ESE 5310: Digital Signal Processing

Lecture 15: March 26, 2024

Design of IIR/FIR Filters

Mean: 67.1
Stdev: 23.4

Linear Filter Design

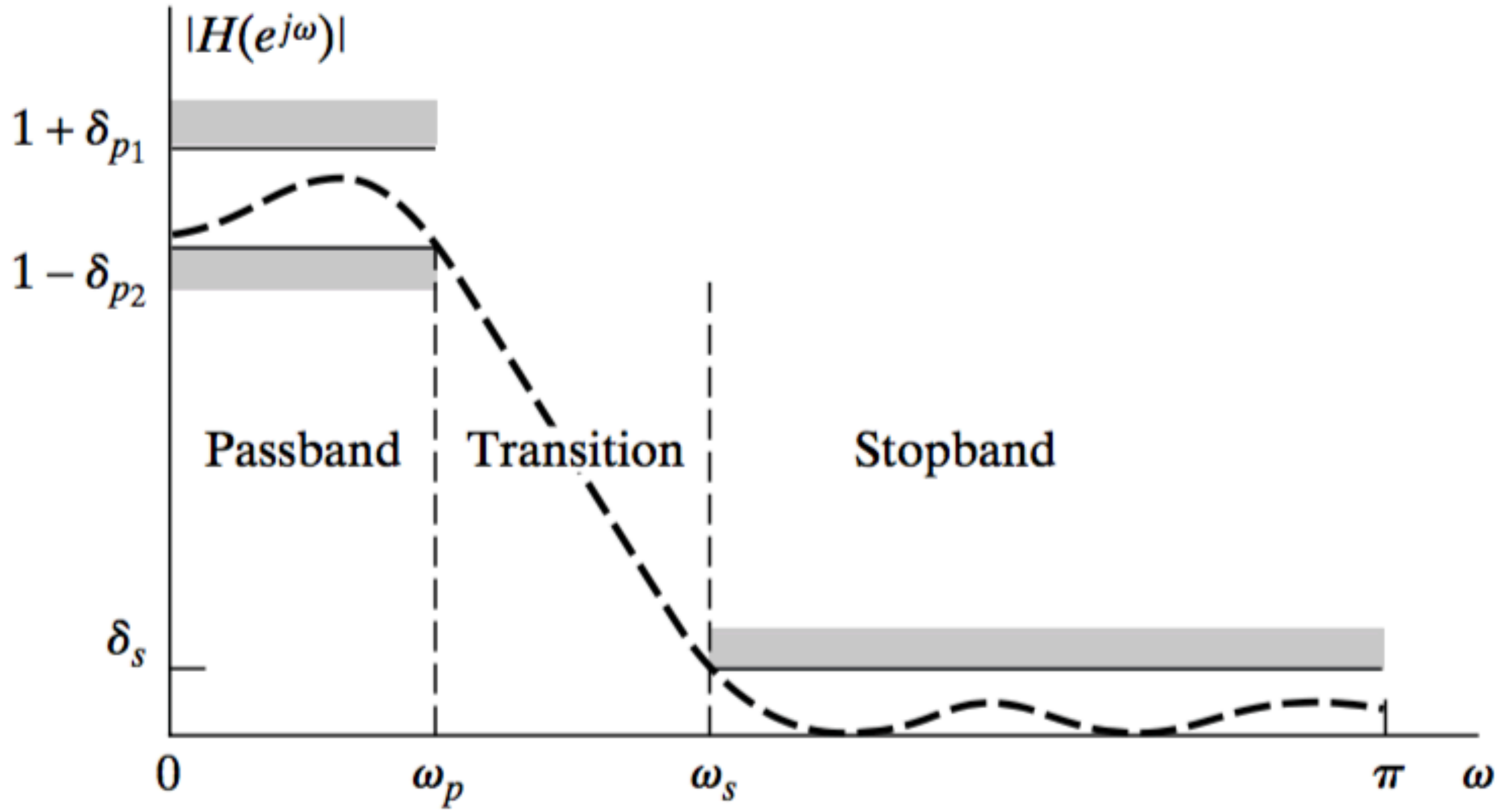
- Used to be an ambiguous process
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- Both classes use finite order of parameters for design
  - Filter order (ie. Length) restricts filter design

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- What does it mean to design a filter?
  - Determine the parameters of a transfer function or difference equation that approximates a desired impulse response ($h[n]$) or frequency response ($H(e^{j\omega})$).

