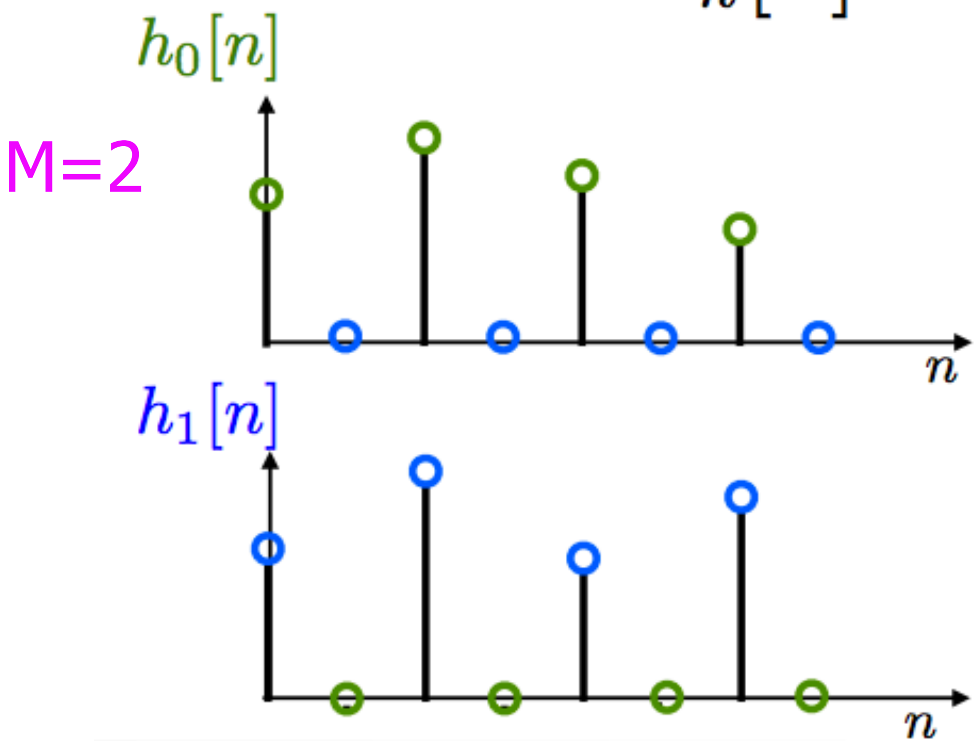
M=2



Polyphase Decomposition

$$h_k[n] \longrightarrow \boxed{\downarrow \mathrm{M}} \longrightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$

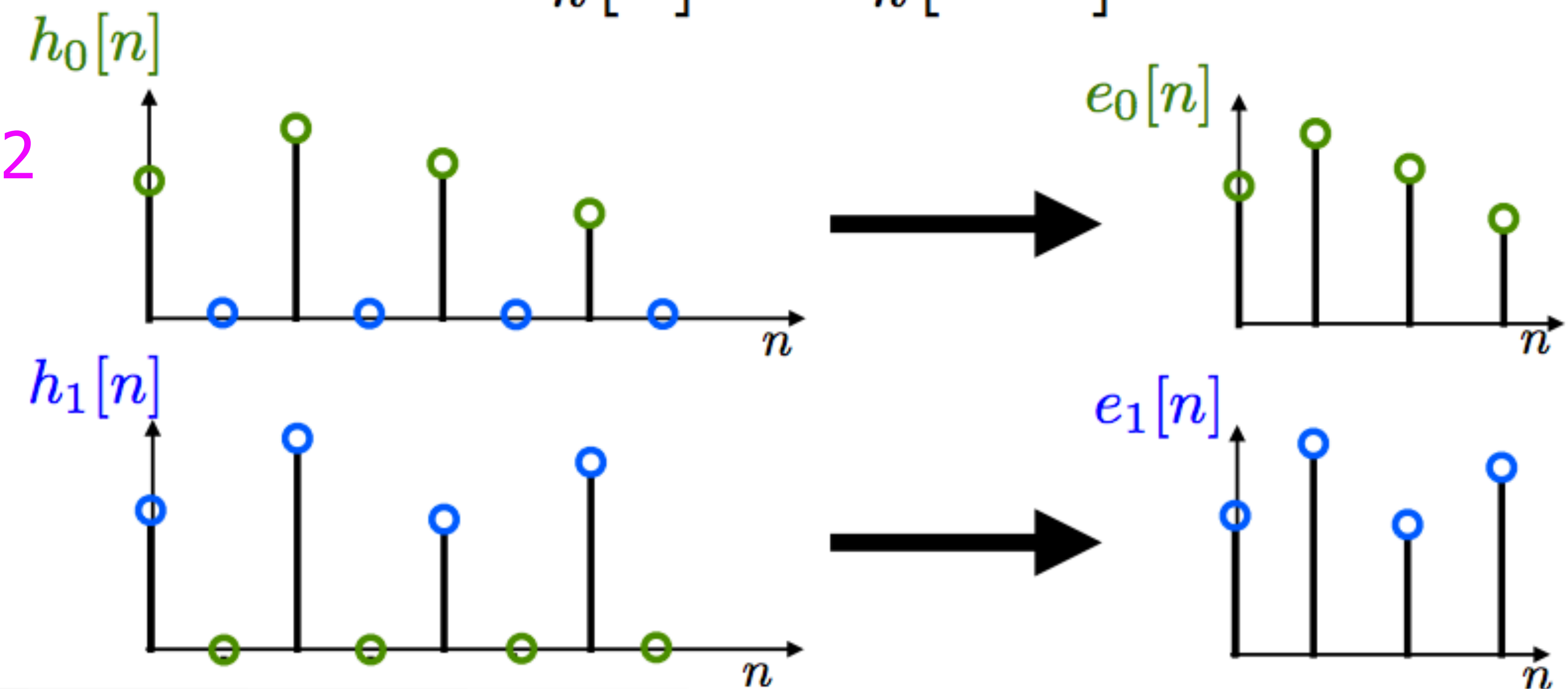


Polyphase Decomposition

$$h_k[n] \longrightarrow \boxed{\downarrow M} \longrightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$

M=2



Polyphase Decomposition

$$e_k[n] \longrightarrow \boxed{\uparrow \mathrm{M}} \longrightarrow h_k[n]$$

Polyphase Decomposition

$$e_k[n] \longrightarrow \boxed{\uparrow \mathrm{M}} \longrightarrow h_k[n]$$

recall upsampling ⇒ scaling

$$H_k(z) = E_k(z^M)$$

Also, recall:

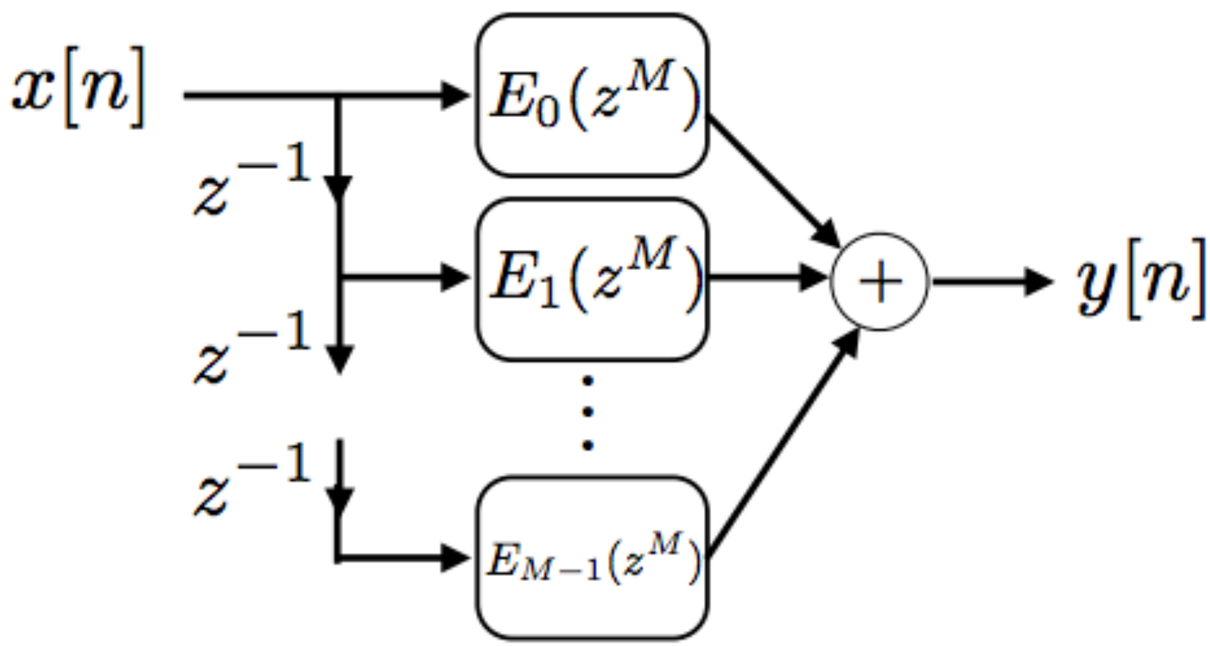
$$h[n] = \sum_{k=0}^{M-1} h_k[n-k]$$

So,

$$\boxed{H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}}$$

Polyphase Decomposition

$$H(z) = \sum_{k=0}^{M-1} E_k(z^M) z^{-k}$$



Polyphase Implementation of Decimation

$$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n] \rightarrow \boxed{\downarrow \mathrm{M}} \rightarrow w[n] = y[nM]$$

