

ESE 5310: Digital Signal Processing

Lecture 12: March 12, 2024

Frequency Response of LTI Systems

Frequency Response of LTI Systems



Lecture Outline

- Frequency Response of LTI Systems
 - Magnitude Response
 - Simple Filters
 - Phase Response
 - Group Delay
 - Example: Zero on Real Axis
 - Example: Higher order systems

Frequency Response of LTI System

- LTI Systems are uniquely determined by their impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[k] * h[k]$$

- We can write the input-output relation also in the z-domain

$$Y(z) = H(z)X(z)$$

- Or we can define an LTI system with its frequency response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- $H(e^{j\omega})$ defines magnitude and phase change at each frequency



Frequency Response of LTI System

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- We can define a magnitude response

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

- And a phase response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$



Phase Response

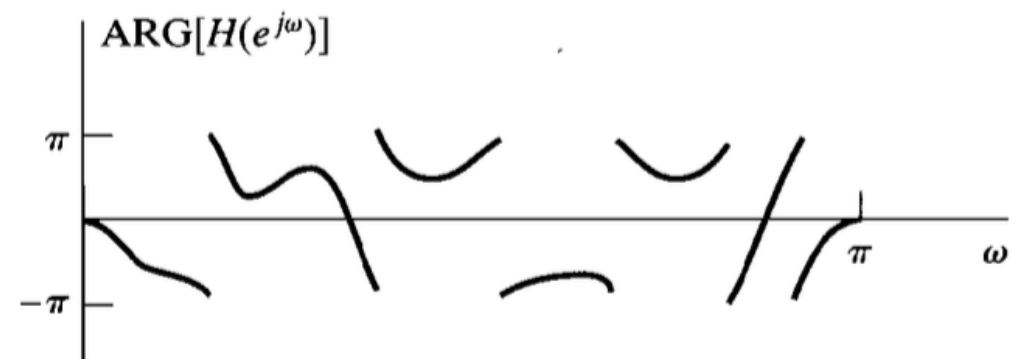
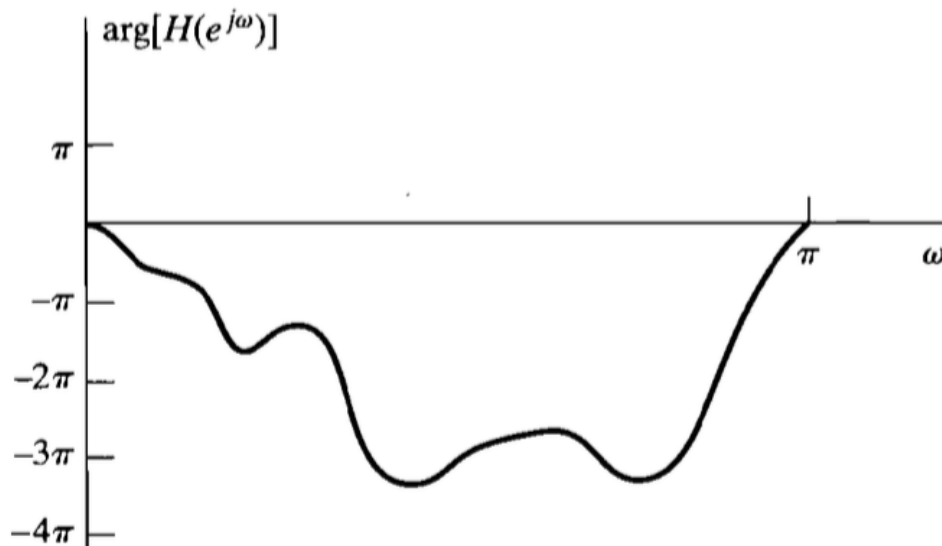
- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$

Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$



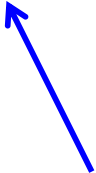
Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

- More later...

Unwrapped phase





Linear Difference Equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^M (1 - c_k z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$



Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

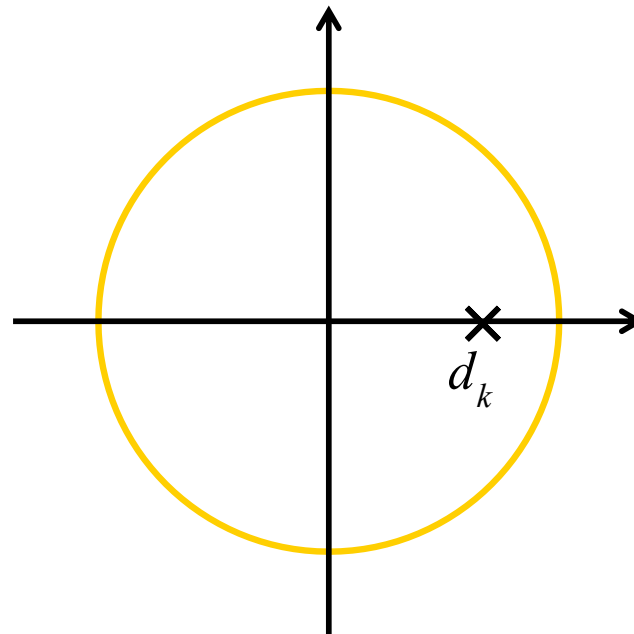
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



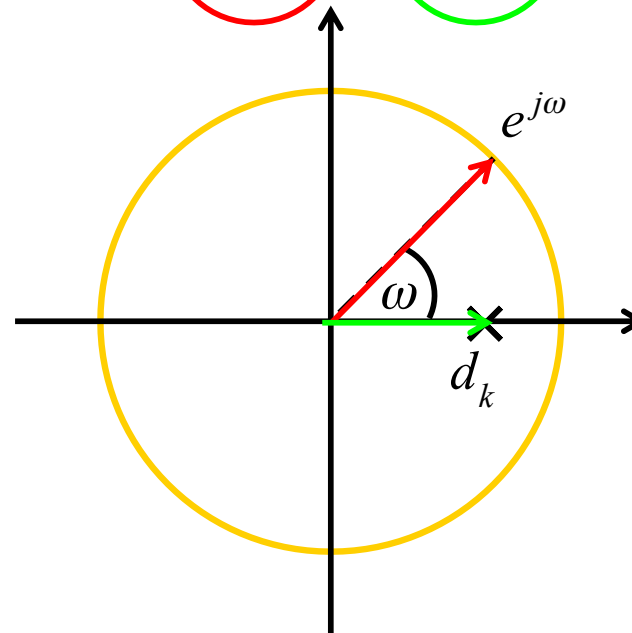
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



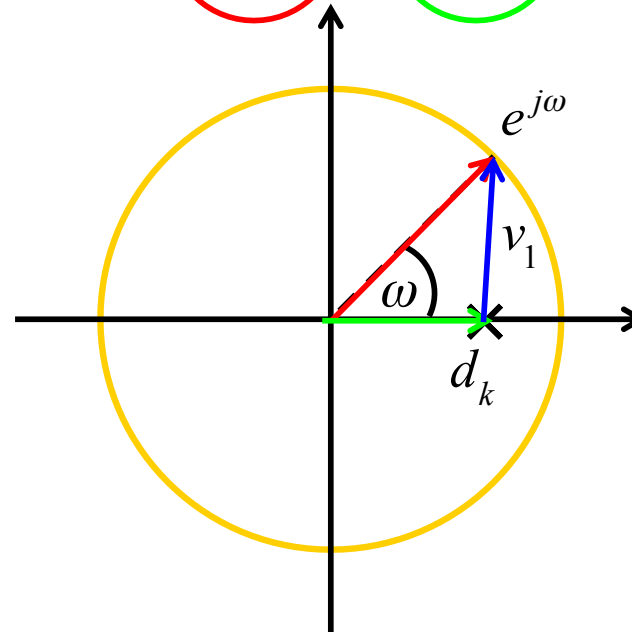
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



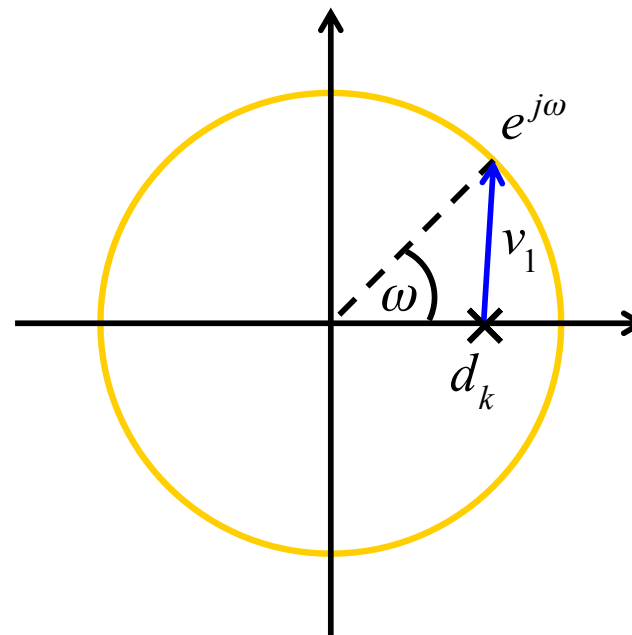
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



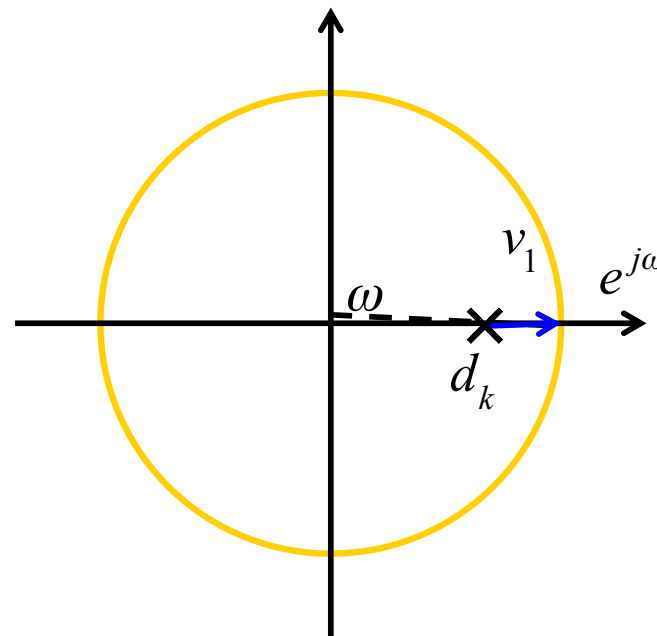
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



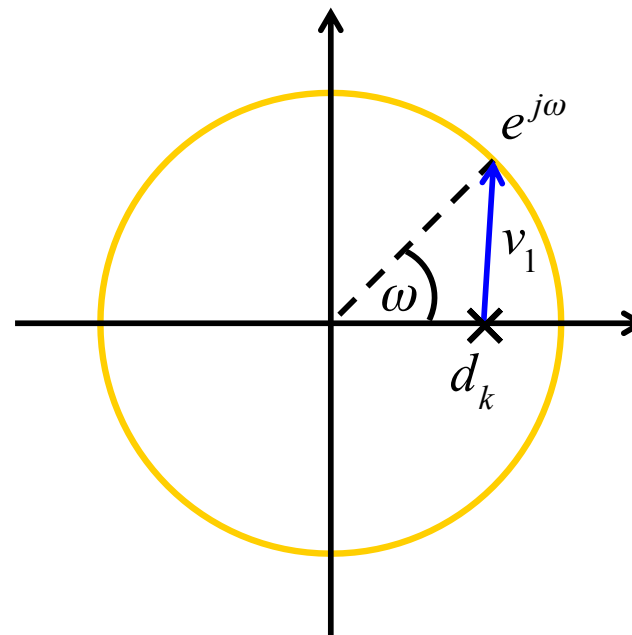
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



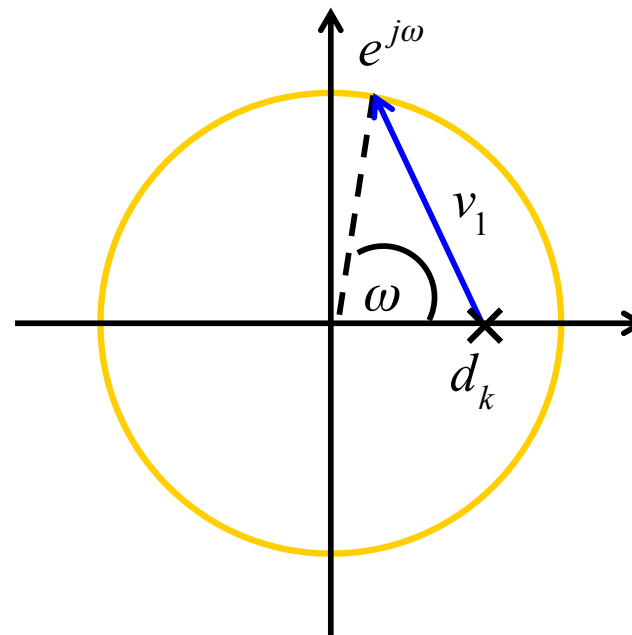
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



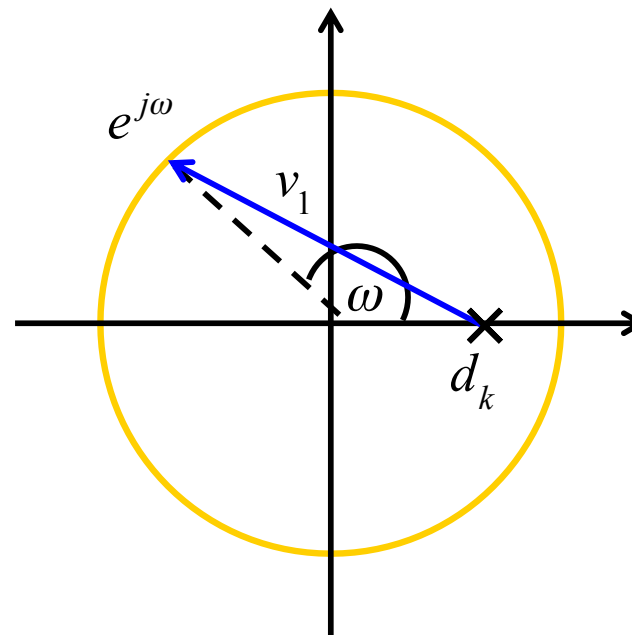
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



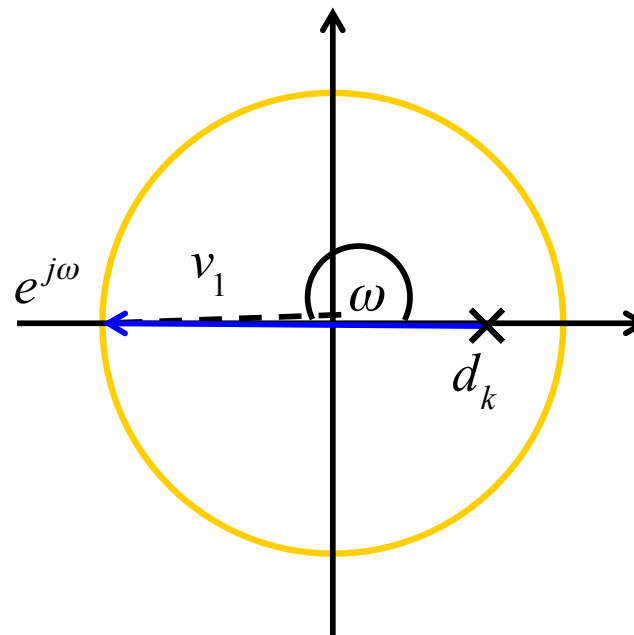
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



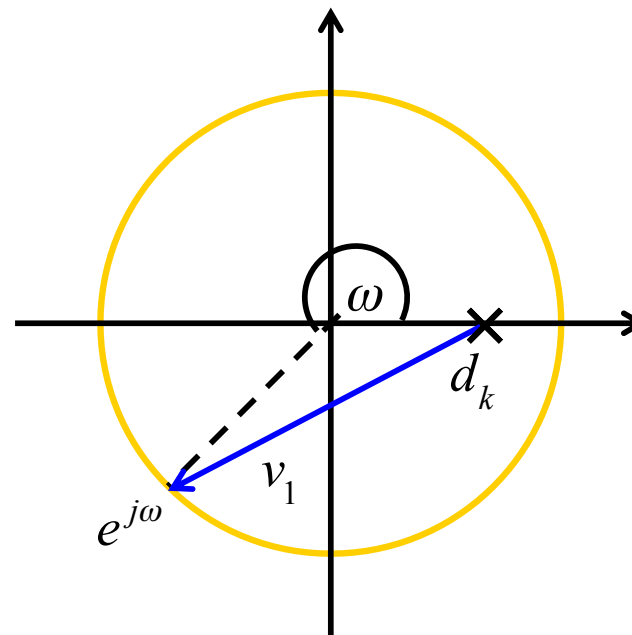
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



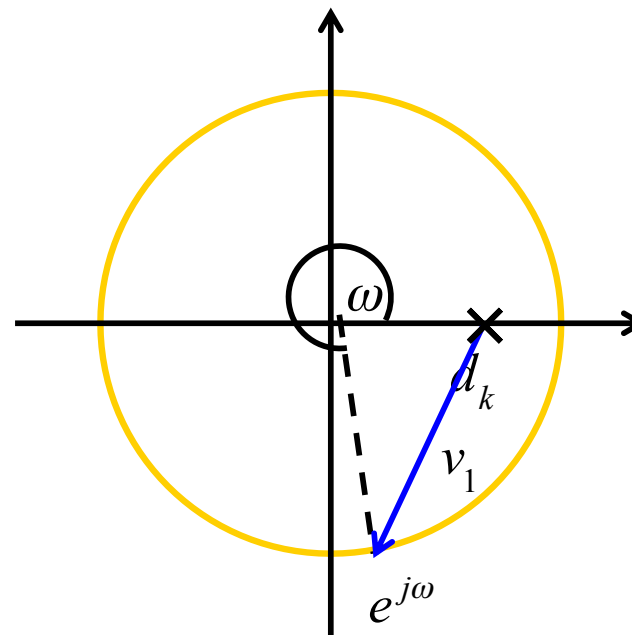
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$



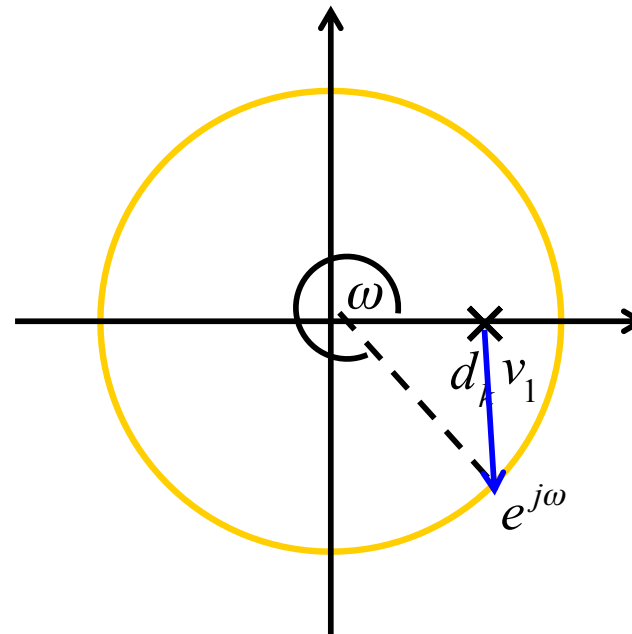
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$





