

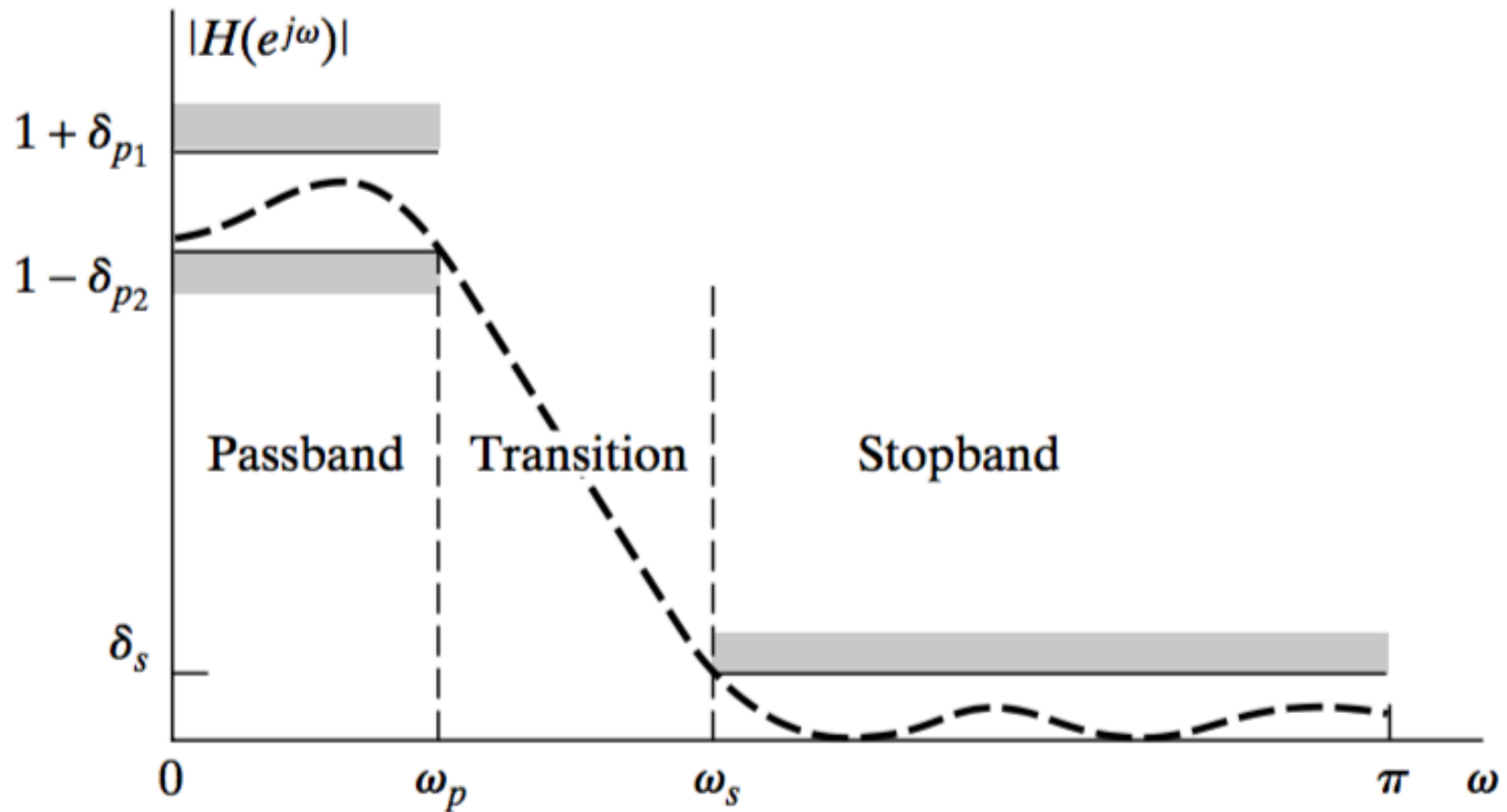
ESE 5310: Digital Signal Processing

Lecture 14: March 19, 2024
Generalized Linear Phase Systems



Generalized Linear Phase Systems

Filter Specifications



Generalized Linear Phase

- An LTI system has generalized linear phase if frequency response $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = A(\omega)e^{-j\omega\alpha+j\beta}, \quad |\omega| < \pi$$

- Where $A(\omega)$ is a real function.
- What is the group delay?

Reminder: Moving Average

□ Moving Average Filter

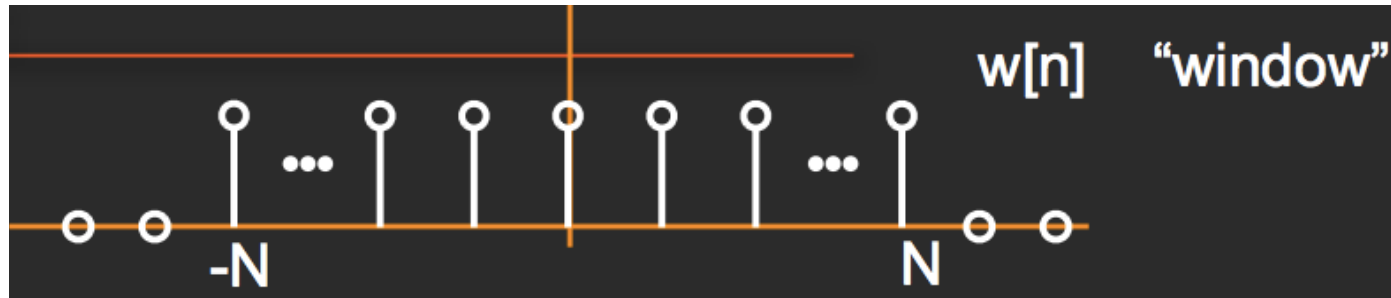
- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response

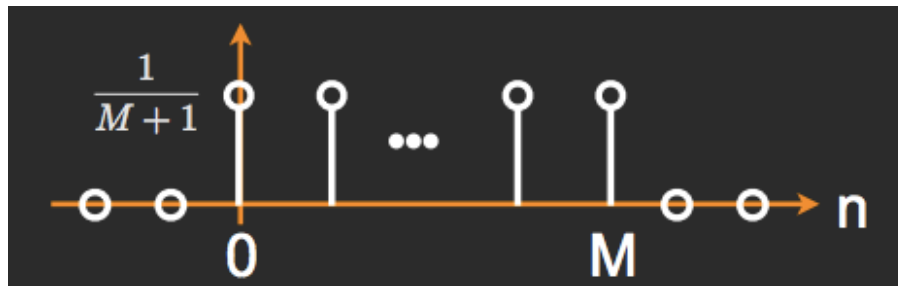


Reminder: Moving Average



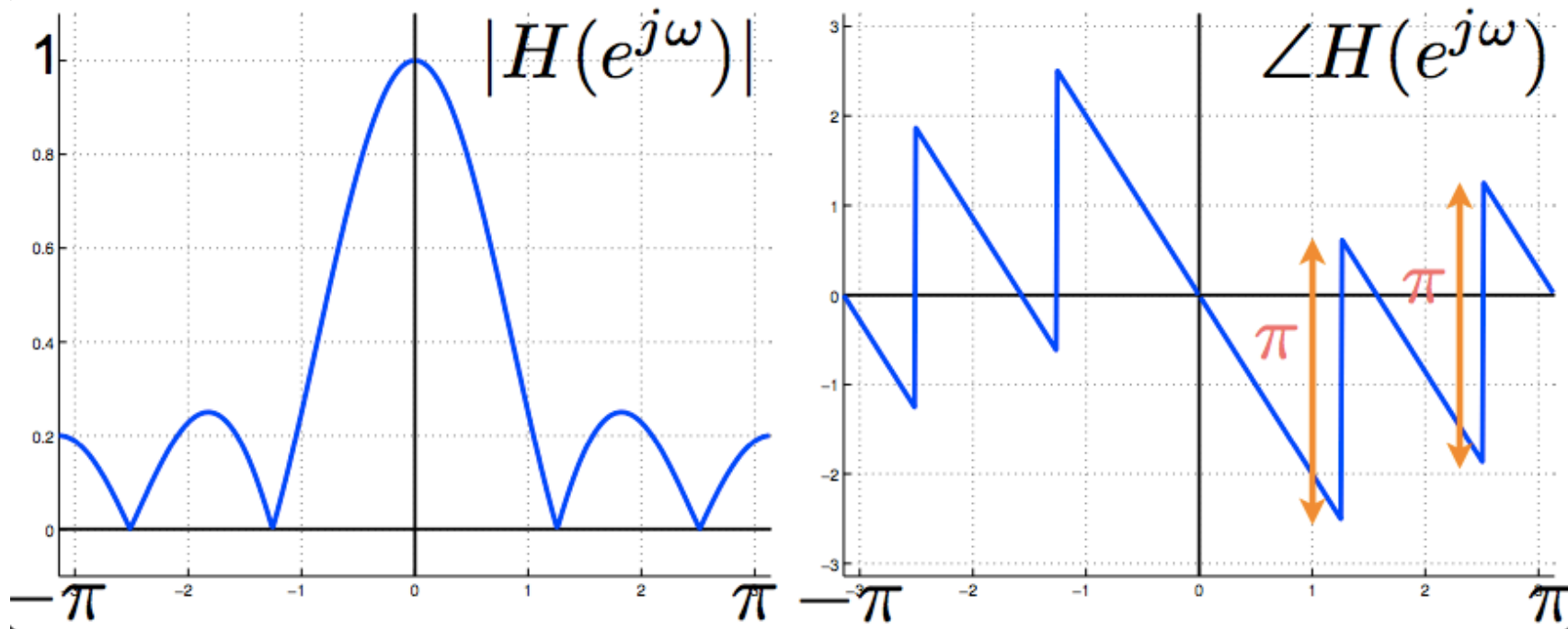
$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin((N + 1/2)\omega)}{\sin(\omega/2)}$$

Scaled & Time
Shifted window



$$\frac{1}{M+1} w[n - M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin((M/2 + 1/2)\omega)}{\sin(\omega/2)}$$

Reminder: Moving Average



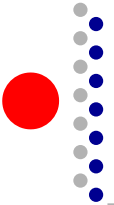


Causal FIR Systems

$$y[n] = b_0x[n] + b_1x[n-1] + \dots + b_Mx[n-M], \quad \text{all } n$$

$$H(z) = b_0 + b_1z^{-1} + \dots + b_Mz^{-M} = b_0 \prod_{k=1}^M (1 - c_k z^{-1})$$

$$h[n] = \begin{cases} b_n, & n = 0, 1, \dots, M \\ 0, & \text{otherwise} \end{cases}$$



Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length $(M+1)$ and some symmetry
- It can be shown if

$$h[n] = \begin{cases} h[M - n], & 0 \leq n \leq M, \\ 0, & \text{otherwise,} \end{cases}$$

- then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

Reminder: Moving Average

□ Moving Average Filter

- Causal: $M_1=0$, $M_2=M$

$$y[n] = \frac{x[n-M] + \dots + x[n]}{M+1}$$

Impulse
response



Causal FIR Systems

- Causal FIR systems have generalized linear phase if they have impulse response length $(M+1)$
- It can be shown if

$$h[n] = \begin{cases} h[M - n], & 0 \leq n \leq M, \\ 0, & \text{otherwise,} \end{cases}$$

- Then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2},$$

- Sufficient conditions to guarantee GLP, not necessary
 - Causal IIR can also have GLP



FIR GLP Systems

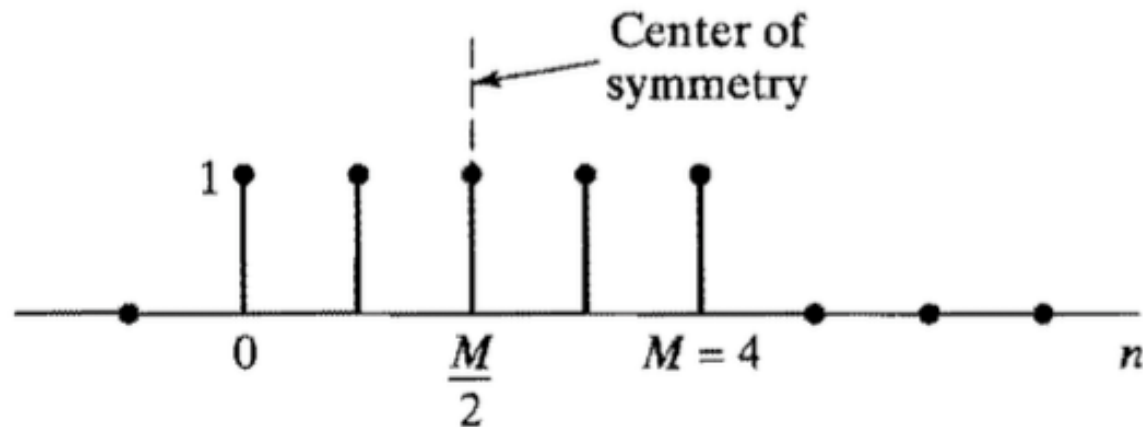
- Four types of FIR GLP
 - Type I
 - Even Symmetry, M even
 - Type II
 - Even Symmetry, M odd
 - Type III
 - Odd Symmetry, M even
 - Type IV
 - Odd Symmetry, M odd



FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

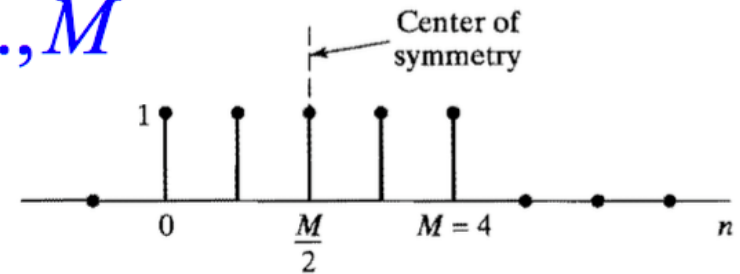




FIR GLP: Type I

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

integer delay

FIR GLP: Type I – Example, $M=4$

Type I Even Symmetry, M even

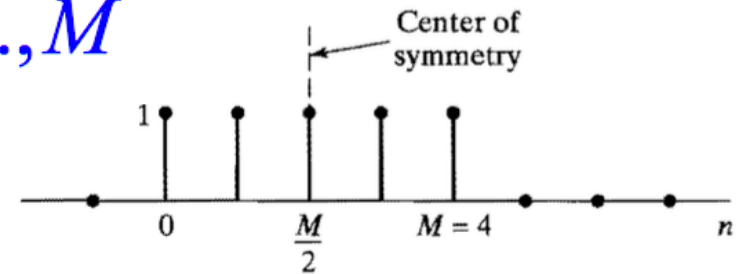
$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} + h[2] + h[1]e^{-j\omega} + h[0]e^{-j2\omega} \right] \\ &= \underbrace{\left[2h[0]\cos(2\omega) + 2h[1]\cos(\omega) + h[2] \right]}_{A(\omega) \text{ (even)}} e^{-j2\omega} \end{aligned}$$

Type I Even Symmetry, M even

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



Then $H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$  integer delay

$$H(e^{j\omega}) = e^{-j\omega M/2} \left(\sum_{k=0}^{M/2} a[k] \cos \omega k \right)$$

$$a[0] = h[M/2],$$

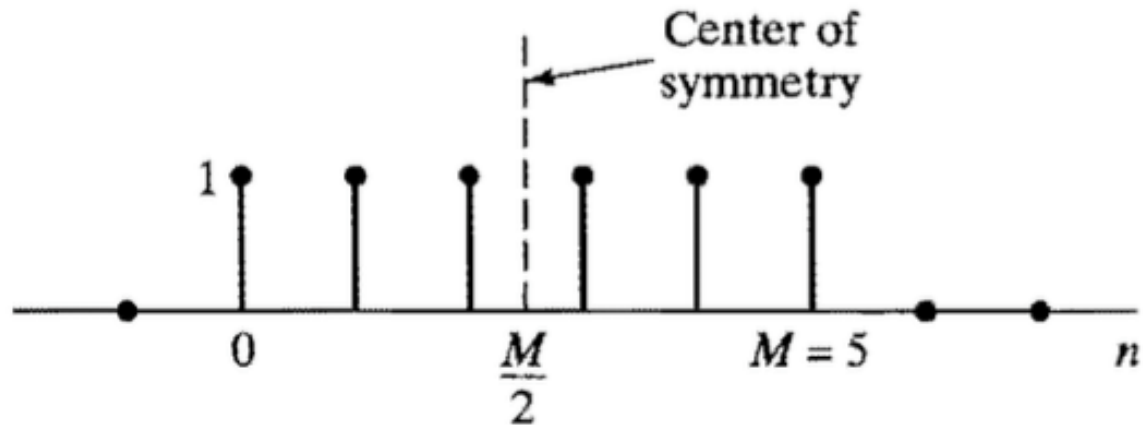
$$a[k] = 2h[(M/2) - k], \quad k = 1, 2, \dots, M/2.$$



FIR GLP: Type II

Type II Even Symmetry, M odd

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$

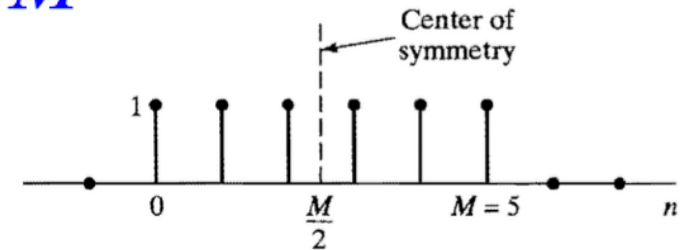




FIR GLP: Type II

Type II Even Symmetry, M odd

$$h[n] = h[M - n], \quad n = 0, 1, \dots, M$$



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

← integer delay

FIR GLP: Type II – Example, $M=3$

Type II Even Symmetry, M odd

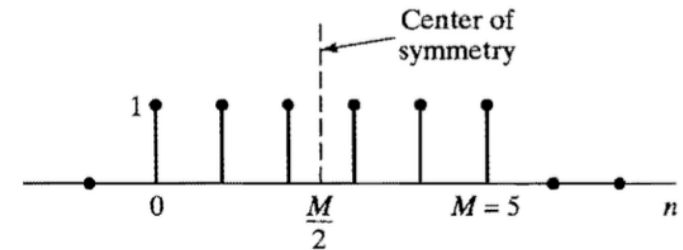
$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n]e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2} \quad \leftarrow \text{integer delay}$$

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega}$$

$$H(e^{j\omega}) = \underbrace{[2h[0]\cos(3\omega / 2) + 2h[1]\cos(\omega / 2)]}_{A(\omega)} e^{-j3\omega/2}$$

FIR GLP: Type II

Type II Even Symmetry, M odd



$$\text{Then } H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Even}} e^{-j\omega M/2}$$

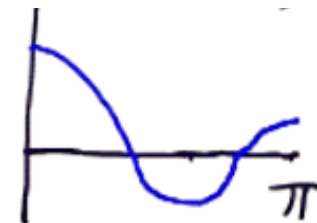
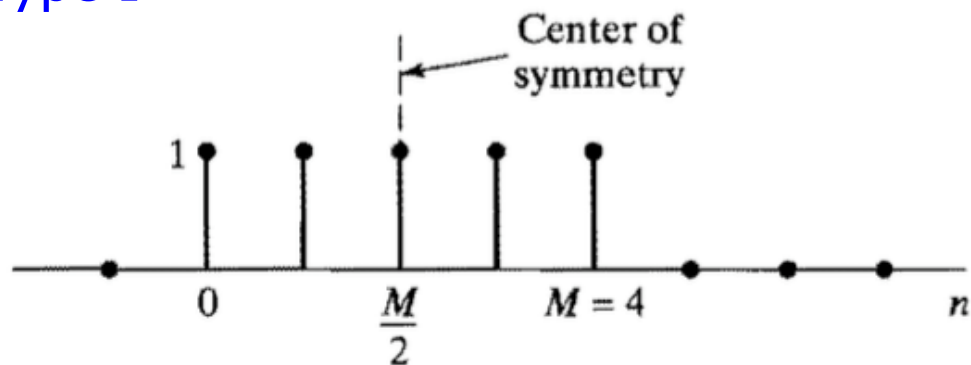
integer delay

$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{(M+1)/2} b[k] \cos \left[\omega \left(k - \frac{1}{2} \right) \right] \right\}$$
$$b[k] = 2h[(M+1)/2 - k], \quad k = 1, 2, \dots, (M+1)/2.$$