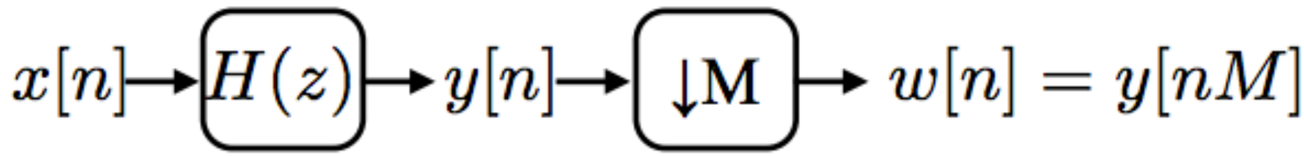
- Problem:
  - Compute all y[n] and then throw away -- wasted computation!

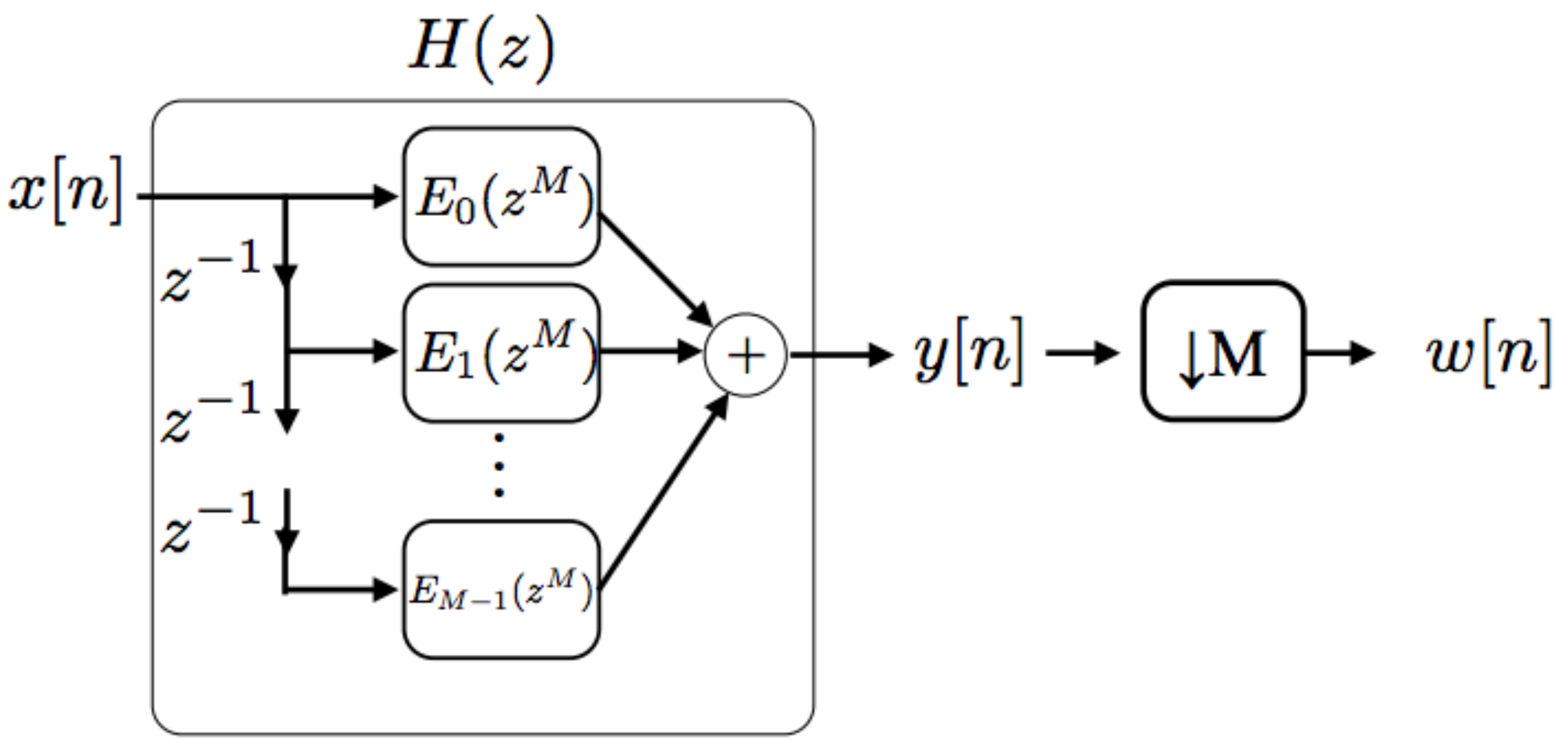
Polyphase Implementation of Decimation

$$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n] \rightarrow \boxed{\downarrow M} \rightarrow w[n] = y[nM]$$

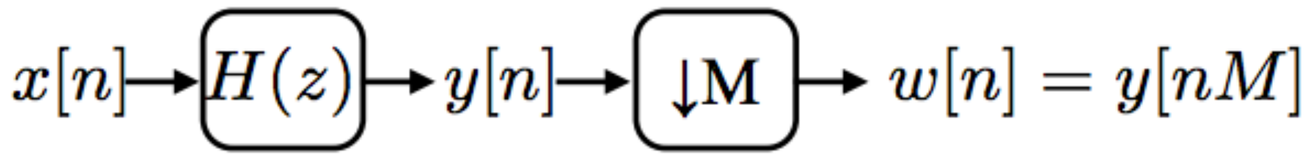
- Problem:
  - Compute all y[n] and then throw away -- wasted computation!
  - For FIR length N → N multiplications/unit time