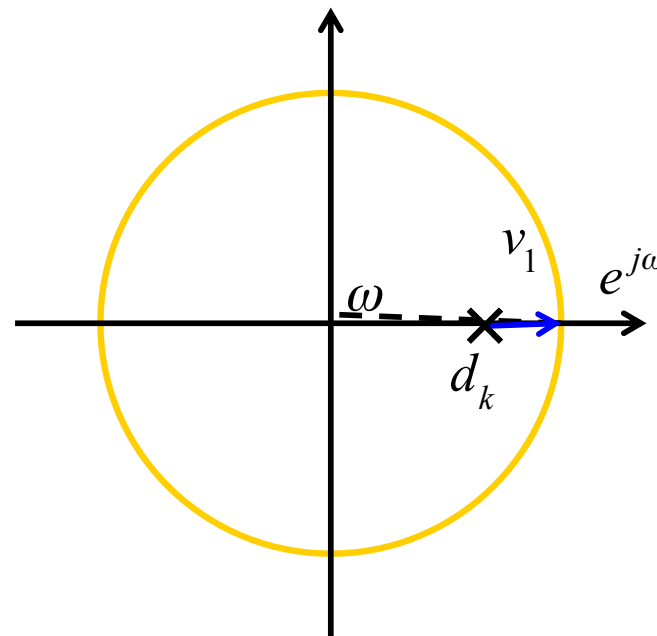
Magnitude Response

Magnitude of products is product of magnitudes

$$|H(e^{j\omega})| = \left| \frac{b_0}{a_0} \right| \cdot \frac{\prod_{k=0}^M |1 - c_k e^{-j\omega}|}{\prod_{k=0}^N |1 - d_k e^{-j\omega}|}$$

Consider one of the poles:

$$|1 - d_k e^{-j\omega}| = |e^{+j\omega} - d_k| = |v_1|$$

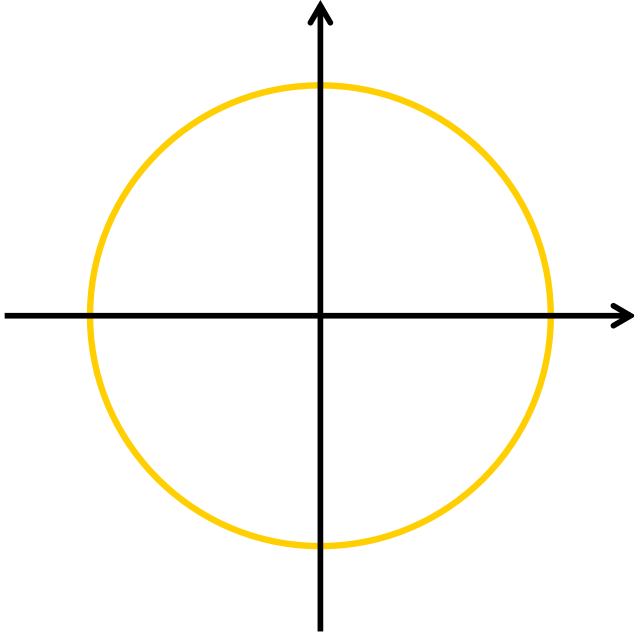




Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$

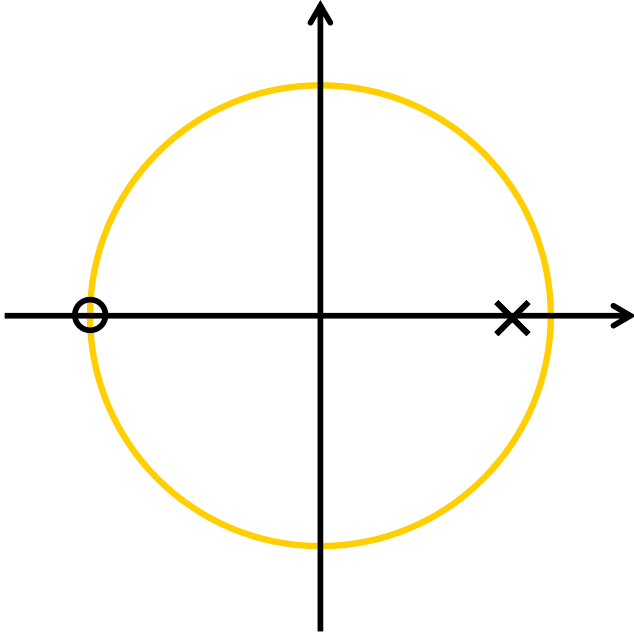




Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

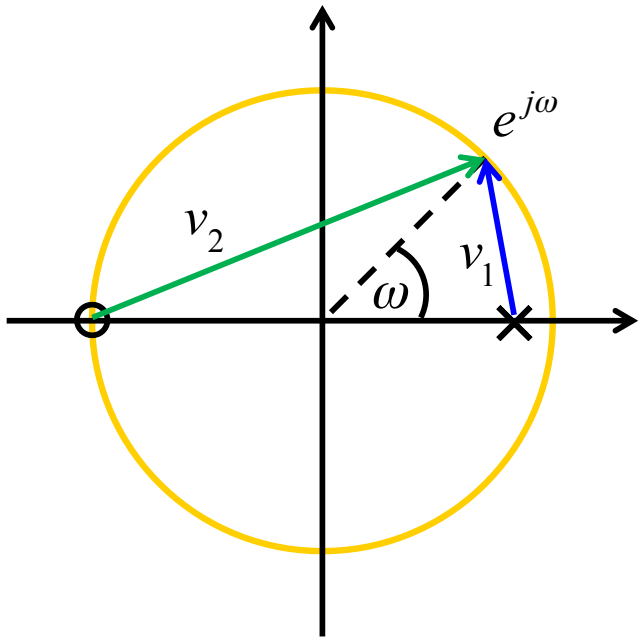
$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$

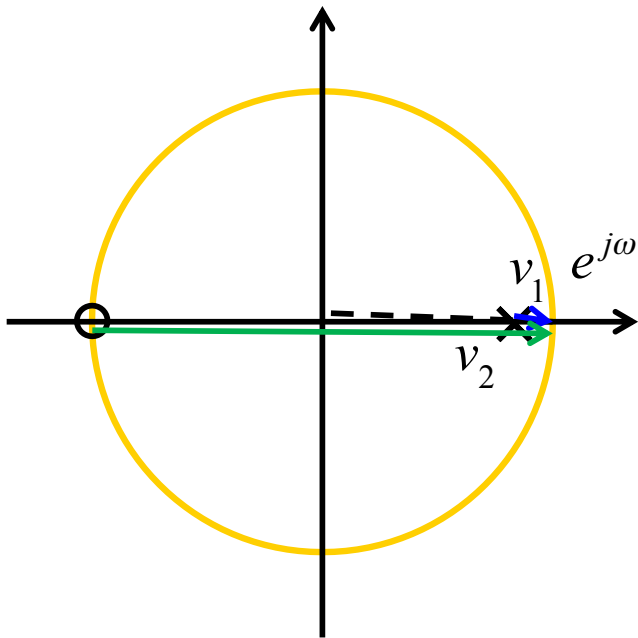




Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

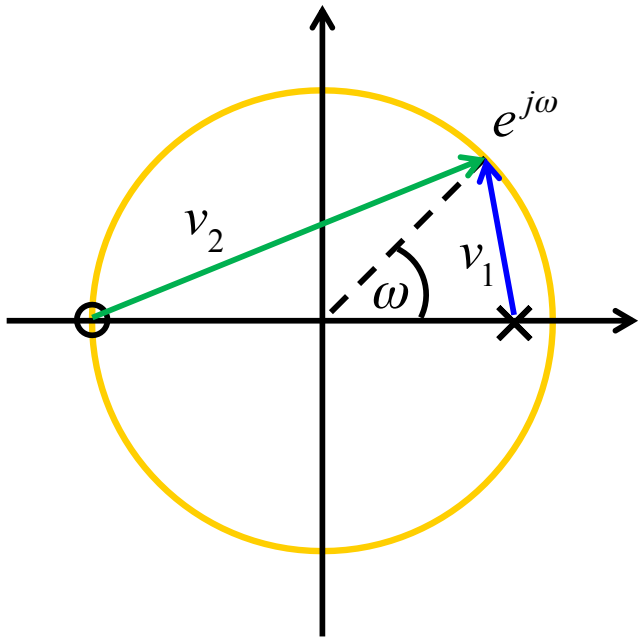
$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

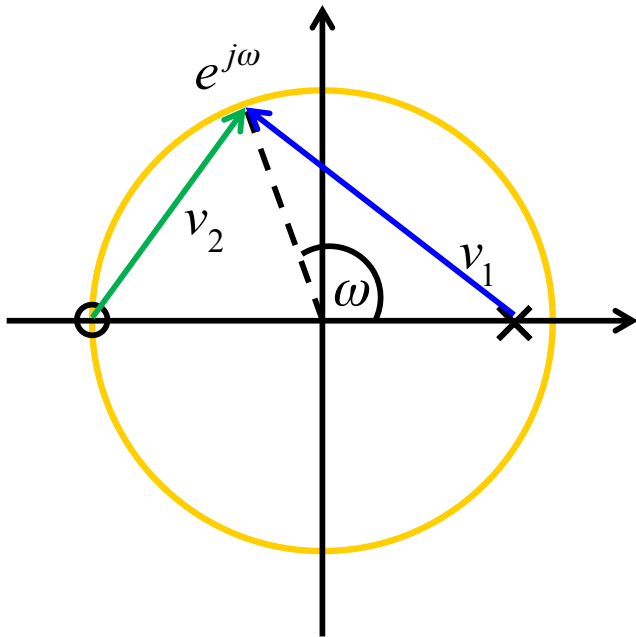
$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

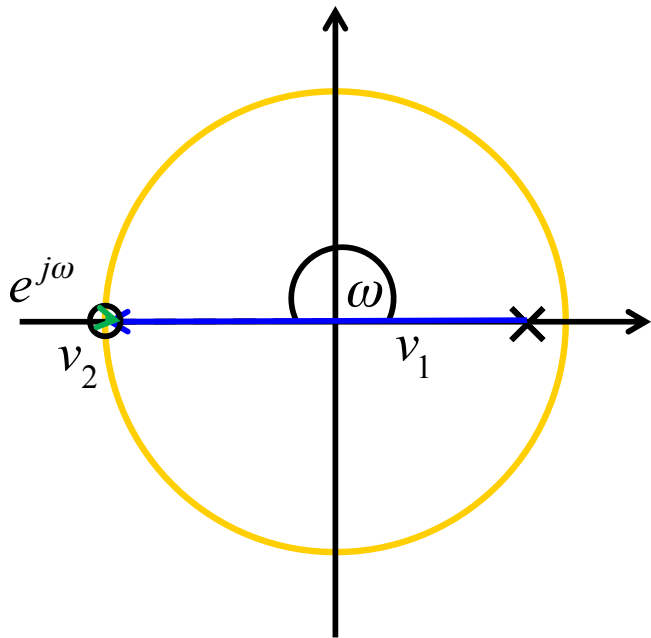
$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$

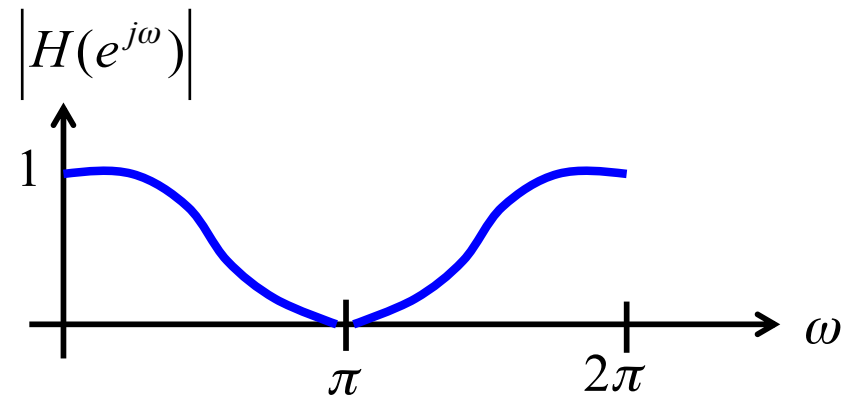
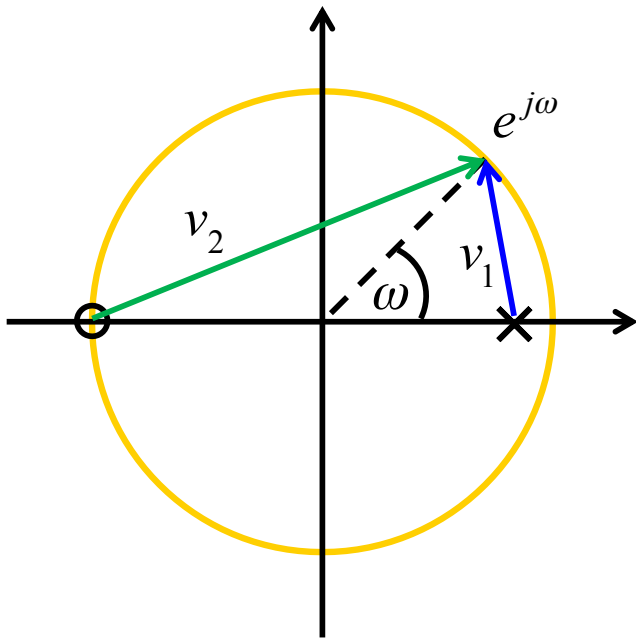




Magnitude Response Example

$$H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(z)| = 0.05 \frac{|v_2|}{|v_1|}$$



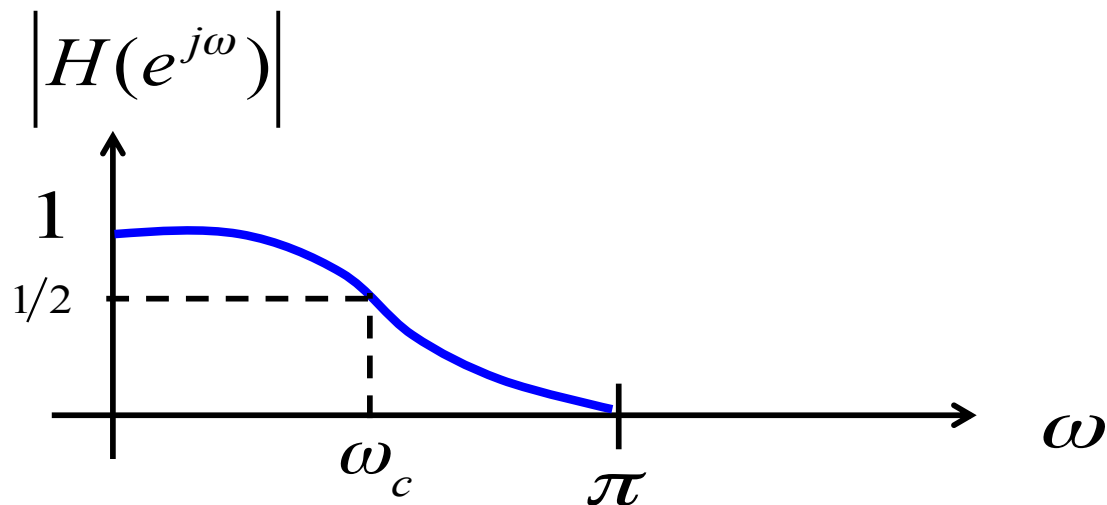


Simple Low Pass Filter

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$

Simple Low Pass Filter

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$



ω_c is the 3dB cutoff frequency

$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$



Simple Low Pass Filter $\rightarrow$ High Pass

$$H_{LP}(z) = \frac{1 - \alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}} \quad |\alpha| < 1$$



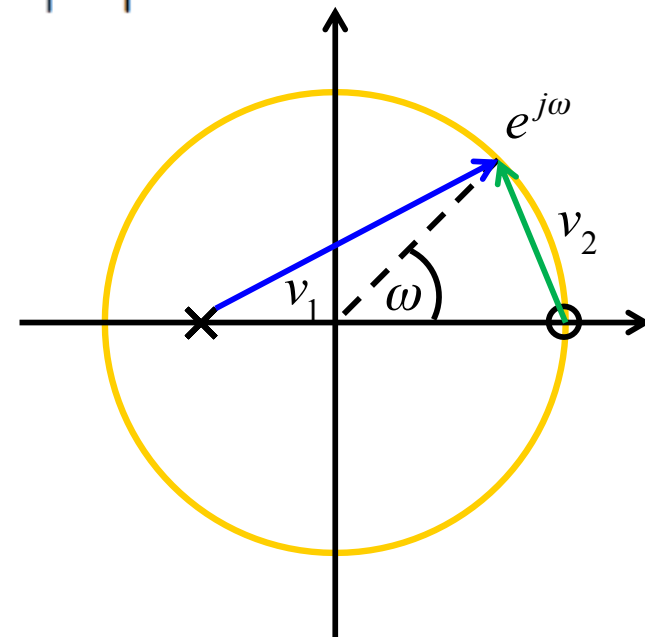
Simple High Pass Filter

$$H_{HP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-1}}{1 + \alpha z^{-1}} \quad |\alpha| < 1$$

Simple High Pass Filter

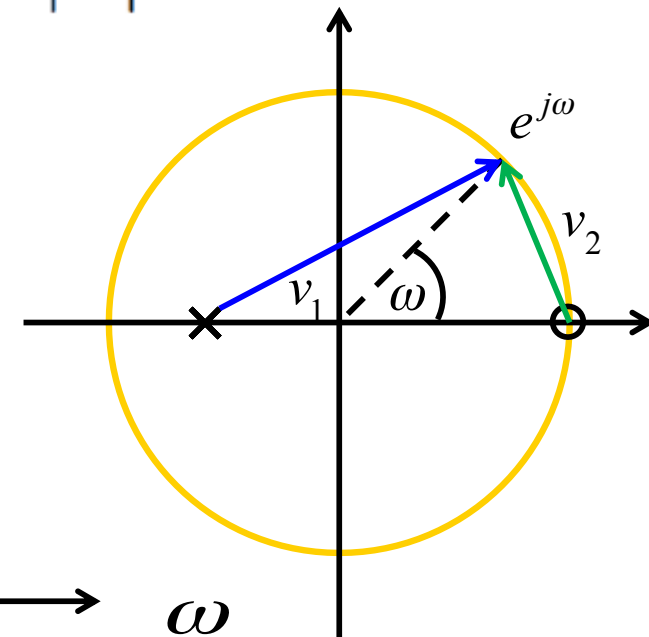
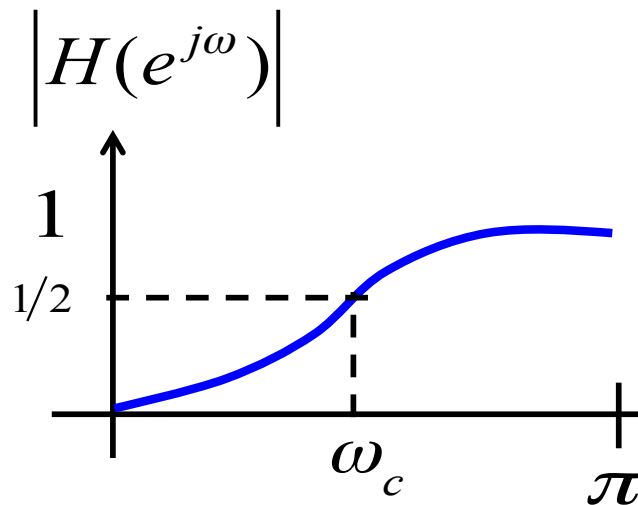
$$H_{HP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-1}}{1 + \alpha z^{-1}}$$

$$|\alpha| < 1$$



Simple High Pass Filter

$$H_{HP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-1}}{1 + \alpha z^{-1}} \quad |\alpha| < 1$$