Filter Specifications



What is a Linear Filter?

- Attenuates certain frequencies
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- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

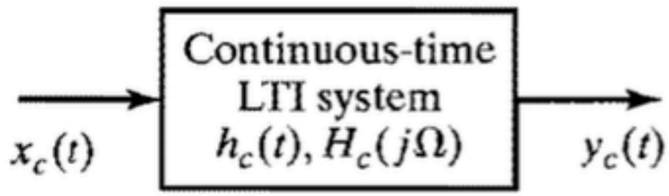
IIR Filter Design

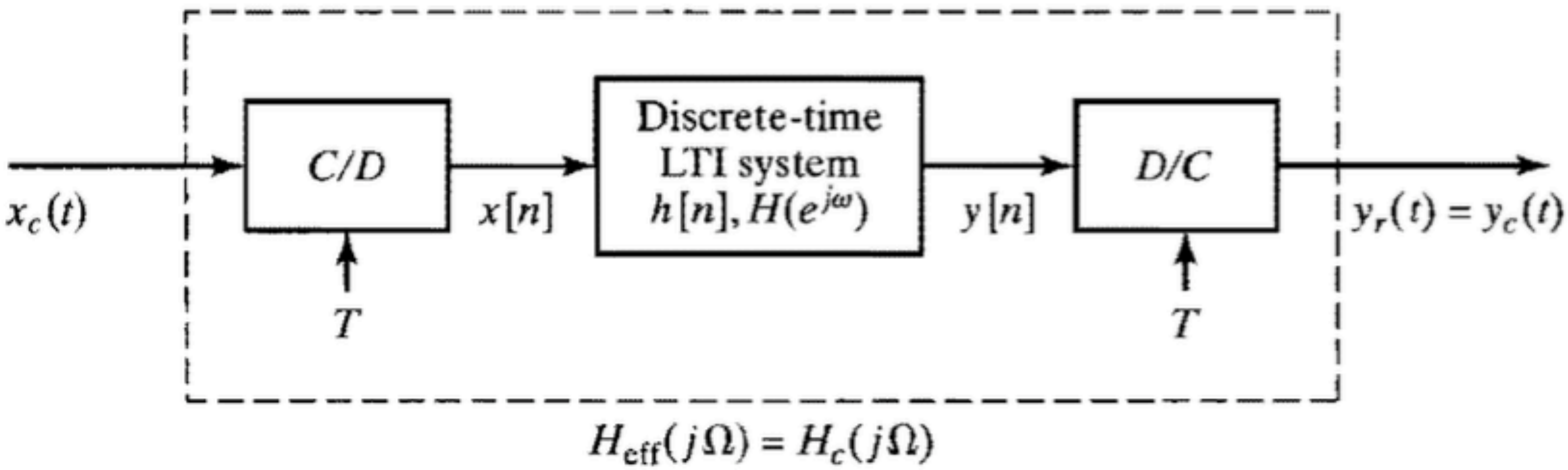
- Transform continuous-time filter into a discrete-time filter meeting specs
  - Pick suitable transformation from s (Laplace variable) to z (or t to n)
  - Pick suitable analog $H_c(s)$ allowing specs to be met, transform to $H(z)$