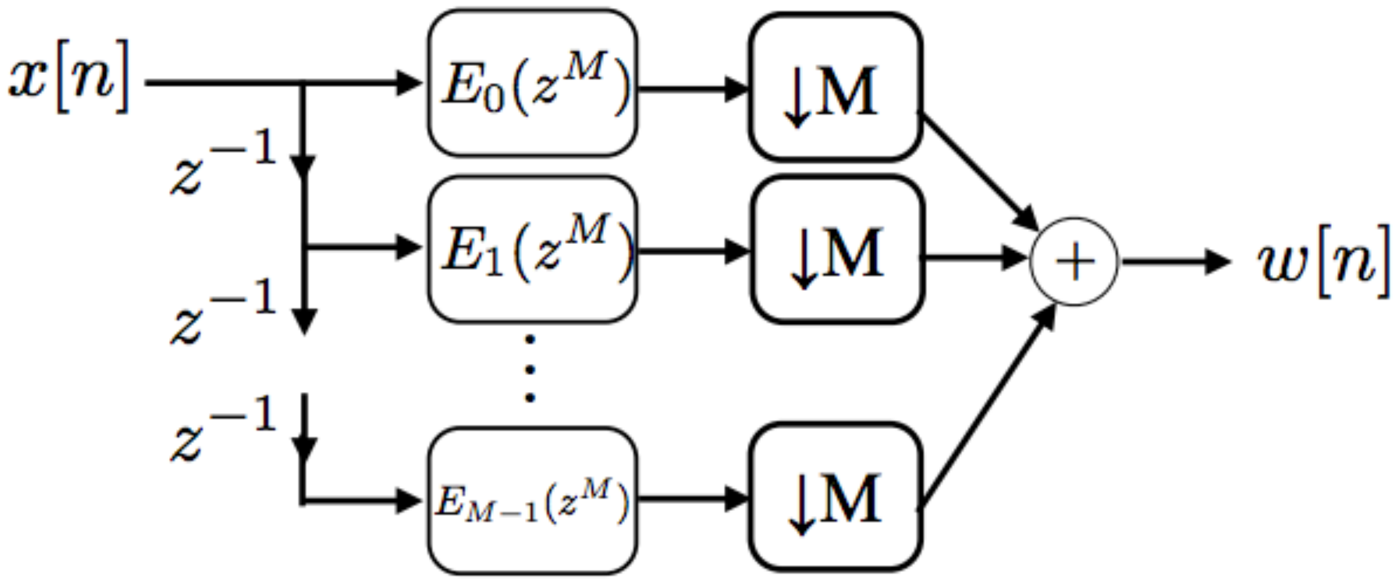
Polyphase Implementation of Decimation



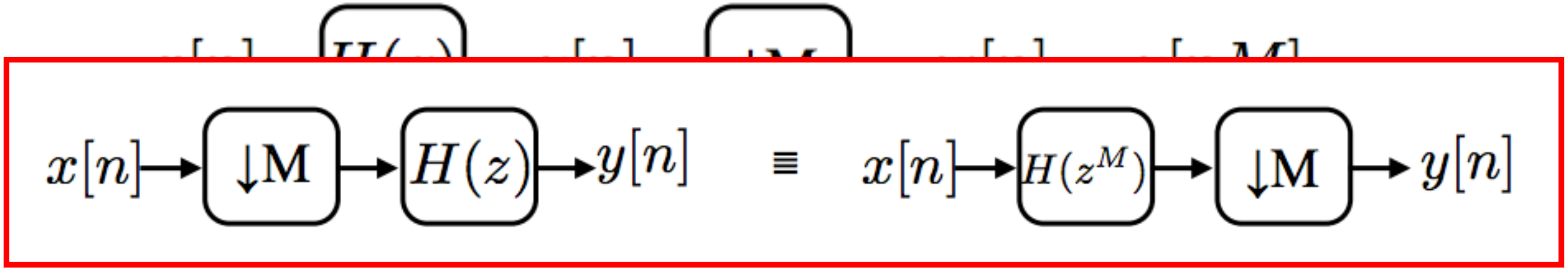


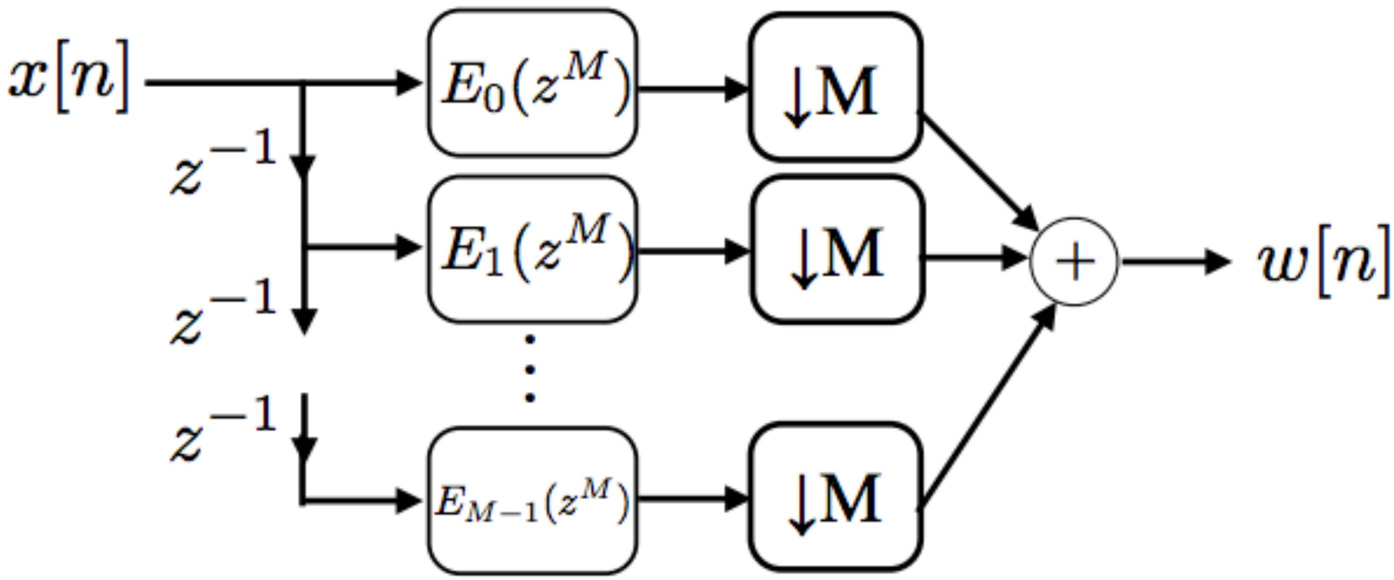
Polyphase Implementation of Decimation



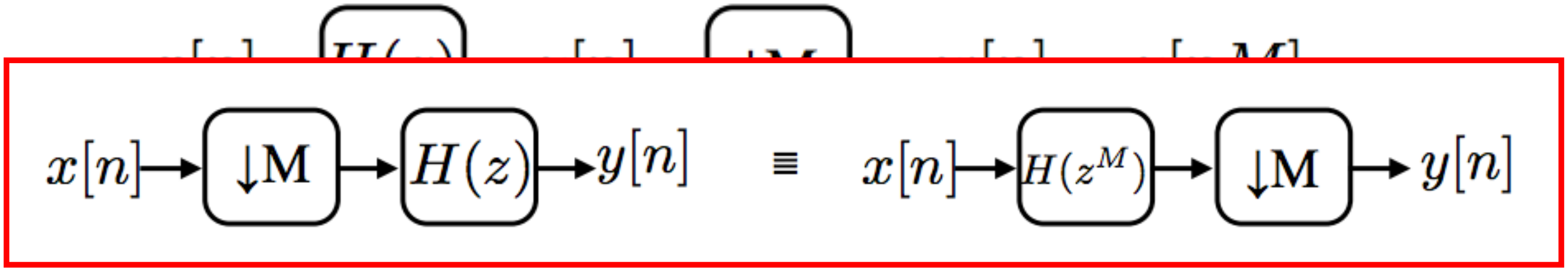


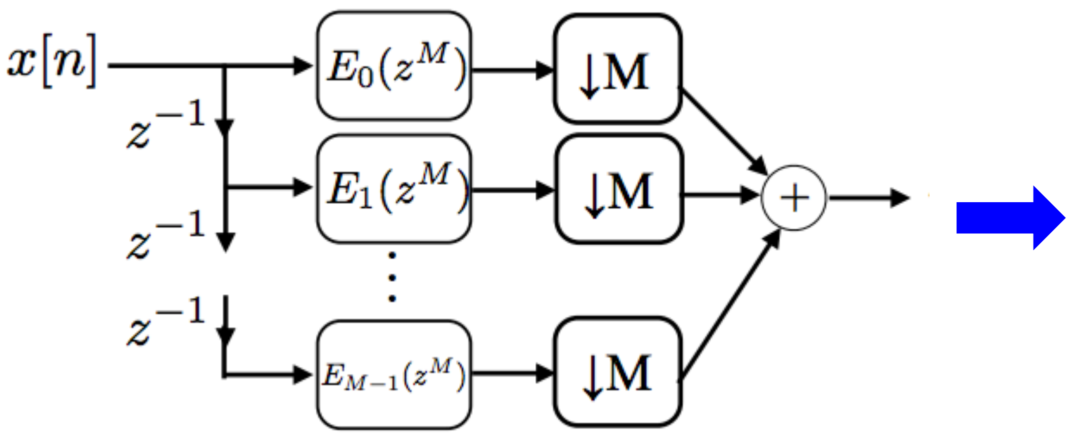
Polyphase Implementation of Decimation



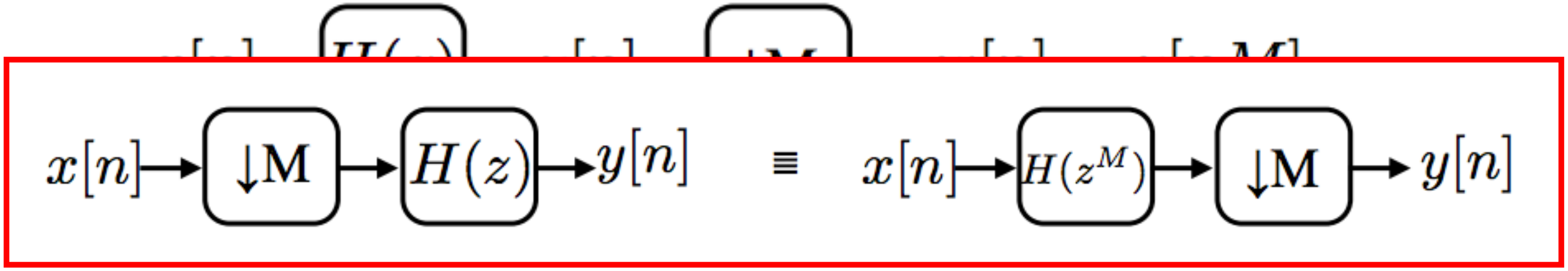


