ESE 5310: Digital Signal Processing

Lecture 13: March 14, 2024

All-Pass Systems, Min Phase Decomposition

Penn ESE 5310 Spring 2024 – Khanna
Adapted from M. Lustig, EECS Berkeley

All Pass Systems

Stability and Causality

LTI System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

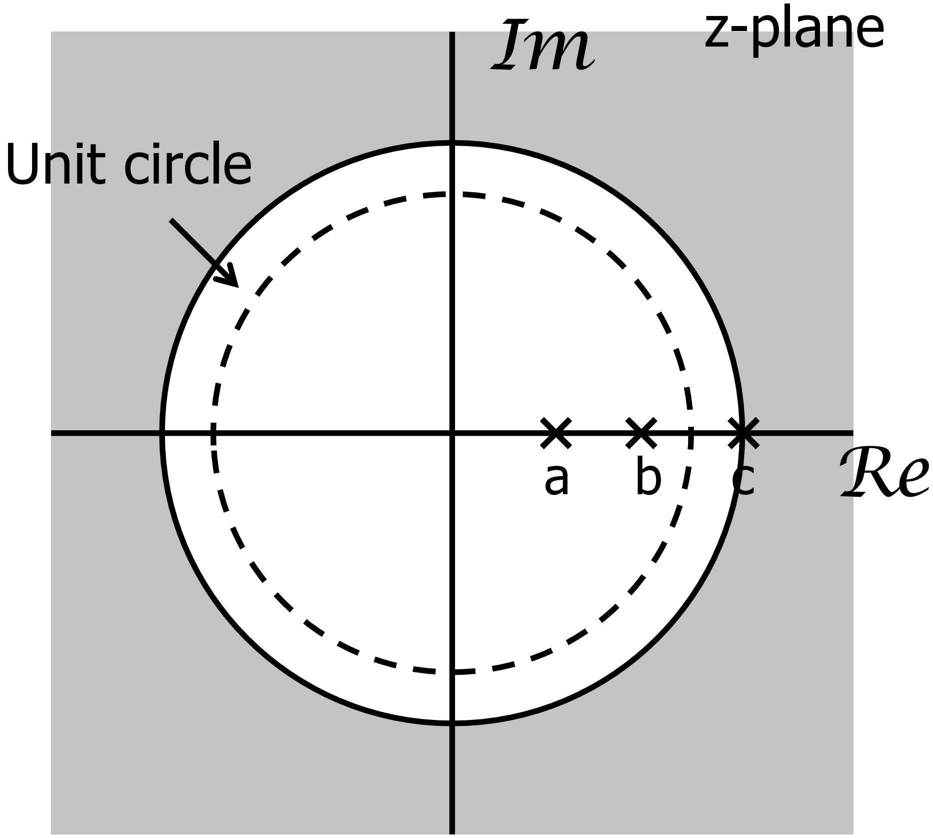
Example: $y[n] = x[n] + 0.1y[n-1]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
  - If diff. eq represents LTI and causal system, ROC is region outside all singularities (i.e poles)
  - If diff. eq represents LTI and stable system, ROC includes unit circle in z-plane

Example: ROC from Pole-Zero Plot

ROC 1: right-sided



LTI System

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Example: $y[n] = x[n] + 0.1y[n-1]$

If stable and causal, all poles inside unit circle

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Transfer function is not unique without ROC
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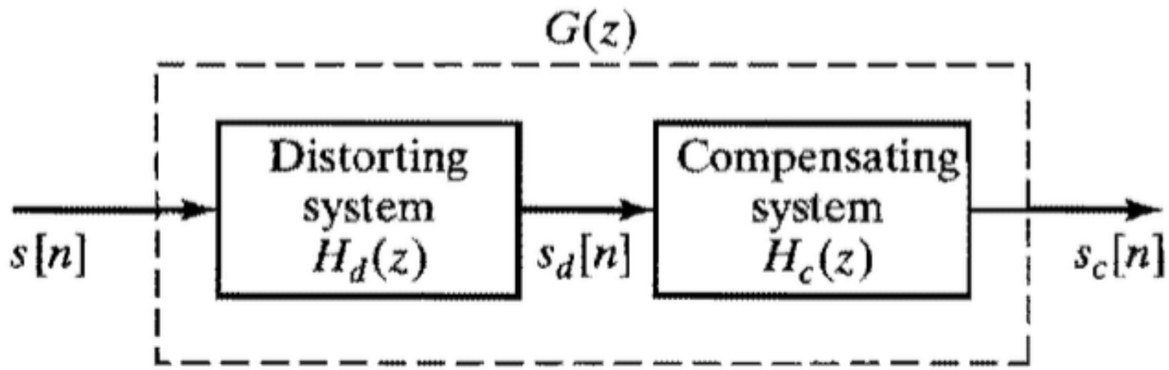
Distortion Compensation

- Have some distortion that we want to compensate for:



Minimum-Phase Systems

- Definition: A stable and causal system H(z) (i.e. poles inside unit circle) whose inverse 1/H(z) is also stable and causal (i.e. zeros inside unit circle)

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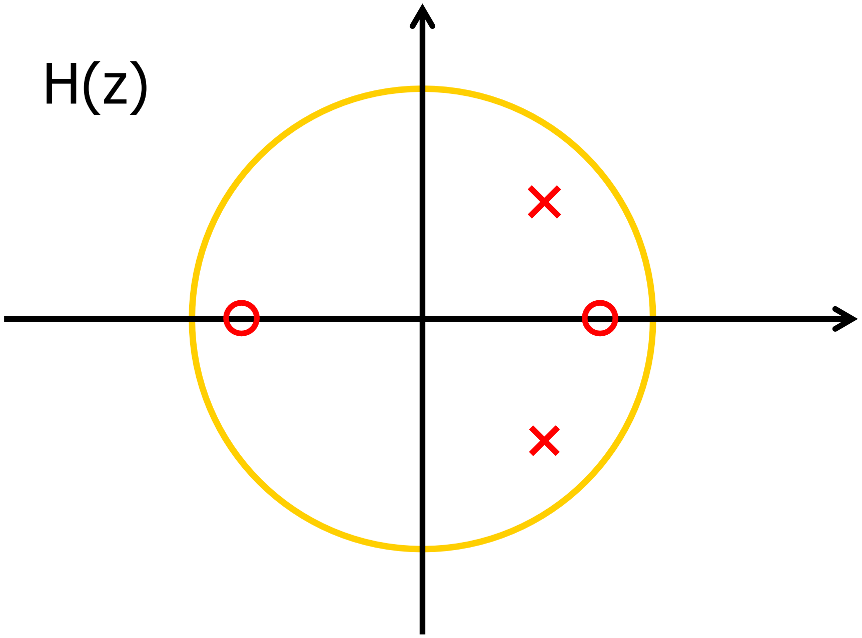
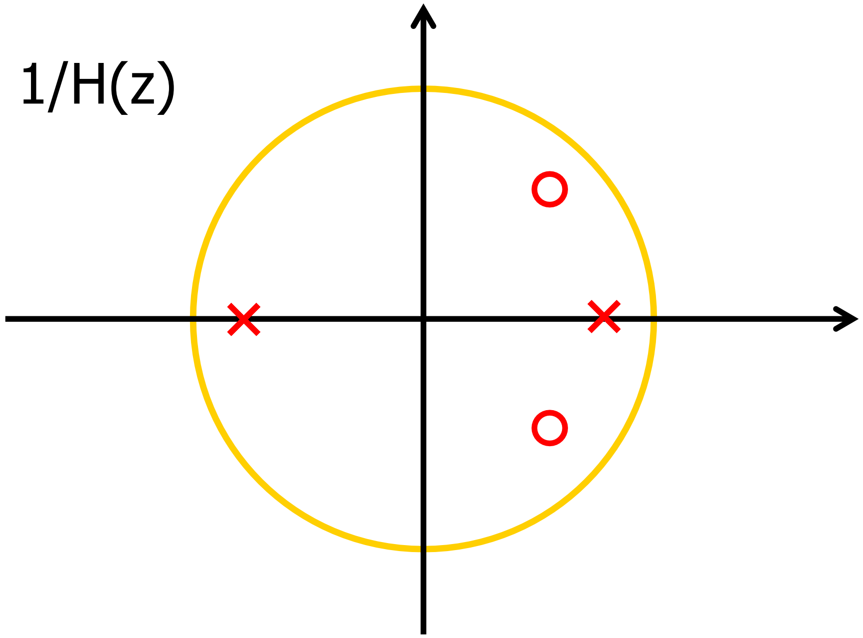
  - All poles and zeros inside unit circle





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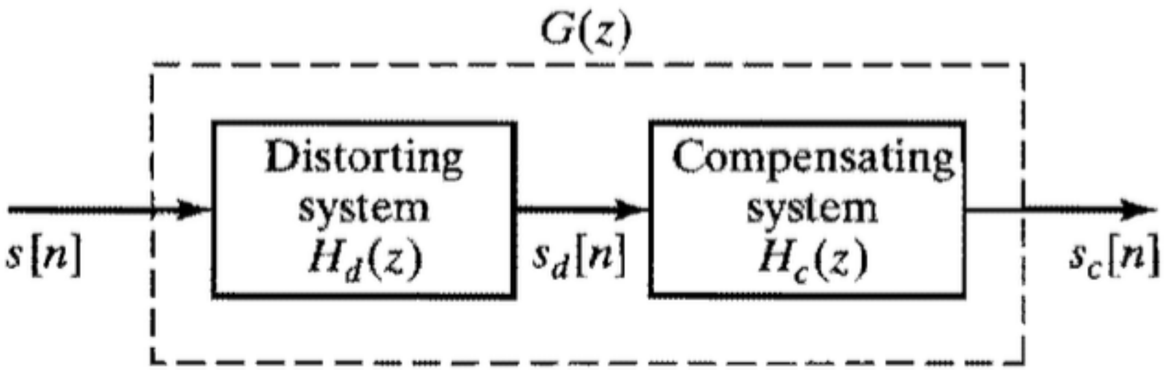

- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- What if not min phase?

All-Pass Systems

All-Pass Filters

- A system is an all-pass system if

$$\left|H(e^{j\omega})\right| = 1, \text{ all } \omega$$

- Its phase response $\theta(\omega)$ may be non-trivial

First Order All-Pass Filter

$$H(z)=\frac{z^{-1}-a^{*}}{1-az^{-1}}$$

First Order All-Pass Filter

$$H(z)=\frac{z^{-1}-a^*}{1-az^{-1}}$$

$$\left|H(e^{j\omega})\right|=\frac{\left|e^{-j\omega}-a^*\right|}{\left|1-ae^{-j\omega}\right|}$$

$$=\frac{\left|e^{-j\omega}(1-a^*e^{j\omega})\right|}{\left|1-ae^{-j\omega}\right|}$$

$$=\frac{\left|1-a^*e^{j\omega}\right|}{\left|1-ae^{-j\omega}\right|}=1$$

General All-Pass Filter

- d_k=real pole
- e_k=complex poles paired w/ conjugate, e_k^*

$$H_{\mathrm{ap}}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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Gain factor

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- Example:

$$d_k = -\frac{3}{4}$$

$$e_k = 0.8e^{j\pi/4}$$

General All-Pass Filter

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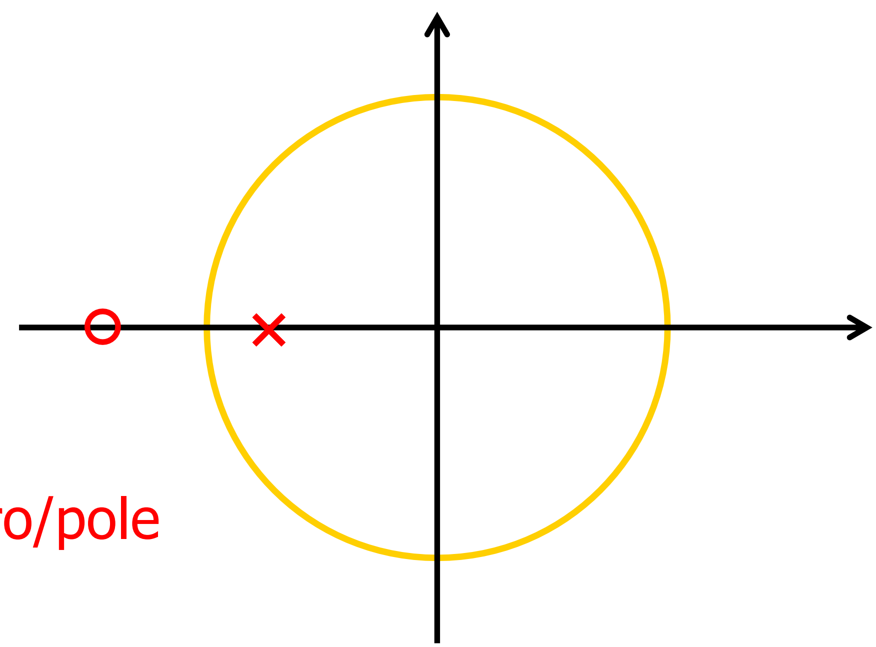

Real zero/pole

General All-Pass Filter

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- Example:

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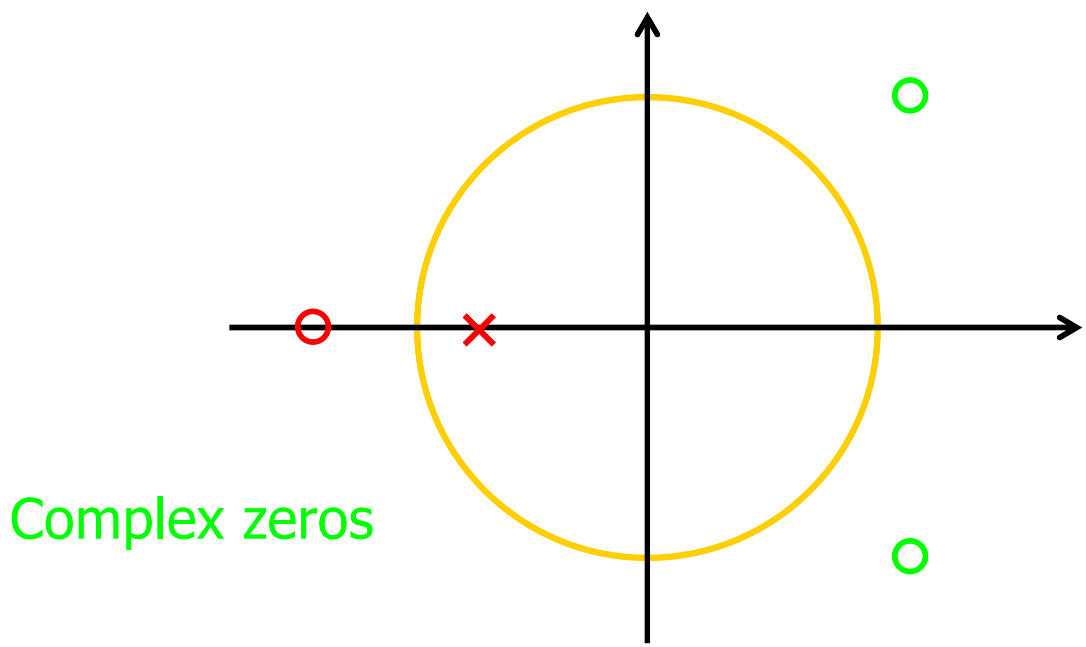

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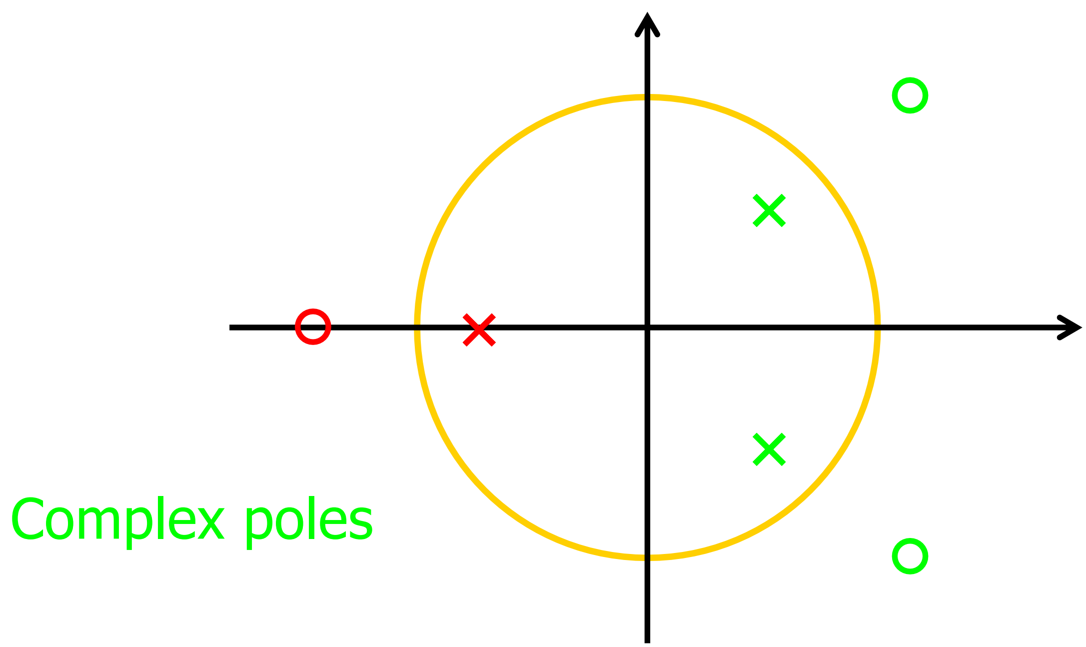


All Pass Filter Phase Response

- First order system
$$H(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$
$$= \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$$

- Phase?

All Pass Filter Phase Response

- phase $\arg\left(\dfrac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right)$

All Pass Filter Phase Response

- phase $$\arg\left(\frac{e^{-j\omega}-re^{-j\theta}}{1-re^{j\theta}e^{-j\omega}}\right)$$

$$=\arg\left(\frac{e^{-j\omega}(1-re^{-j\theta}e^{j\omega})}{1-re^{j\theta}e^{-j\omega}}\right)$$

$$=\arg(e^{-j\omega})+\arg(1-re^{-j\theta}e^{j\omega})-\arg(1-re^{j\theta}e^{-j\omega})$$

Group Delay Math

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{M} \text{grd}[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \text{grd}[1 - d_k e^{-j\omega}]$$

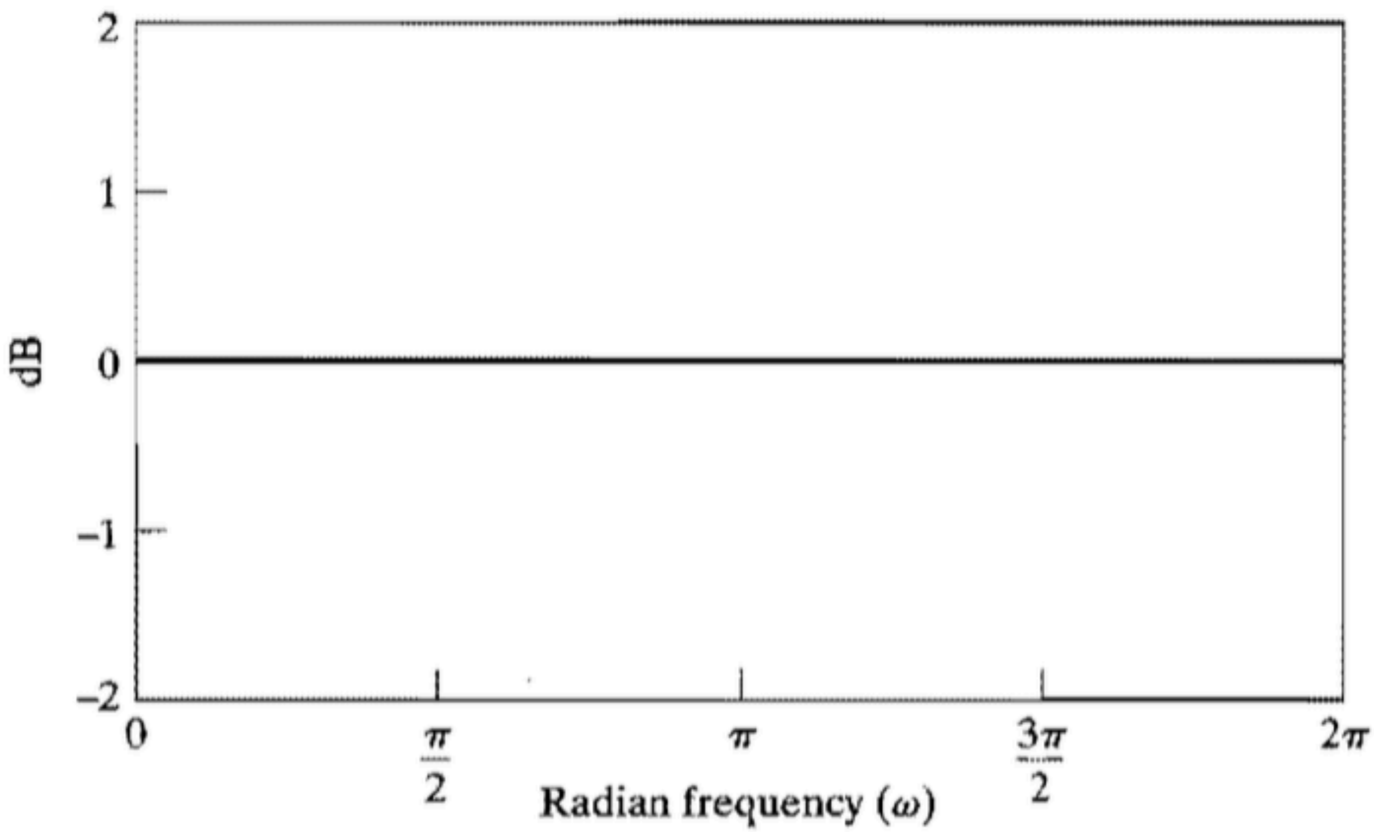
- Look at each factor:

$$\arg[1 - re^{j\theta}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right)$$

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

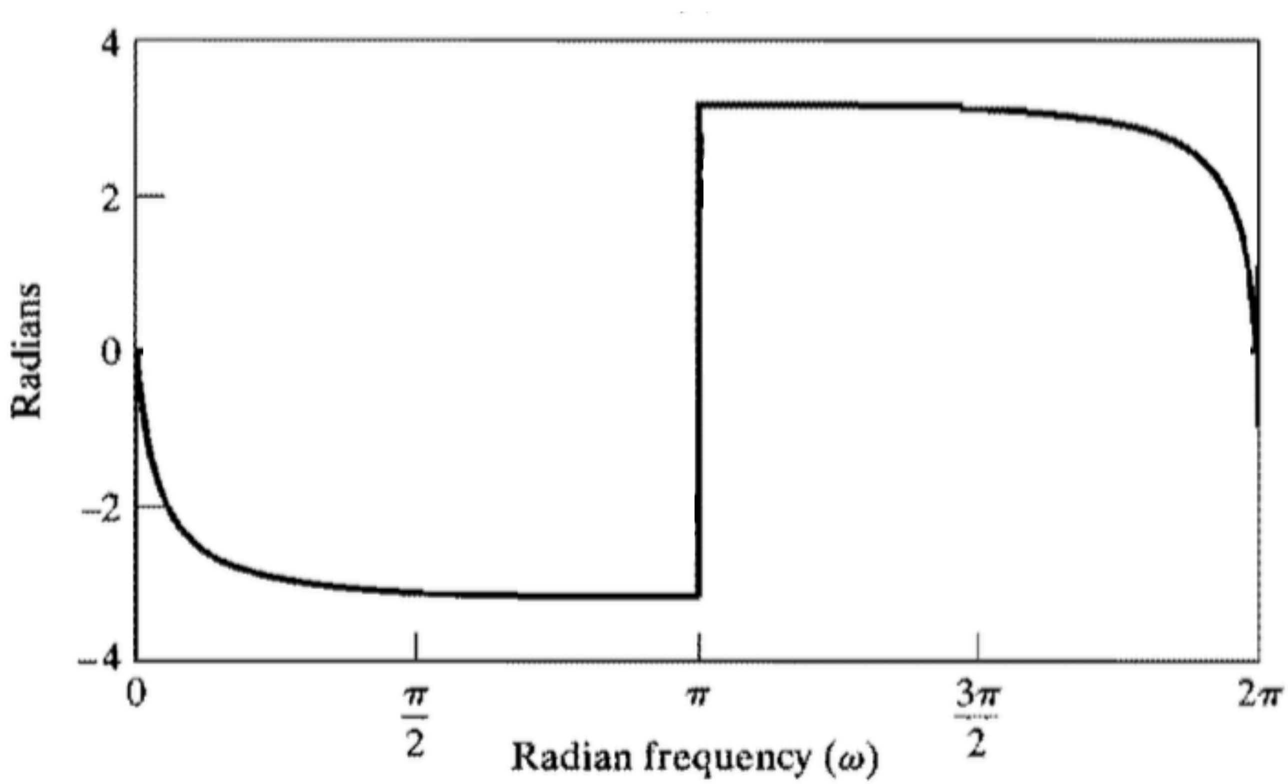
First Order Example

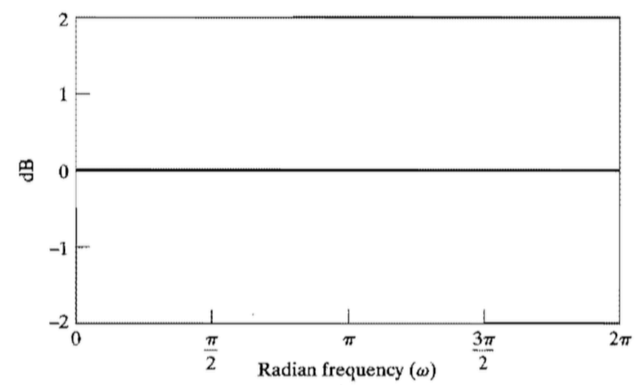
- Magnitude:



First Order Example

- Phase:



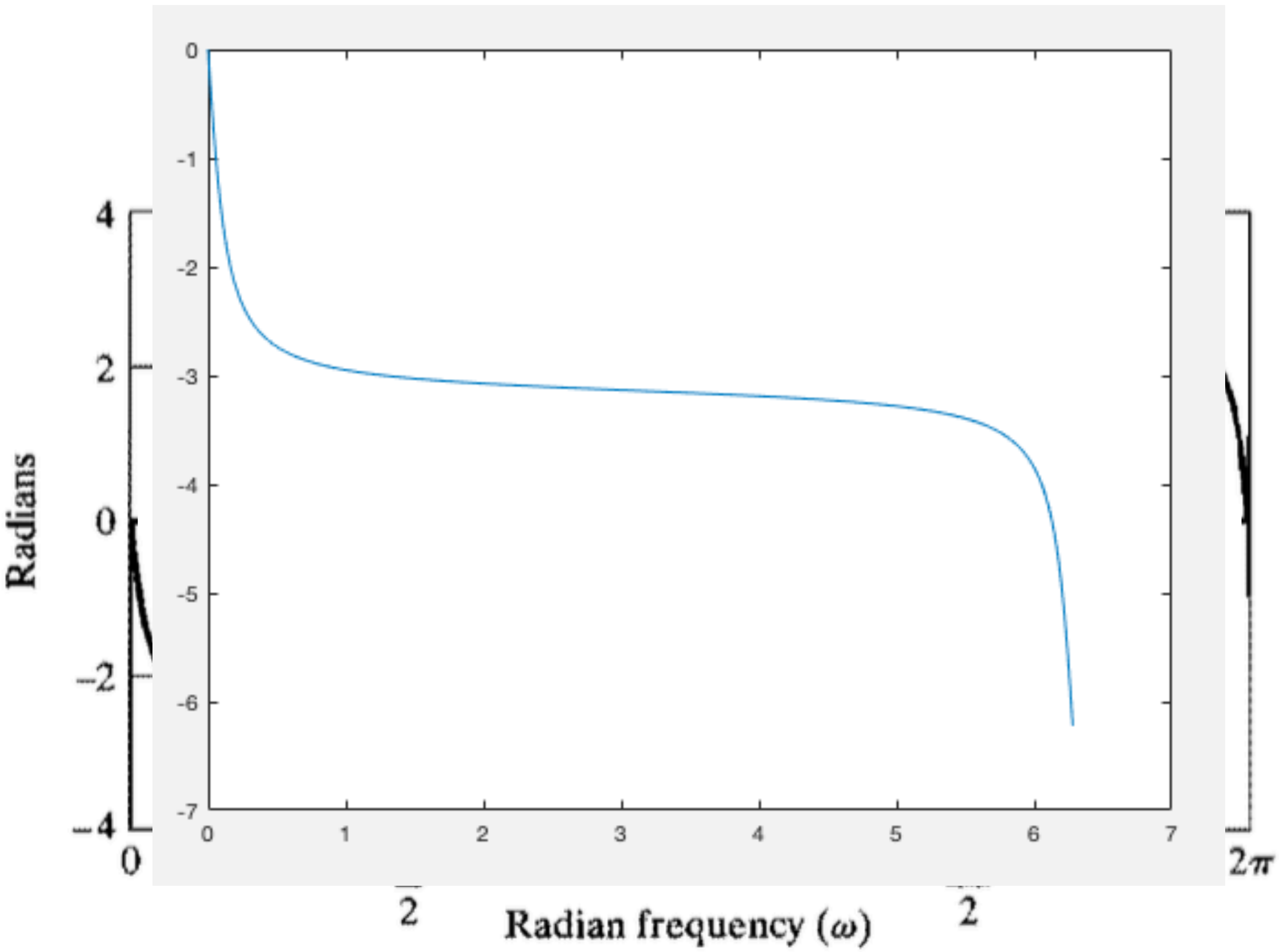


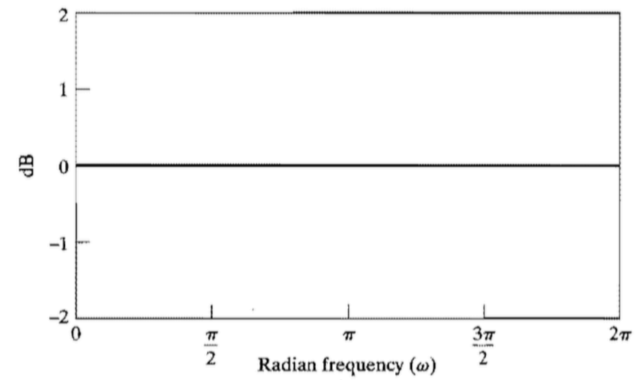
$z = 0.9$

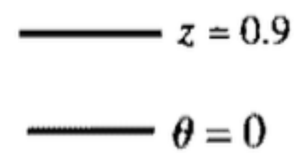
$\theta = 0$

First Order Example

- Phase:

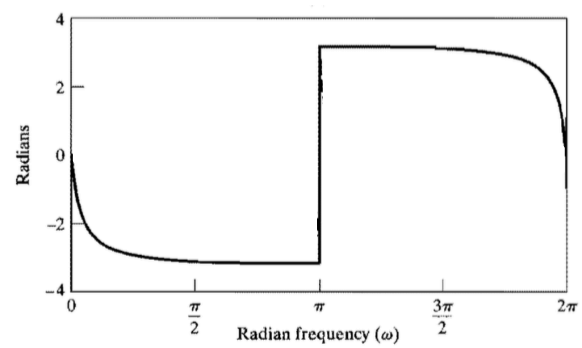


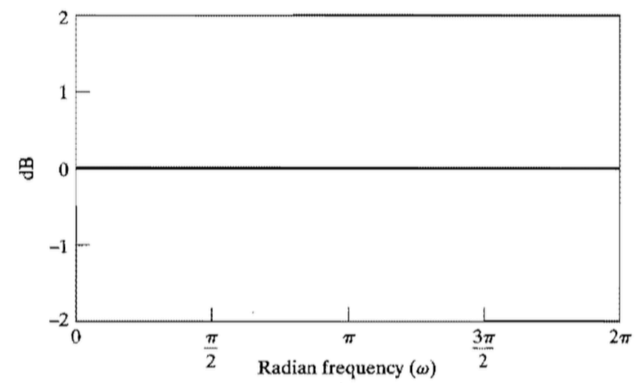


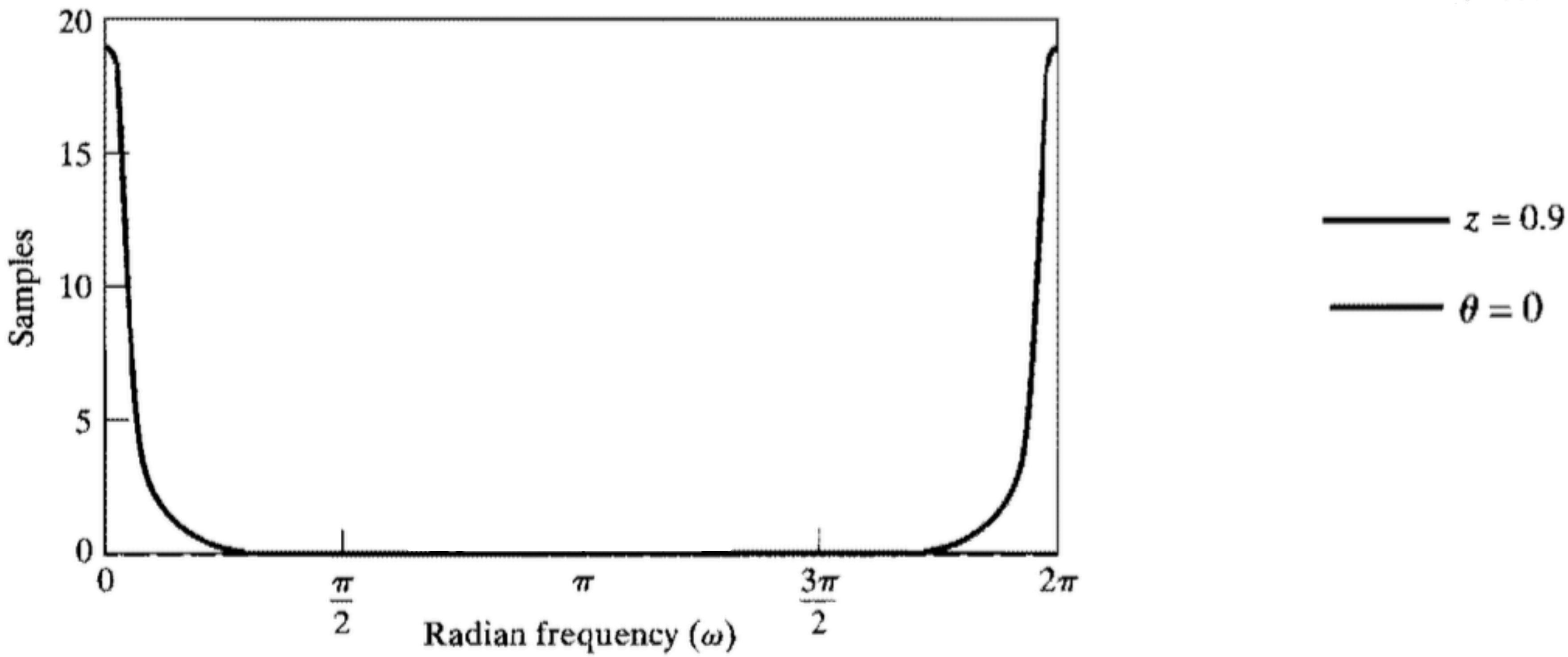


First Order Example

- Group Delay:







All Pass Filter Phase Response

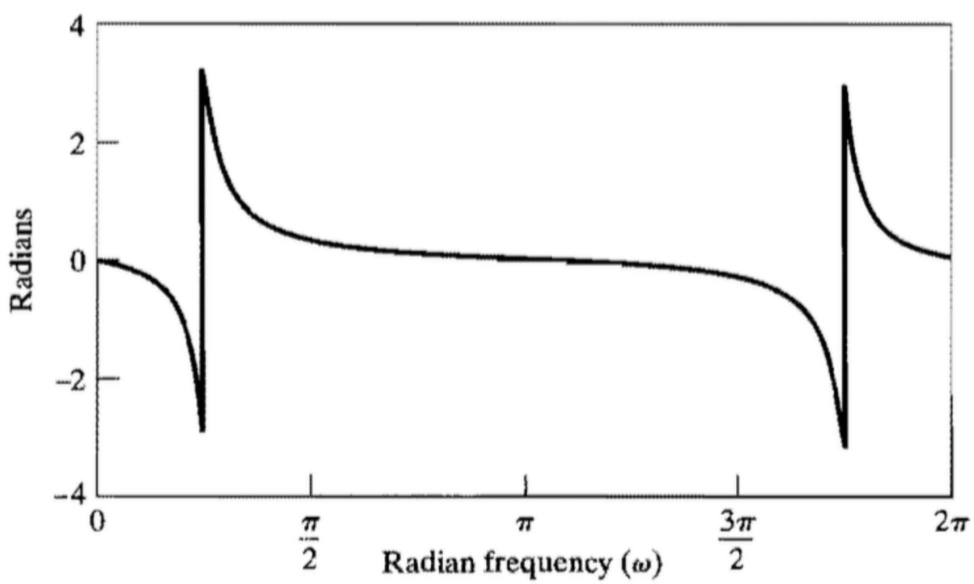
- Second order system with poles at $z = re^{j\theta}, re^{-j\theta}$

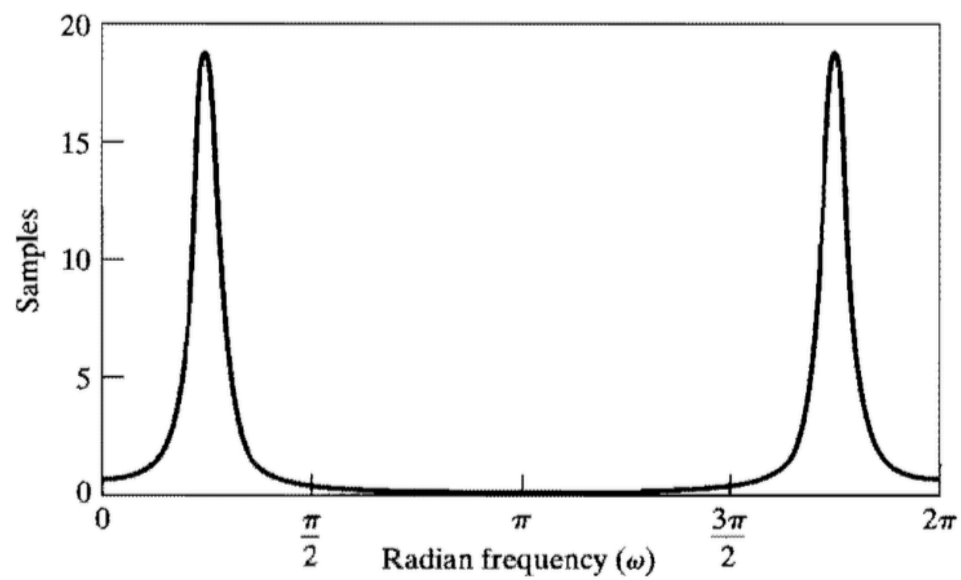
$$\angle\left[\frac{(e^{-j\omega}-re^{-j\theta})(e^{-j\omega}-re^{j\theta})}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}\right] = -2\omega - 2\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

$$-2\arctan\left[\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right].$$

Second Order Example

- Poles at $z = 0.9e^{\pm j\pi/4}$ (zeros at conjugates)