- We've seen this before… impulse invariance

Impulse Invariance

- Want to implement continuous-time system in discrete-time





Impulse Invariance

- With $H_c(j\Omega)$ bandlimited, choose

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi$$

- With the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T$$

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$$\boxed{h[n] = Th_c(nT)}$$

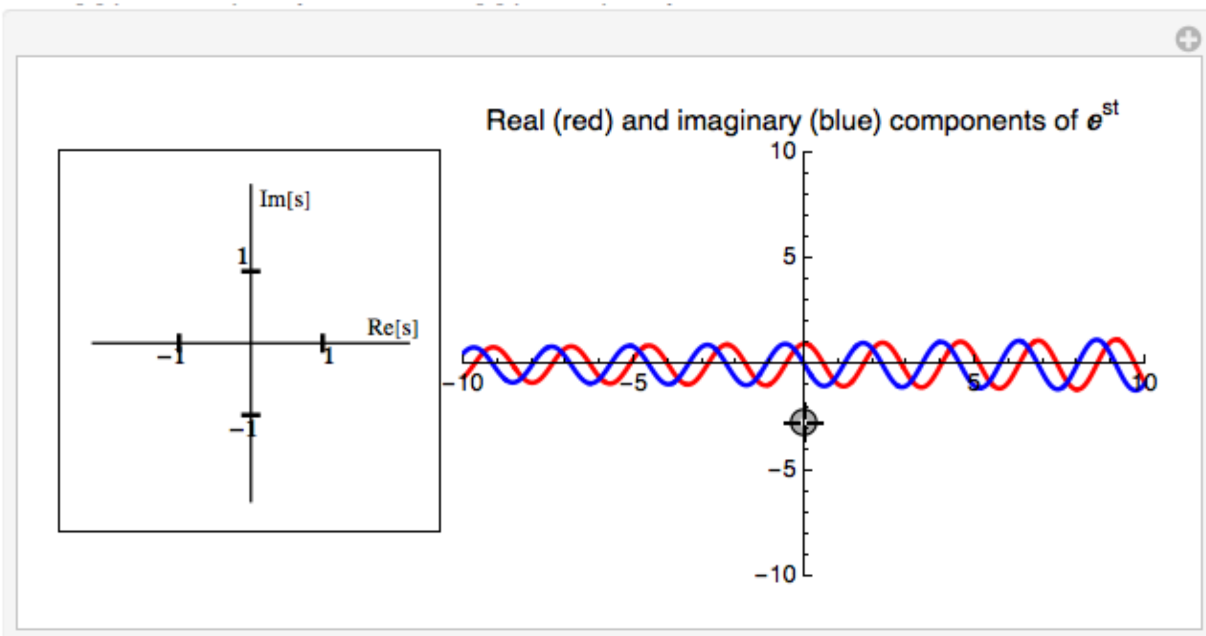
Laplace Transform

- The Laplace transform takes a function of time, t, and transforms it to a function of a complex variable, s.
- Because the transform is invertible, no information is lost and it is reasonable to think of a function f(t) and its Laplace transform F(s) as two views of the same phenomenon.
- Each view has its uses and some features of the phenomenon are easier to understand in one view or the other.