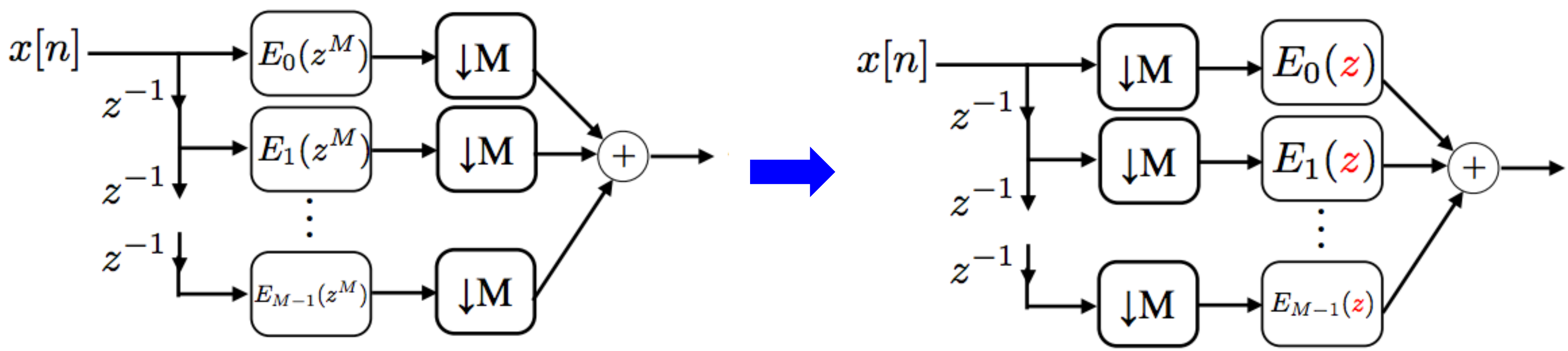
Polyphase Implementation of Decimation



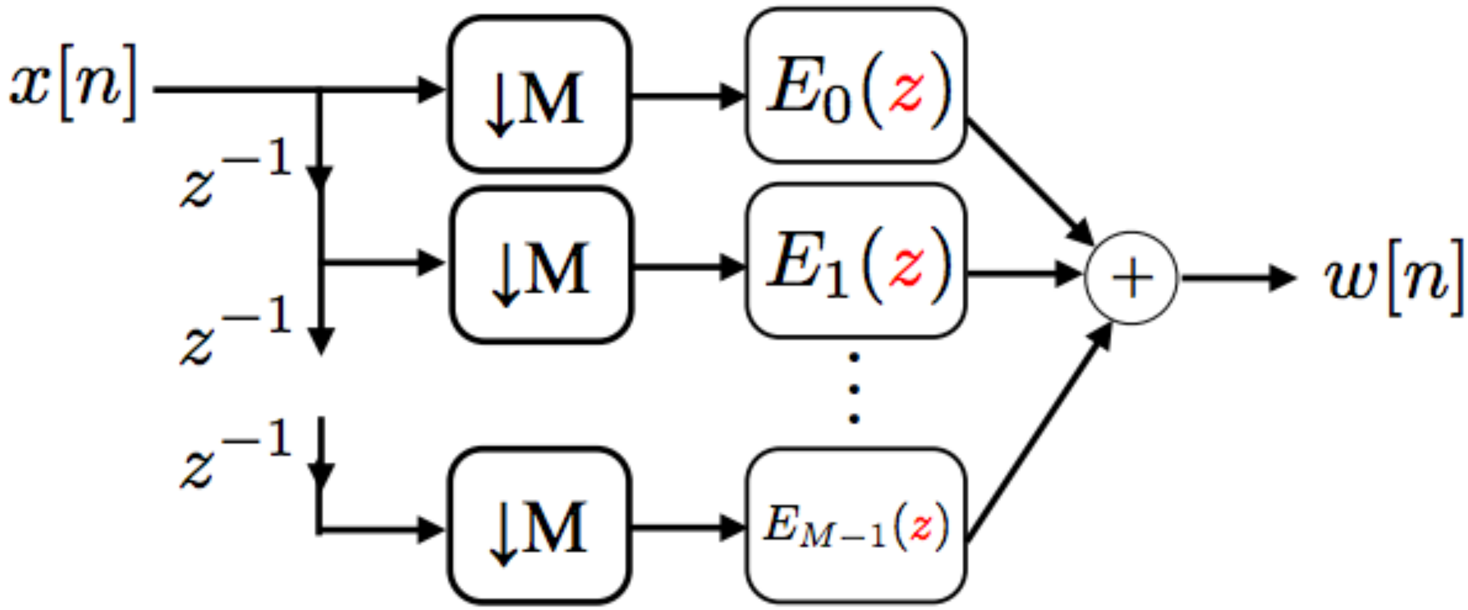


Polyphase Implementation of Decimation

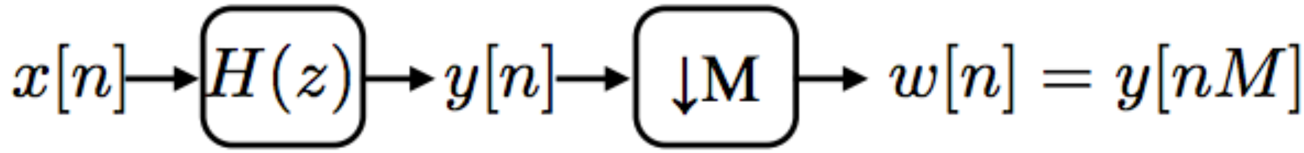




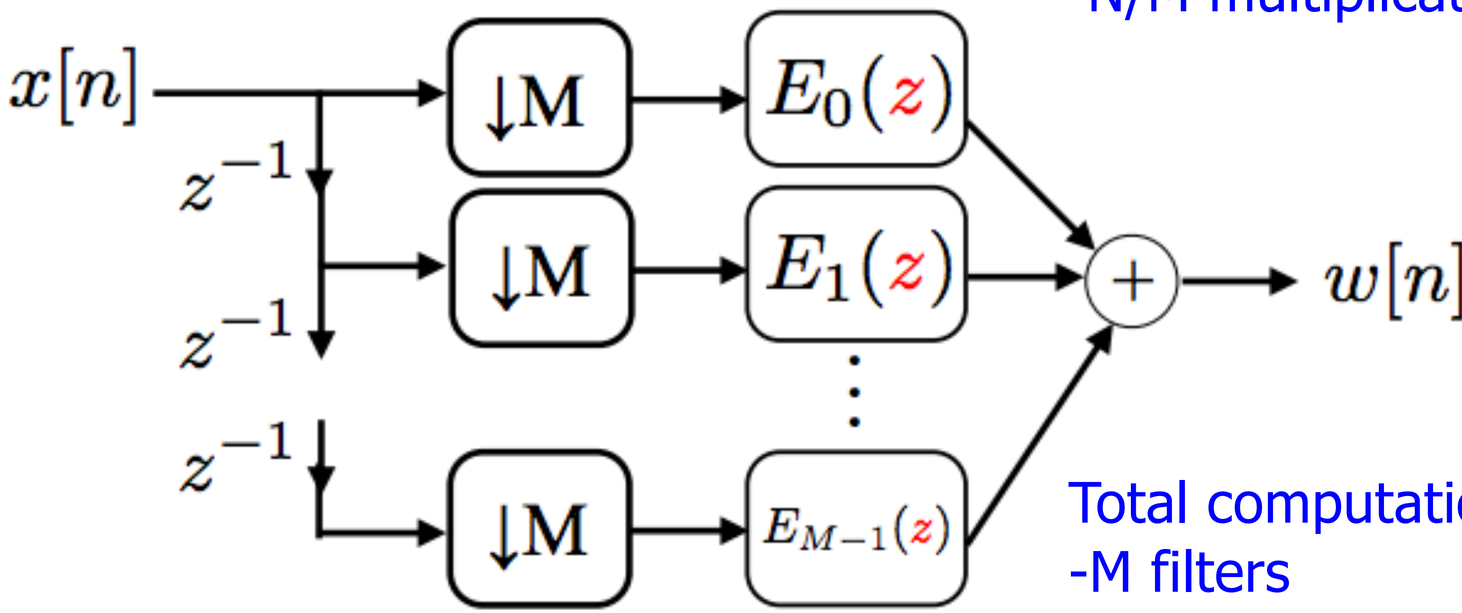
Polyphase Implementation of Decimation



Polyphase Implementation of Decimation

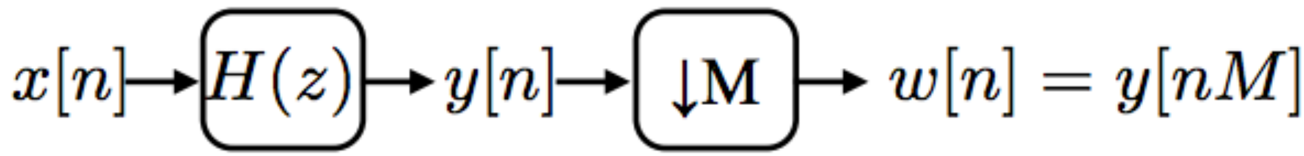