ω_c is the 3dB cutoff frequency

$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)}$$

Simple Band-Stop (Notch) Filter

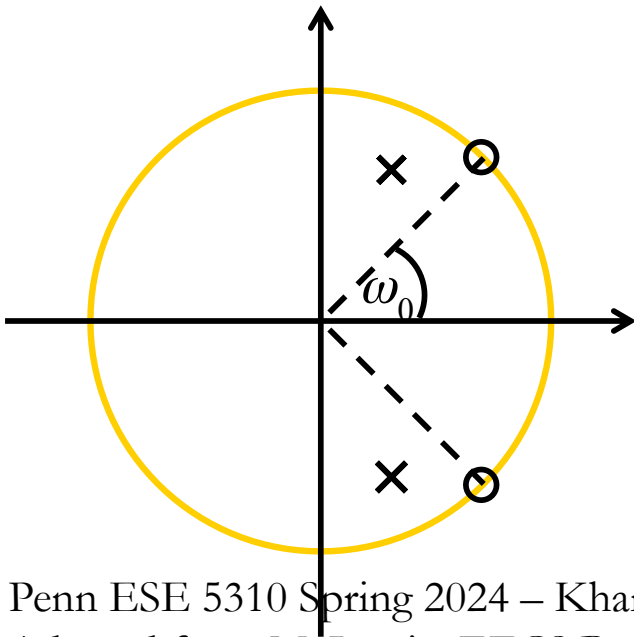
$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$
 $\cos(\omega_0) = \beta$

Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

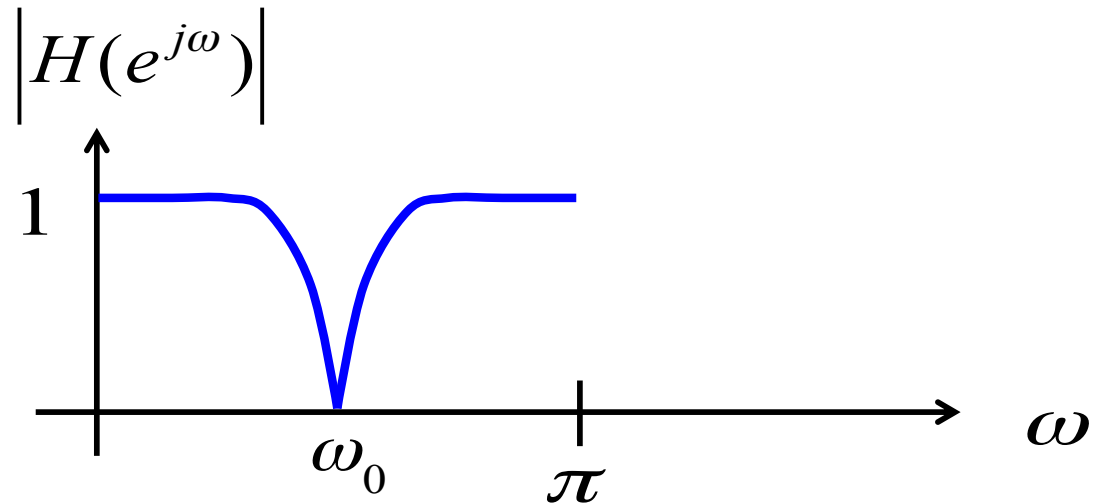
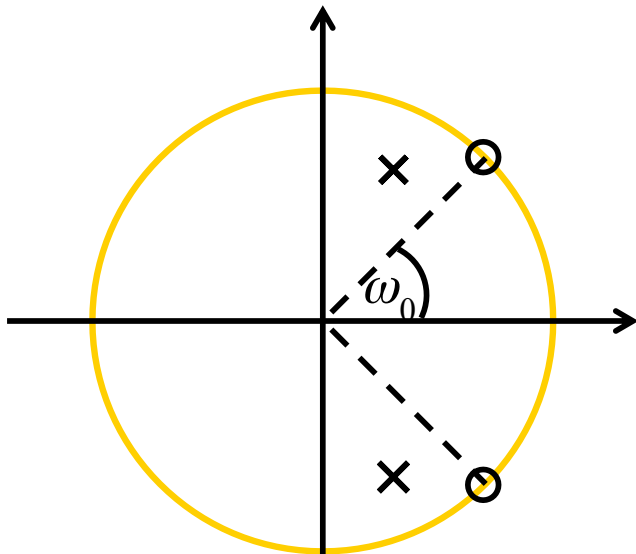
Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$
 $\cos(\omega_0) = \beta$



Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$
 $\cos(\omega_0) = \beta$

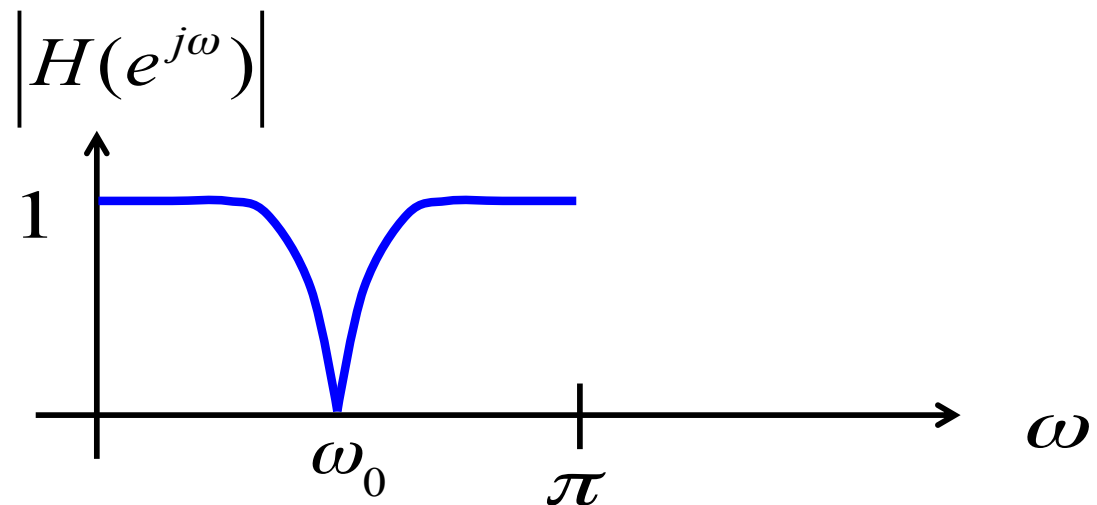
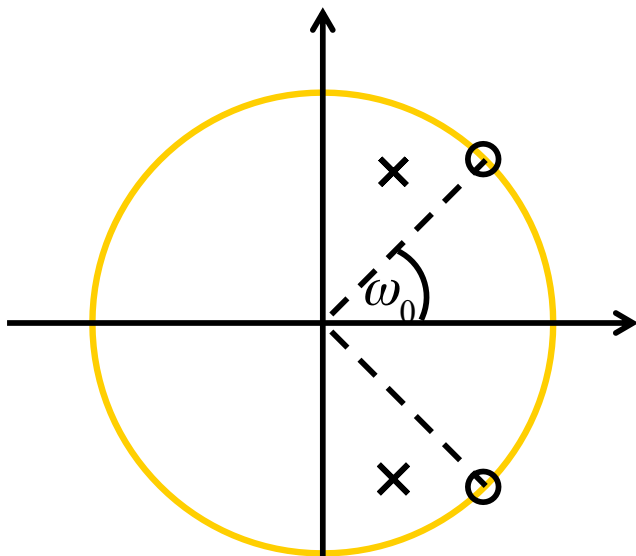


Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note:

$$H_{BS}(\mp 1) = \frac{1 + \alpha}{2} \frac{2 \pm 2\beta}{(1 + \alpha)(1 \pm \beta)} = 1$$

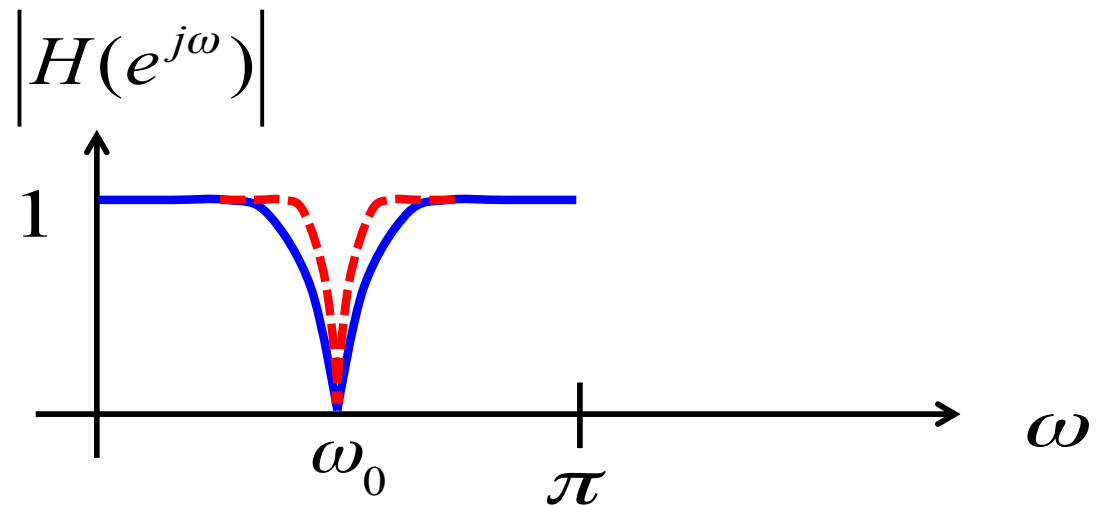
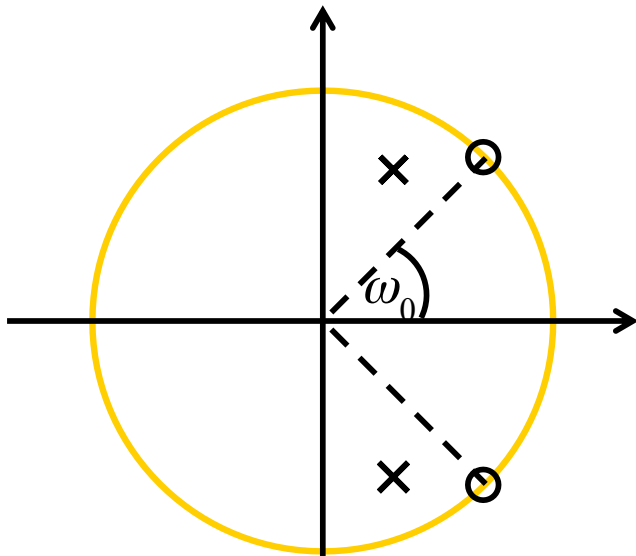


Simple Band-Stop (Notch) Filter

$$H_{BS}(z) = \frac{1 + \alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

Note: As $\alpha \rightarrow 1$ poles approach zeros

$$H_{BS}(\mp 1) = \frac{1 + \alpha}{2} \frac{2 \pm 2\beta}{(1 + \alpha)(1 \pm \beta)} = 1$$



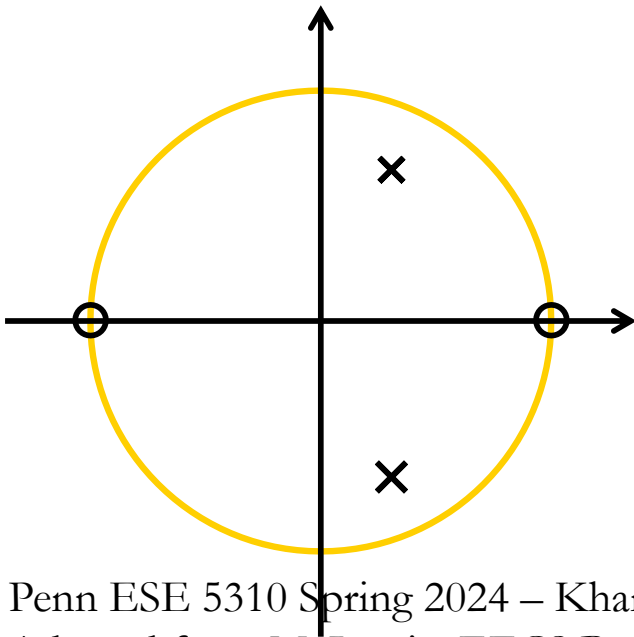


Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

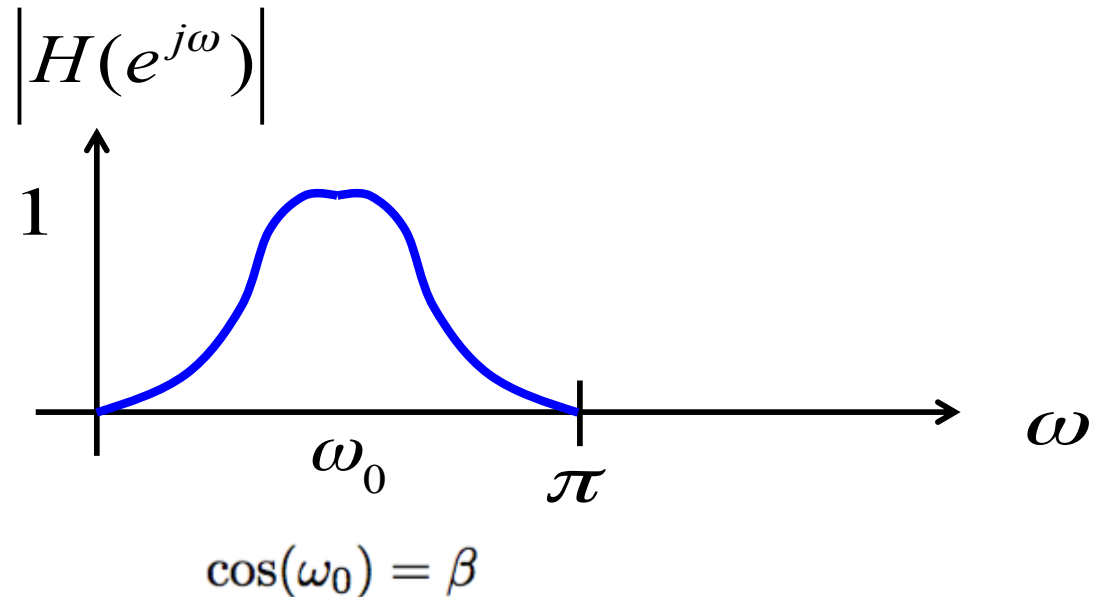
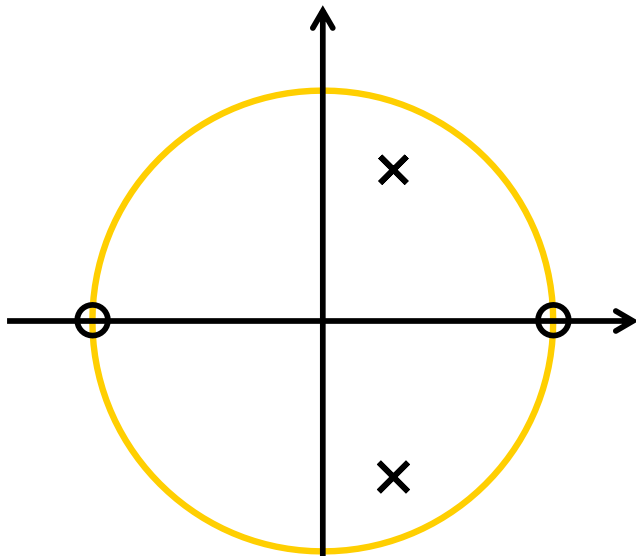
Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$



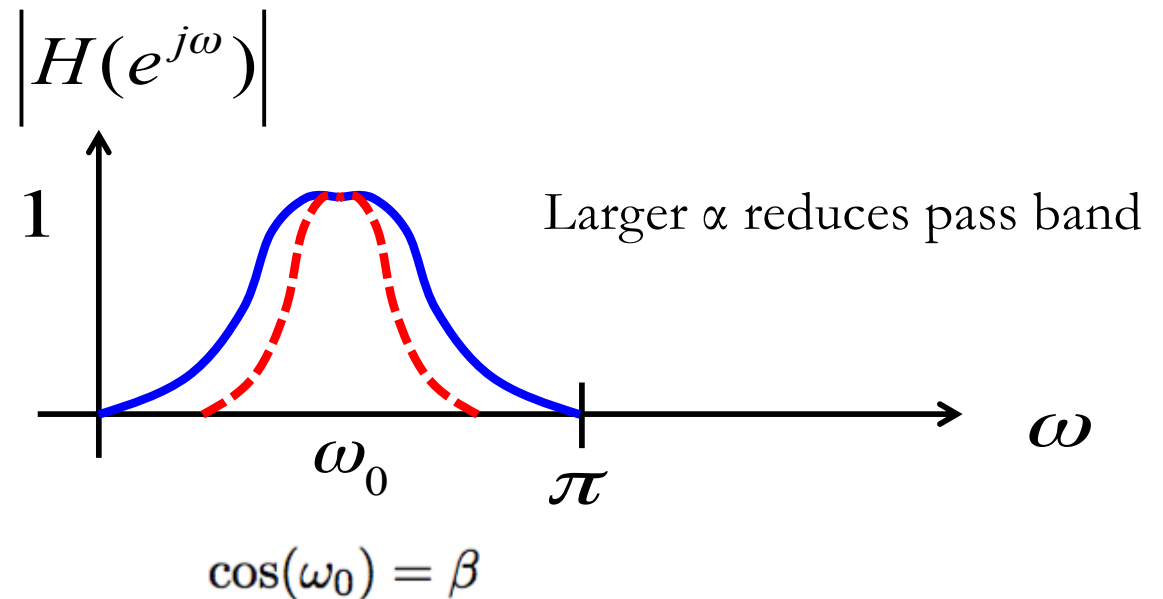
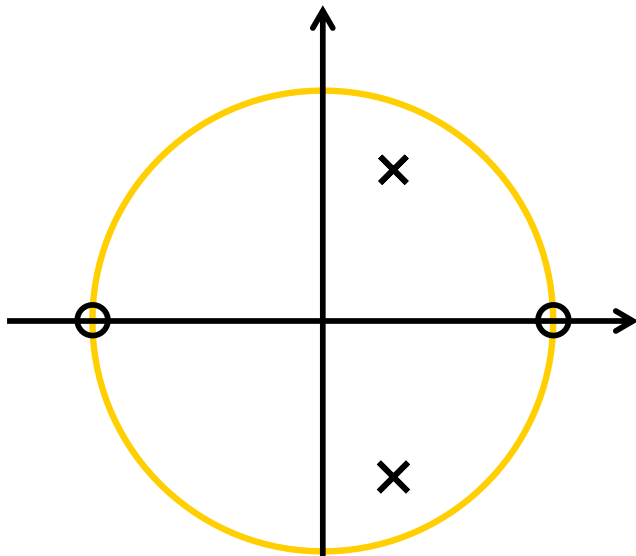
Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$