S-Plane

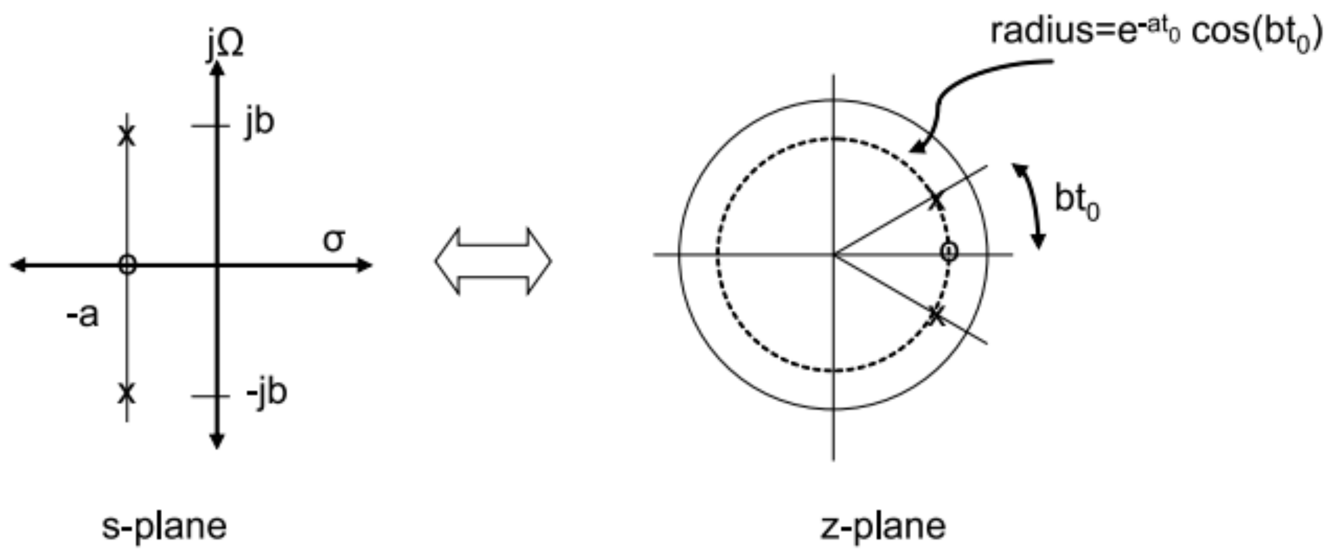
- $s=\sigma+j\Omega$

- Wolfram Demo



- http://pilot.cnxproject.org/content/collection/col10064/latest/module/m10060/latest

S-Plane Mapping to Z-Plane



Example

$$\textit{Example}: \text{ If } \boxed{H_c(s) = \frac{A_k}{s - p_k}}$$

Laplace: $$e^{at} \xleftrightarrow{L} \frac{1}{s-a}$$

Z-transform: $$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}$$

Example

Example: If $\boxed{H_c(s) = \frac{A_k}{s - p_k}}$ (e.g. one term in PF expansion)

$h_c(t) = A_k e^{p_k t}, \ \ t \geq 0; \quad h[n] = T_d A_k e^{p_k T_d n} = T_d A_k \left(e^{p_k T_d}\right)^n$

$\therefore \ \boxed{H(z) = T_d A_k \frac{1}{1 - e^{p_k T_d} z^{-1}}}$ Pole mapping is $z \leftarrow e^{sT_d}$

Zeros do *not* map the same way; *not* the general mapping of s to z

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Zeros do *not* map the same way; *not* the general mapping of s to z

- Stability, causality, preserved.
- $j\Omega$ axis mapped linearly to unit-circle, with aliasing
- No control of zeros or of phase

Impulse Invariance

- Sampling the impulse response is equivalent to mapping the s-plane to the z-plane using:
  - $z = e^{sTd} = r\, e^{j\omega}$
- The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times

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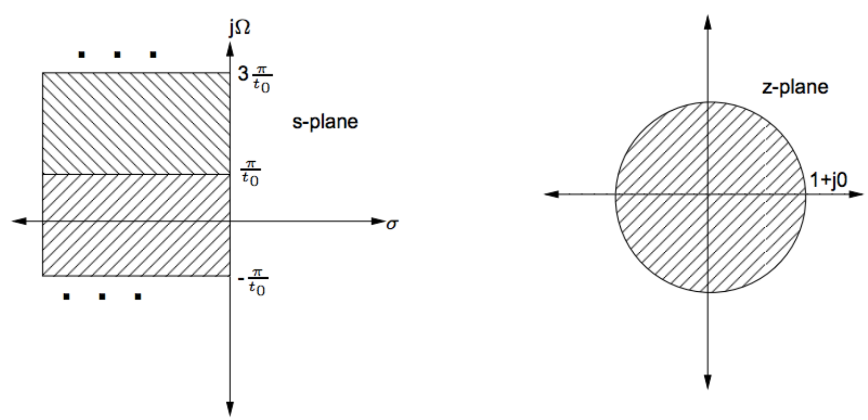
- The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior

Mapping



Impulse Invariance

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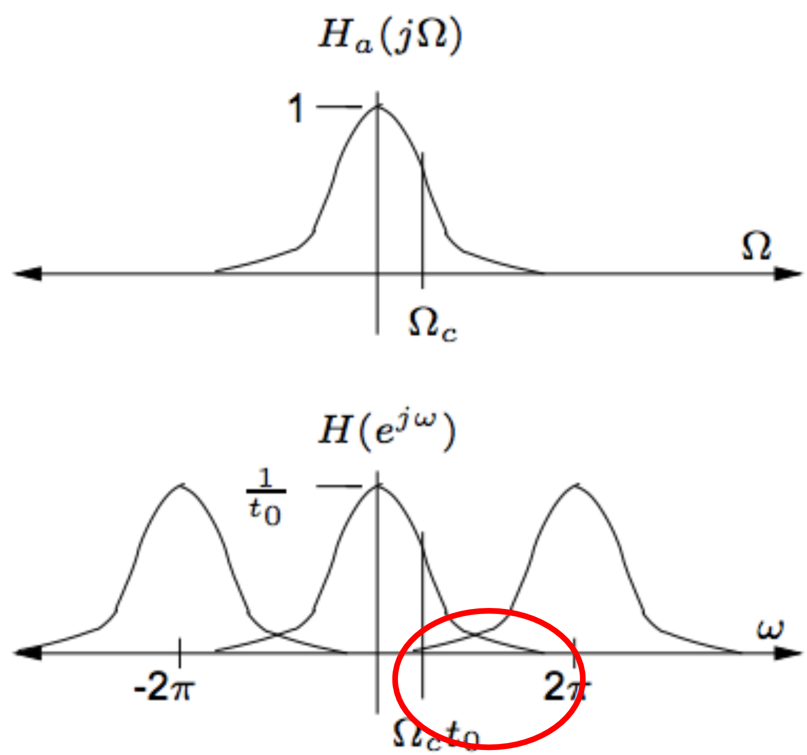
- The entire Ω axis of the s-plane wraps around the unit circle of the z-plane an infinite number of times
- The left half s-plane maps to the interior of the unit circle and the right half plane to the exterior
- This means stable analog filters (poles in LHP) will transform to stable digital filters (poles inside unit circle)
- This is a many-to-one mapping of strips of the s-plane to regions of the z-plane
  - Not a conformal mapping
  - The poles map according to $z = e^{sT_d}$, but the zeros do not always

Impulse Invariance

- Limitation of Impulse Invariance: overlap of images of the frequency response. This prevents it from being used for high-pass filter design



Bilinear Transformation

- The technique uses an algebraic transformation between the variables s and z that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane.

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

$$H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right).$$

Bilinear Transformation

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

- Substituting $s = \sigma + j\,\Omega$ and $z = e^{j\omega}$

Bilinear Transformation

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right);$$

- Substituting s = σ+ j Ω and z = $e^{j\omega}$

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right),$$

$$s = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j \sin \omega/2)}{2e^{-j\omega/2}(\cos \omega/2)}\right] = \frac{2j}{T_d}\tan(\omega/2).$$

Bilinear Transformation

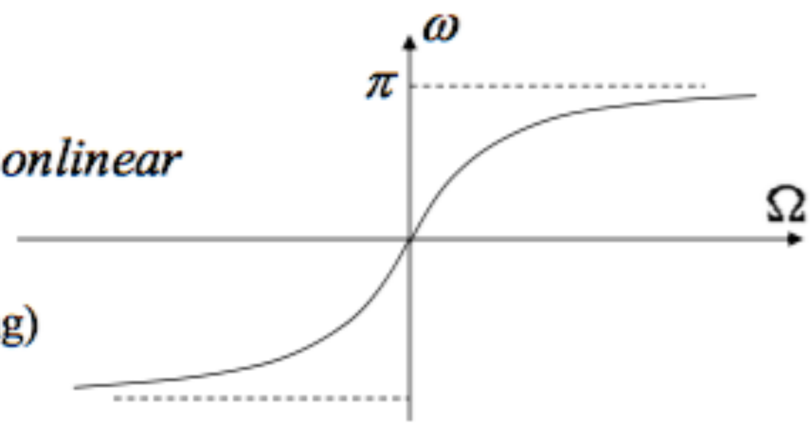
$$\Omega = \frac{2}{T_d}\tan(\omega/2),$$

$$\omega = 2\arctan(\Omega T_d/2).$$

No aliasing, but mapping nonlinear

(Impulse invariance:

linear mapping, but with aliasing)