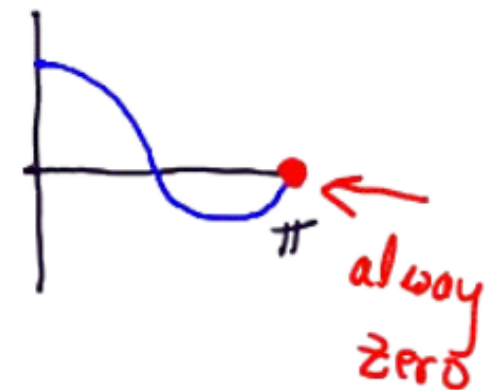
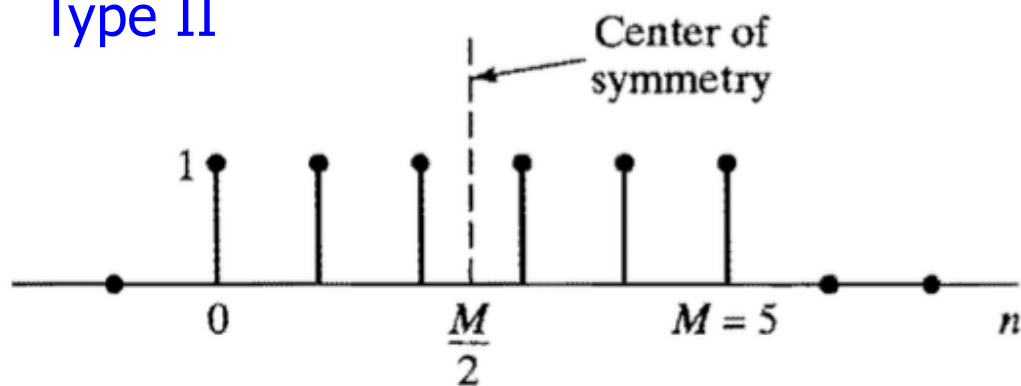


FIR GLP: Type I and II

Type I

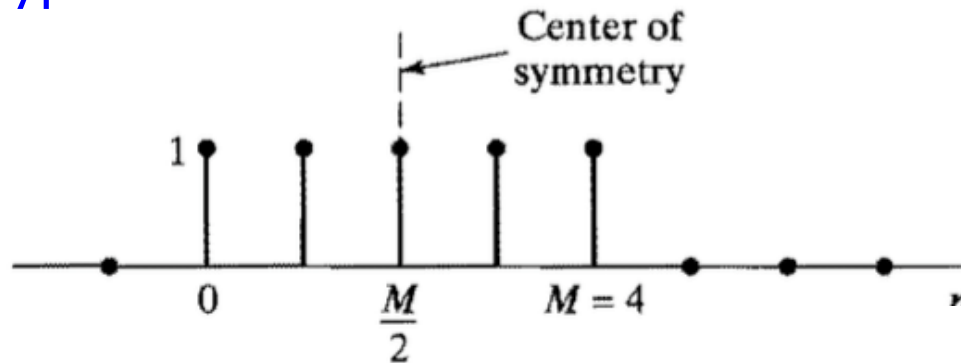


Type II

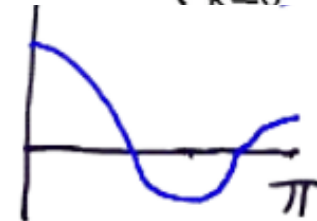


FIR GLP: Type I and II

Type I

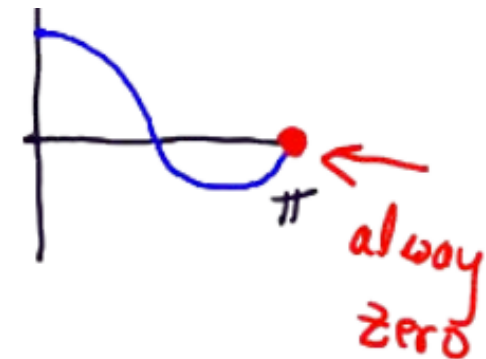
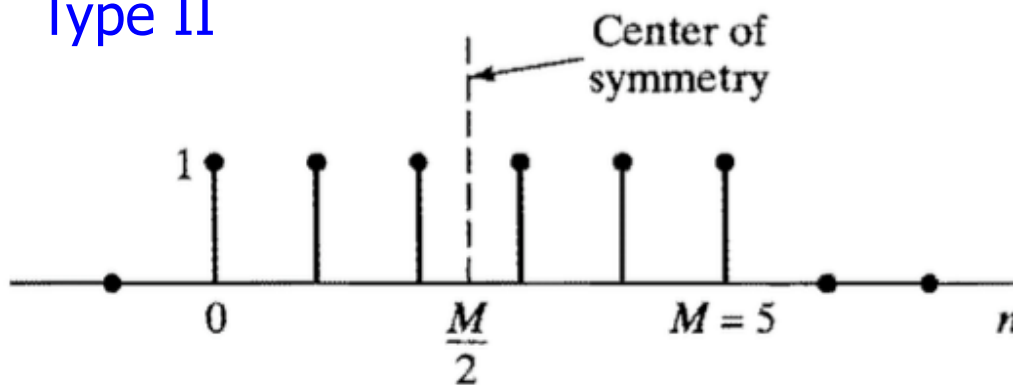


$$H(e^{j\omega}) = e^{-j\omega M/2} \left(\sum_{k=0}^{M/2} a[k] \cos \omega k \right)$$



$$H(e^{j\omega}) = e^{-j\omega M/2} \left\{ \sum_{k=1}^{(M+1)/2} b[k] \cos \left[\omega \left(k - \frac{1}{2} \right) \right] \right\}$$

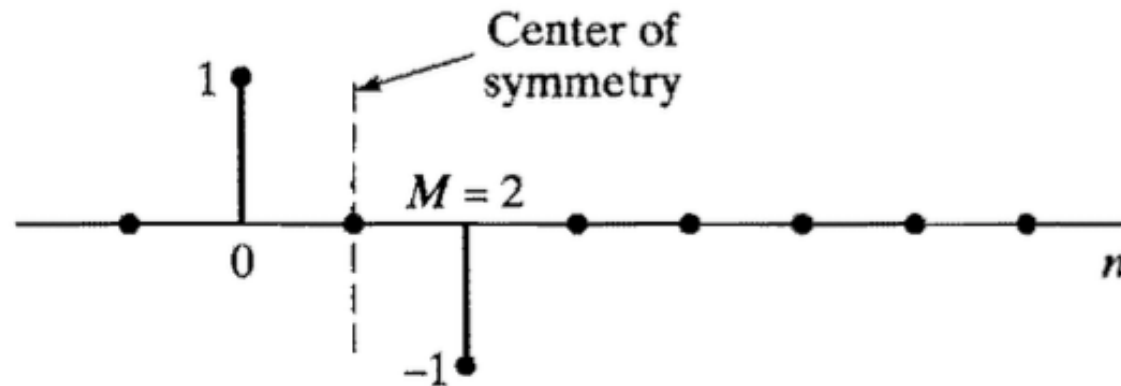
Type II



FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

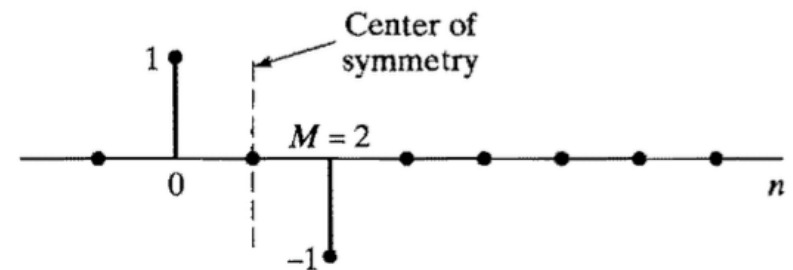




FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$



$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

FIR GLP: Type III – Example, $M=4$

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} \\ &= e^{-j2\omega} \left[h[0]e^{j2\omega} + h[1]e^{j\omega} - h[1]e^{-j\omega} - h[0]e^{-j2\omega} \right] \\ &= \underbrace{\left[2h[0]\sin(2\omega) + 2h[1]\sin(\omega) \right]}_{A(\omega) \text{ (odd)}} j e^{-j2\omega} \end{aligned}$$

FIR GLP: Type III

Type III Odd Symmetry, M even

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (\text{note } h[M/2] = 0)$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

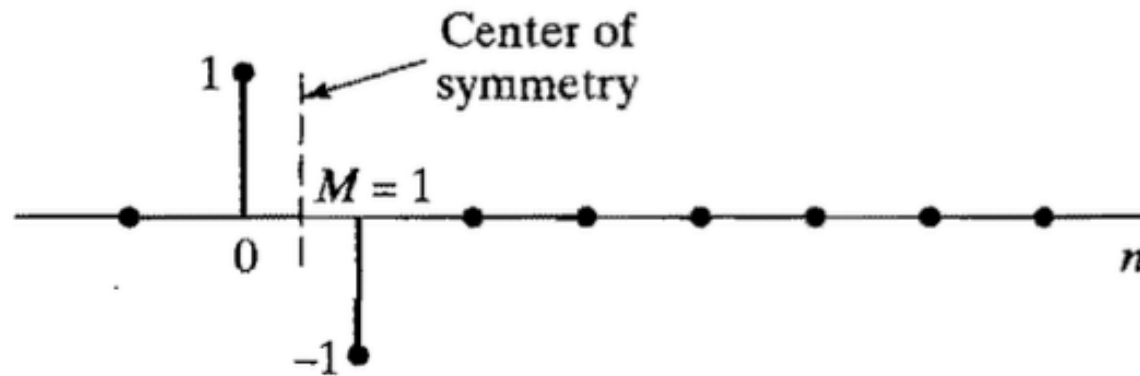
$$H(e^{j\omega}) = j e^{-j\omega M/2} \left[\sum_{k=1}^{M/2} c[k] \sin \omega k \right]$$

$$c[k] = 2h[(M/2) - k], \quad k = 1, 2, \dots, M/2.$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$

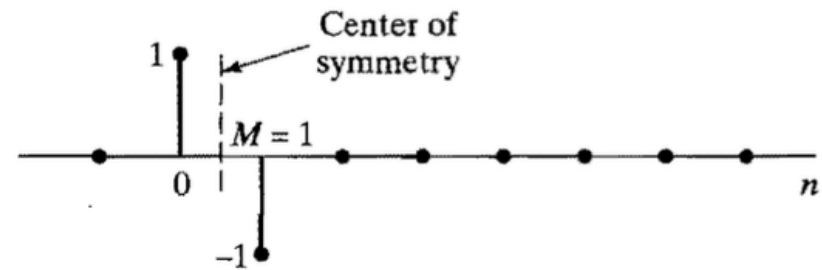




FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$



$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

fractional delay

FIR GLP: Type IV – Example, M=3

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$

$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

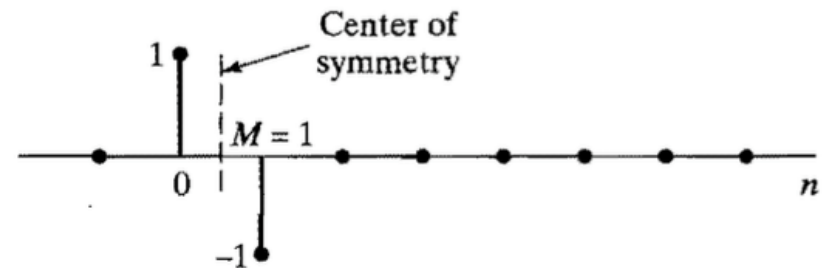
fractional delay

$$H(e^{j\omega}) = \underbrace{[2h[0]\sin(3\omega / 2) + 2h[1]\sin(\omega / 2)]}_{A(\omega)} j e^{-j3\omega/2}$$