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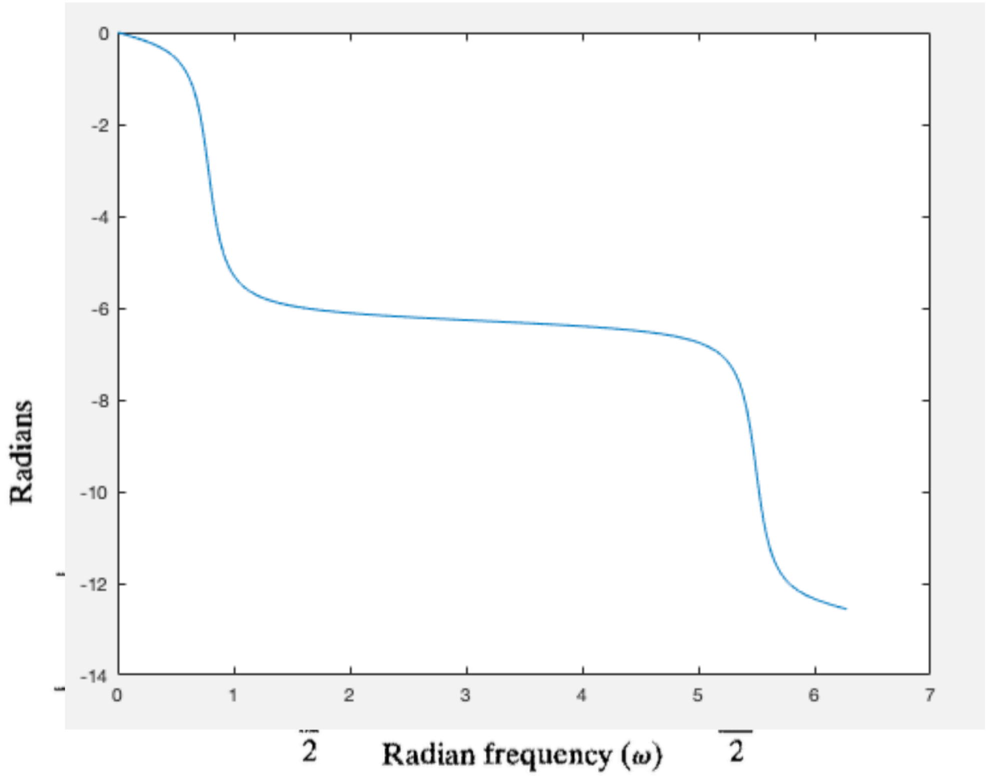

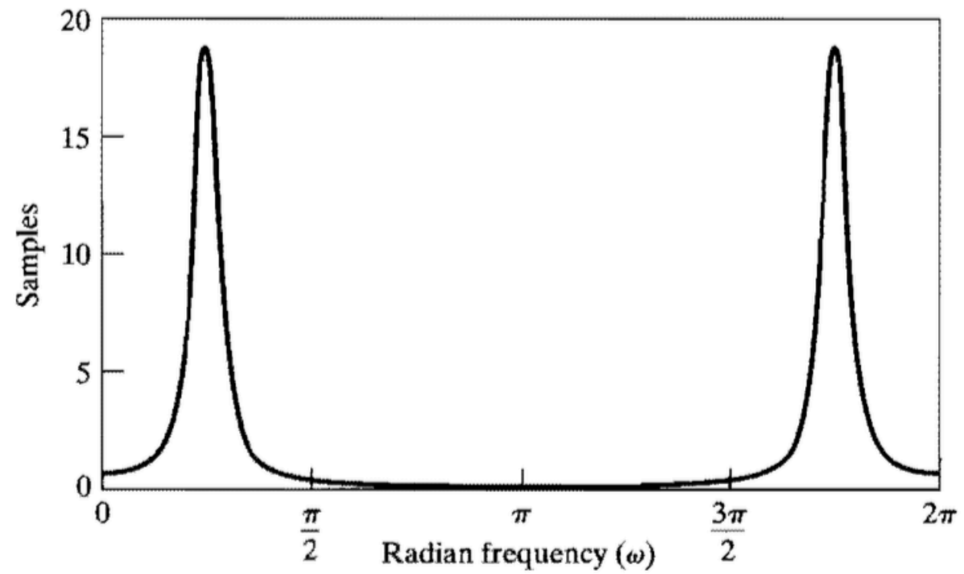

All-pass Example

- d_k=real pole, e_k=complex poles paired w/ conjugate, e_k^*

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

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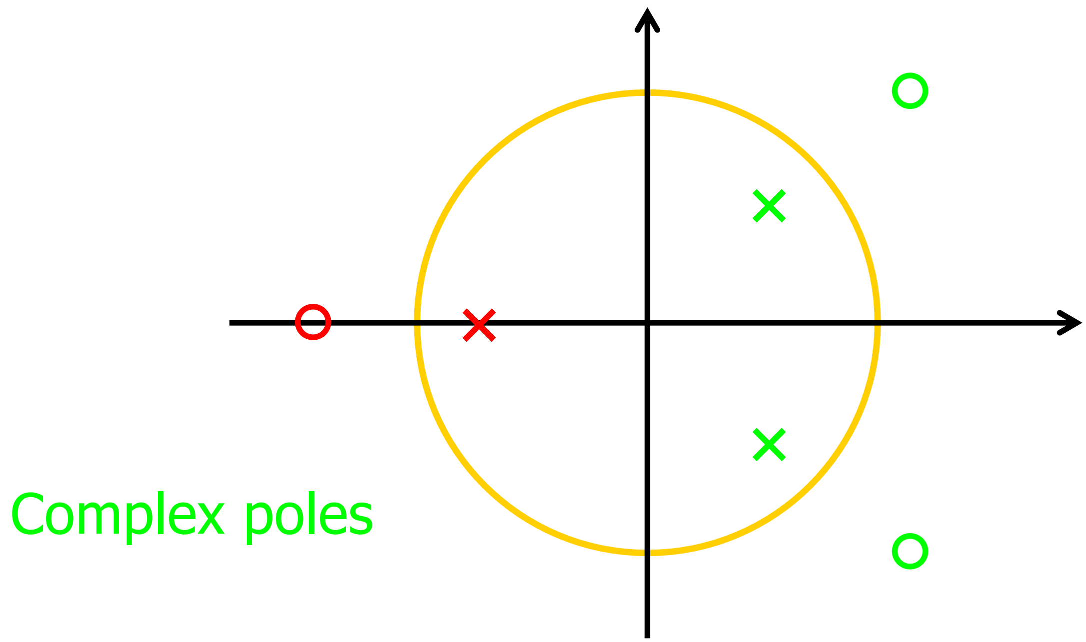
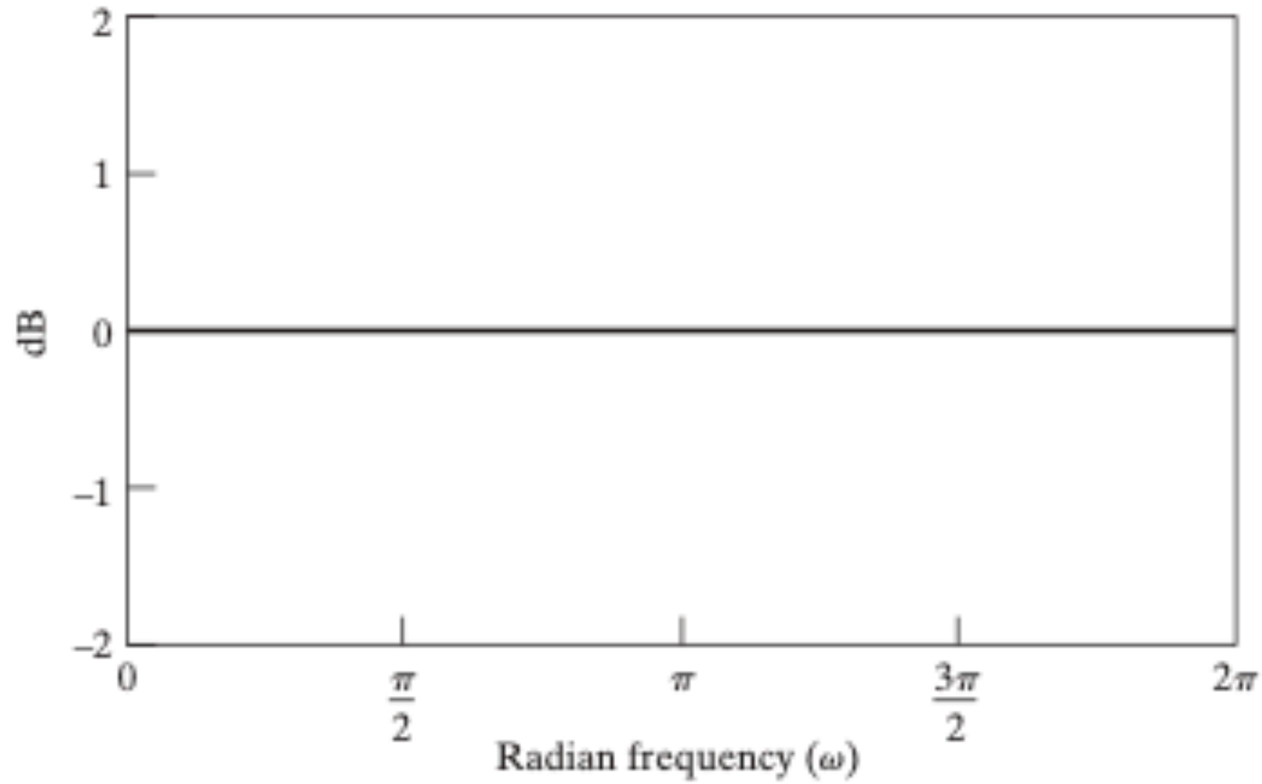
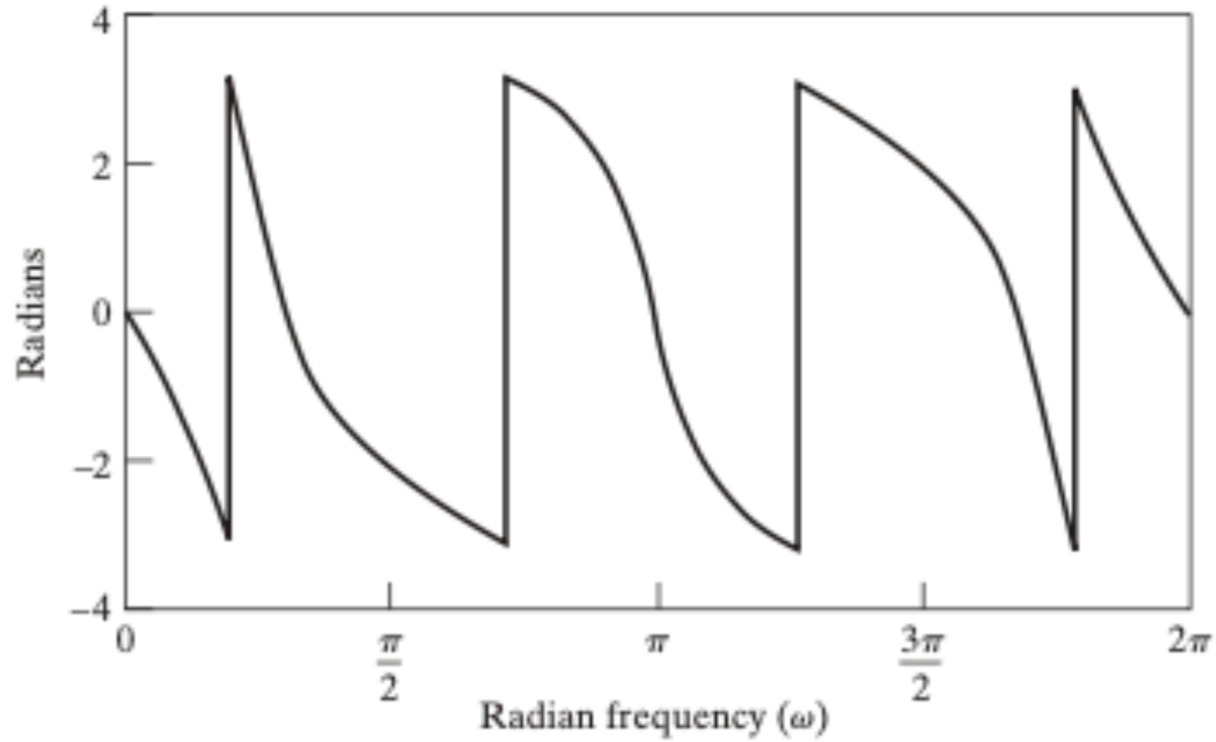