Simple Band-Pass Filter

$$H_{BP}(z) = \frac{1 - \alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}} \quad \begin{array}{l} |\alpha| < 1 \\ |\beta| < 1 \end{array}$$

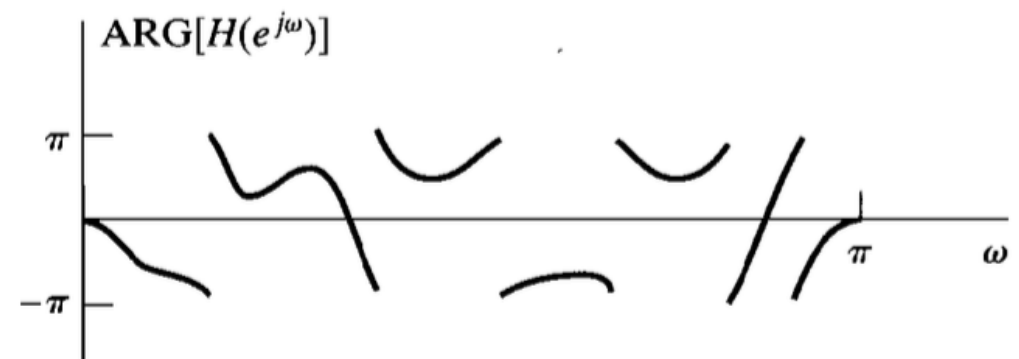
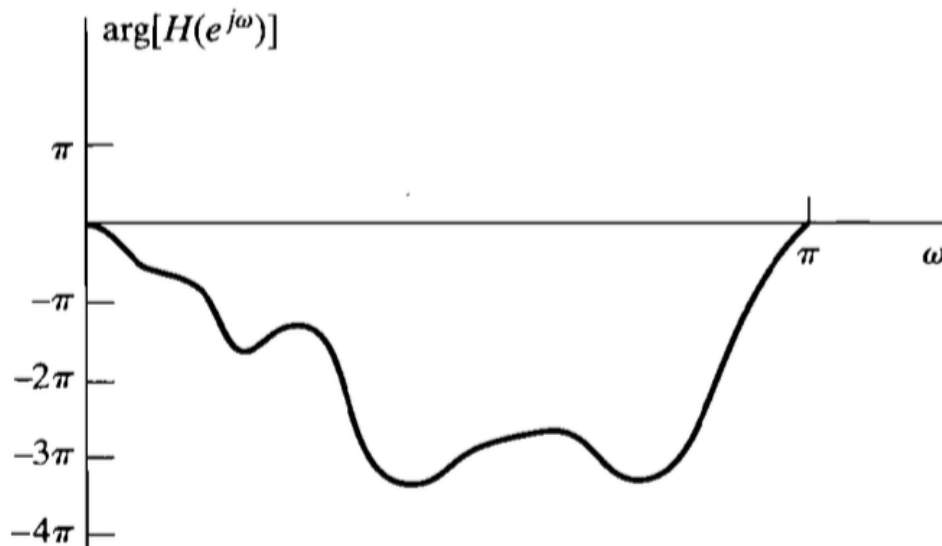




Phase Response

- Limit the range of the phase response

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi.$$





Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$



Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase

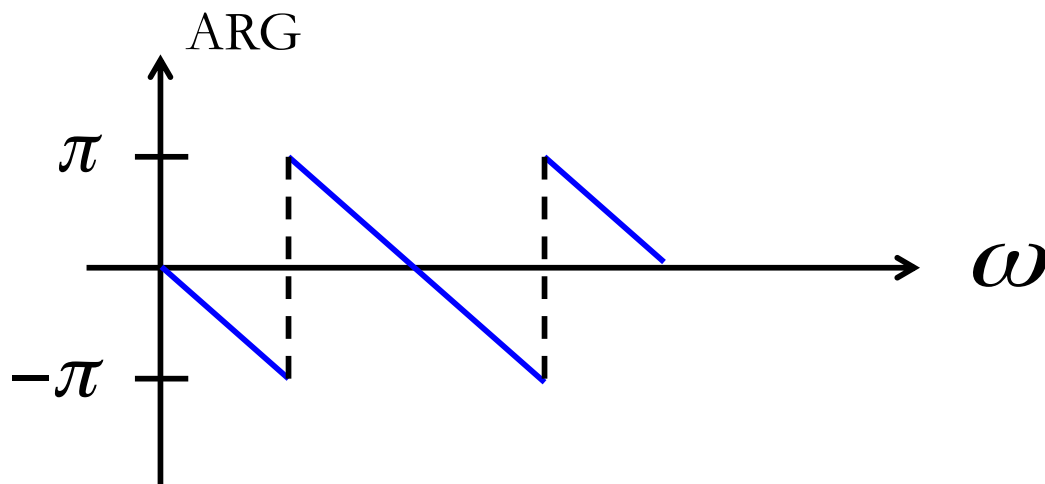
Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

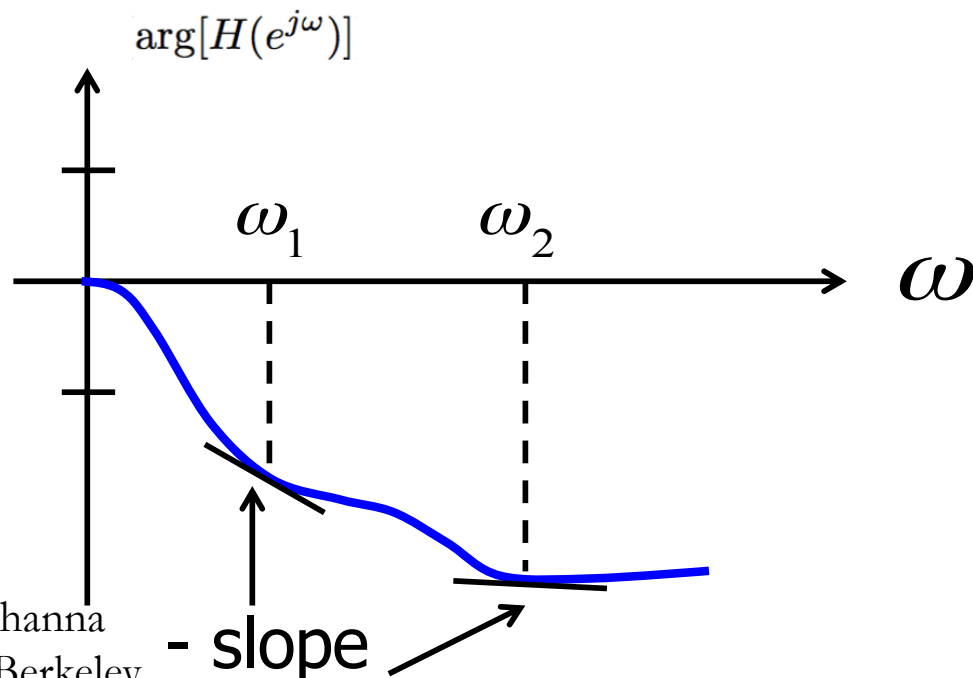
ARG is the wrapped phase
arg is the unwrapped phase



Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$



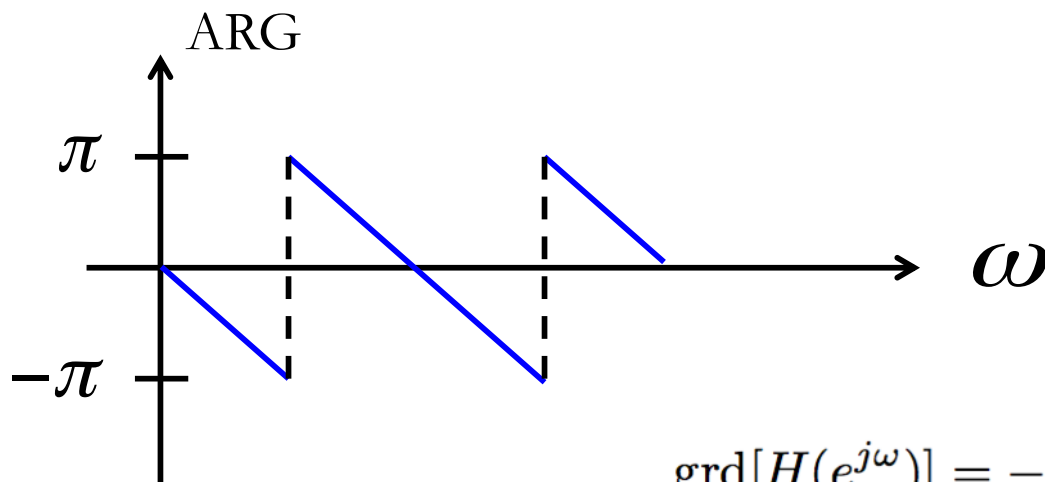
Phase Response Example

$$H(e^{j\omega}) = e^{j\omega n_d} \leftrightarrow h[n] = \delta[n - n_d]$$

$$|H(e^{j\omega})| = 1$$

$$\arg[H(e^{j\omega})] = -\omega n_d$$

ARG is the wrapped phase
arg is the unwrapped phase



$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$$

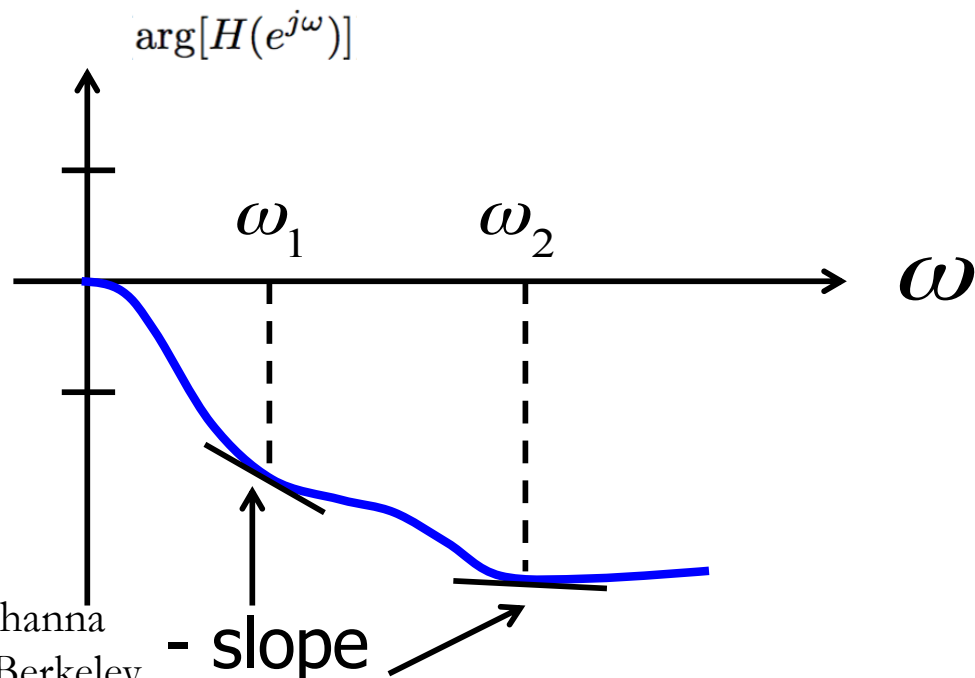
For linear phase system, group delay is n_d



Group Delay

- General phase response at a given frequency can be characterized with group delay, which is related to phase

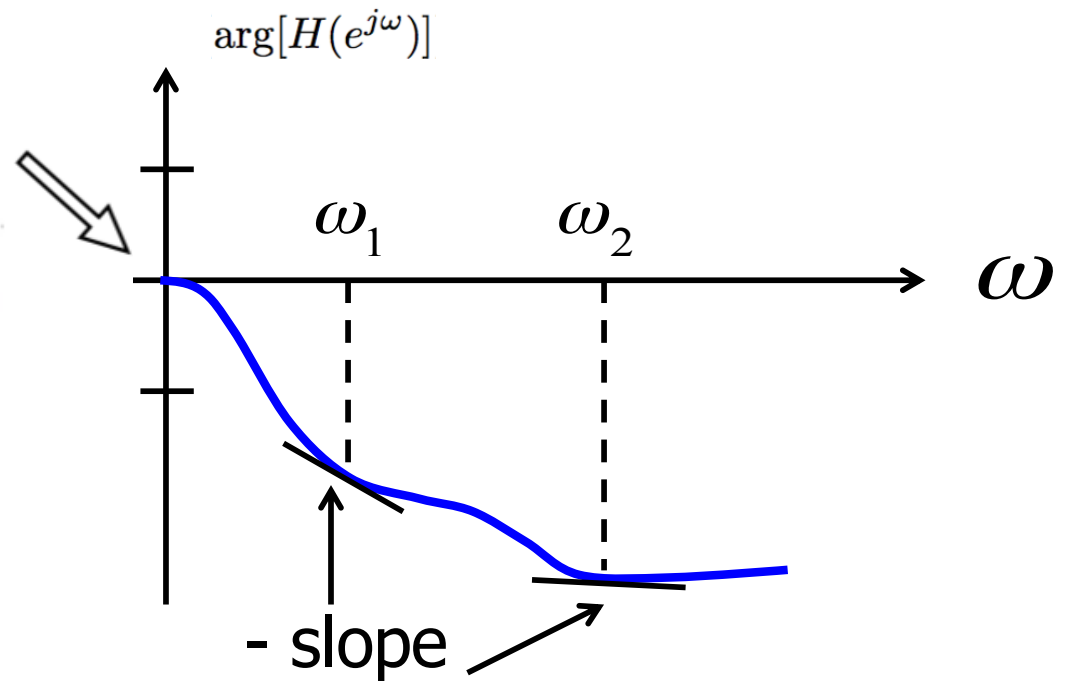
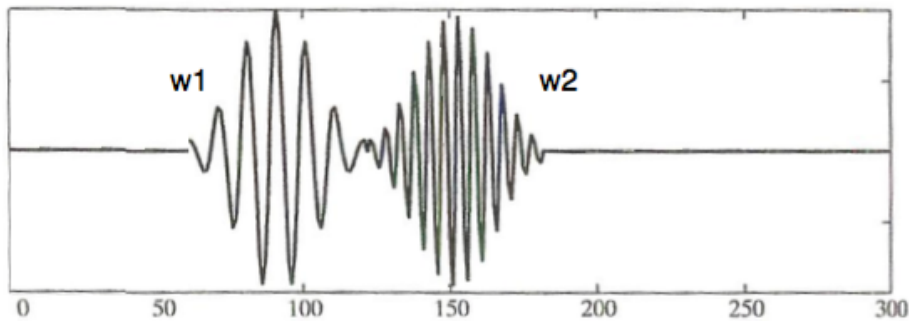
$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$



Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

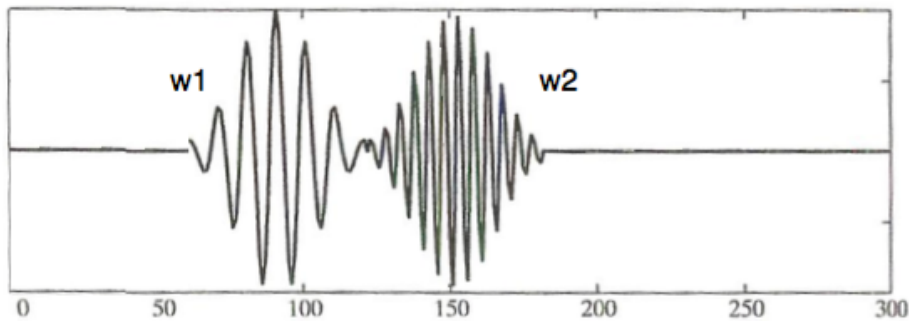
Input



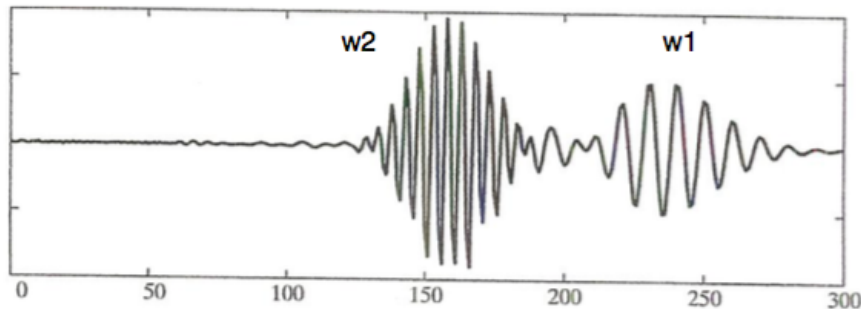
Group Delay

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$$

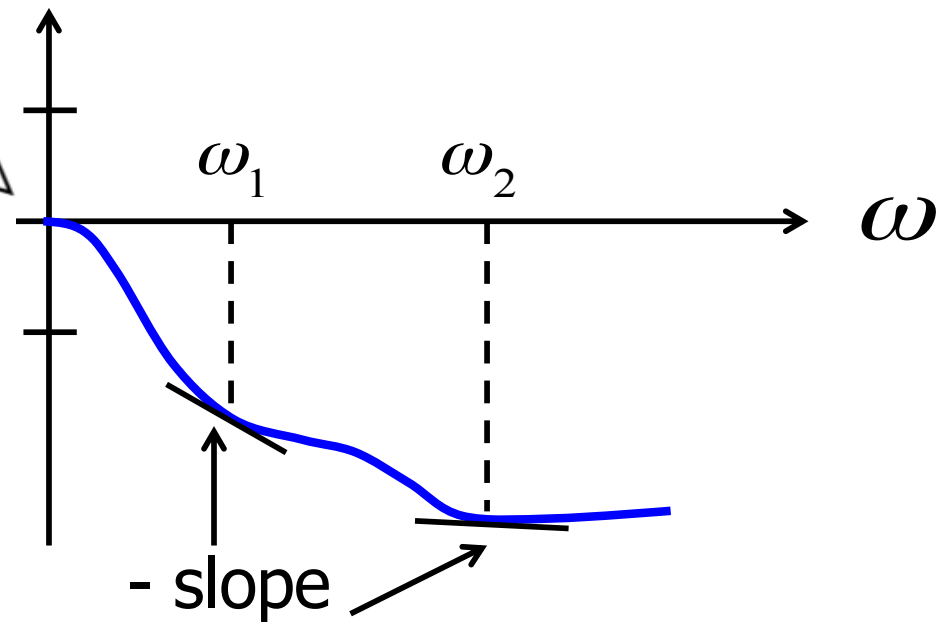
Input



Output



$\arg[H(e^{j\omega})]$





Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

Group Delay Math

$$H(z) = \frac{b_0 \prod_{k=1}^M (1 - c_k z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

$$H(e^{j\omega}) = \frac{b_0 \prod_{k=1}^M (1 - c_k e^{-j\omega})}{a_0 \prod_{k=1}^N (1 - d_k e^{-j\omega})}$$

arg of products is sum of args

$$\arg[H(e^{j\omega})] = \sum_{k=1}^M \arg[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \arg[1 - d_k e^{-j\omega}]$$

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$



Group Delay Math

- Look at each factor:

$$[1 - re^{j\theta}e^{-j\omega}] =$$



Group Delay Math

□ Look at each factor:

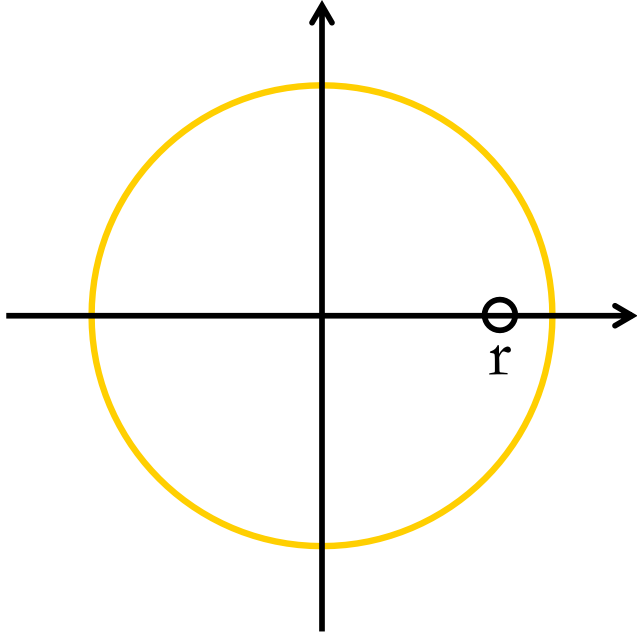
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

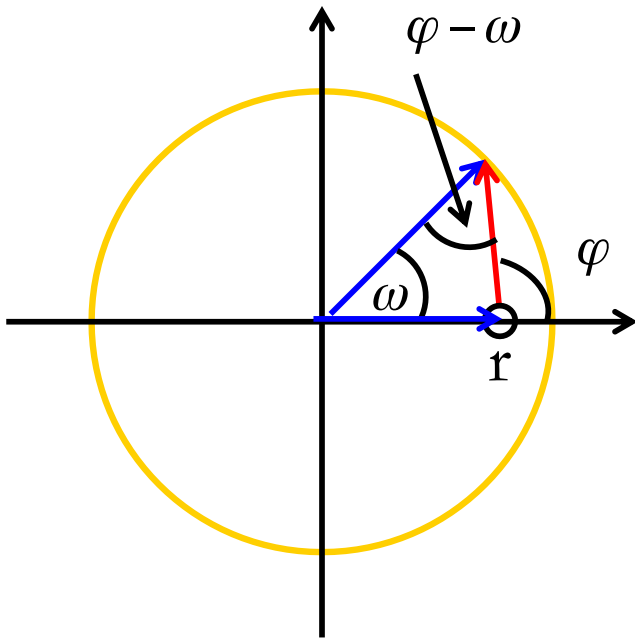
$$\arg[1 - re^{-j\omega}]$$



Example: Zero on Real Axis

- Geometric Interpretation for ($\theta=0$)

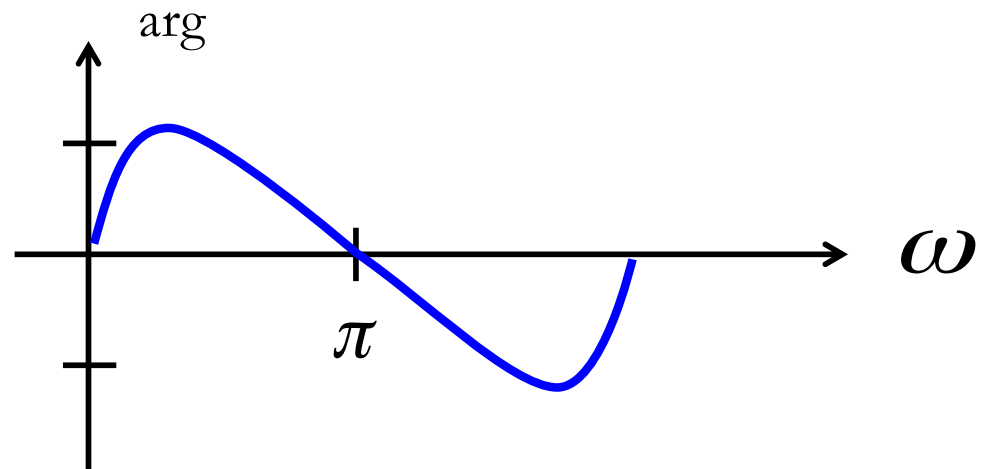
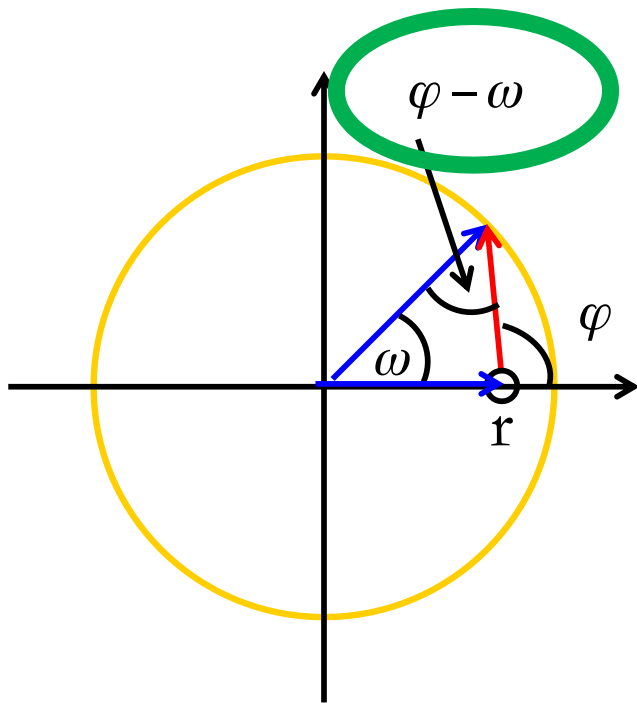
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

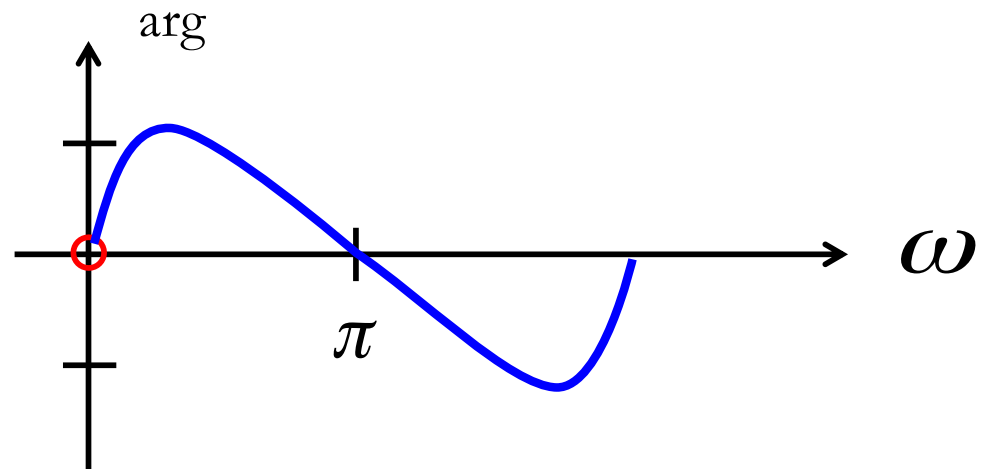
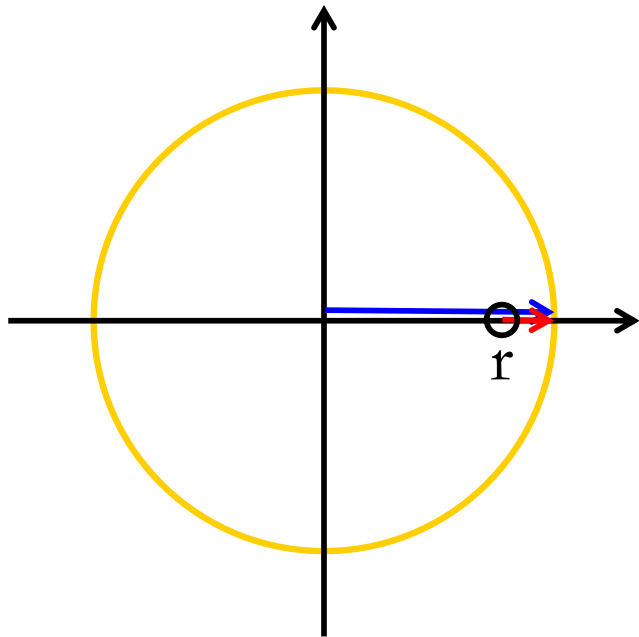


Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

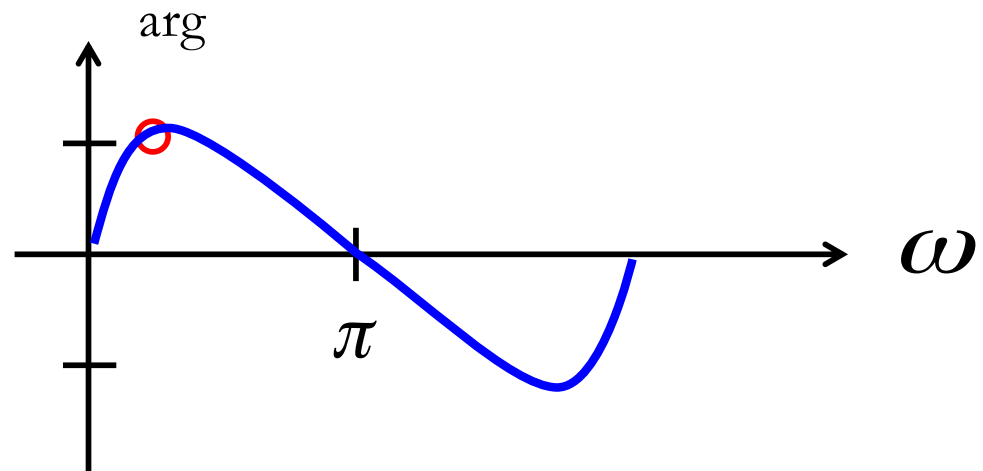
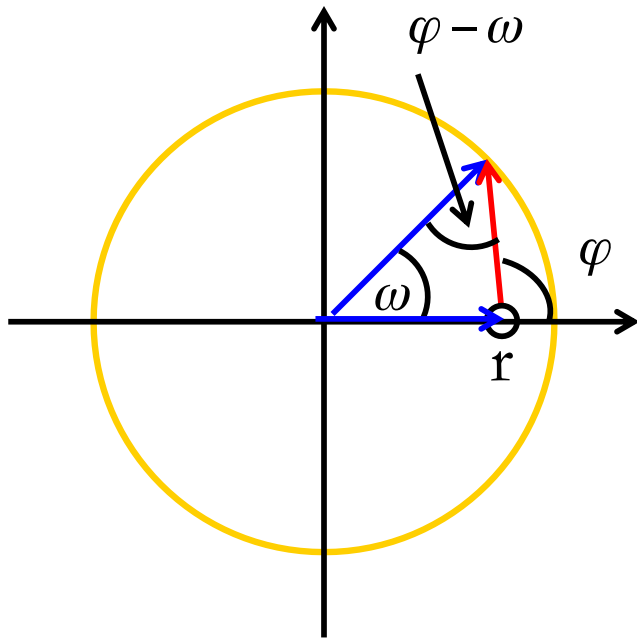
$\omega = 0$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

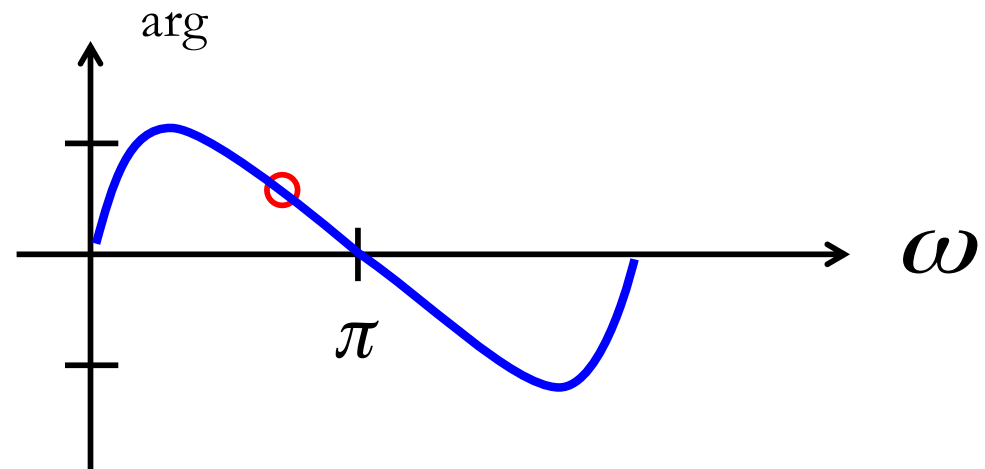
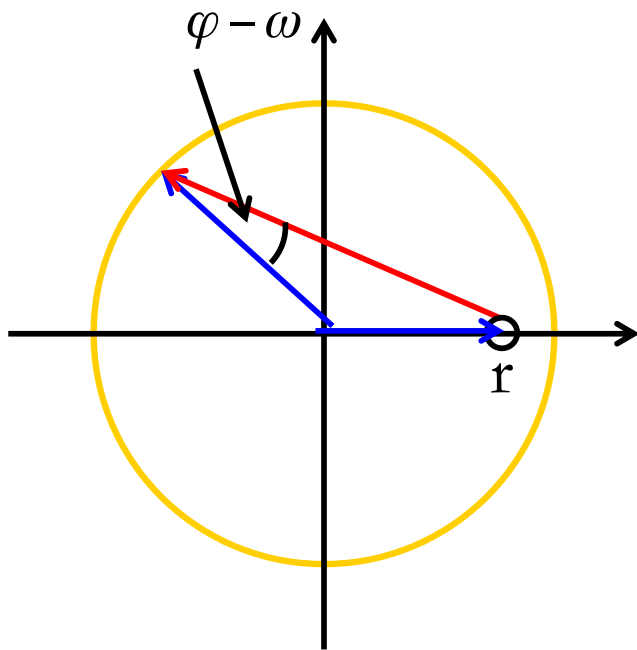
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

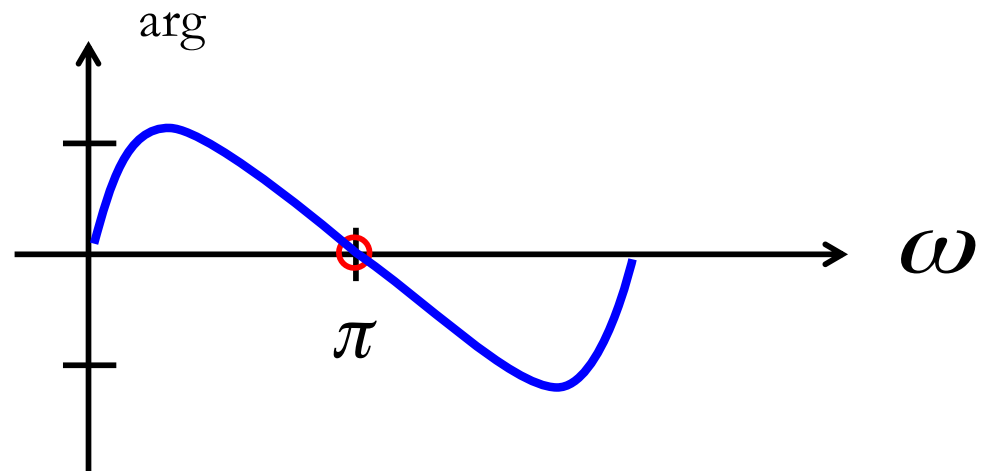
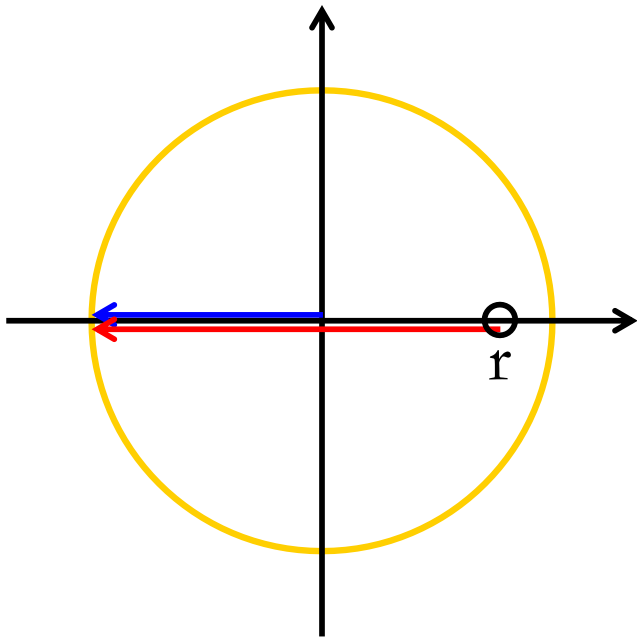


Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$

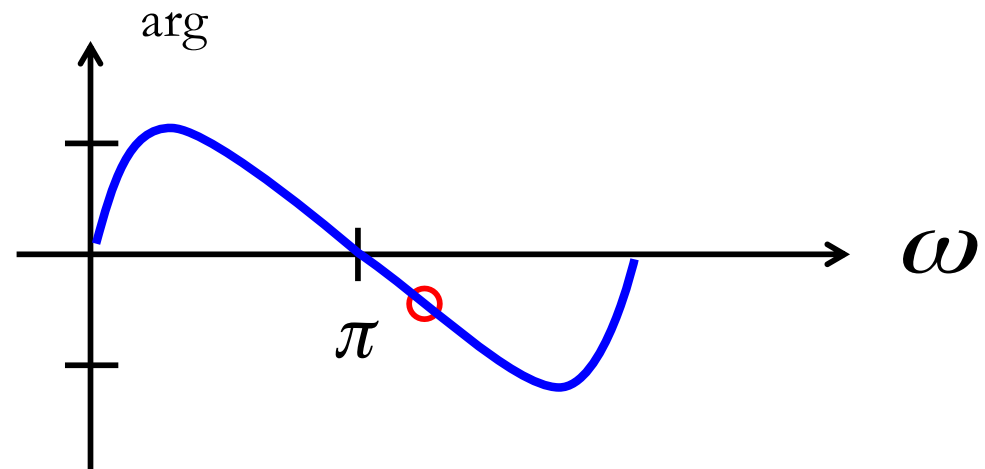
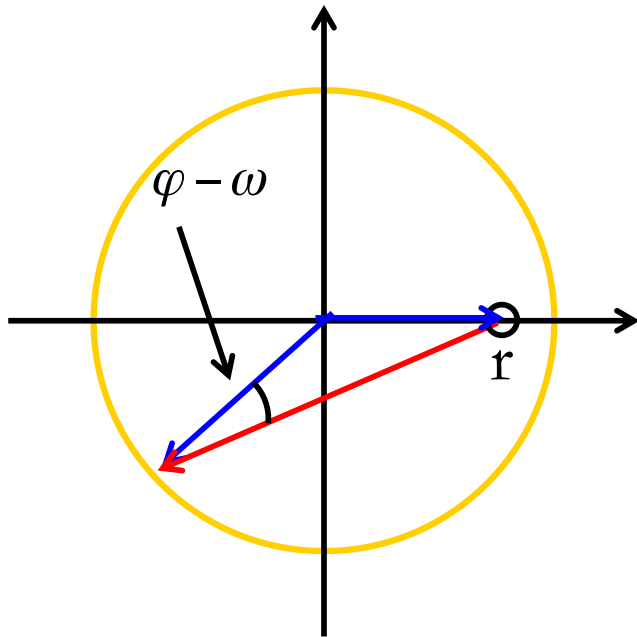
$$\omega = \pi$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