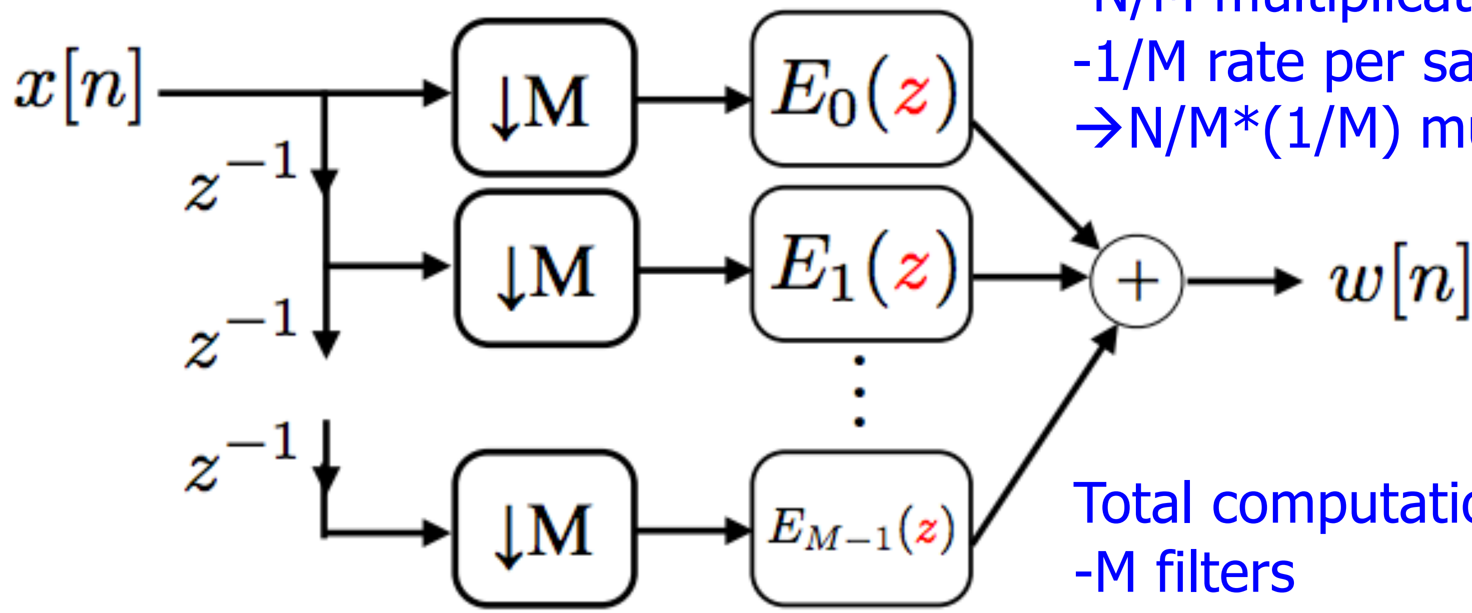
Each filter computation:
-N/M multiplications



Total computation:
-M filters
→N mults??

Polyphase Implementation of Decimation

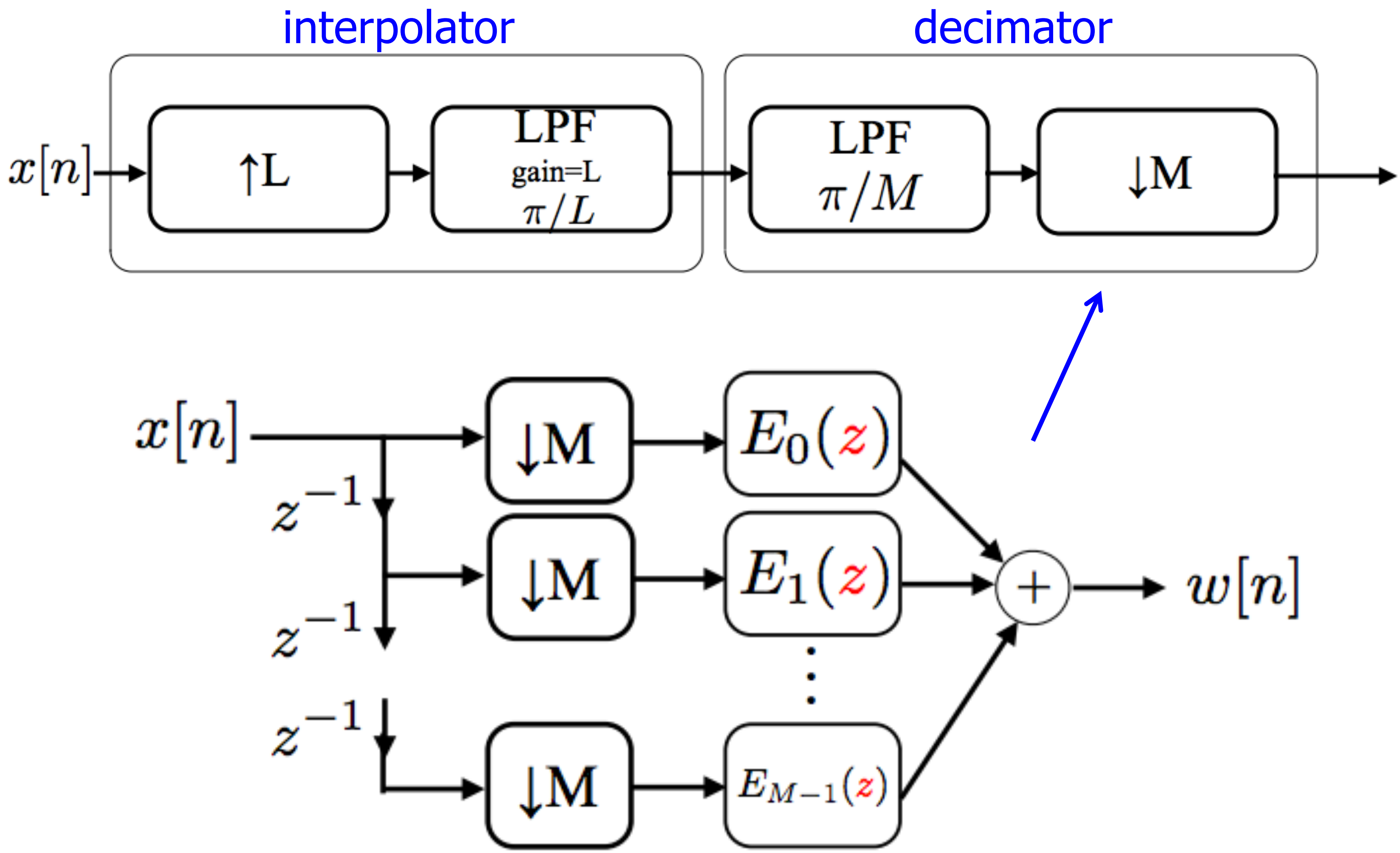




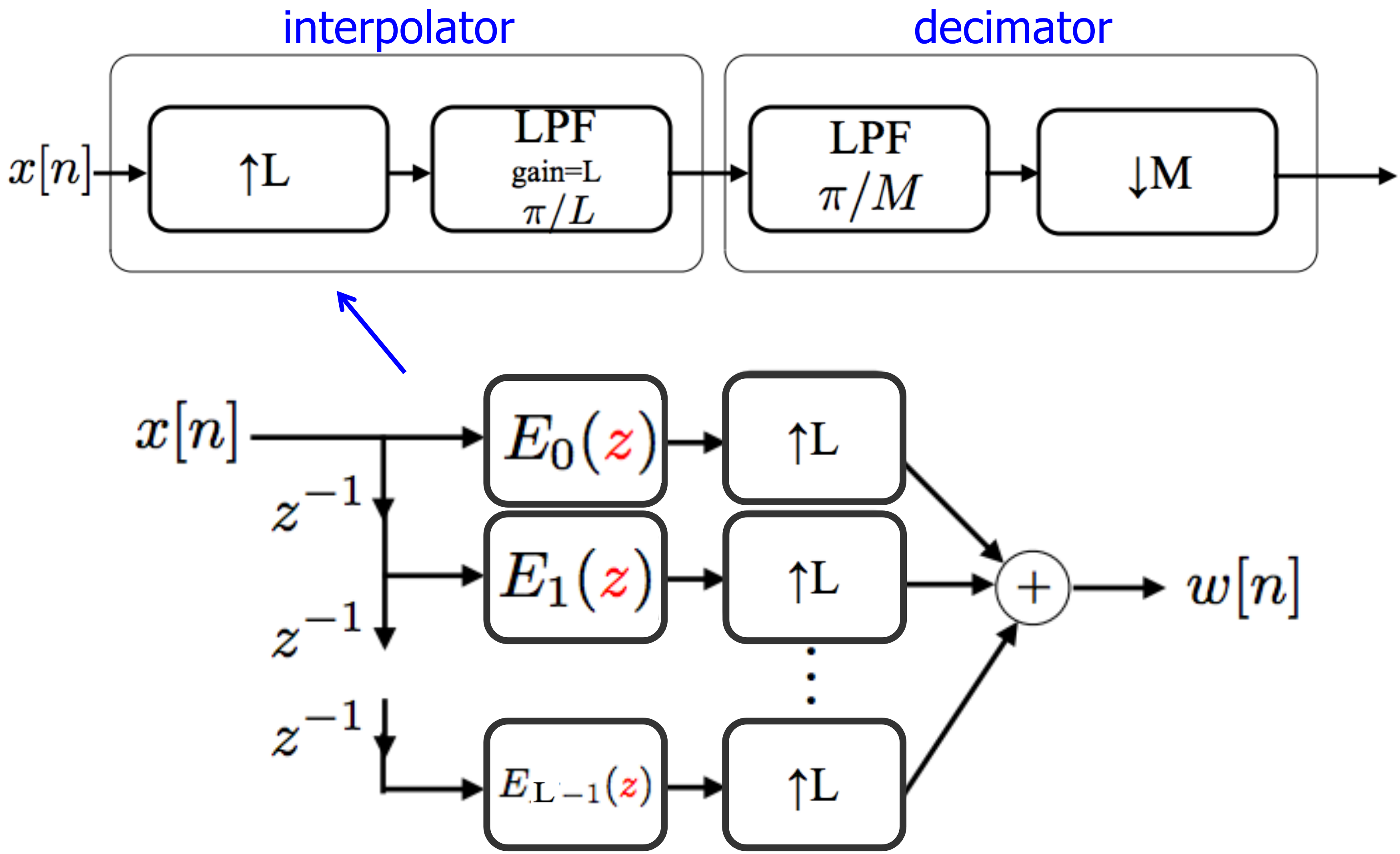
Each filter computation:
-N/M multiplications
-1/M rate per sample
→N/M*(1/M) mults/unit time