All-pass Example: Magnitude



All-pass Example: Phase



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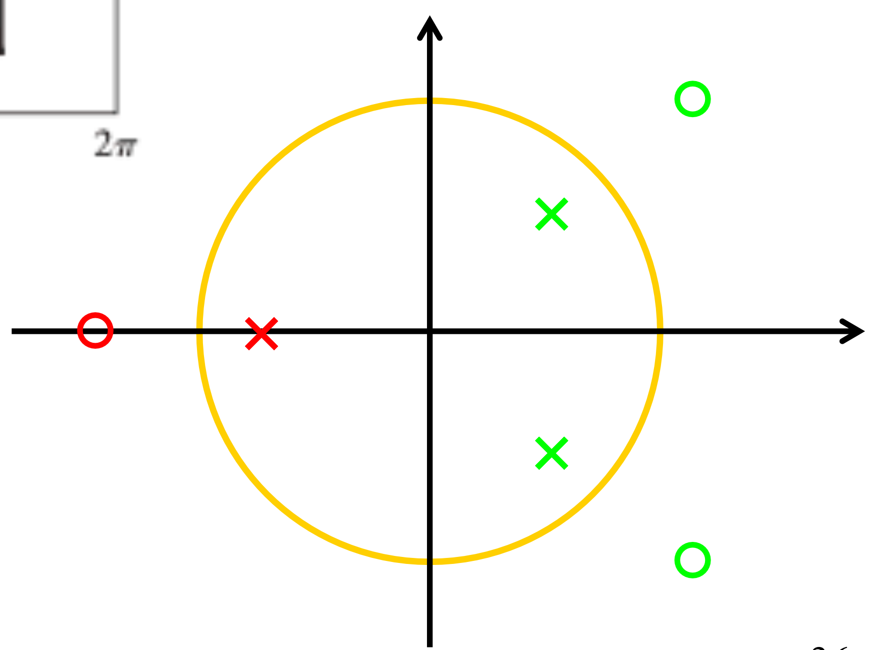
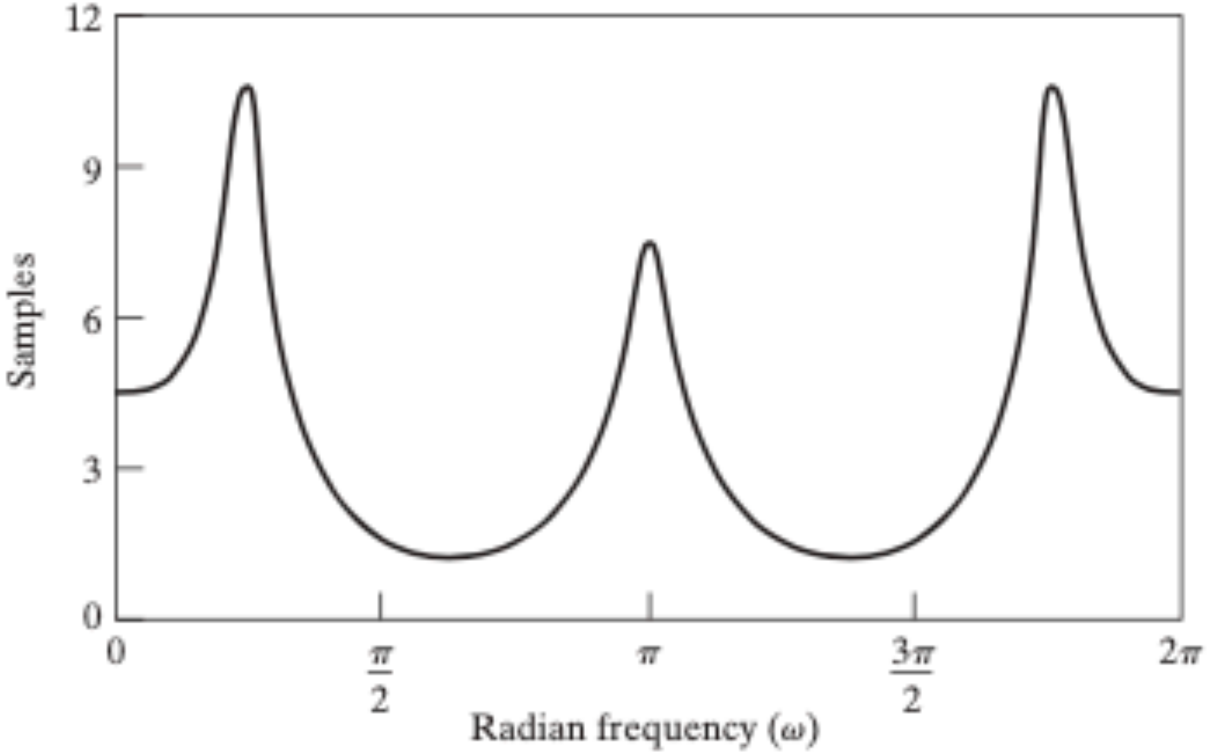


All-pass Example: Group Delay



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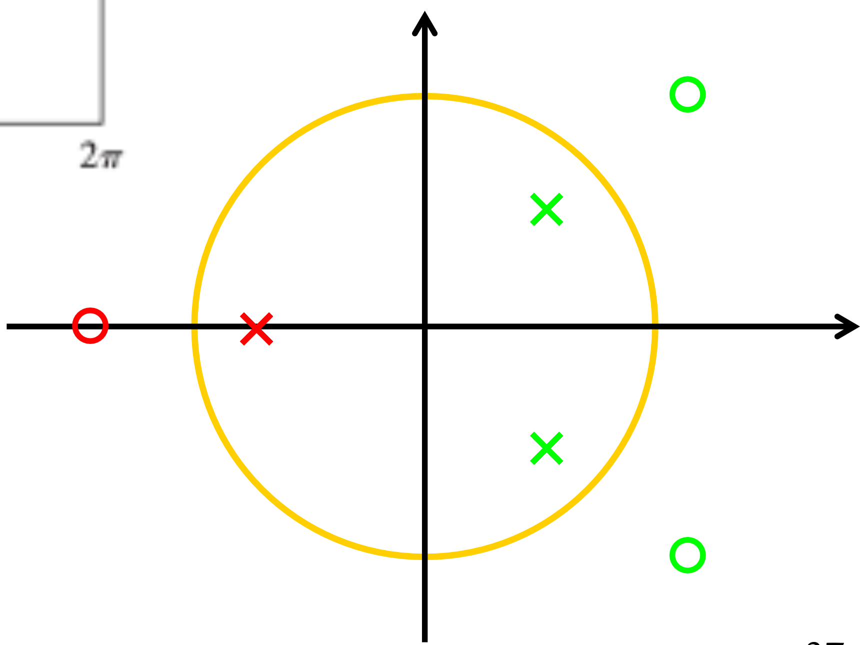

All-Pass Properties

- Claim: For a stable, causal ($r < 1$) all-pass system:
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
    - Unwrapped phase always non-positive and decreasing
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$
    - Group delay always positive

Pg 309 in text book

  - Intuition
    - delay is positive → system is causal
    - Phase negative → phase lag

Minimum Phase Systems

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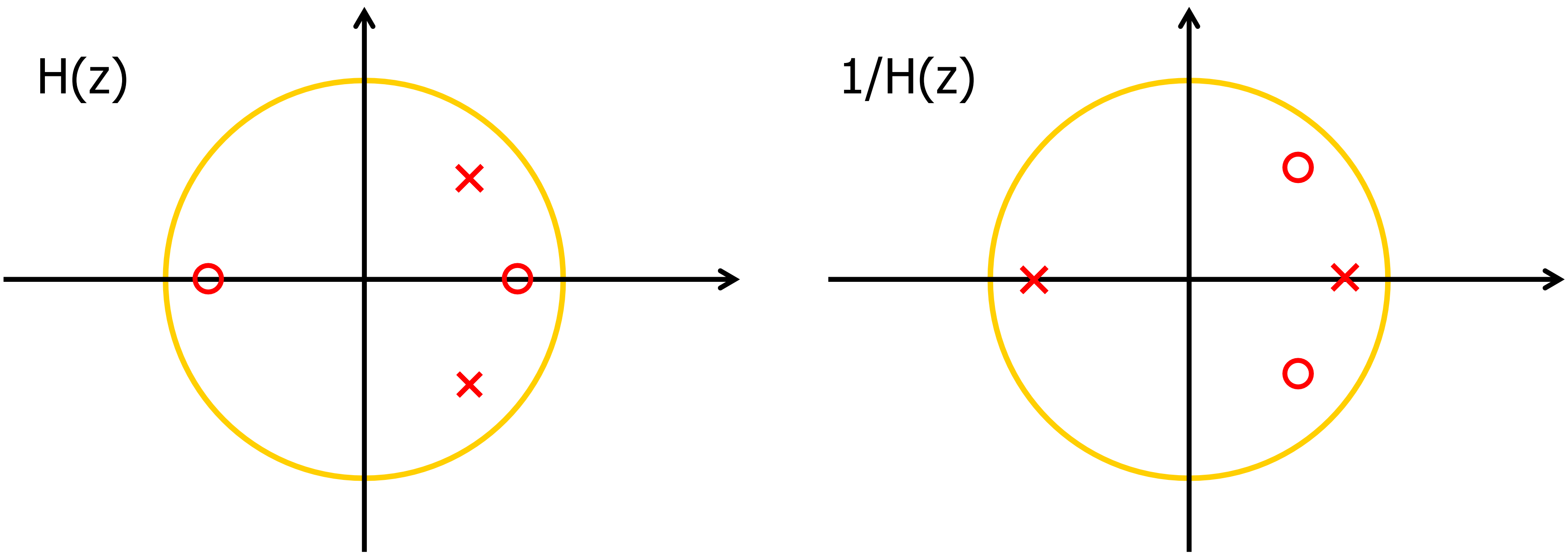

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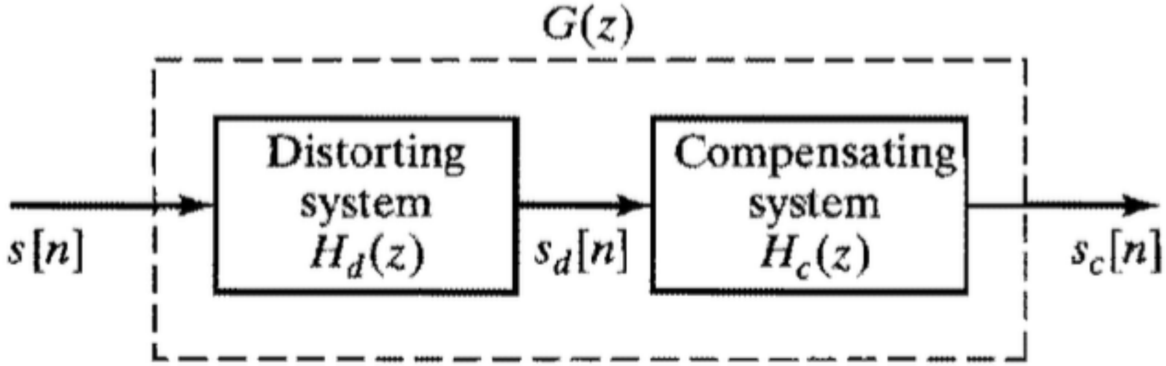


- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- What if not min phase?

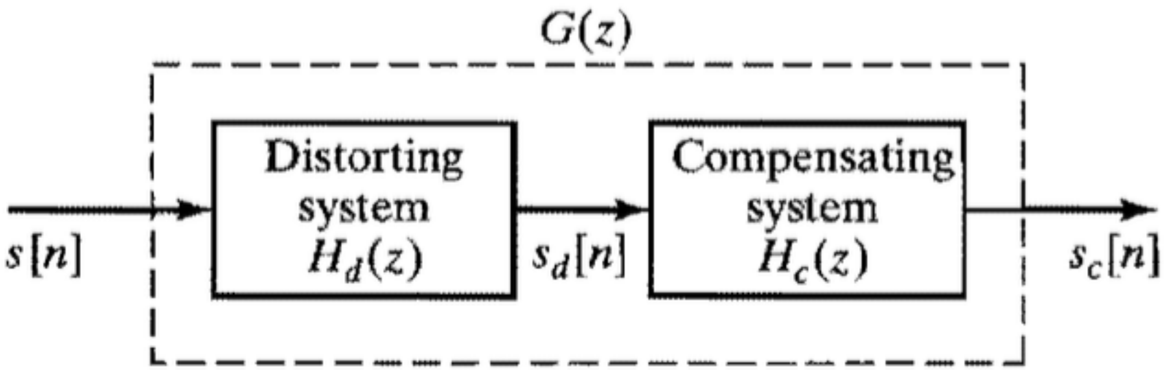
All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

Min-Phase Decomposition Purpose

- Have some distortion that we want to compensate for:



- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ ← also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

All-Pass Min-Phase Decomposition

- Any stable, causal system can be decomposed to:

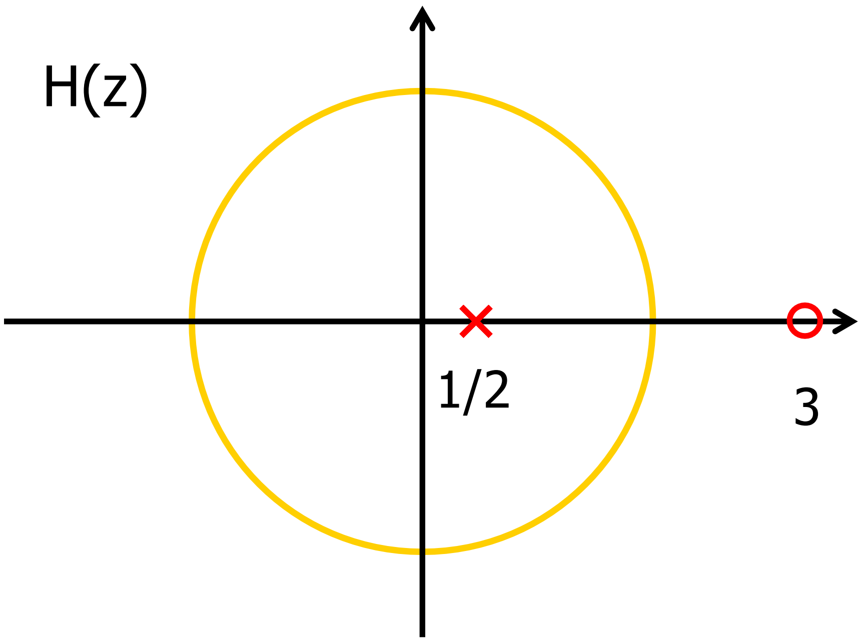
$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

- Approach:
  - (1) First construct H_{ap} with all zeros outside unit circle
  - (2) Compute

$$H_{\min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Min-Phase Decomposition Example