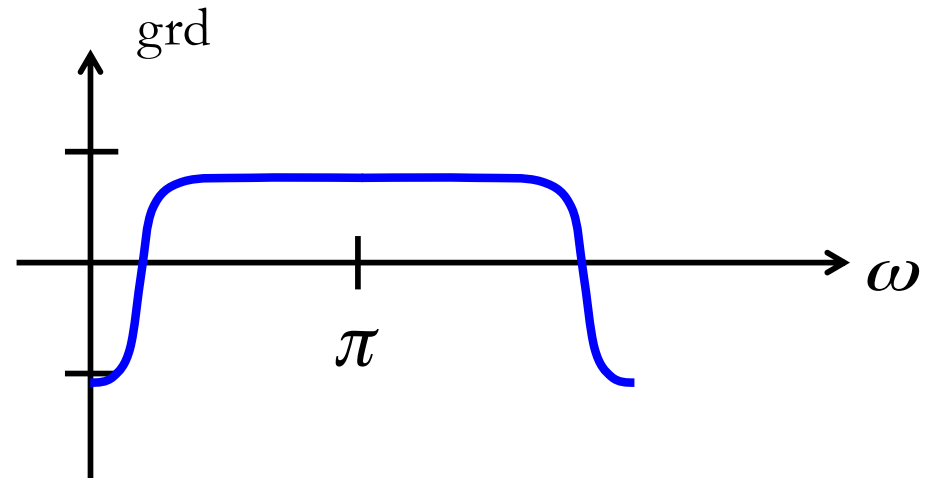
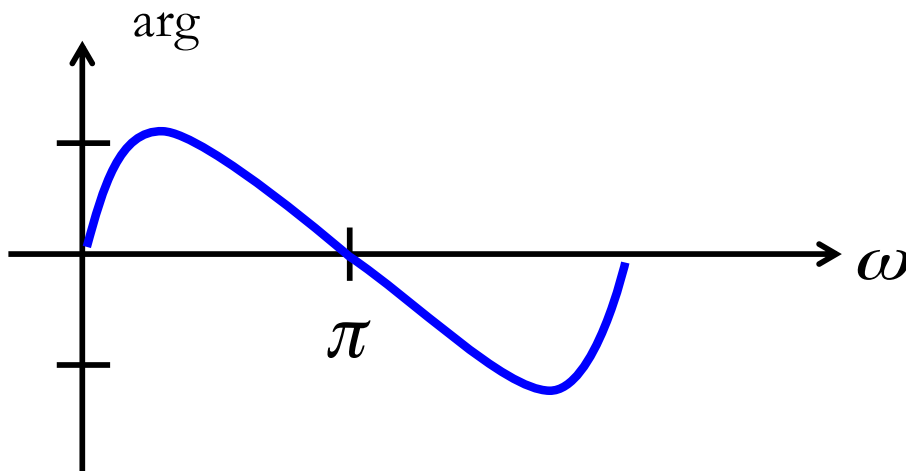
$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Example: Zero on Real Axis

□ Geometric Interpretation for ($\theta=0$)

$$\arg[1 - re^{-j\omega}] = \arg[(e^{j\omega} - r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega} - r]}_{\varphi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$



Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^M \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^N \text{grd}[1 - d_k e^{-j\omega}]$$

□ Look at each factor:

$\theta \neq 0?$

$$\arg[1 - re^{j\theta} e^{-j\omega}] = \tan^{-1} \left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)} \right)$$

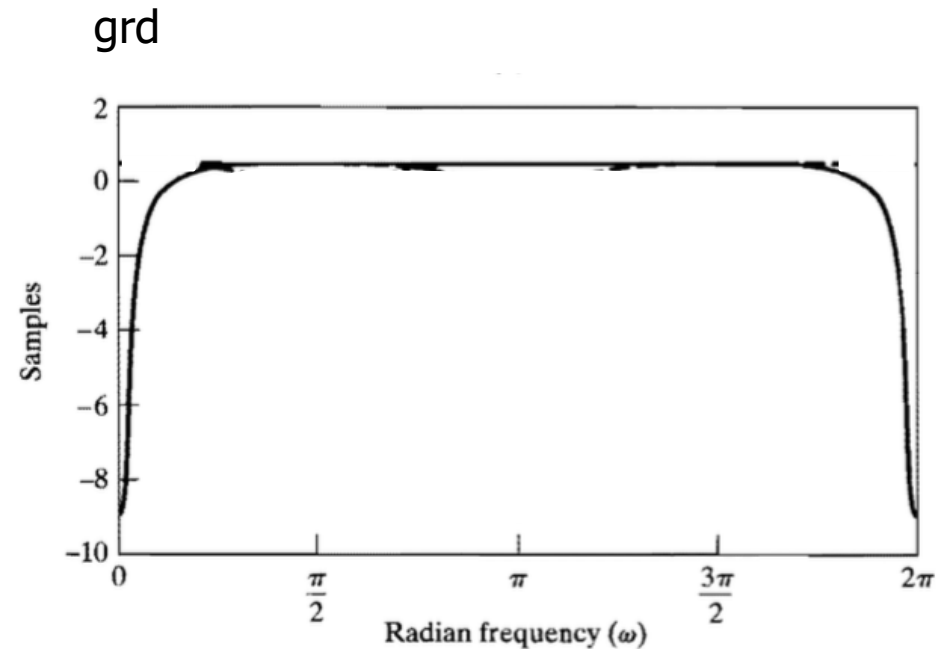
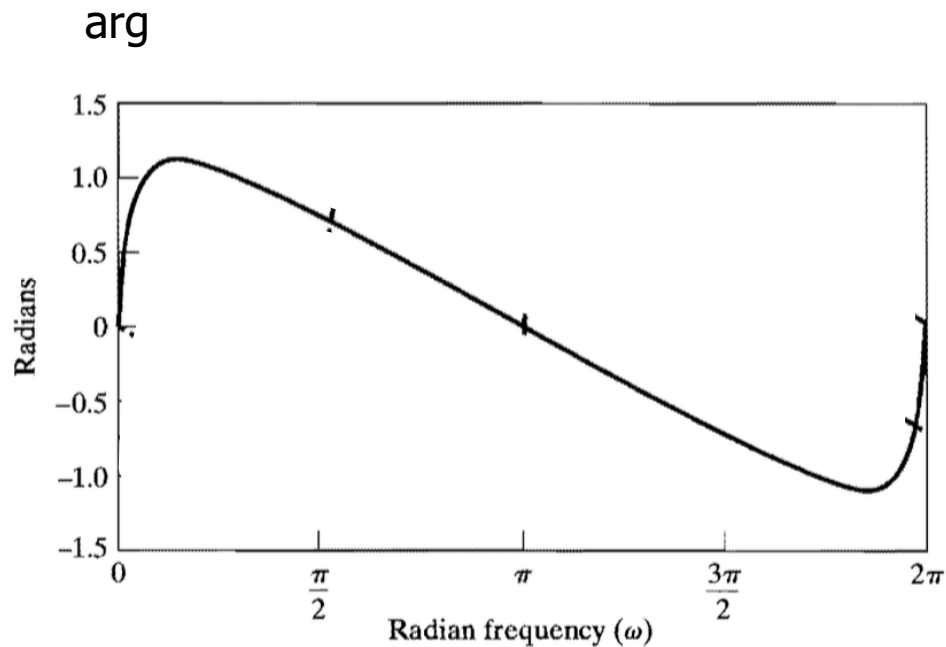
$$\text{grd}[1 - re^{j\theta} e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta} e^{-j\omega}|^2}$$



Example: Zero on Real Axis

□ For $\theta=0$

— $\theta = 0$



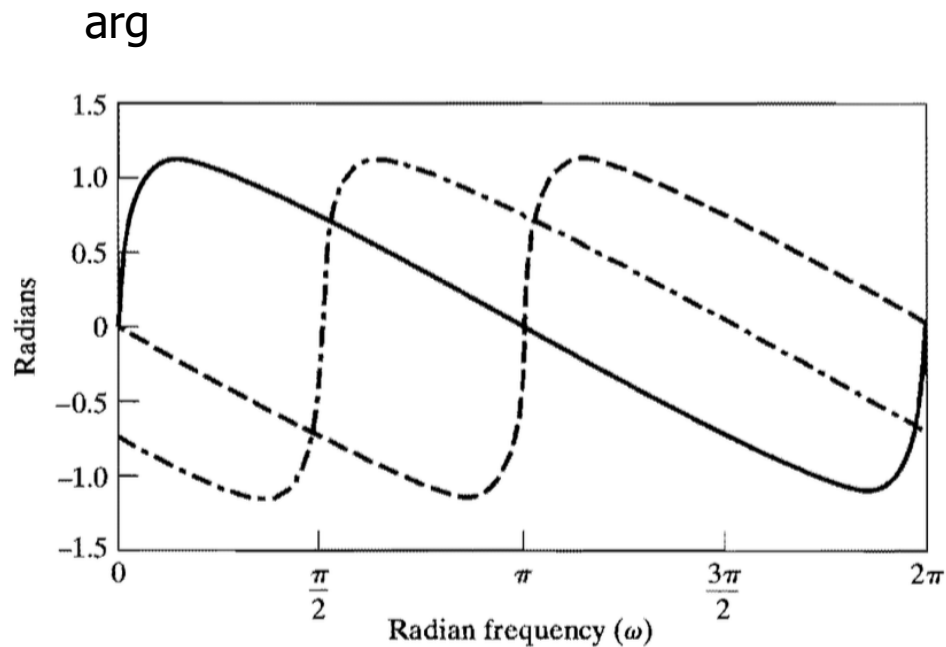
$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

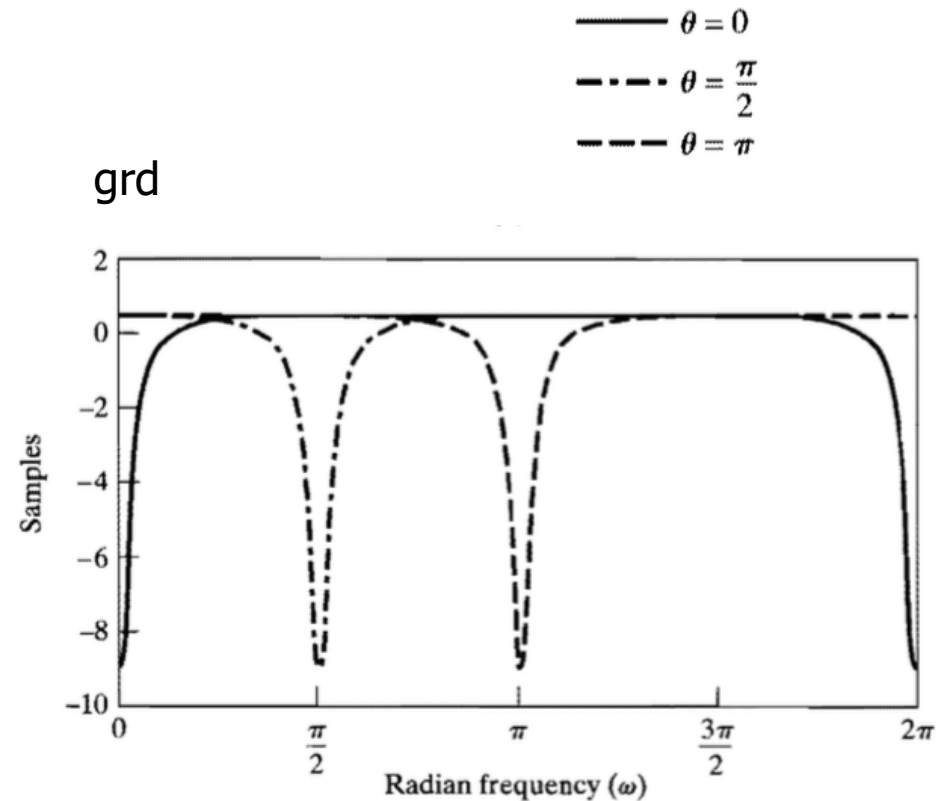


Example: Zero on Real Axis

□ For $\theta \neq 0$



$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r \sin(\omega - \theta)}{1 - r \cos(\omega - \theta)}\right)$$



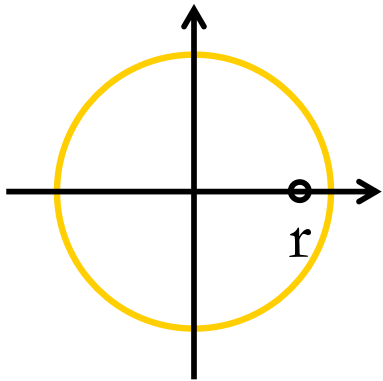
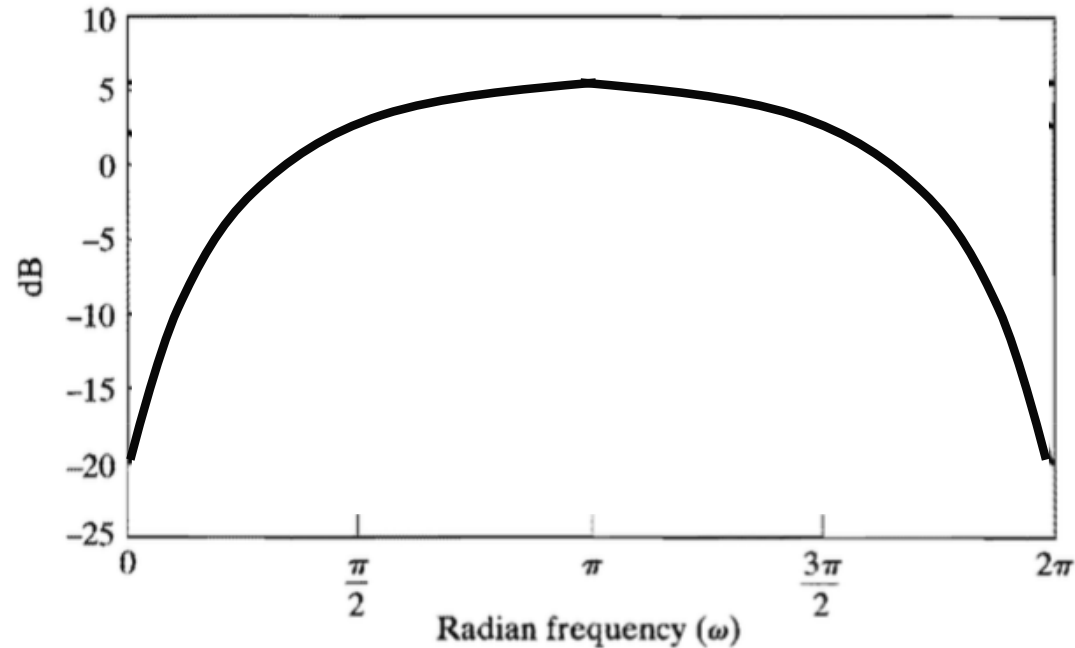
$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r \cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

Example: Zero on Real Axis

□ Magnitude Response

— $\theta = 0$

$$1 - re^{j\theta} e^{-j\omega} = 1 - re^{-j\omega}$$

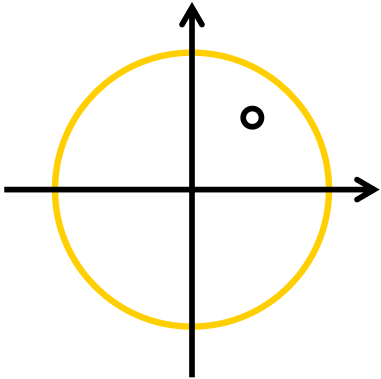
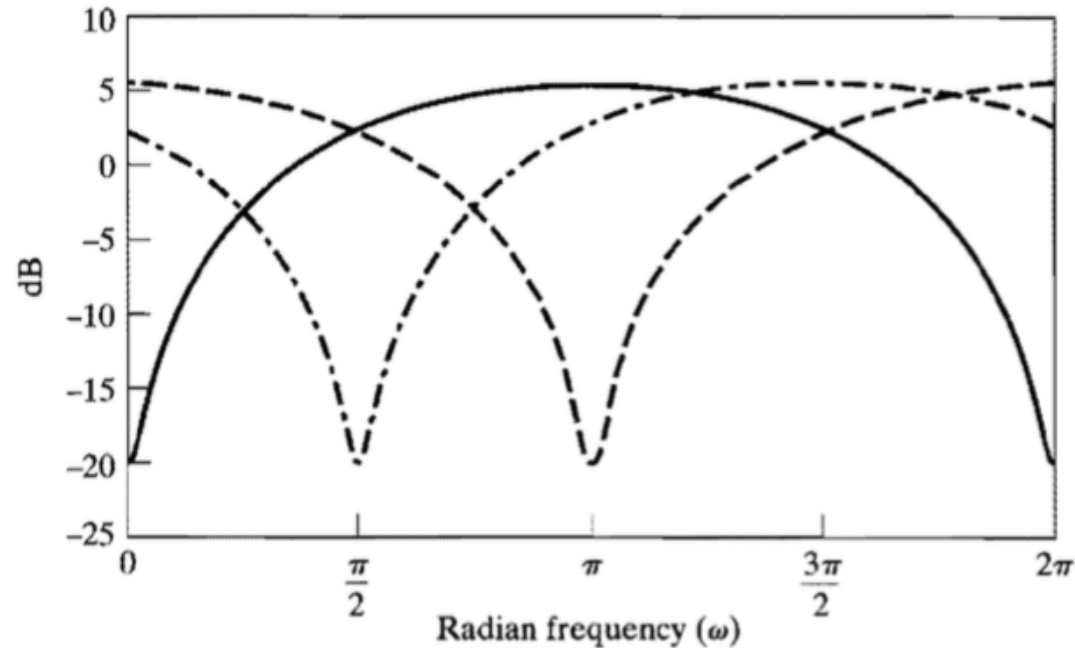