FIR GLP: Type IV

Type IV Odd Symmetry, M Odd

$$h[n] = -h[M - n], \quad n = 0, 1, \dots, M \quad (M / 2 \text{ not an integer})$$



$$H(e^{j\omega}) = \sum_{n=0}^M h[n] e^{-j\omega n} = \underbrace{A(\omega)}_{\text{Real, Odd}} e^{-j\omega M/2 + j\pi/2}$$

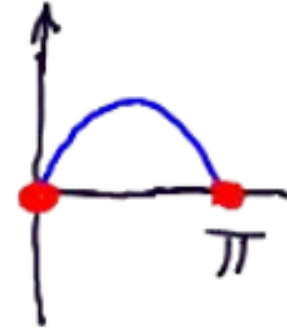
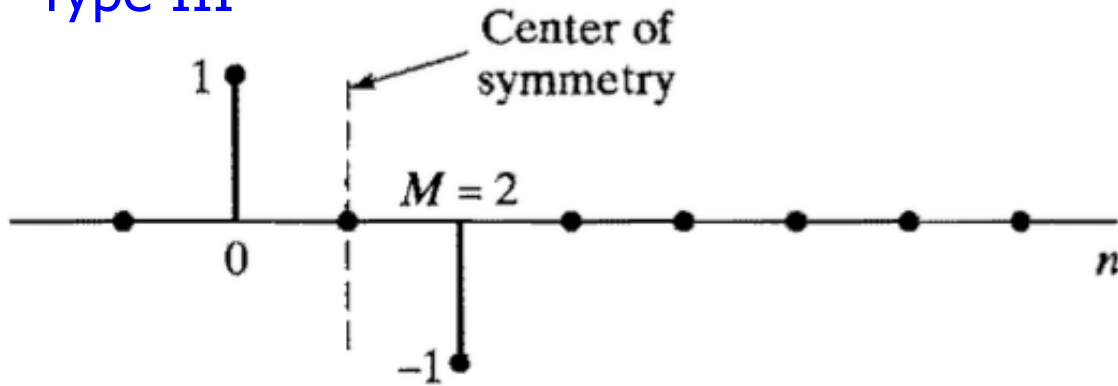
fractional delay

$$H(e^{j\omega}) = j e^{-j\omega M/2} \left[\sum_{k=1}^{(M+1)/2} d[k] \sin \left[\omega \left(k - \frac{1}{2} \right) \right] \right]$$

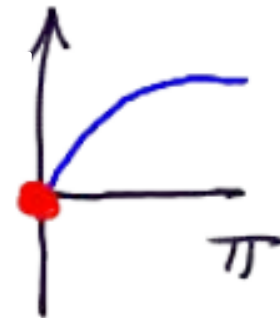
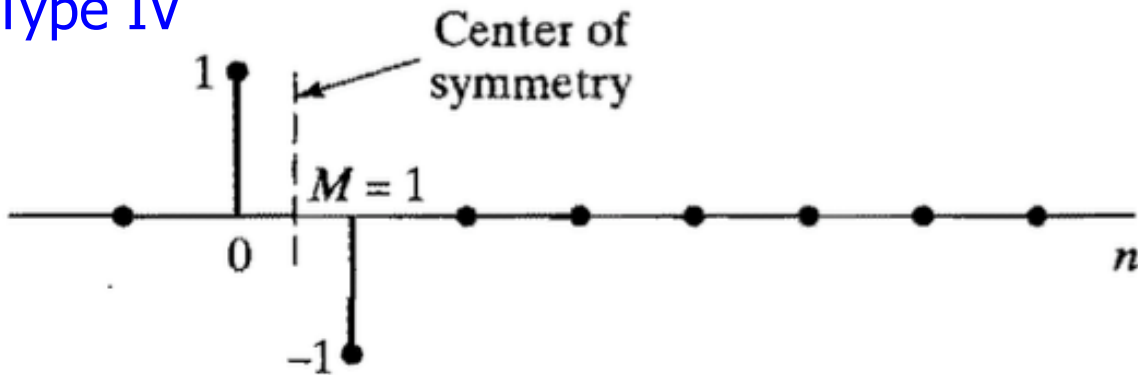
$$d[k] = 2h[(M+1)/2 - k], \quad k = 1, 2, \dots, (M+1)/2.$$

FIR GLP: Type III and IV

Type III



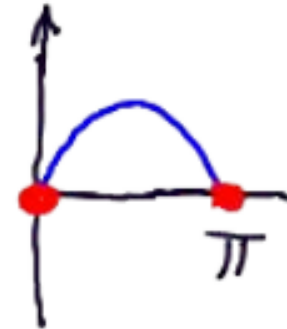
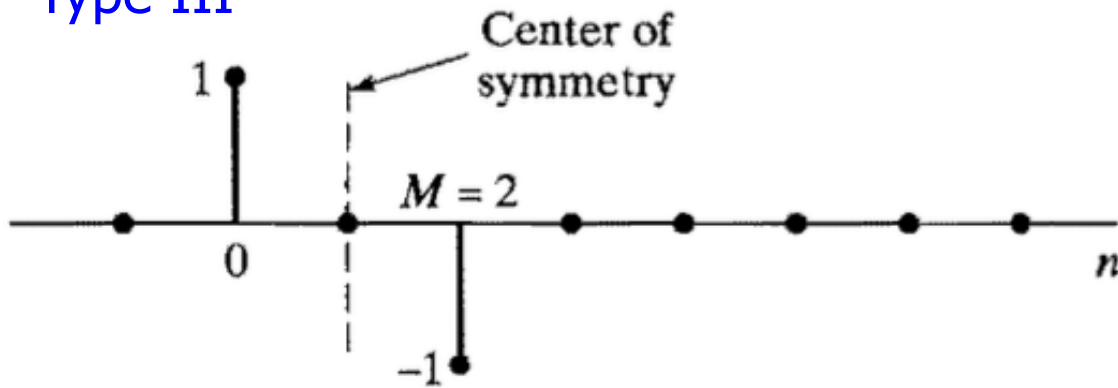
Type IV



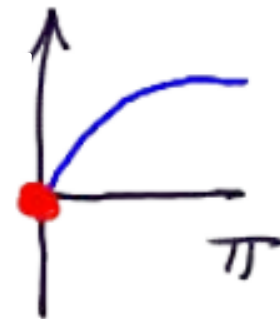
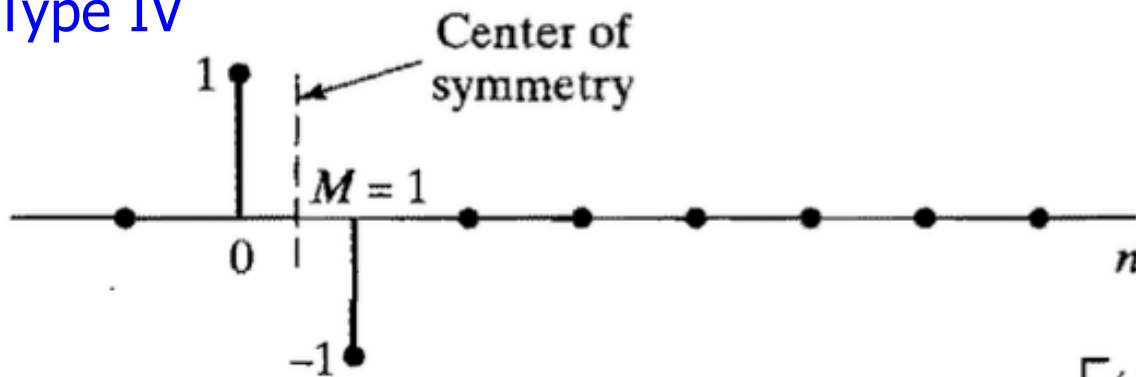
FIR GLP: Type III

$$H(e^{j\omega}) = je^{-j\omega M/2} \left[\sum_{k=1}^{M/2} c[k] \sin \omega k \right]$$

Type III



Type IV



$$H(e^{j\omega}) = je^{-j\omega M/2} \left[\sum_{k=1}^{(M+1)/2} d[k] \sin \left[\omega \left(k - \frac{1}{2} \right) \right] \right]$$



Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$\begin{aligned} H(z) &= \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M} \\ &= z^{-M} H(z^{-1}). \end{aligned}$$

If z_0 is a zero then z_0^{-1} is also a zero.



Zeros of GLP System – Type I and II

- ❑ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

- ❑ If $h[n]$ is real, with the conjugate symmetry property

$$H(z) = H^*(z^*)$$

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

If z_0 is a zero then z_0^{-1} is also a zero.

□ If $h[n]$ is real,

If z_0 is a zero then z_0^* is also a zero.