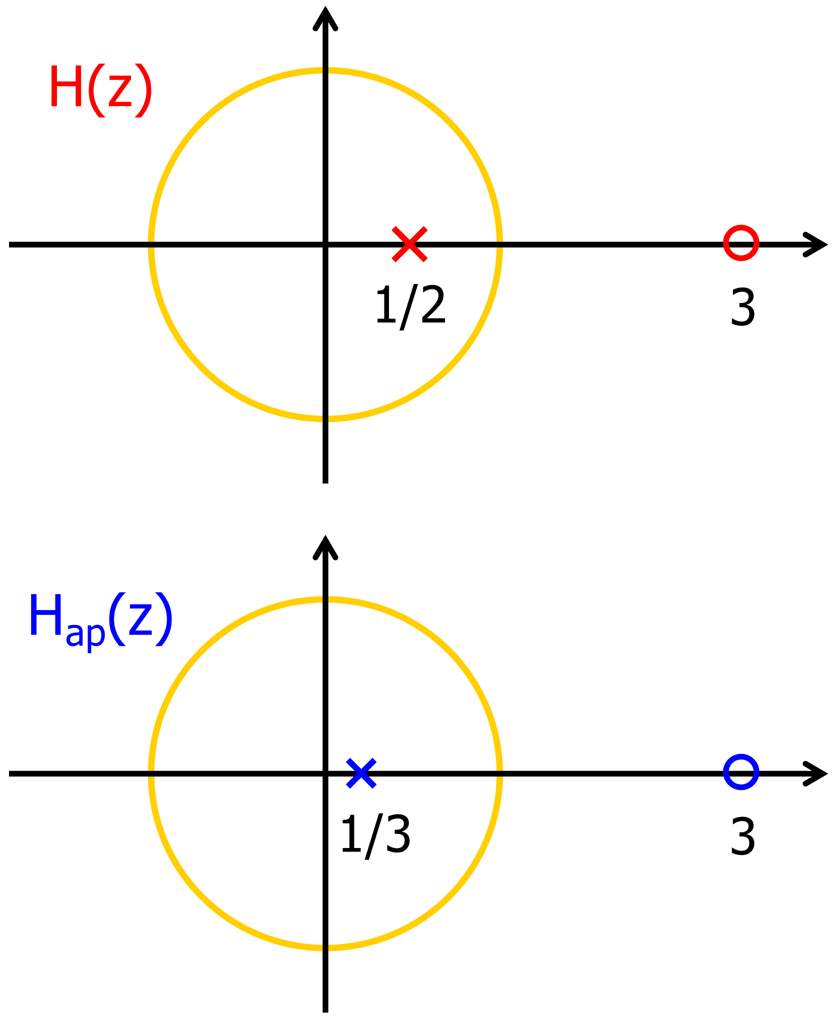
$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$



Min-Phase Decomposition Example

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

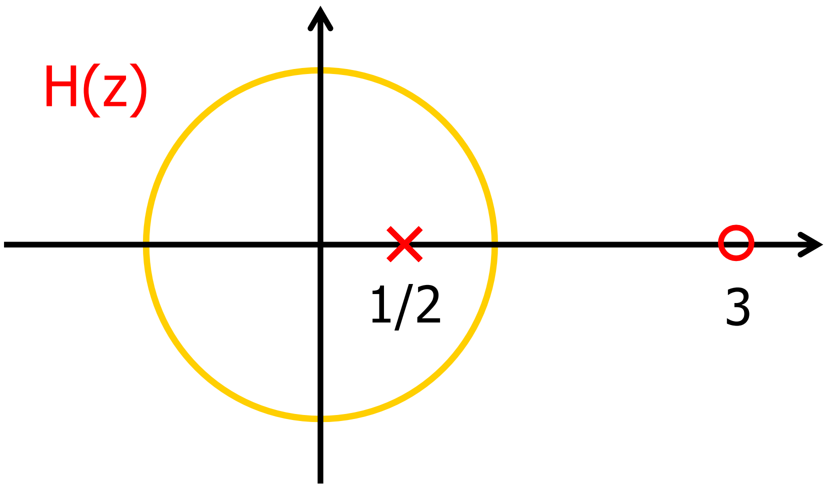
- Set $\quad H_{ap}(z) = \dfrac{z^{-1} - a^*}{1 - az^{-1}}$

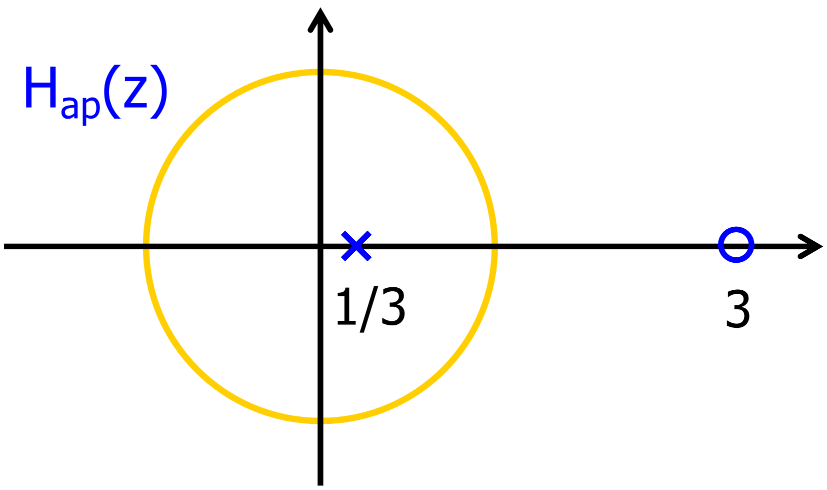


Min-Phase Decomposition Example

$$H(z)=\frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z)=\dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$





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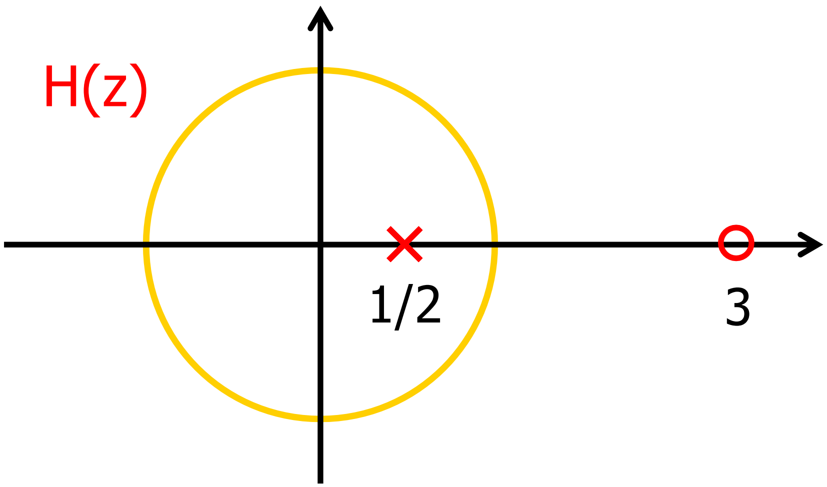
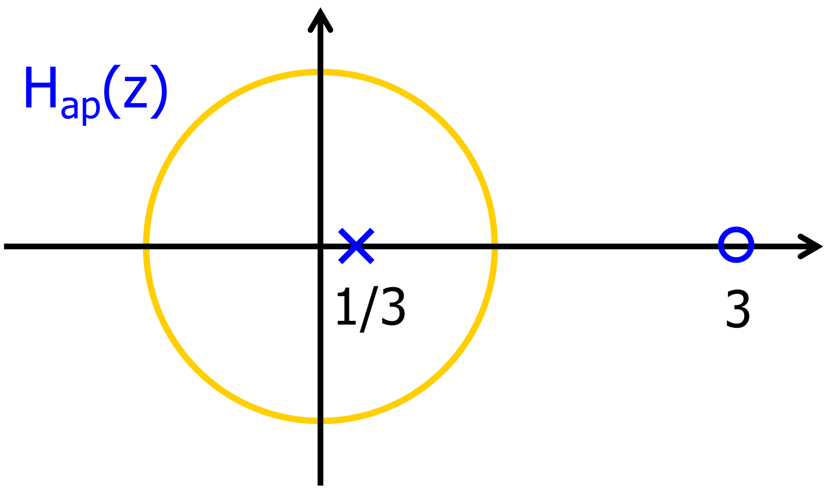


- Set $$H_{ap}(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}$$



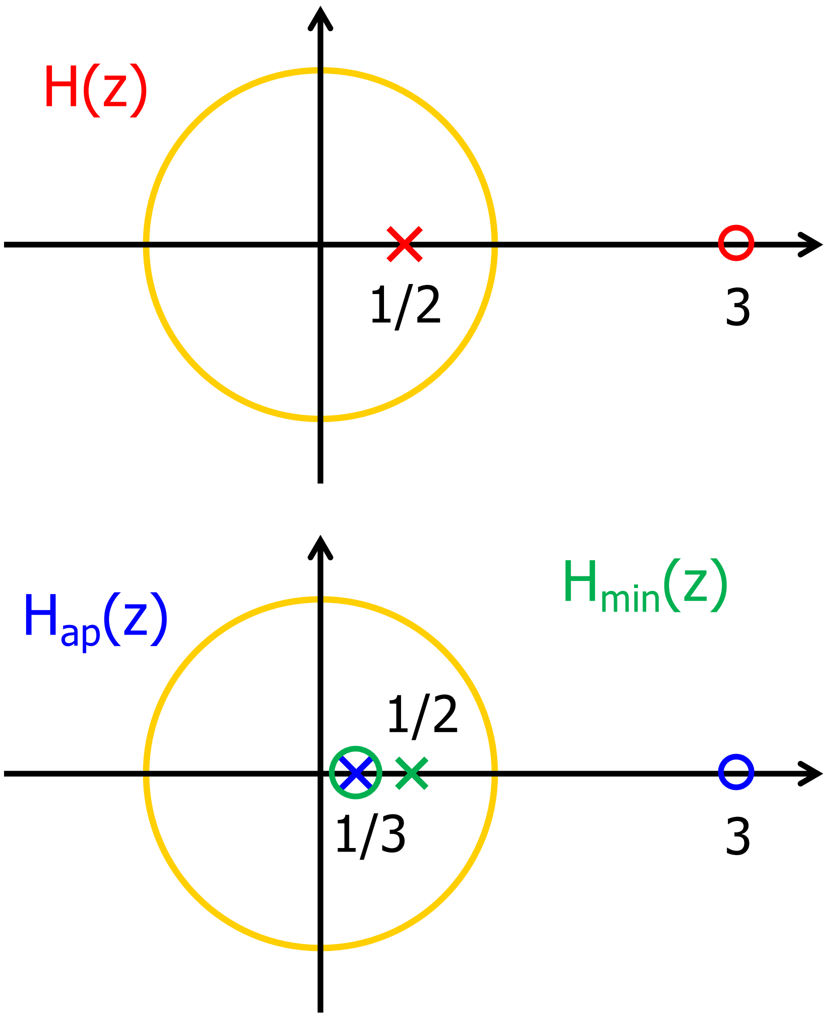
$$H_{\min}(z) = -3\frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

Min-Phase Decomposition Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$

- Set $H_{ap}(z) = \dfrac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$

$$H_{\min}(z) = -3\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$



Min-Phase Decomposition Purpose

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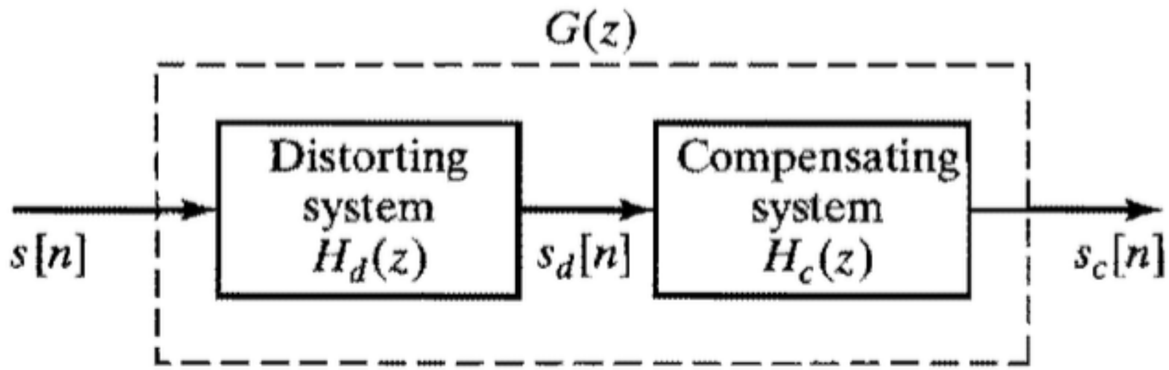
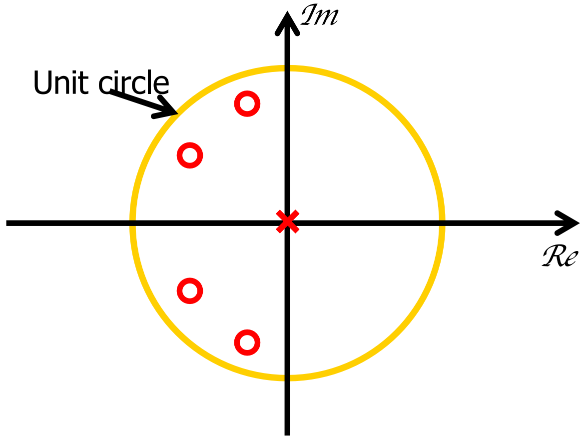


- If $H_d(z)$ is min phase, easy:
  - $H_c(z)=1/H_d(z)$ $\leftarrow$ also stable and causal
- Else, decompose $H_d(z)=H_{d,min}(z)\ H_{d,ap}(z)$
  - $H_c(z)=1/H_{d,min}(z) \rightarrow H_d(z)H_c(z)=H_{d,ap}(z)$
    - Compensate for magnitude distortion