Example: Zero on Real Axis

□ Magnitude Response

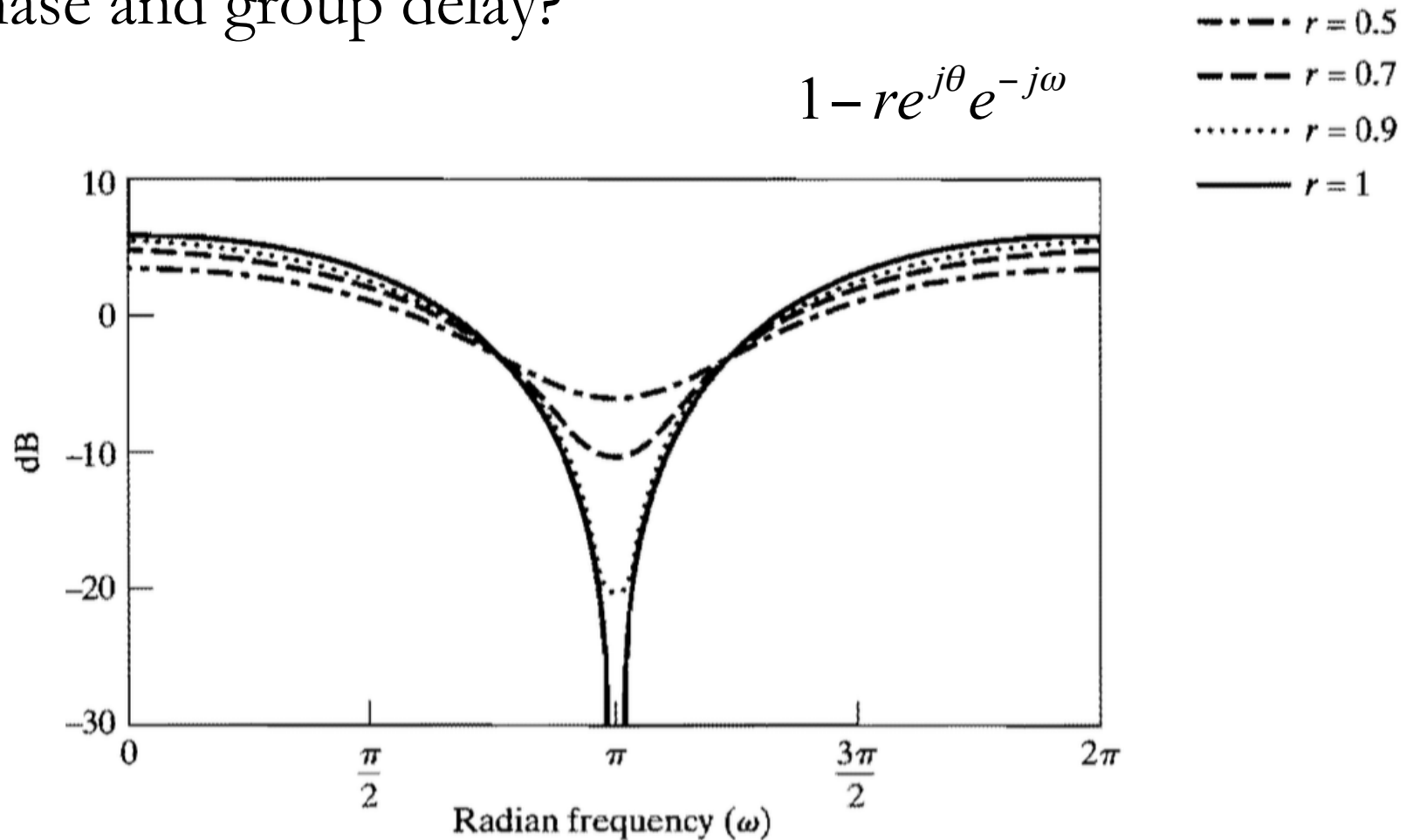
$$1 - re^{j\theta}e^{-j\omega}$$

— $\theta = 0$
- · - $\theta = \frac{\pi}{2}$
- - - $\theta = \pi$



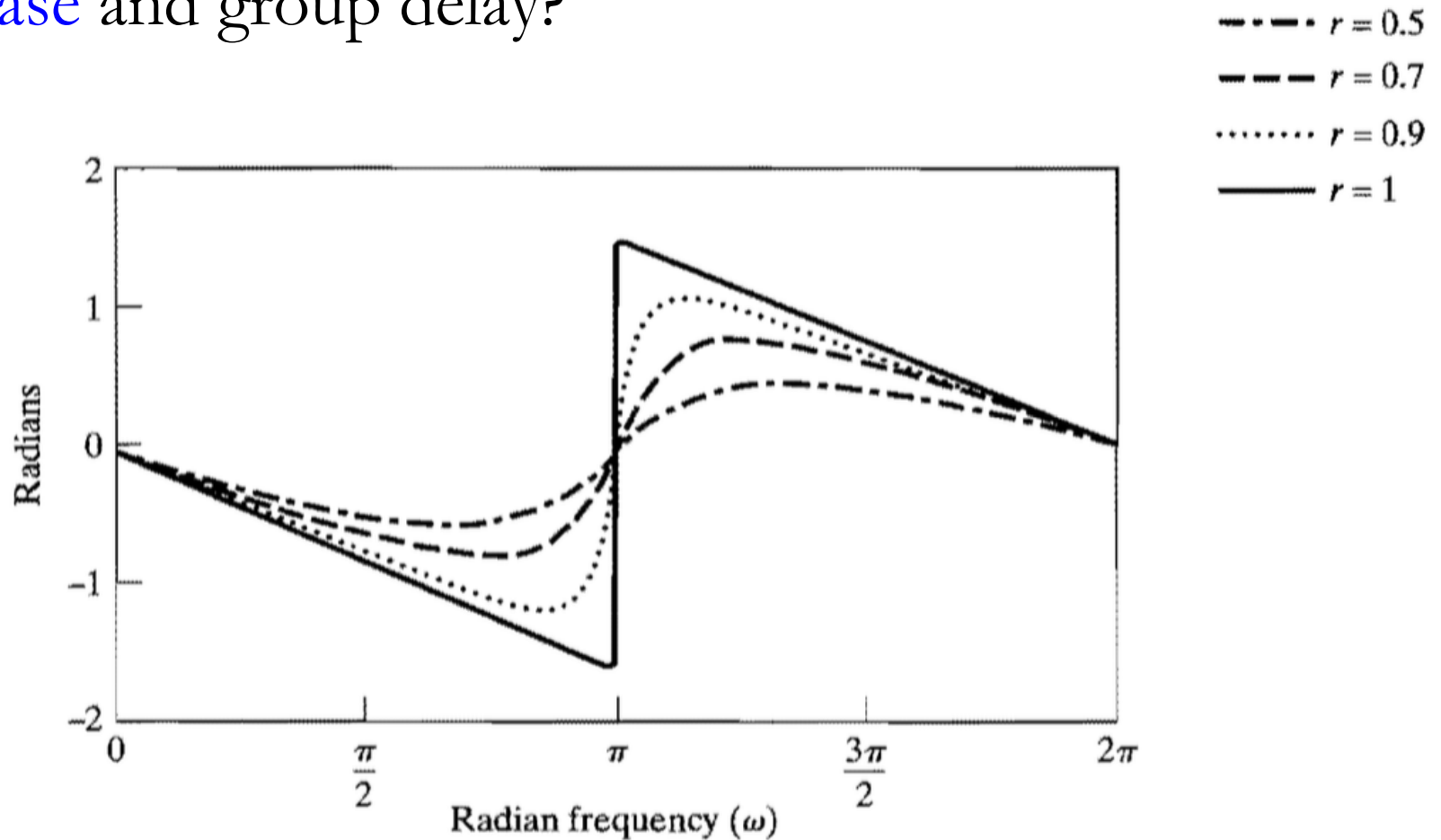
Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, phase and group delay?



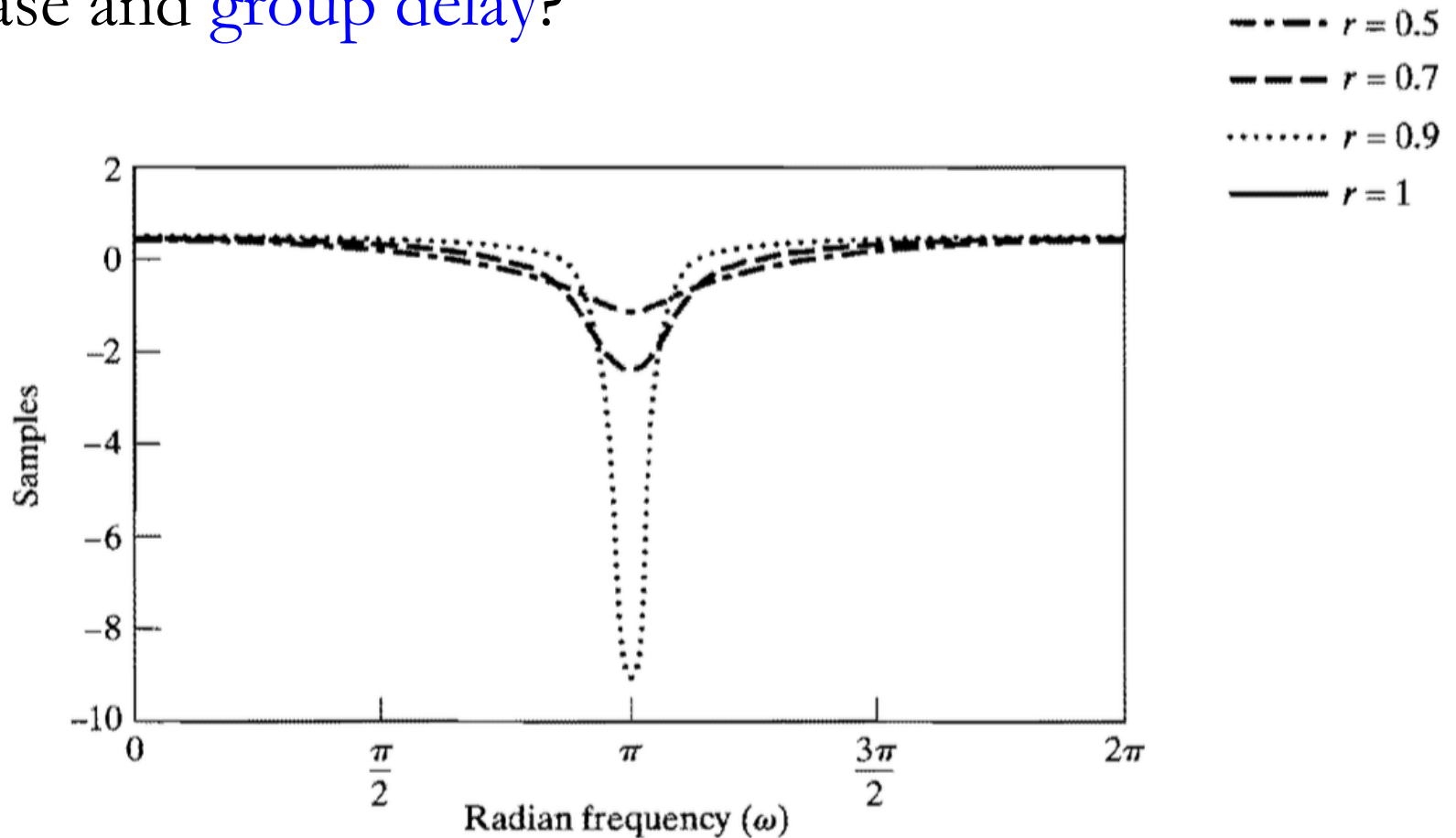
Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, **phase** and group delay?



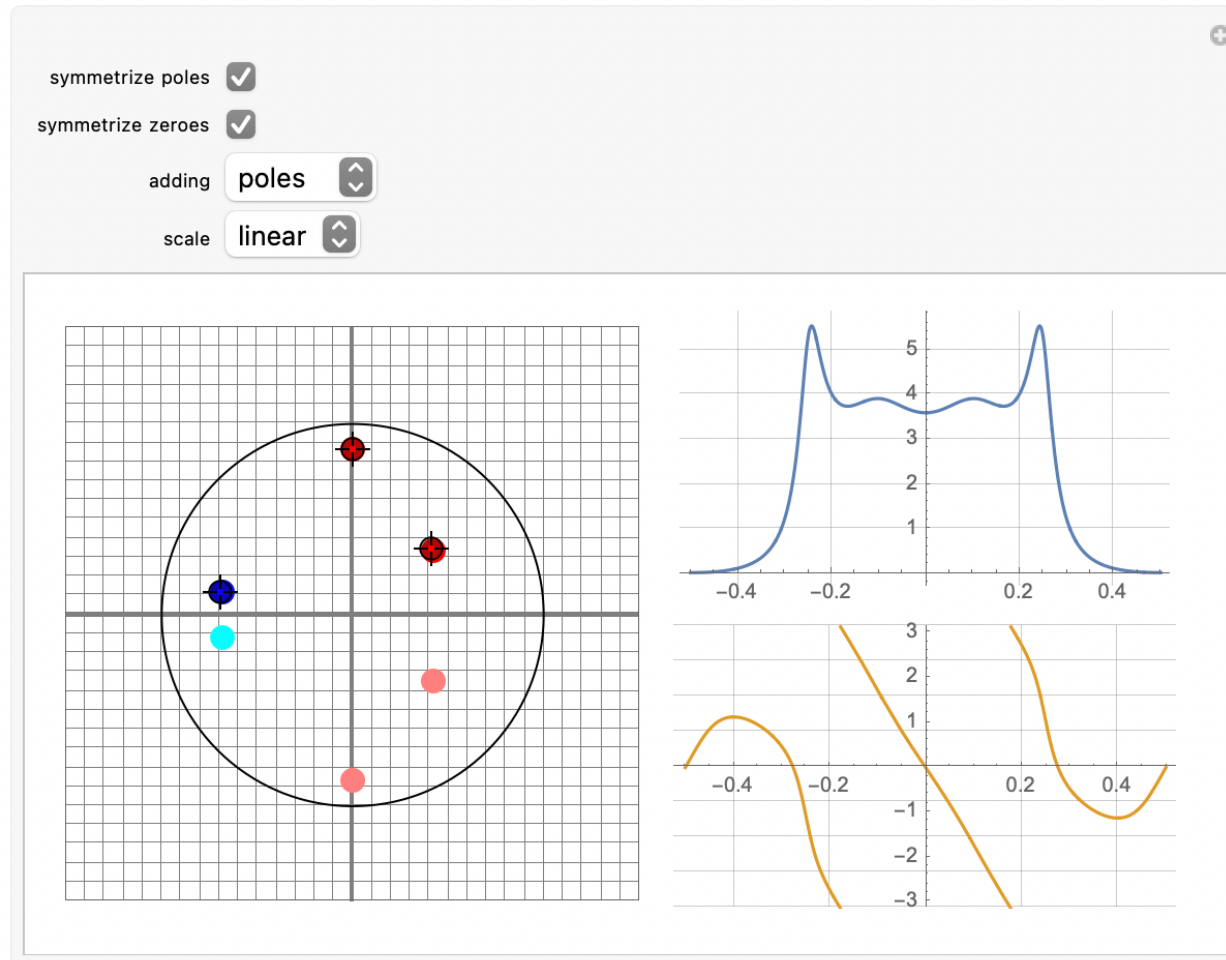
Example: Zero on Real Axis

- For $\theta=\pi$, how does zero location effect magnitude, phase and **group delay**?



Transfer Function from Poles and Zeroes

Transfer Function from Poles and Zeroes

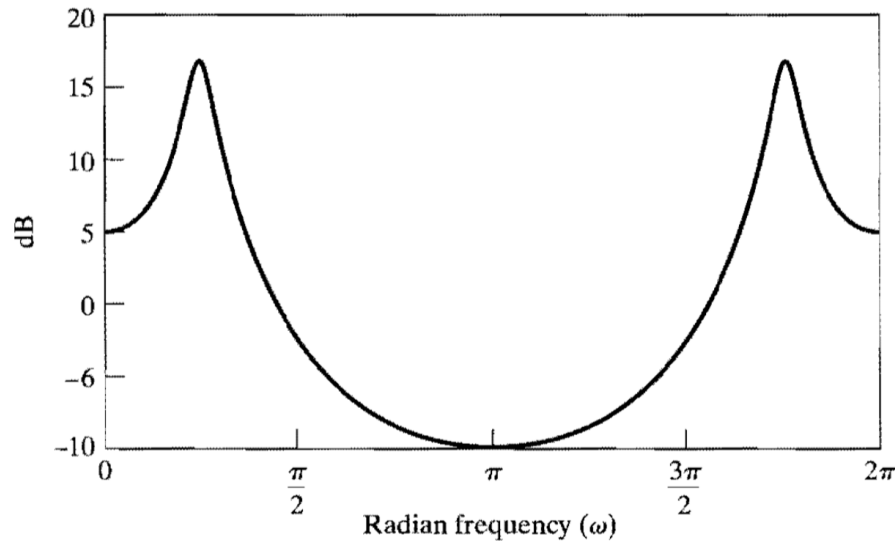


2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

$$r=0.9, \theta=\pi/4$$

magnitude



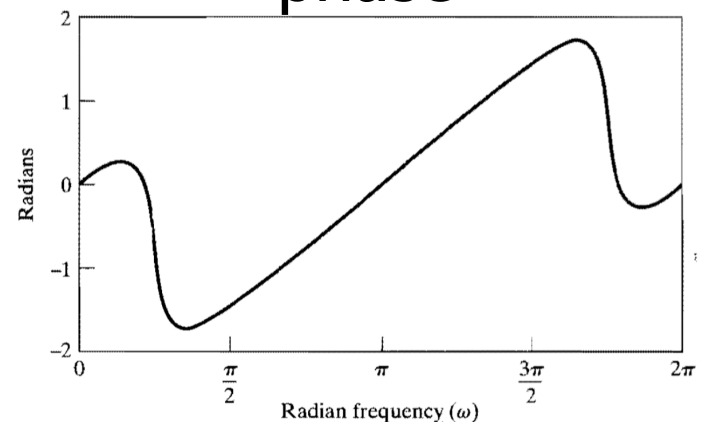
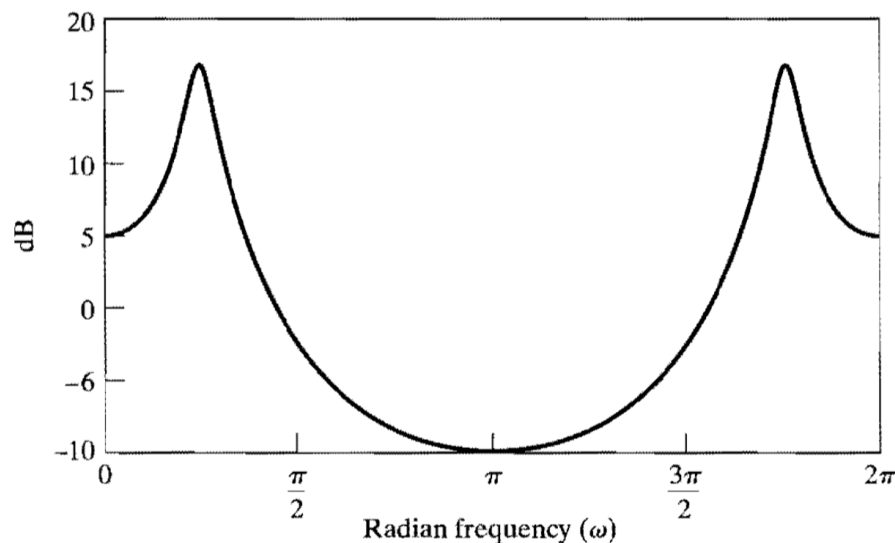
2nd Order IIR with Complex Poles