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - \underbrace{re^{j\theta} z^{-1}}_{z_0})(1 - \underbrace{re^{-j\theta} z^{-1}}_{(z_0)^*})(1 - \underbrace{r^{-1}e^{j\theta} z^{-1}}_{(z_0^{-1})^*})(1 - \underbrace{r^{-1}e^{-j\theta} z^{-1}}_{z_0^{-1}})$$

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

□ If zero is on unit circle (r=1)

$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

Zeros of GLP System – Type I and II

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

□ If zero is on unit circle ($r=1$)

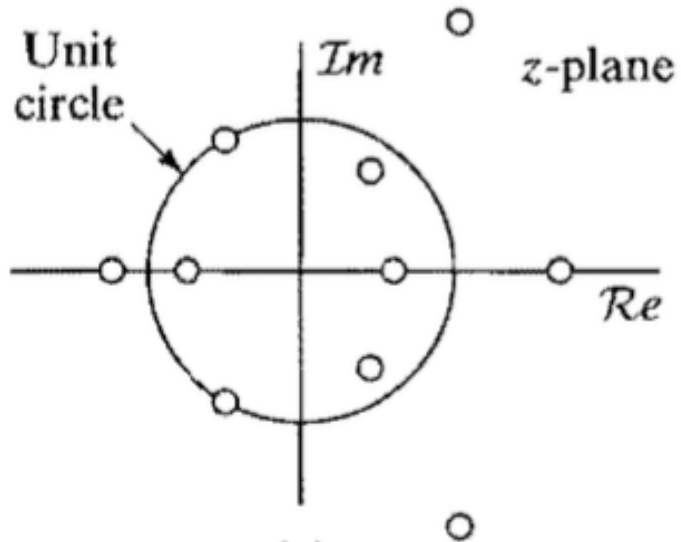
$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

□ If zero is real and not on unit circle ($\theta=0$)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

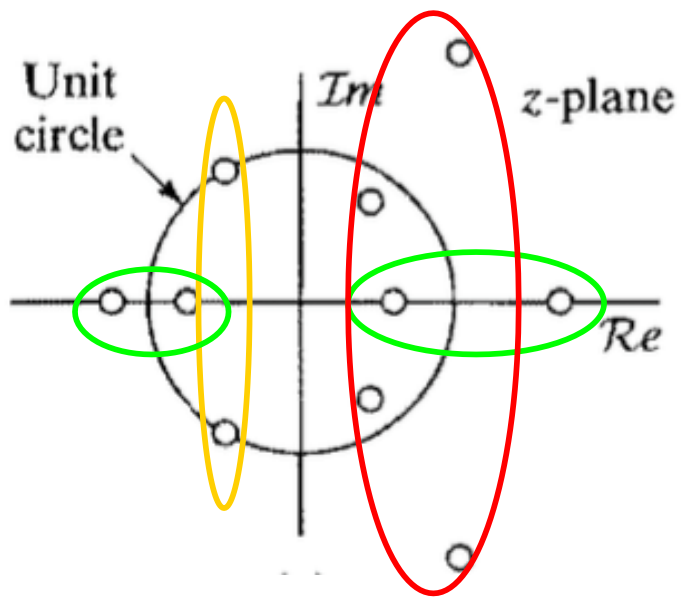
Zeros of GLP System – Type I

Type I



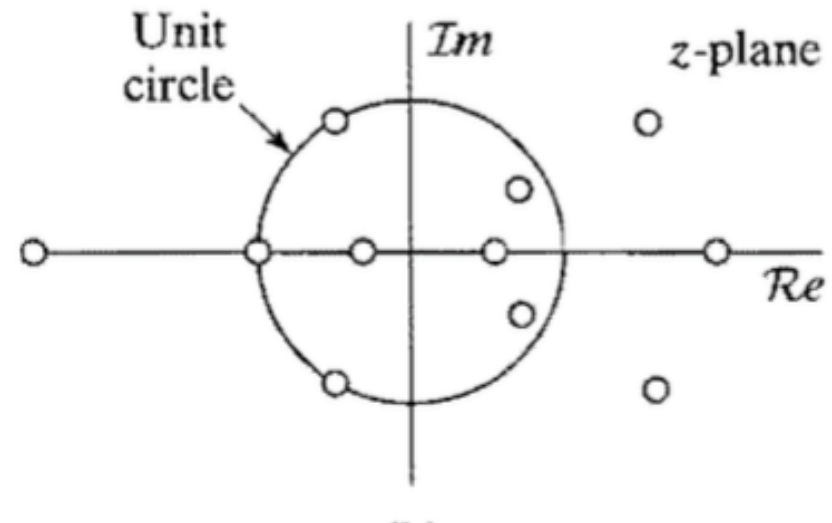
Zeros of GLP System – Type I

Type I



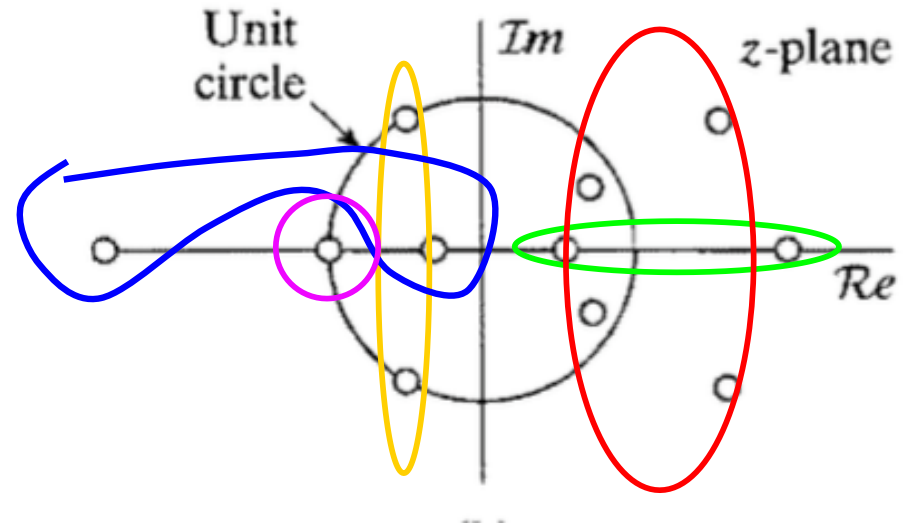
Zeros of GLP System – Type II

Type II



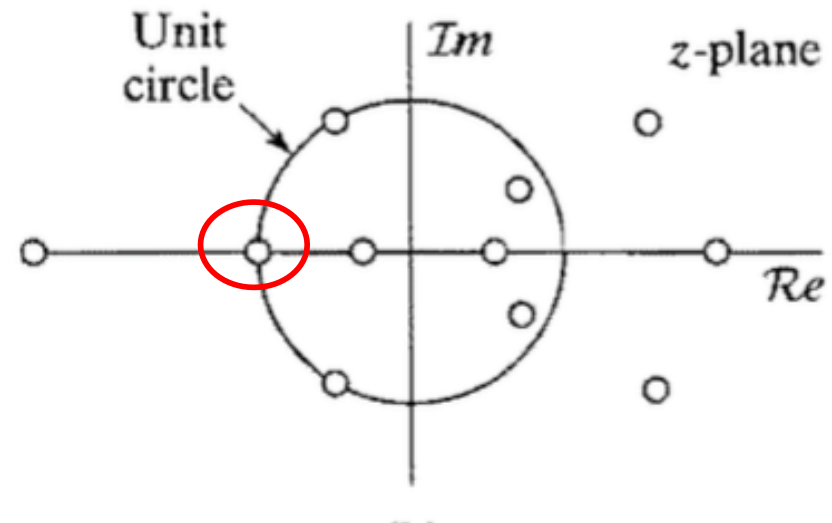
Zeros of GLP System – Type II

Type II

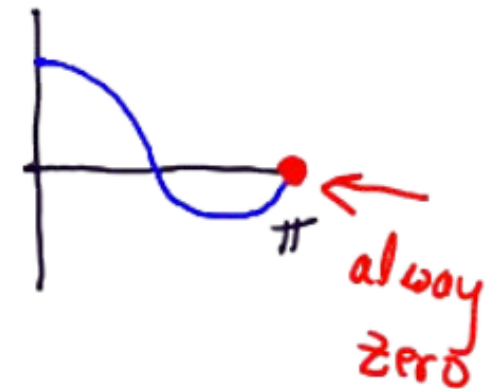
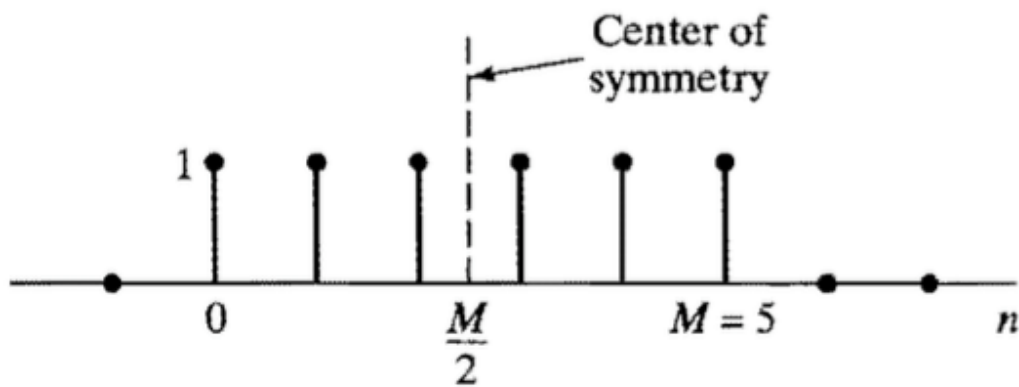
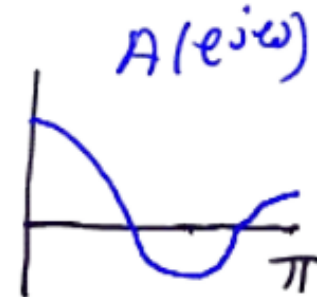
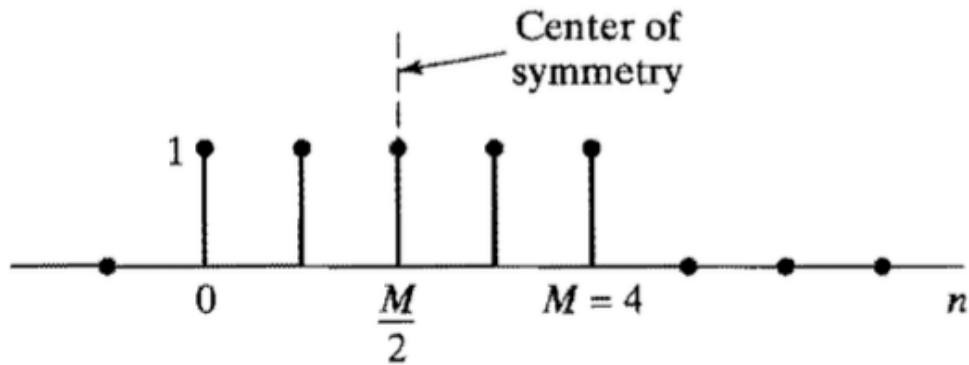


Zeros of GLP System – Type II

Type II



FIR GLP: Type I and II



Zeros of GLP System – Type II

□ FIR GLP System Function

$$\begin{aligned} H(z) &= \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M} \\ &= z^{-M} H(z^{-1}). \end{aligned}$$

Consider $z = -1$:

Zeros of GLP System – Type II

□ FIR GLP System Function

$$\begin{aligned} H(z) &= \sum_{n=0}^M h[M-n]z^{-n} = \sum_{k=M}^0 h[k]z^k z^{-M} \\ &= z^{-M} H(z^{-1}). \end{aligned}$$

Consider $z = -1$: $H(-1) = (-1)^{-M} H(-1)$

$\Rightarrow$ for M odd, $z = -1$ must be a zero (Type II)



Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

$$H(z) = -z^{-M} H(z^{-1}).$$

If z_0 is a zero then z_0^{-1} is also a zero.