Minimum Phase to Max Phase

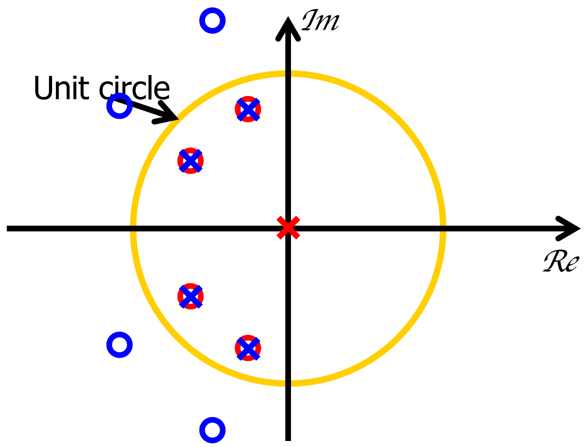
Min phase



Minimum Phase to Max Phase

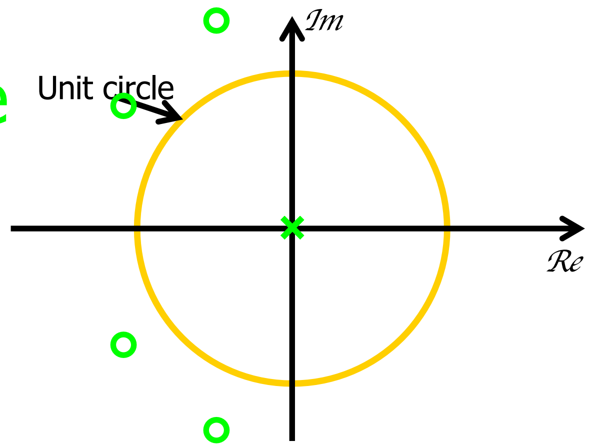
Min phase

x All Pass



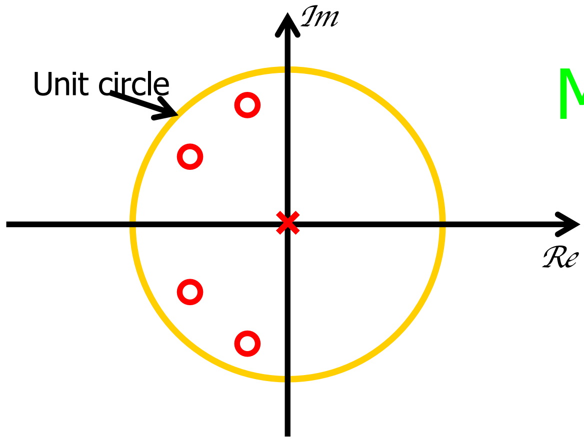
Minimum Phase to Max Phase

Max phase

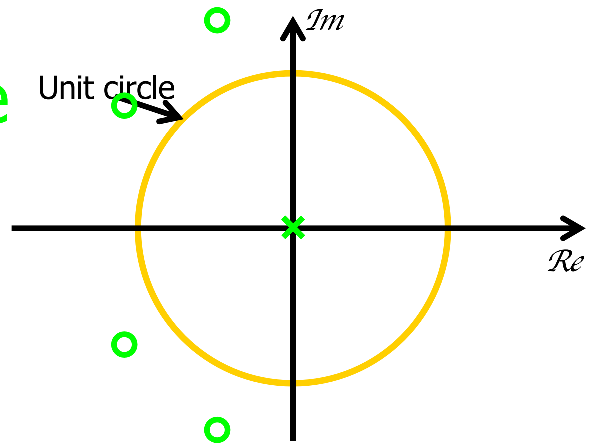


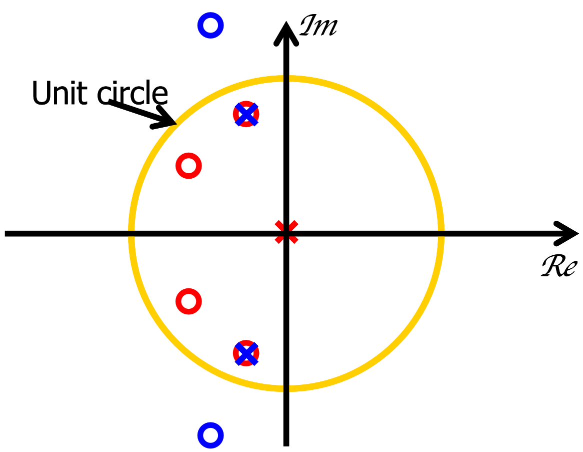
Minimum Phase

Min phase



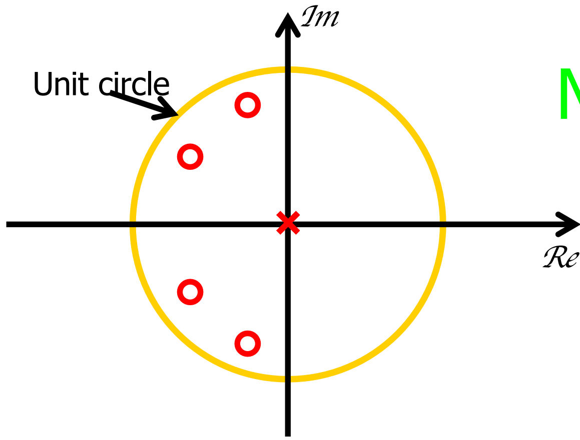
Max phase



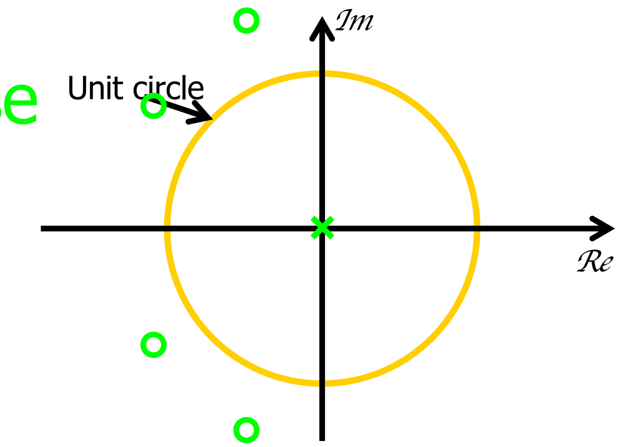


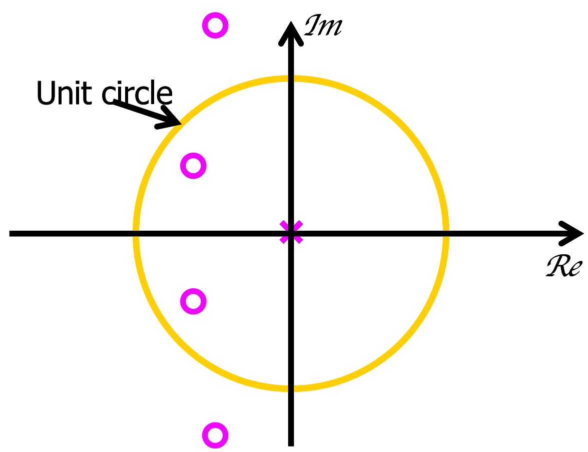
Minimum Phase

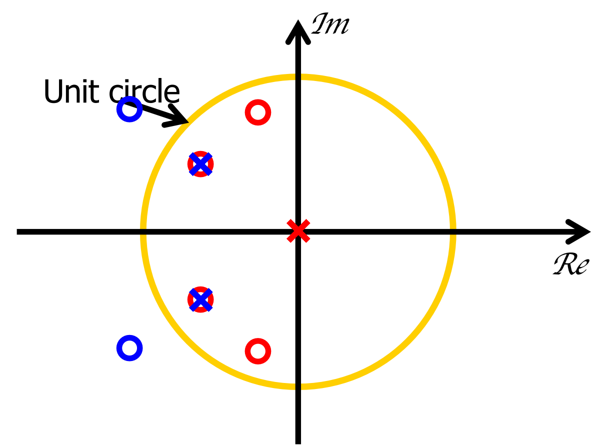
Min phase



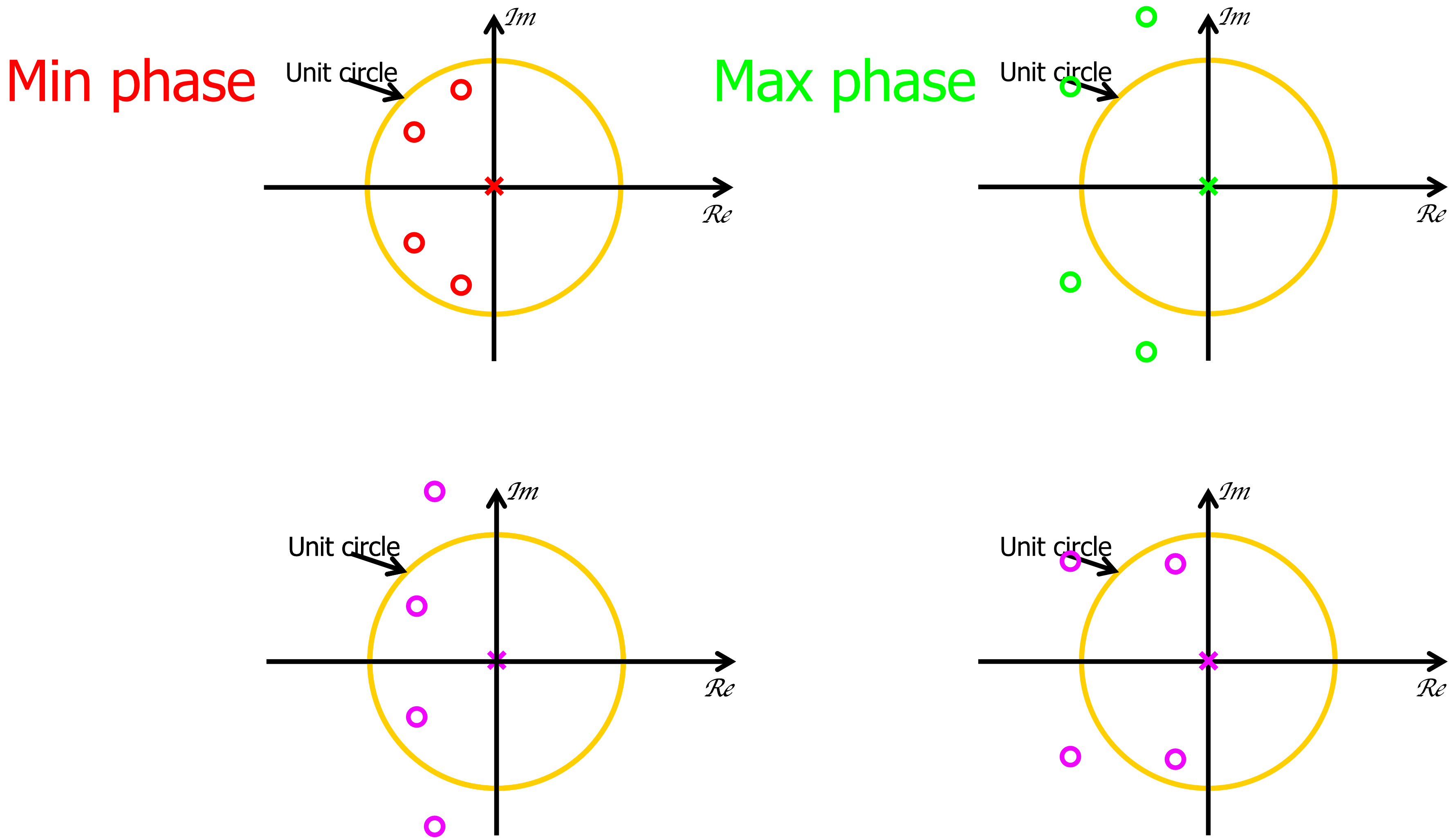
Max phase







Minimum Phase



Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\text{grd}[H_{ap}(e^{j\omega})] > 0$

Minimum Phase Lag Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$

- $\arg[H_{max}(e^{j\omega})] = \arg[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \arg[H_{min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

$$= \arg[H_{min}(e^{j\omega})] + \quad \leq 0$$

Minimum Group Delay Property

- All pass properties
  - $\arg[H_{ap}(e^{j\omega})] \leq 0$
  - $\mathrm{grd}[H_{ap}(e^{j\omega})] > 0$

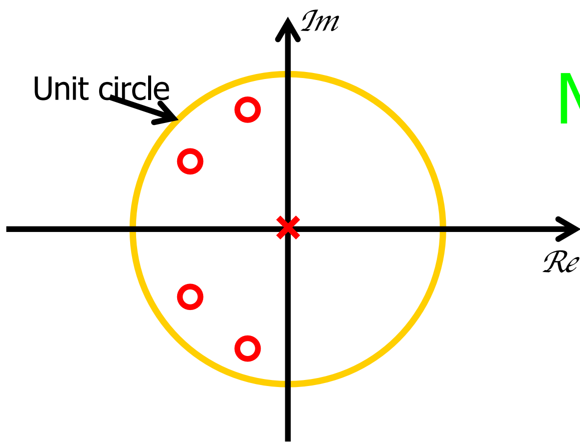
- $\mathrm{grd}[H_{max}(e^{j\omega})] = \mathrm{grd}[H_{min}(e^{j\omega}) * H_{ap}(e^{j\omega})]$

$$= \mathrm{grd}[H_{min}(e^{j\omega})] + \mathrm{grd}[H_{ap}(e^{j\omega})]$$

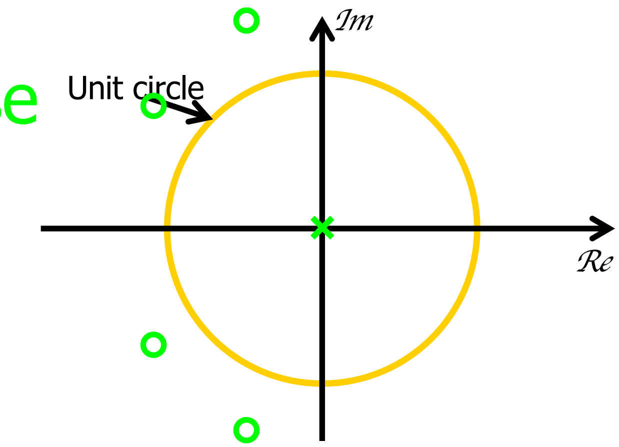
$$= \mathrm{grd}[H_{min}(e^{j\omega})] + \quad \geq 0$$