Total computation:
-M filters
→N/M mults/unit time

Polyphase Implementation of Decimator



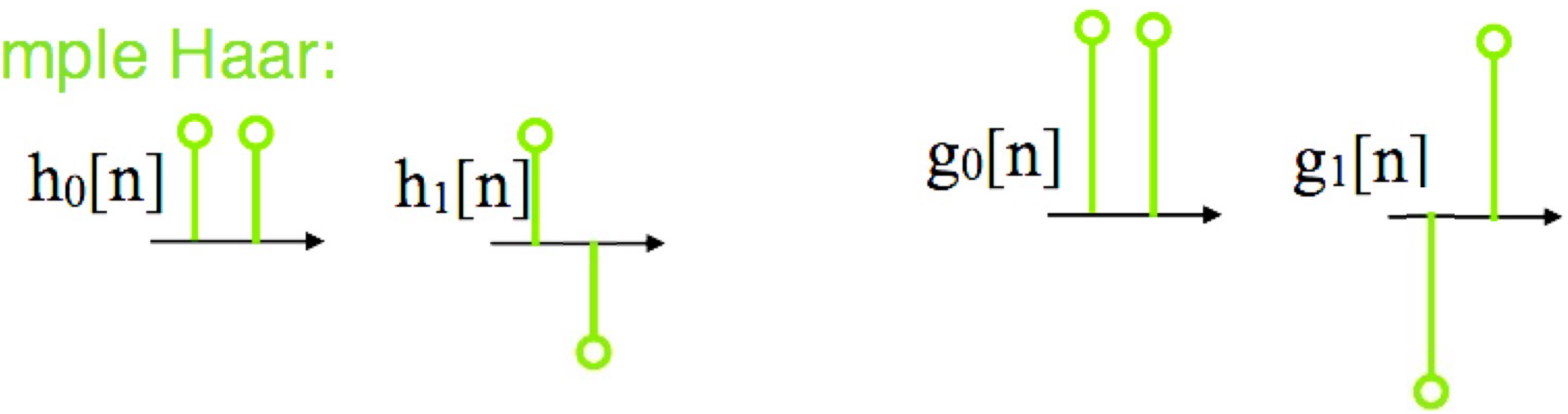
Polyphase Implementation of Interpolation



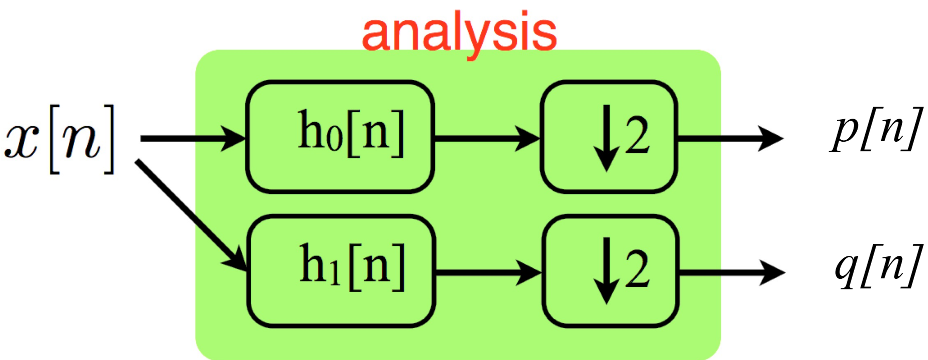
Haar Filter Example

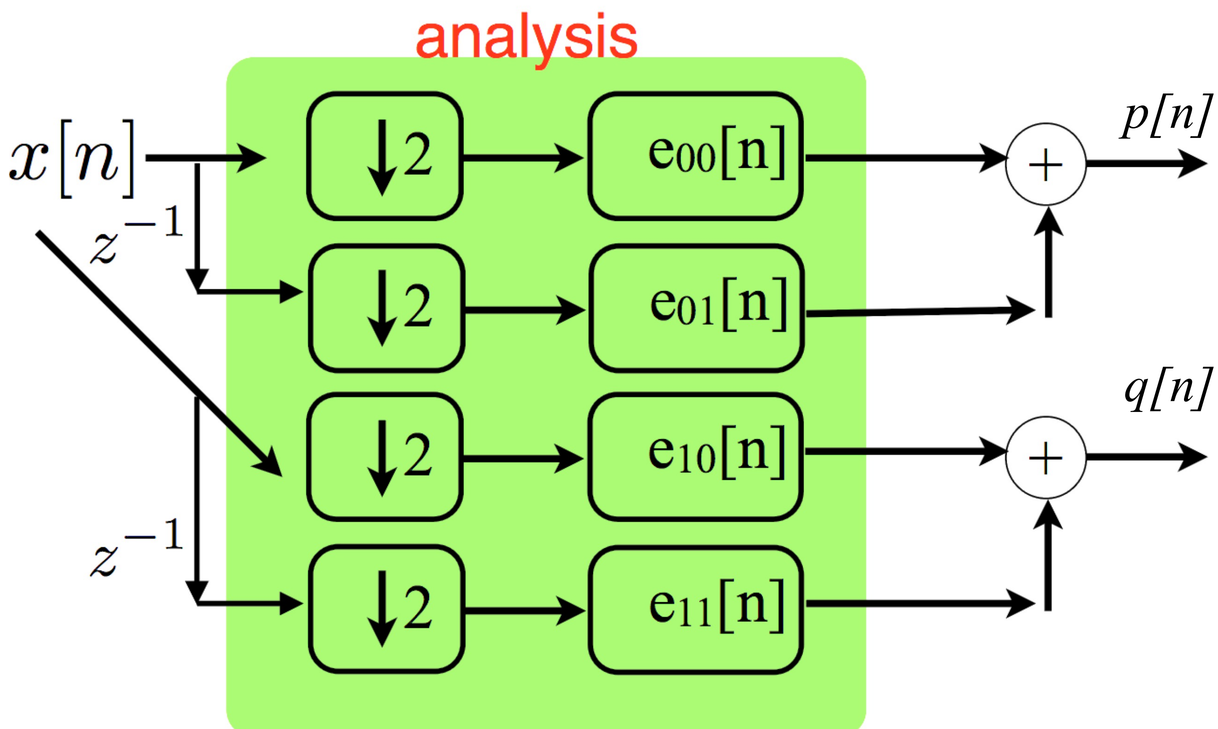
$$
\begin{aligned}
H_1(e^{j\omega}) &= H_0(e^{j(\omega-\pi)}) \\
G_0(e^{j\omega}) &= 2H_0(e^{j\omega}) \\
G_1(e^{j\omega}) &= -2H_1(e^{j\omega})
\end{aligned}
$$

Example Haar:



Polyphase Filter Bank

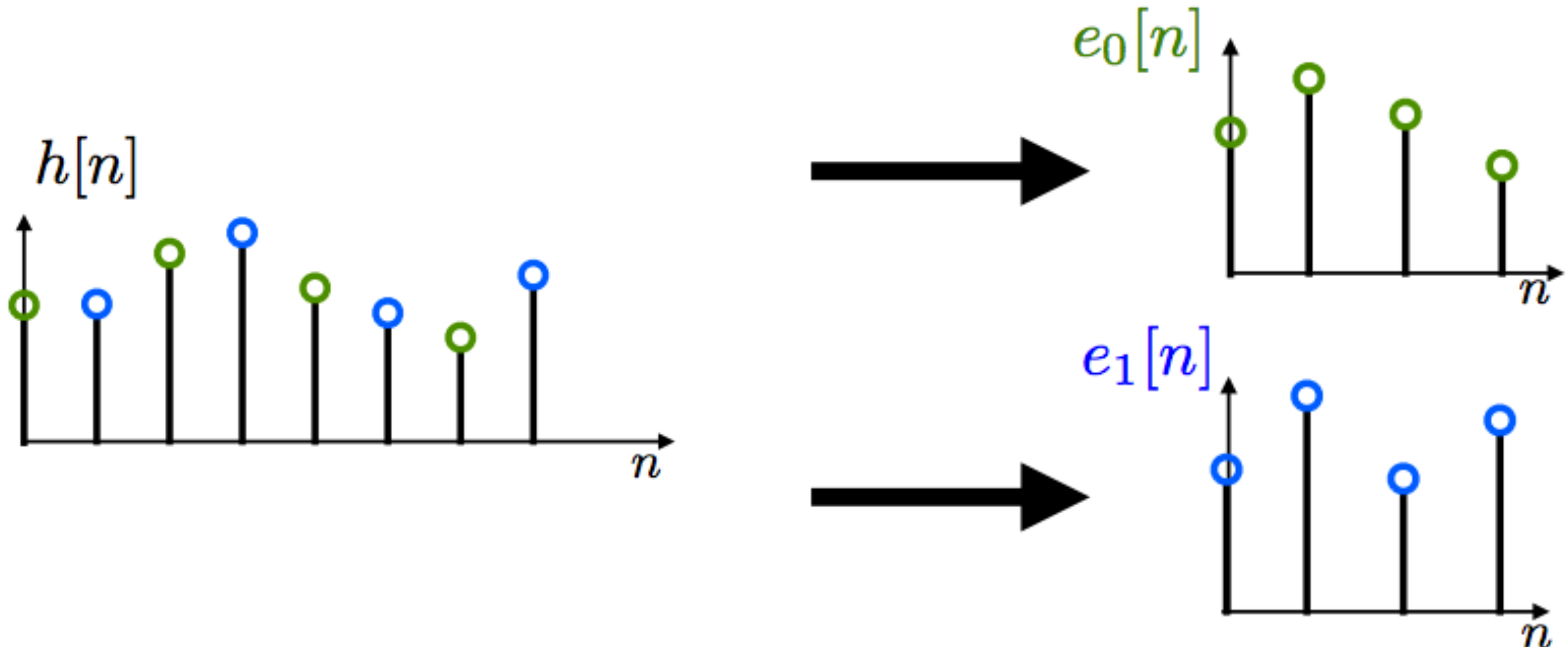




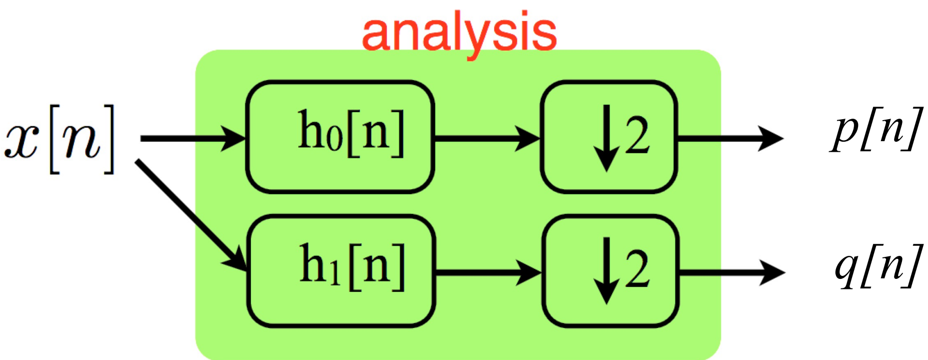
Polyphase Decomposition

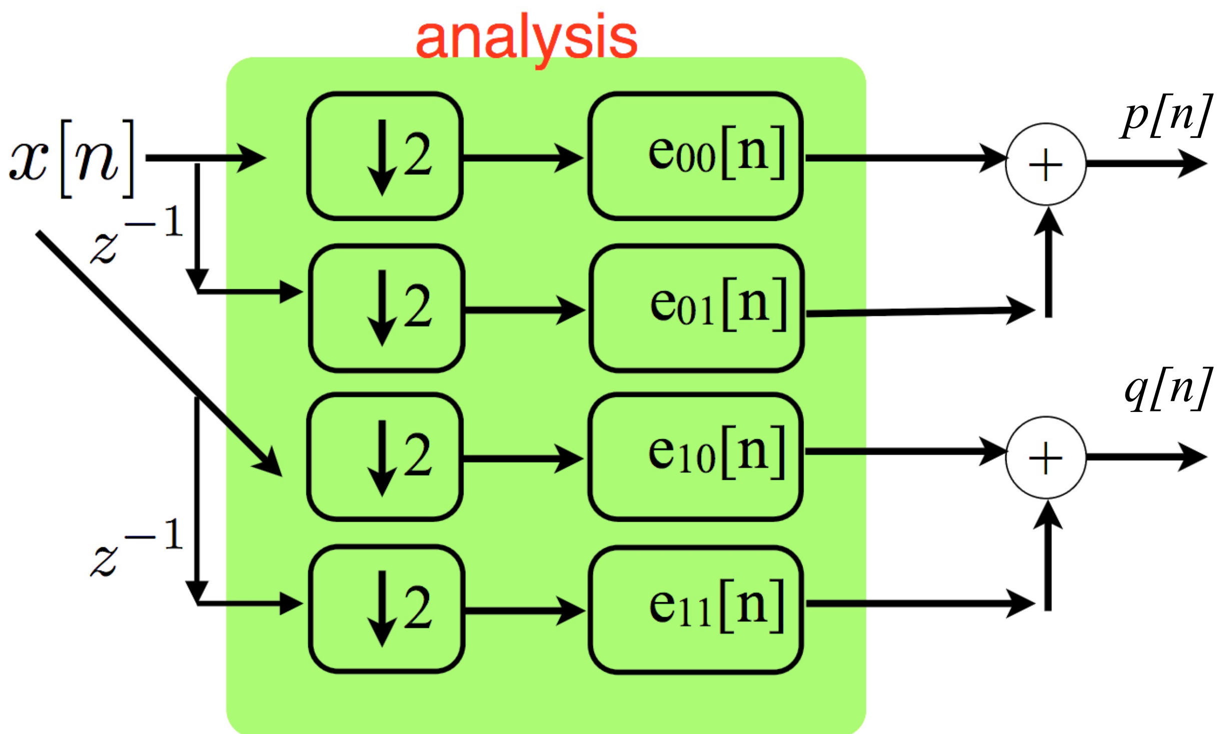
$$h_k[n] \longrightarrow \boxed{\downarrow \mathrm{M}} \longrightarrow e_k[n]$$