Zeros of GLP System – Type III and IV

□ FIR GLP System Function

$$H(z) = \sum_{n=0}^M h[n]z^{-n}$$

Real system → zeros occur in conjugate-reciprocal groups of 4

$$(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})(1 - r^{-1}e^{j\theta}z^{-1})(1 - r^{-1}e^{-j\theta}z^{-1})$$

□ If zero is on unit circle ($r=1$)

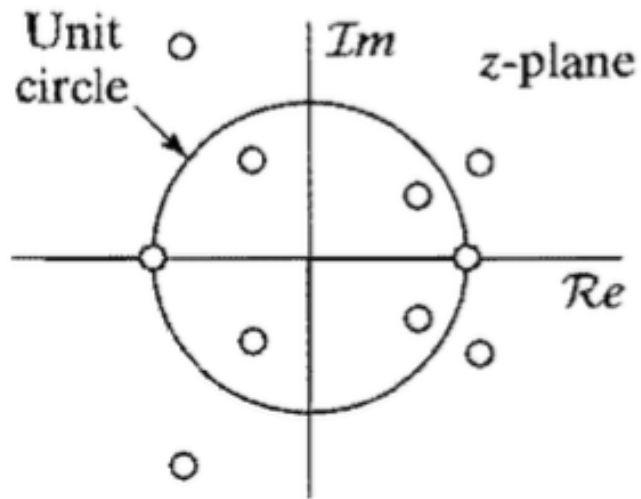
$$(1 - e^{j\theta}z^{-1})(1 - e^{-j\theta}z^{-1}).$$

□ If zero is real and not on unit circle ($\theta=0$)

$$(1 \pm rz^{-1})(1 \pm r^{-1}z^{-1}).$$

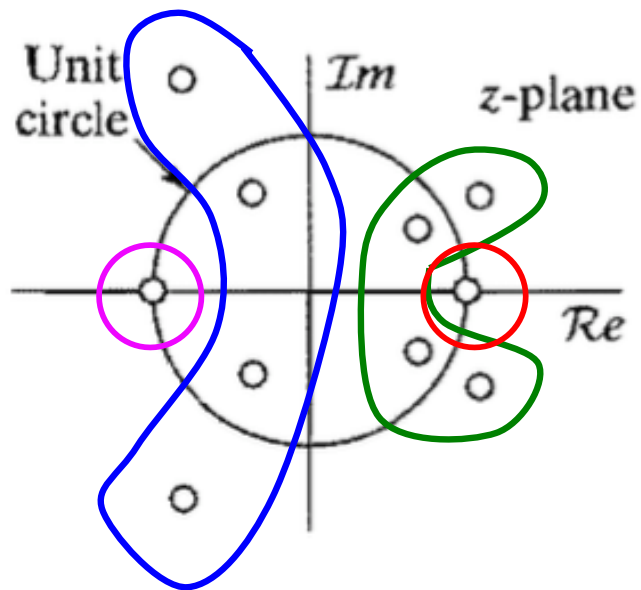
Zeros of GLP System – Type III

Type III



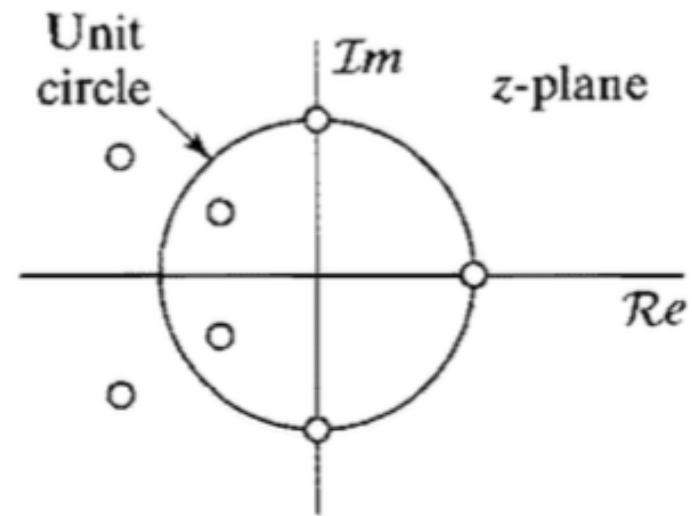
Zeros of GLP System – Type III

Type III



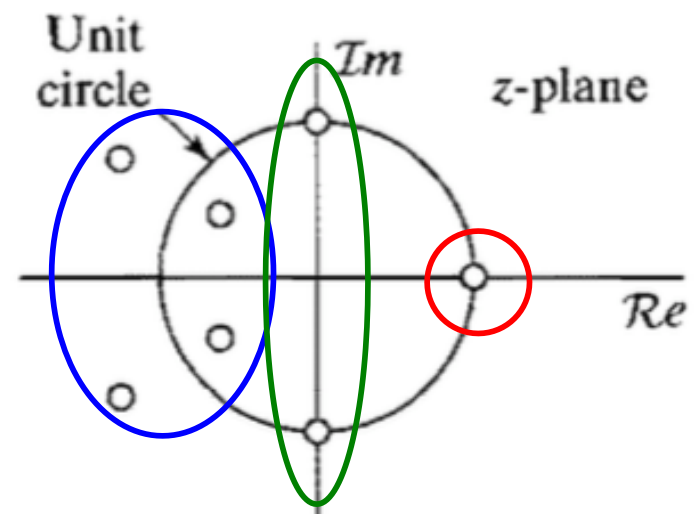
Zeros of GLP System – Type IV

Type IV



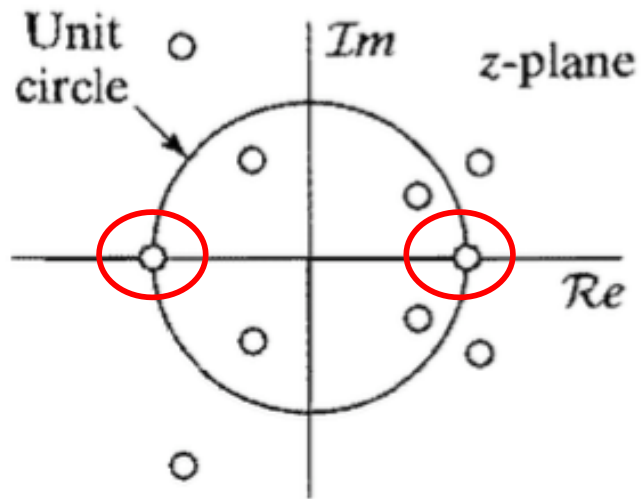
Zeros of GLP System – Type IV

Type IV

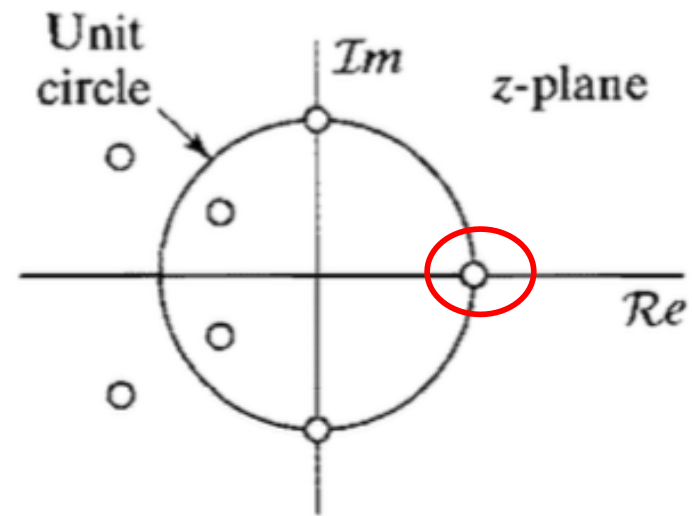


Zeros of GLP System – Type III and IV

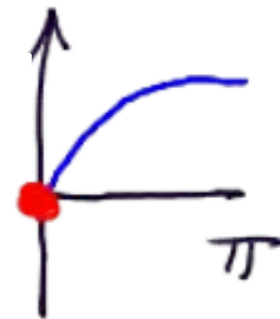
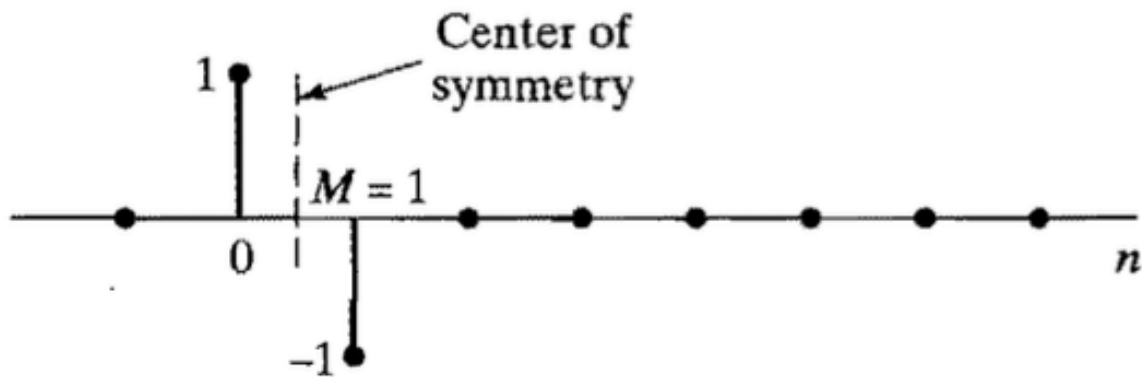
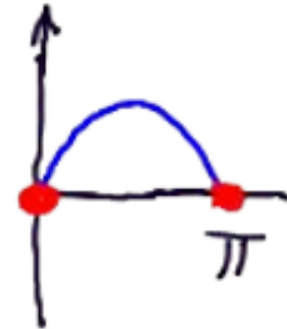
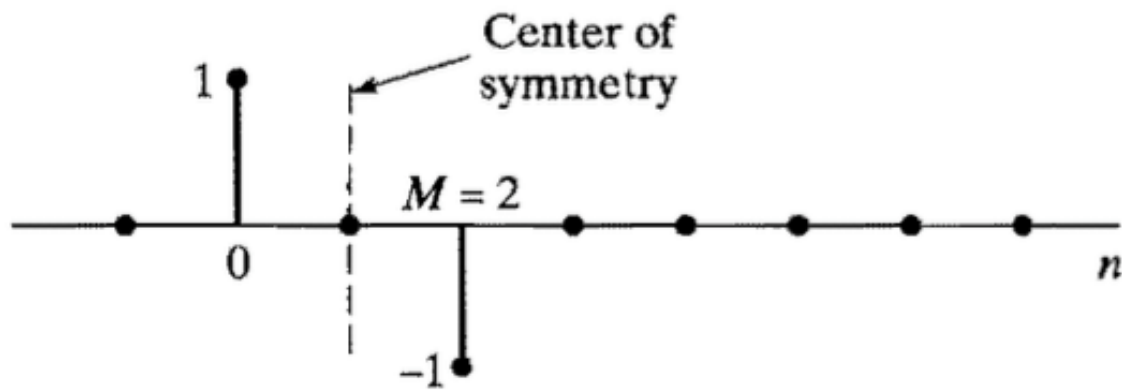
Type III