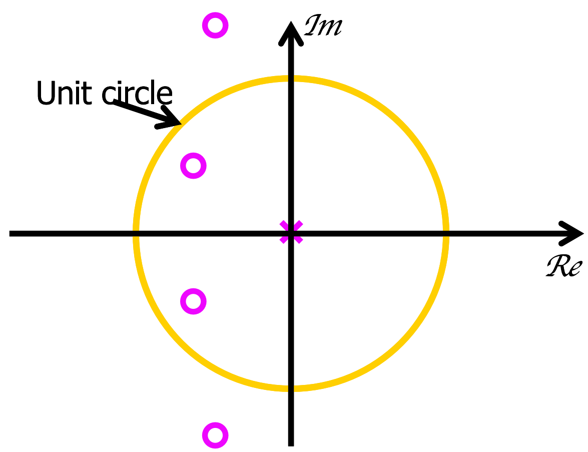
Minimum Phase

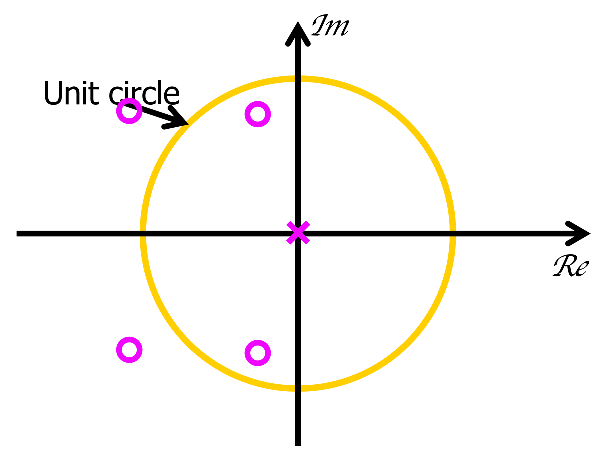
Min phase



Max phase

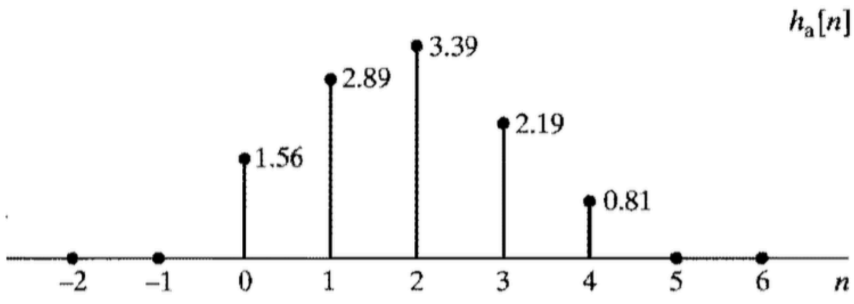




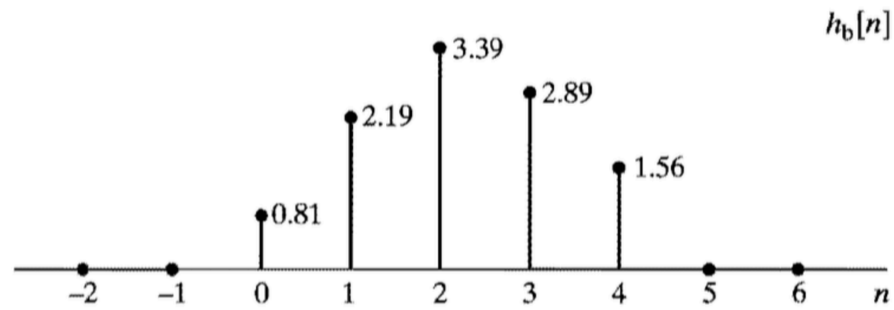


Minimum Energy-Delay Property

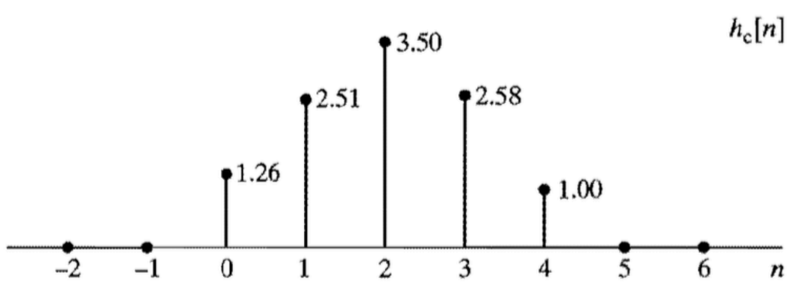
Min phase **Max phase**



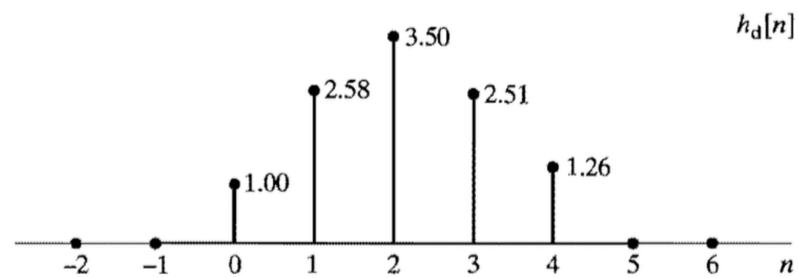
(a)



(b)



(c)

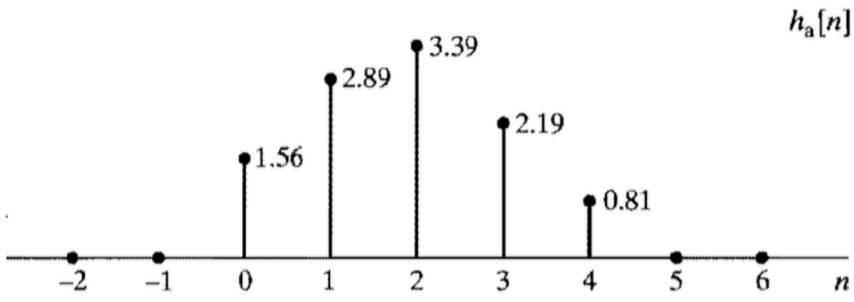


(d)

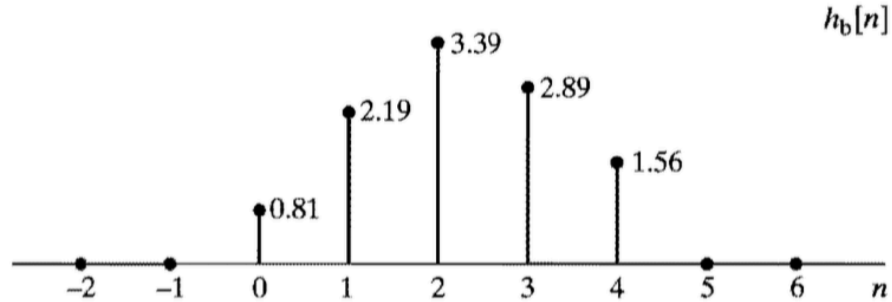
Total energy: $E = \sum_{n=0}^{\infty} |h[n]|^2$

Minimum Energy-Delay Property

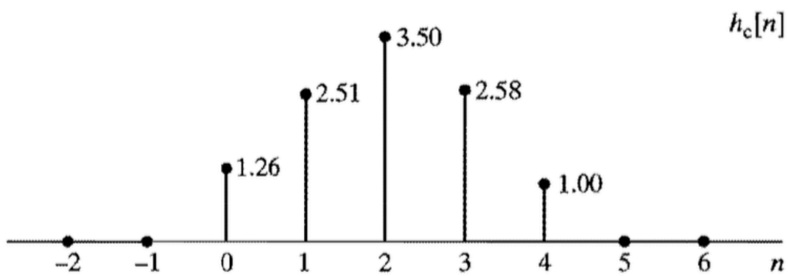
Min phase **Max phase**



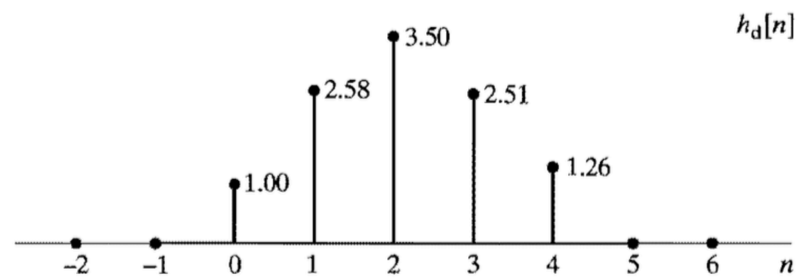
(a)



(b)



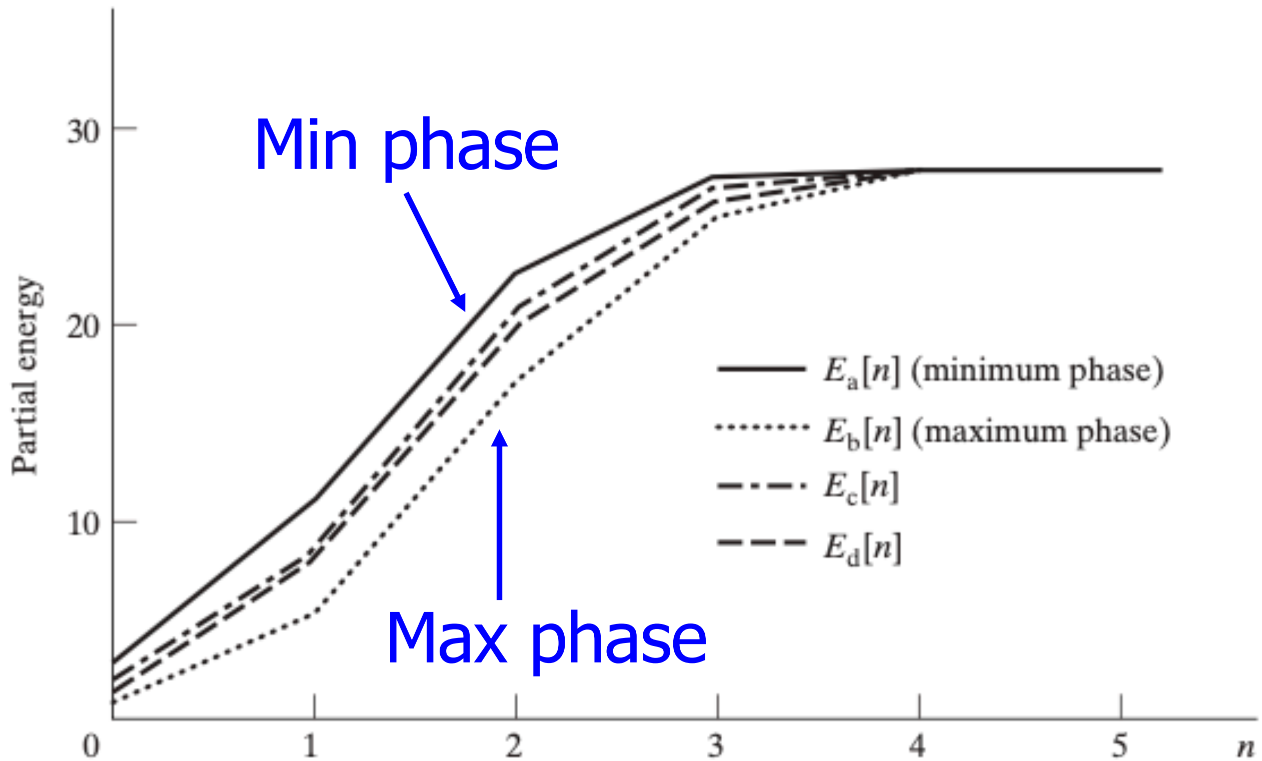
(c)



(d)

Partial energy: $E[n] = \sum_{m=0}^{n} |h[m]|^2$

Minimum Energy-Delay Property



Partial energy: $E[n] = \sum_{m=0}^{n} |h[m]|^2$

Big Ideas

- Frequency Response of LTI Systems
  - Magnitude Response, Phase Response, Group Delay
- Review: LTI Stability and Causality
  - If all poles inside unit circle
- All Pass Systems
  - Used for delay compensation
- Minimum Phase Systems
  - Can compensate for magnitude distortion
  - Minimum energy-delay property

Admin

- HW 6 due Sunday 3/17
- Midterm
  - Thursday 3/21 in person during class in DRLB A8
  - Covers lectures 1-13
    - Doesn't cover data converters and noise shaping (lec 11)
  - Old exams online
    - Disclaimer: different exam coverage for some years
  - TA Review session Monday 3/18 in DRLB A8
    - Recording posted in Canvas after
  - Closed book/notes
  - Can bring 1 two-sided 8.5"x11" sheet and non-cell phone calculator
- Proj 1 posted 3/21
  - Due 4/2