$$H(z) = \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}$$

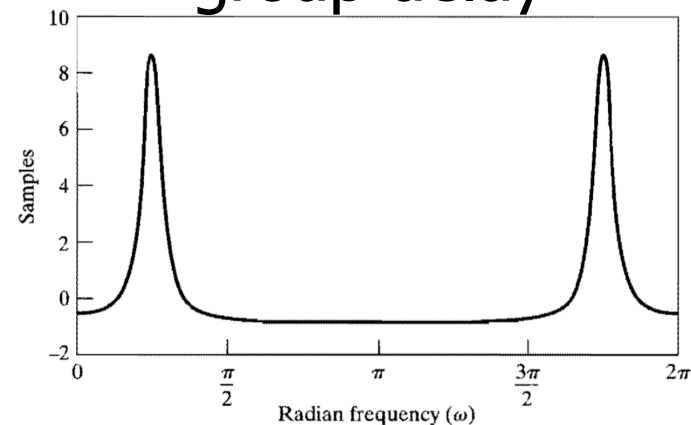
$$r=0.9, \theta=\pi/4$$

phase

magnitude

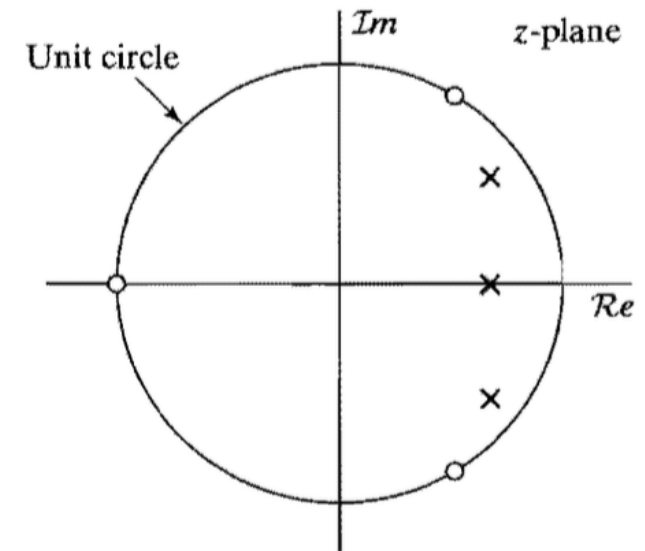


group delay



3rd Order IIR Example

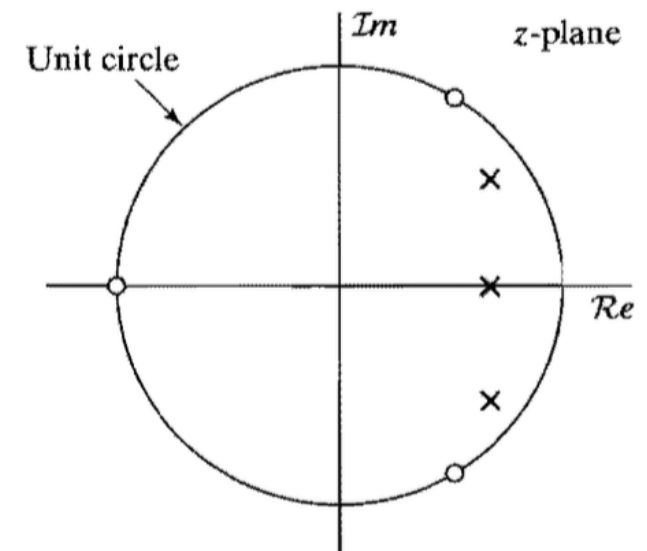
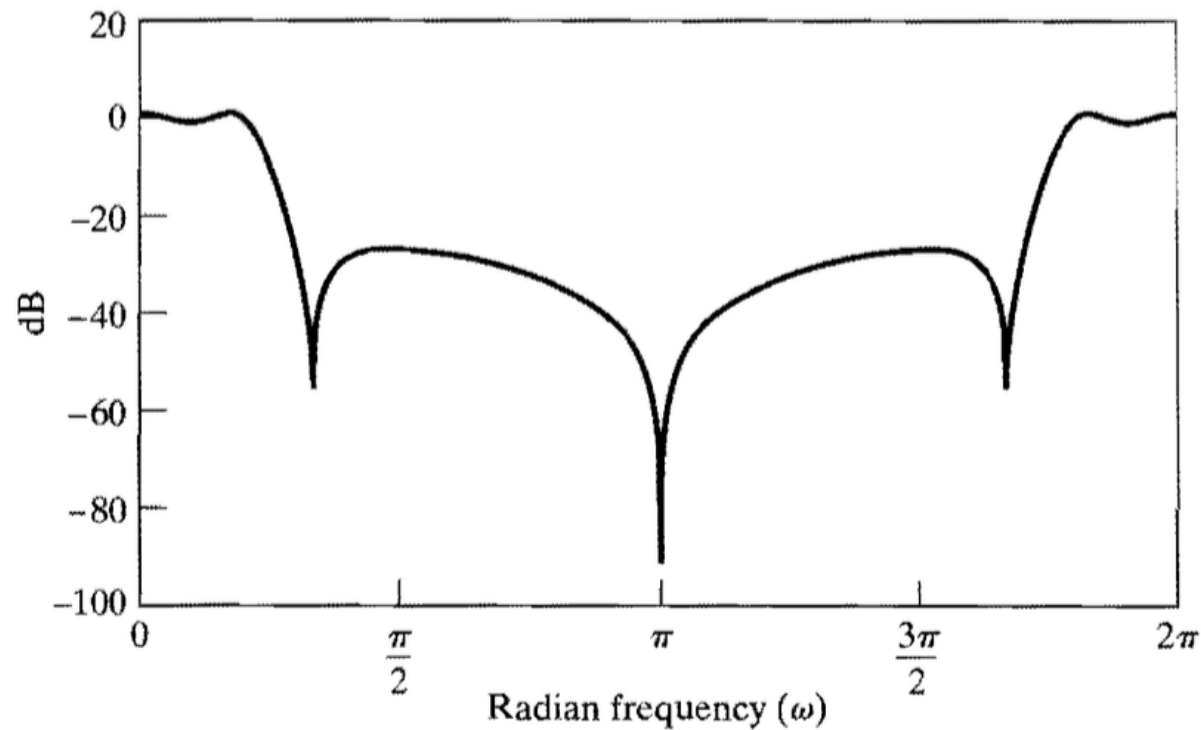
$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



What kind of filter is it?

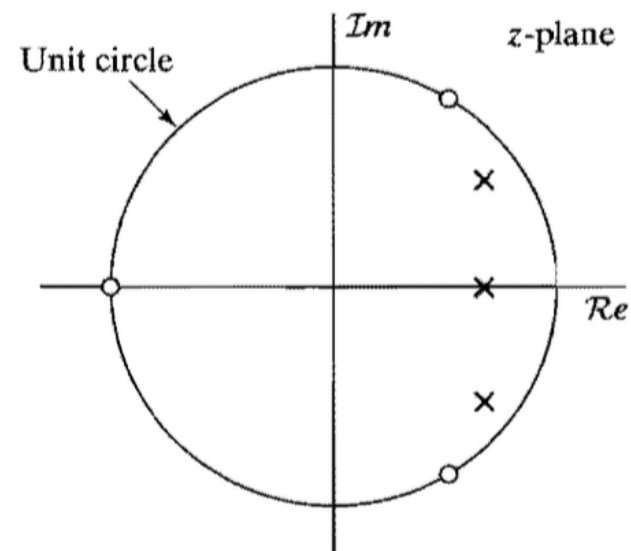
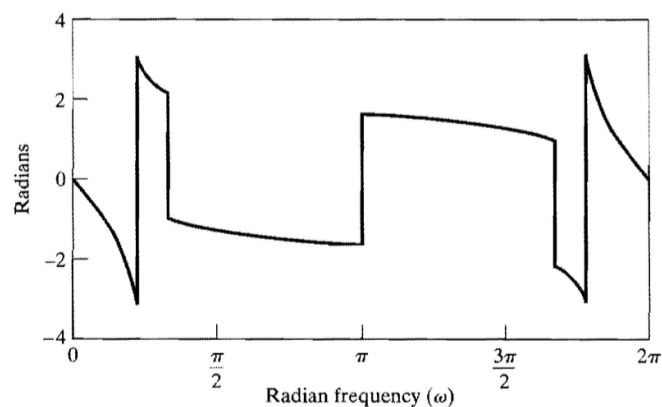
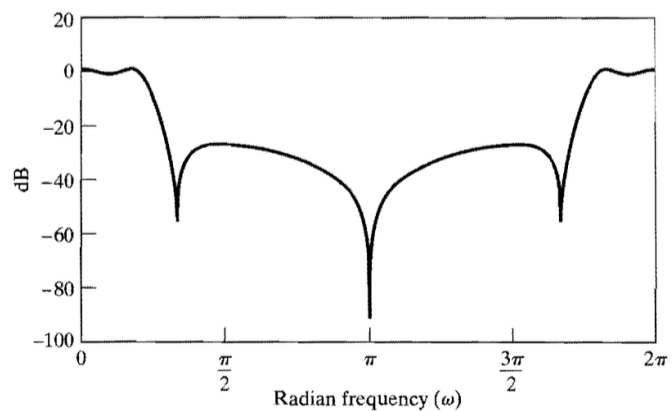
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



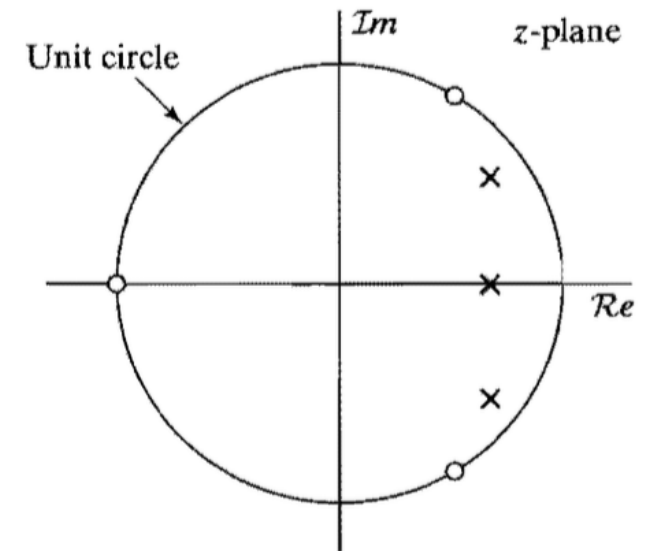
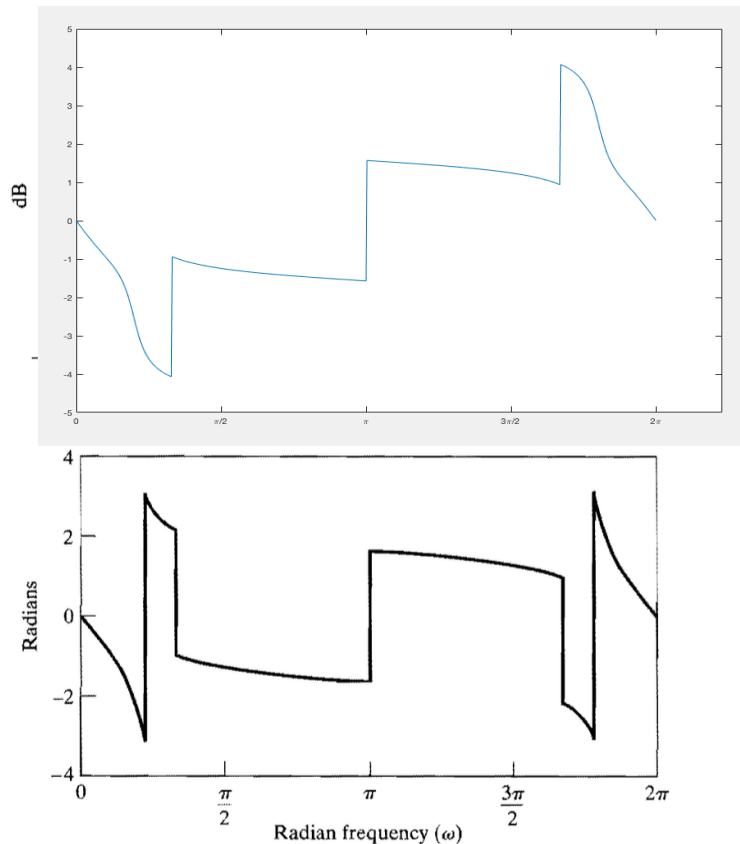
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



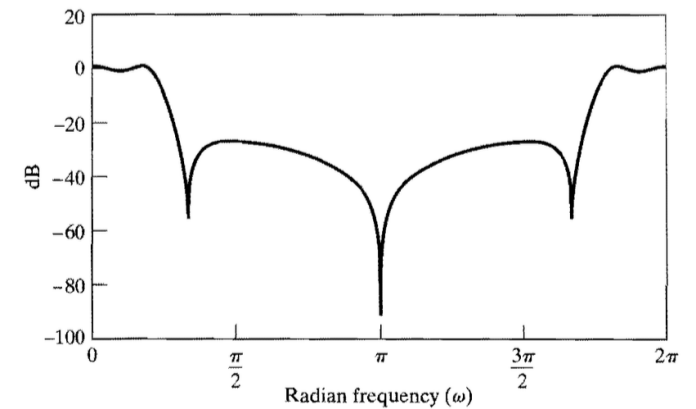
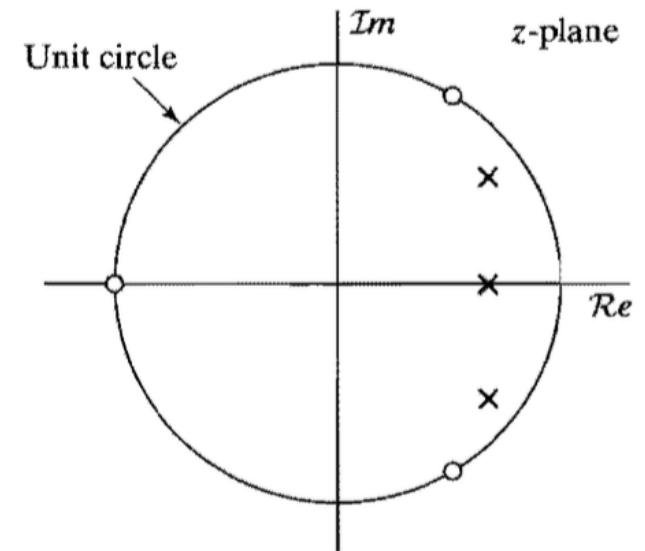
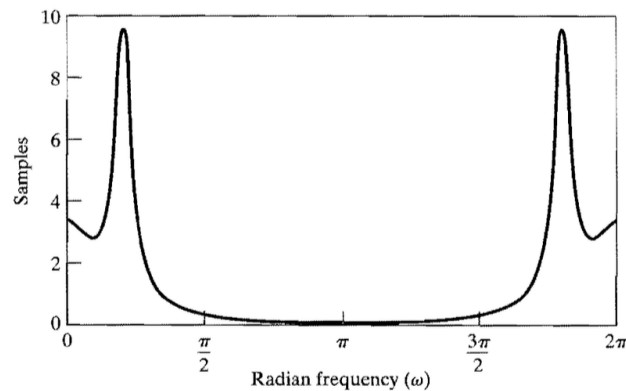
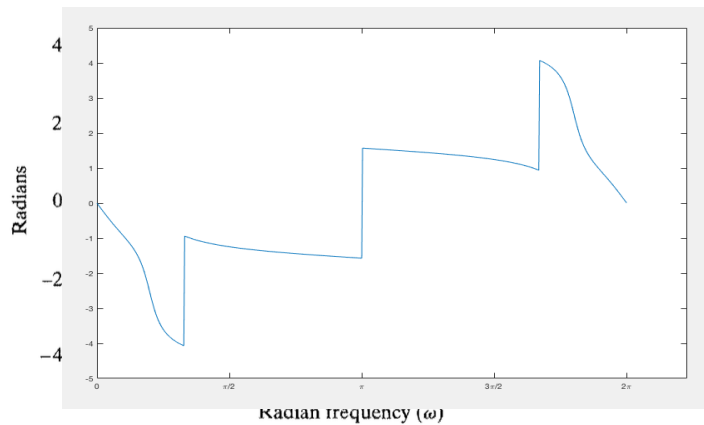
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



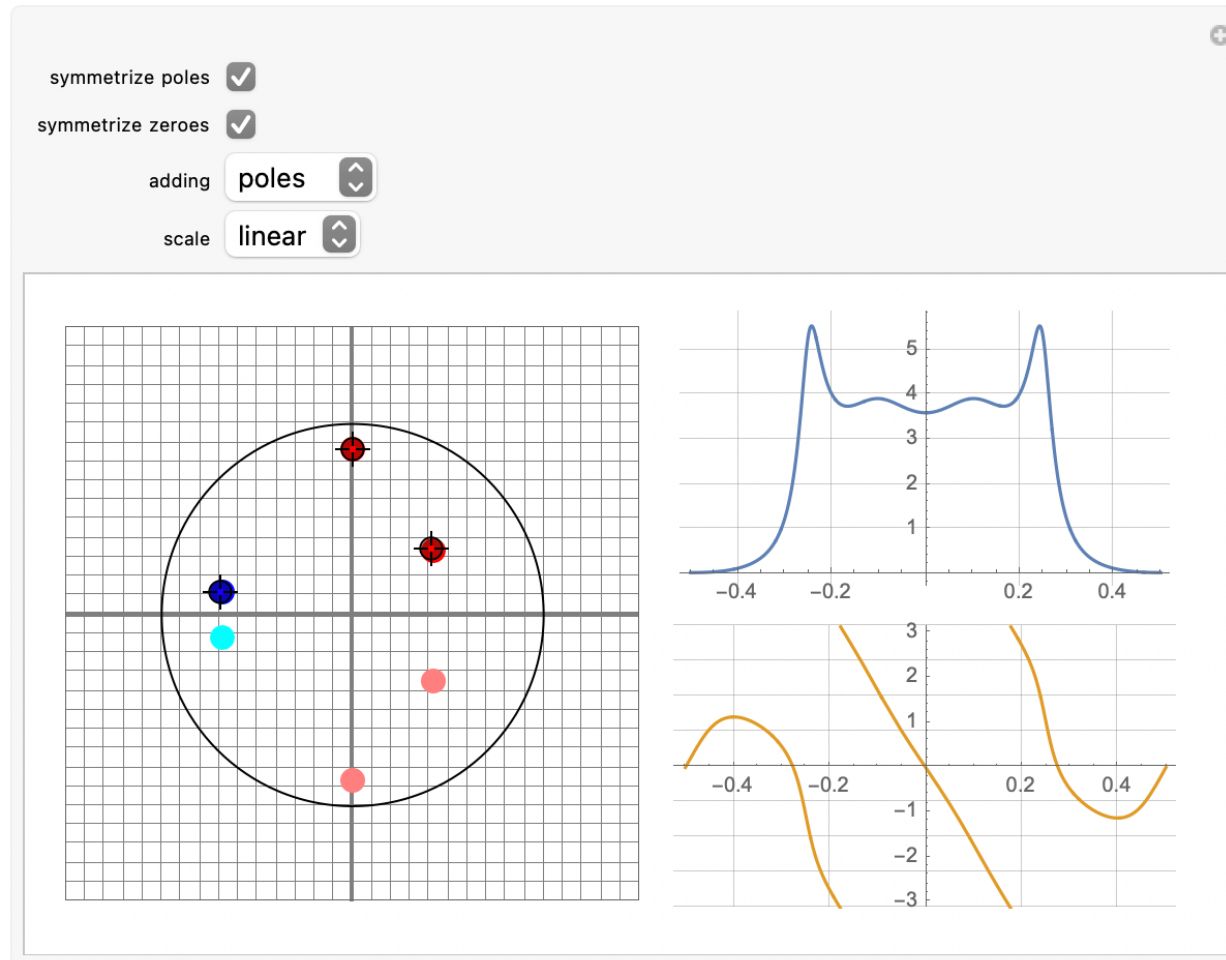
3rd Order IIR Example

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}$$



Transfer Function from Poles and Zeroes

Transfer Function from Poles and Zeroes





Big Ideas

- Frequency Response of LTI Systems
 - Magnitude Response
 - Simple Filters
 - Phase Response
 - Group Delay
 - Example: Zero on Real Axis
 - Example: Higher order systems



Admin

- ❑ HW 6 due Sunday 3/17
- ❑ Midterm next week Thursday
 - Covers lectures 1-13 (except lec 11)
 - Old exams on previous years' websites
 - Allowed 1 8.5x11 sheet