Type IV



FIR GLP: Type III and IV





GLP and Min Phase Systems

- Any FIR linear-phase system can be decomposed into:

$$H(z) = H_{\min}(z)H_{uc}(z)H_{\max}(z)$$

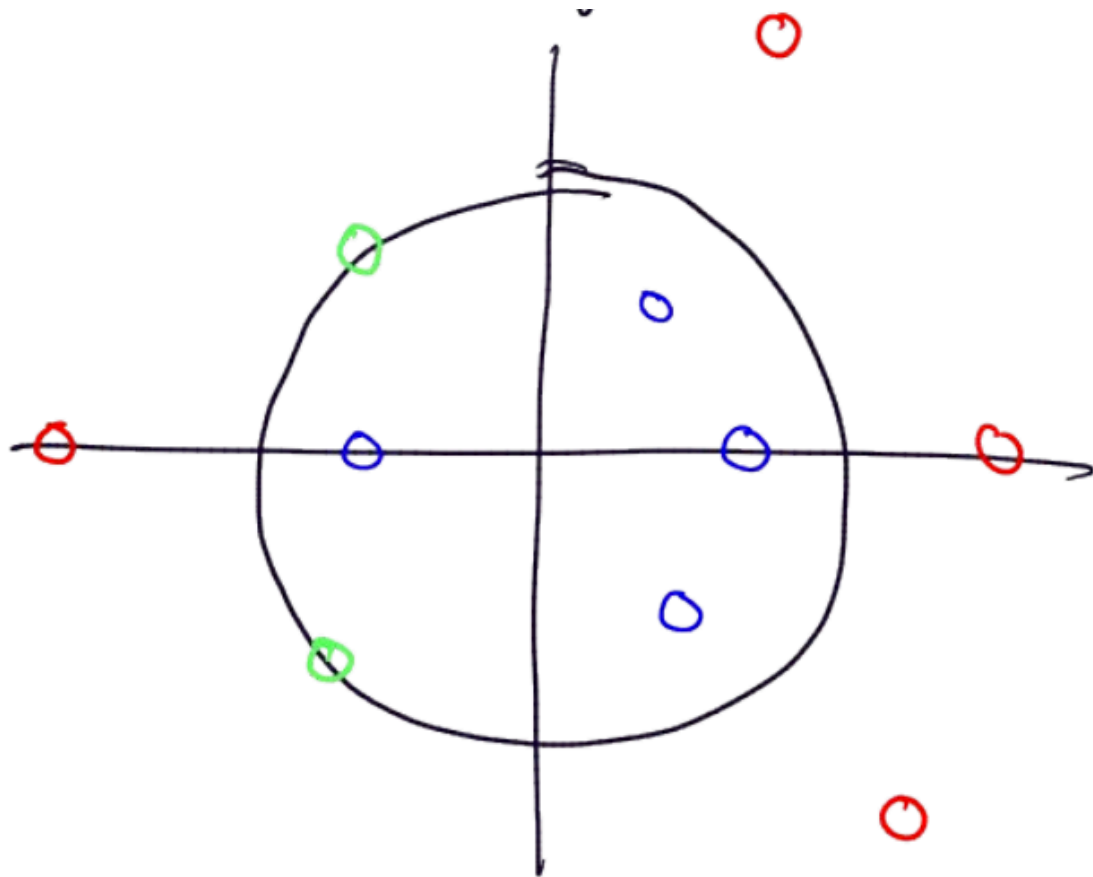
- A min phase system, system containing only zeros on unit circle, and max phase system

GLP and Min Phase Systems

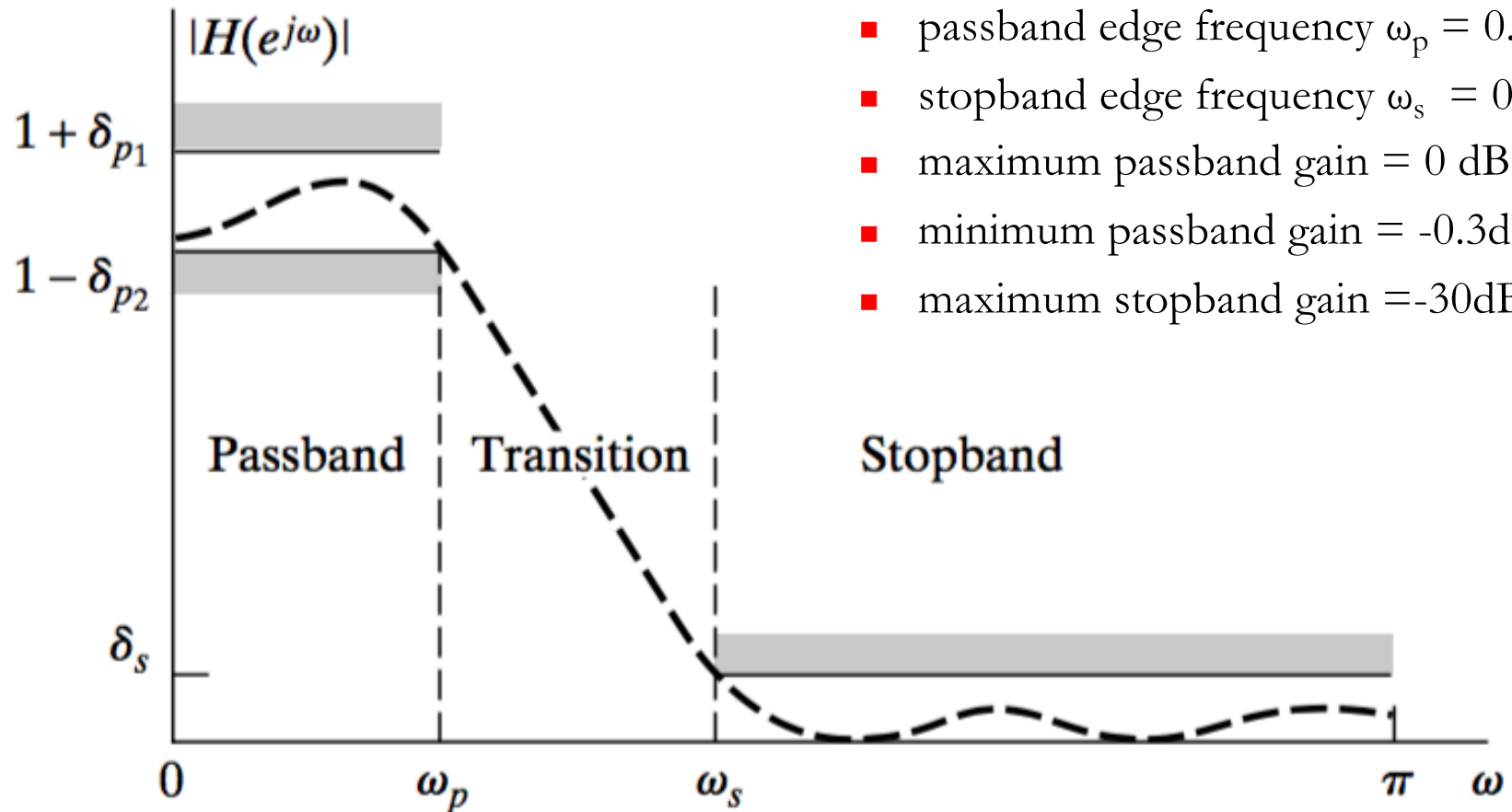
- Any FIR linear-phase system can be decomposed into:

$$H(z) =$$

- A min phase system with all poles inside the unit circle, and zeros on unit circle, and a constant gain.



Filter Specifications



Design specifications

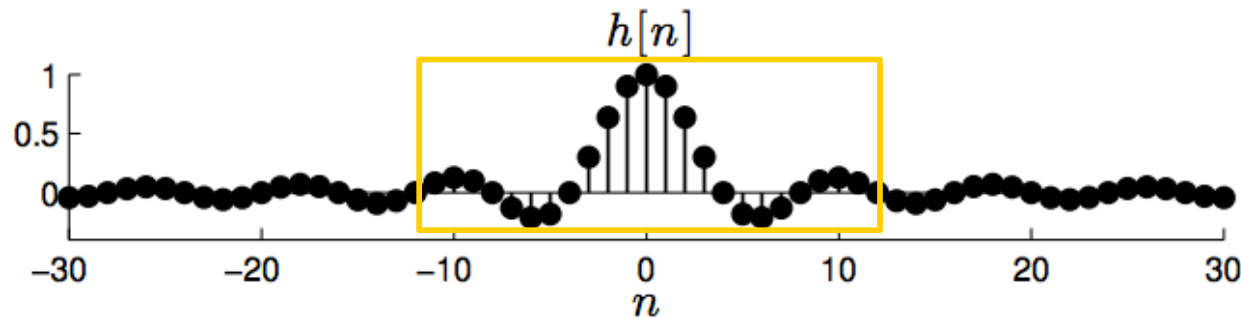
- passband edge frequency $\omega_p = 0.5\pi$
- stopband edge frequency $\omega_s = 0.6\pi$
- maximum passband gain = 0 dB
- minimum passband gain = -0.3dB
- maximum stopband gain = -30dB

Example: Ideal Low-Pass Filter

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm\pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