$$e_k[n] = h_k[nM]$$



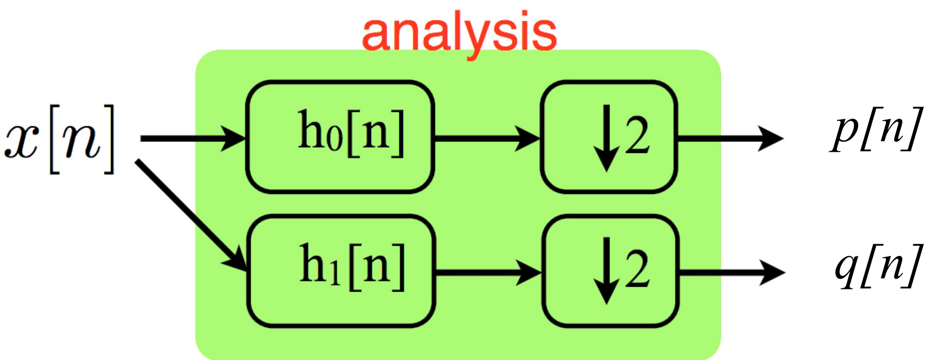
Polyphase Filter Bank



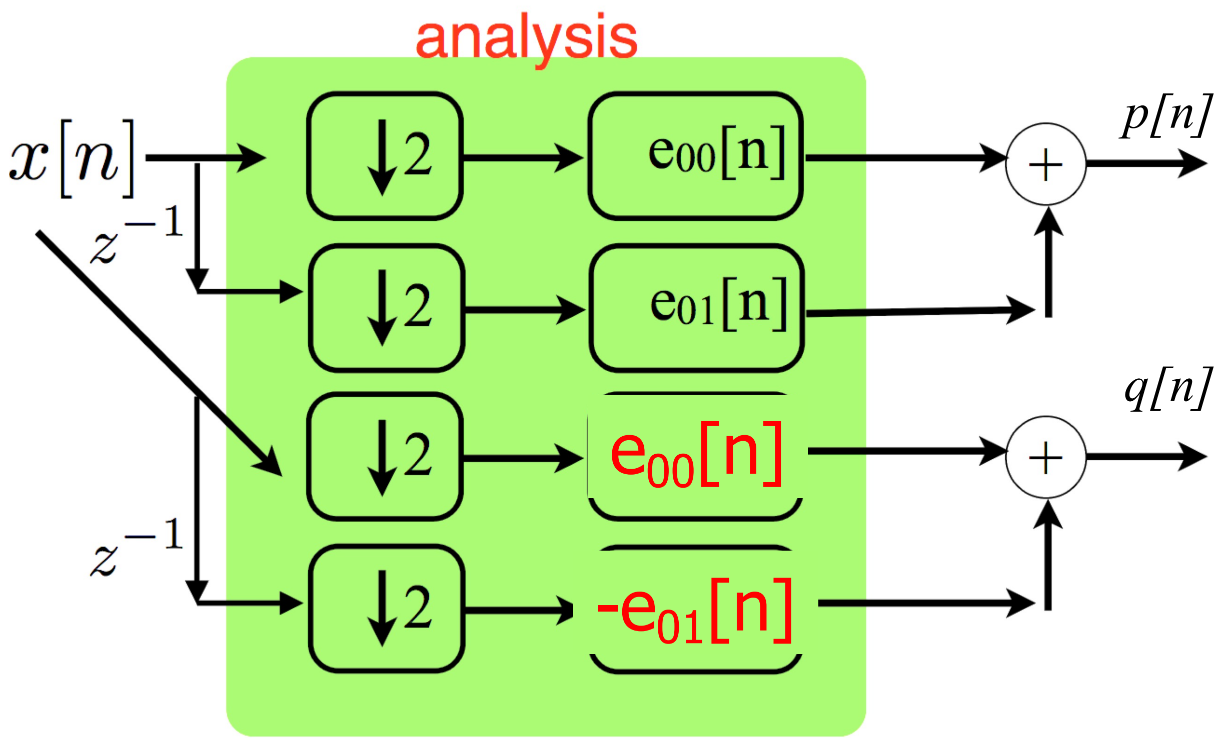


$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{aligned}
$$

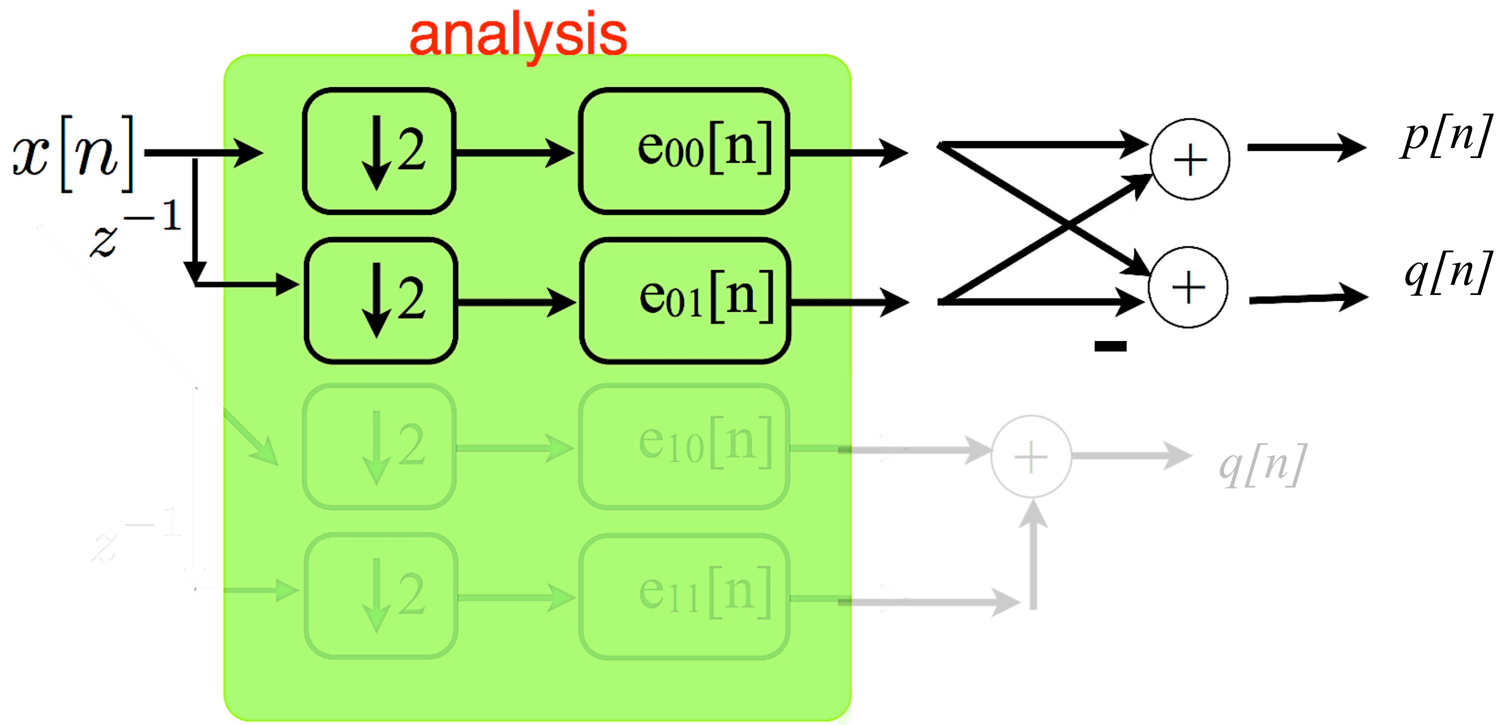
Polyphase Filter Bank



$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{aligned}
$$



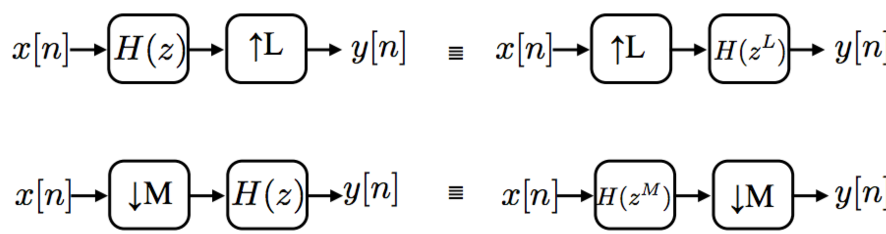
Polyphase Filter Bank

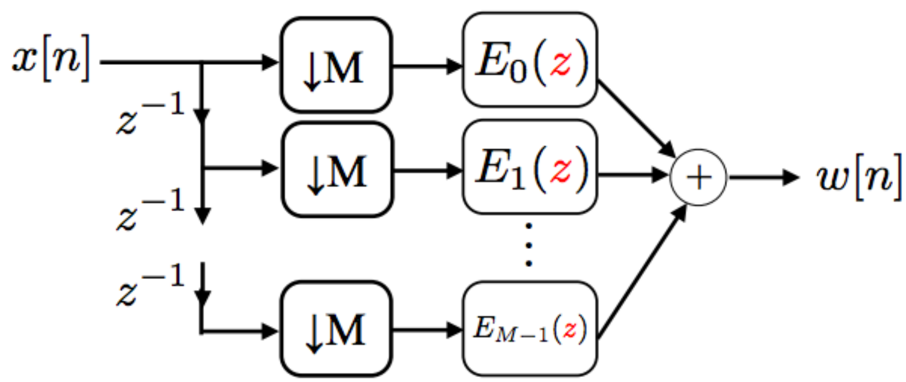


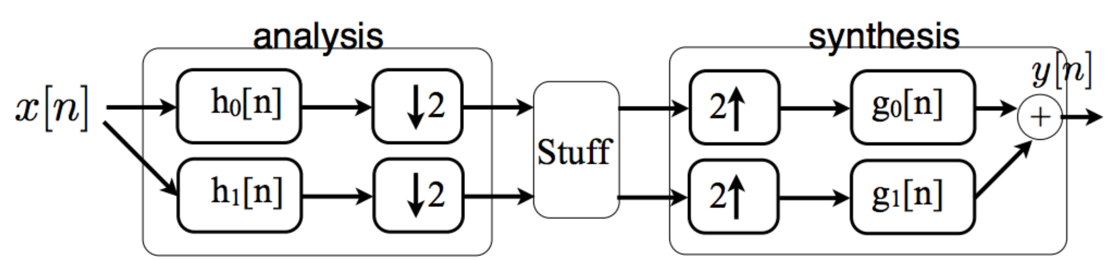
$$
\begin{aligned}
e_{00} &= h_0[2n] \\
e_{01} &= h_0[2n+1] \\
e_{10} &= e_{00}[n] \\
e_{11} &= -e_{01}[n]
\end{aligned}
$$

Big Ideas

- Multi-rate systems enable more efficient processing
  - Interchanging Operations
  - Polyphase Decomposition
  - Multi-Rate Filter Banks







Admin

- HW 4 due Sunday