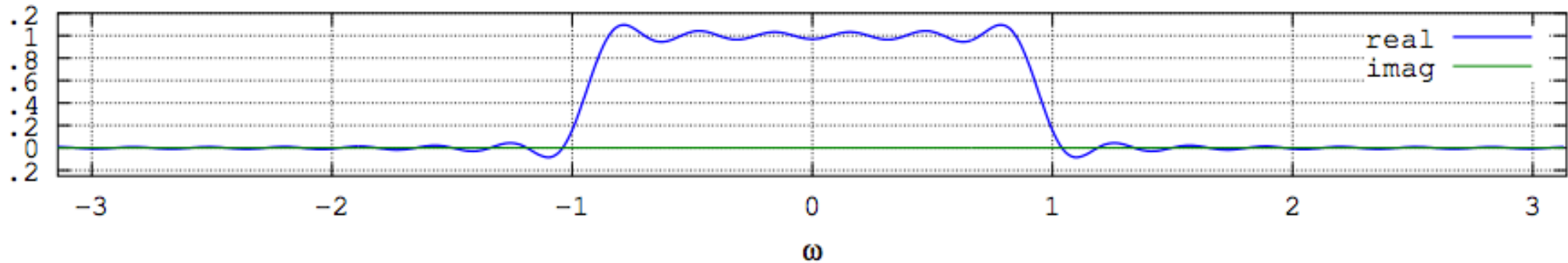
$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$



Truncate
and shift

$$h_{LP}[n] = w_N[n - N] \cdot h[n - N]$$

Example: Practical LP Filter



- ❑ Pass band smeared and rippled
 - Smearing determined by width of main lobe
 - Rippling determined by size of side lobes

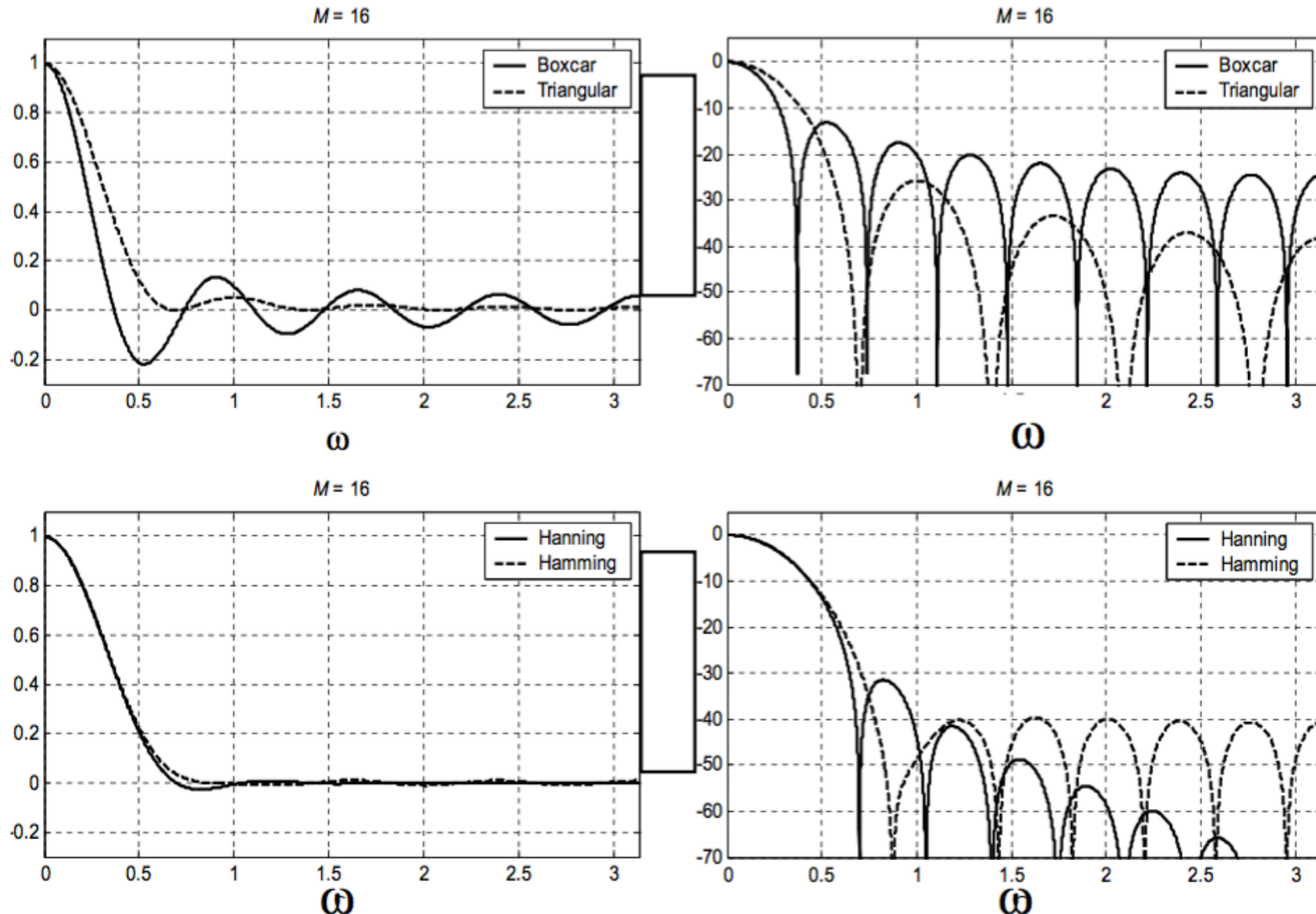
- ❑ Different windows for different main/side lobe characteristics



Tapered Windows

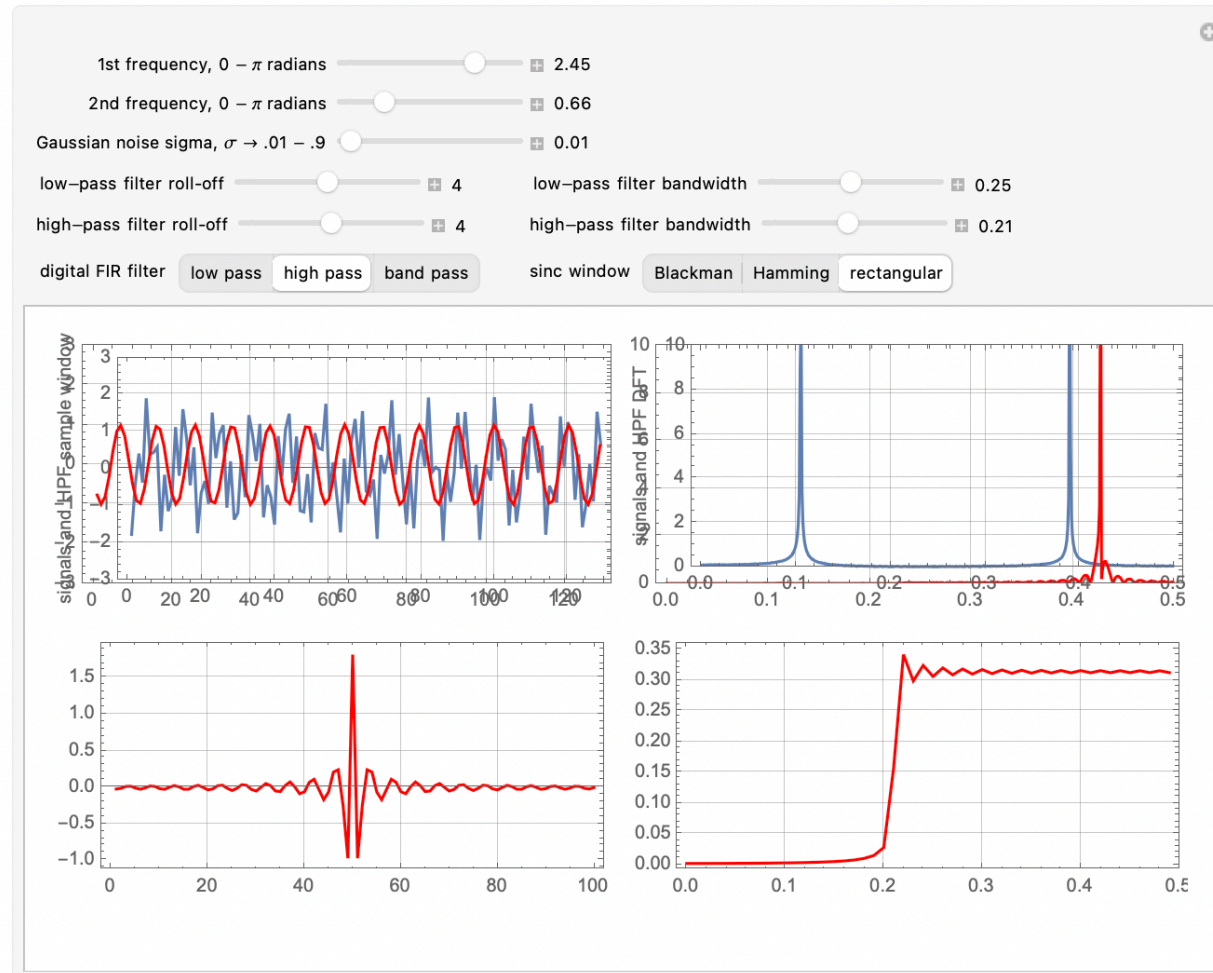
Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2} \left[1 + \cos\left(\frac{\pi n}{M/2}\right) \right] & n \leq M/2 \\ 0 & n > M/2 \end{cases}$	<code>hann(M+1)</code>	
Hanning	$w[n] = \begin{cases} \frac{1}{2} \left[1 + \cos\left(\frac{\pi n}{M/2 + 1}\right) \right] & n \leq M/2 \\ 0 & n > M/2 \end{cases}$	<code>hanning(M+1)</code>	
Hamming	$w[n] = \begin{cases} 0.54 + 0.46 \cos\left(\frac{\pi n}{M/2}\right) & n \leq M/2 \\ 0 & n > M/2 \end{cases}$	<code>hamming(M+1)</code>	

Tradeoff – Ripple vs. Transition Width



Filter Design Demo

Digital Filters with Windowed Sinc Finite Impulse Response



<https://demonstrations.wolfram.com/DigitalFiltersWithWindowedSincFiniteImpulseResponse/>



Big Ideas

- ❑ Generalized Linear Phase Systems
 - Useful for design of causal FIR filters



Admin

❑ Midterm

- Thursday 3/21 in person during class in DRLB A8
- Covers lectures 1-13
 - Doesn't cover data converters and noise shaping
- Old exams online
- TA Review session recording in Canvas
- Closed book/notes
- Can bring 1 two-sided 8"x11" sheet and non-cell phone calculator

❑ Proj 1 posted on Thursday 3/21

- Due 4/2