Example: Notch Filter

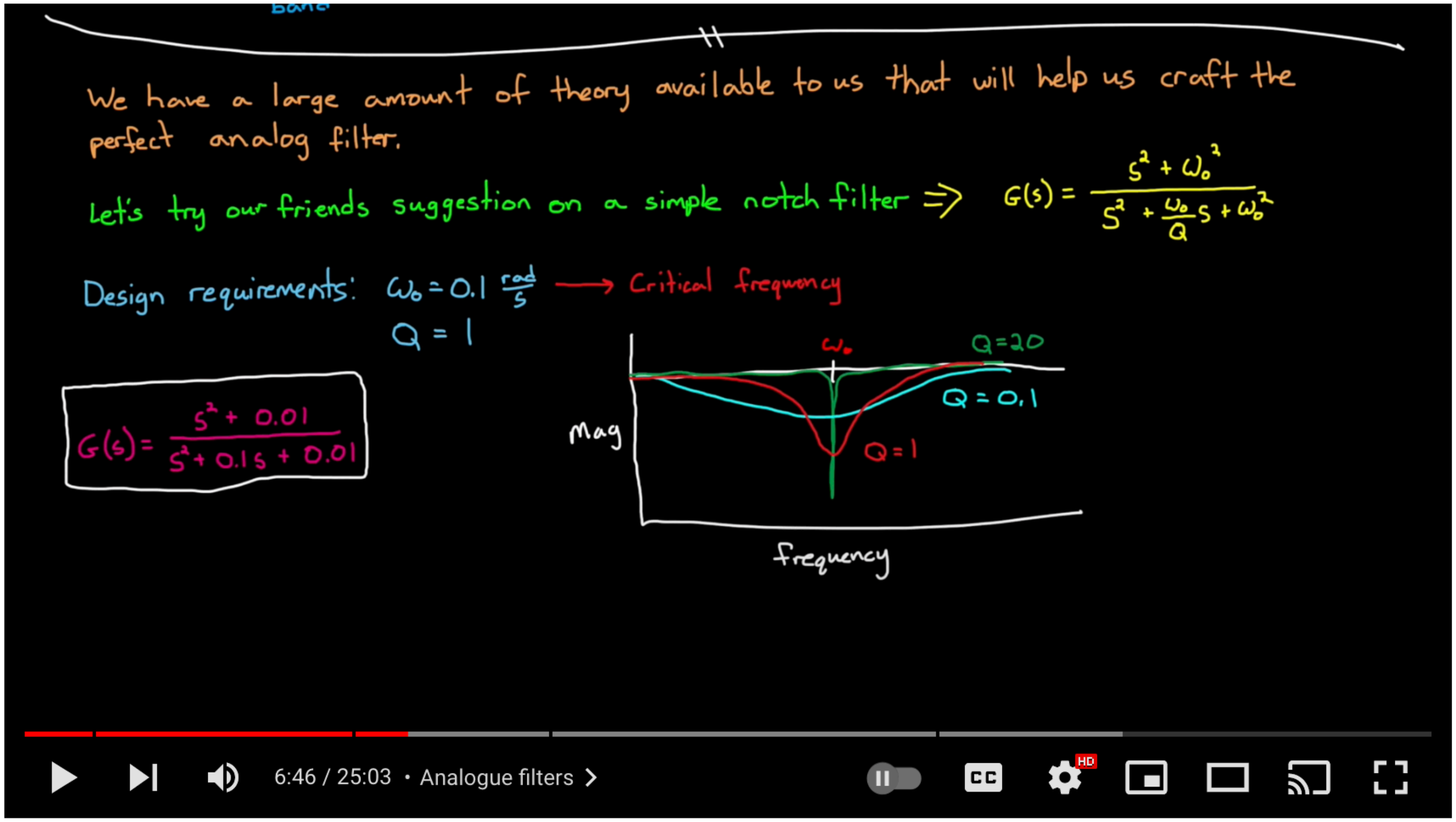
- The continuous time filter with:

$$H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + \frac{\Omega_0}{Q}s + \Omega_0^2}$$

$$\Omega = \frac{2}{T_d}\tan(\omega/2).$$

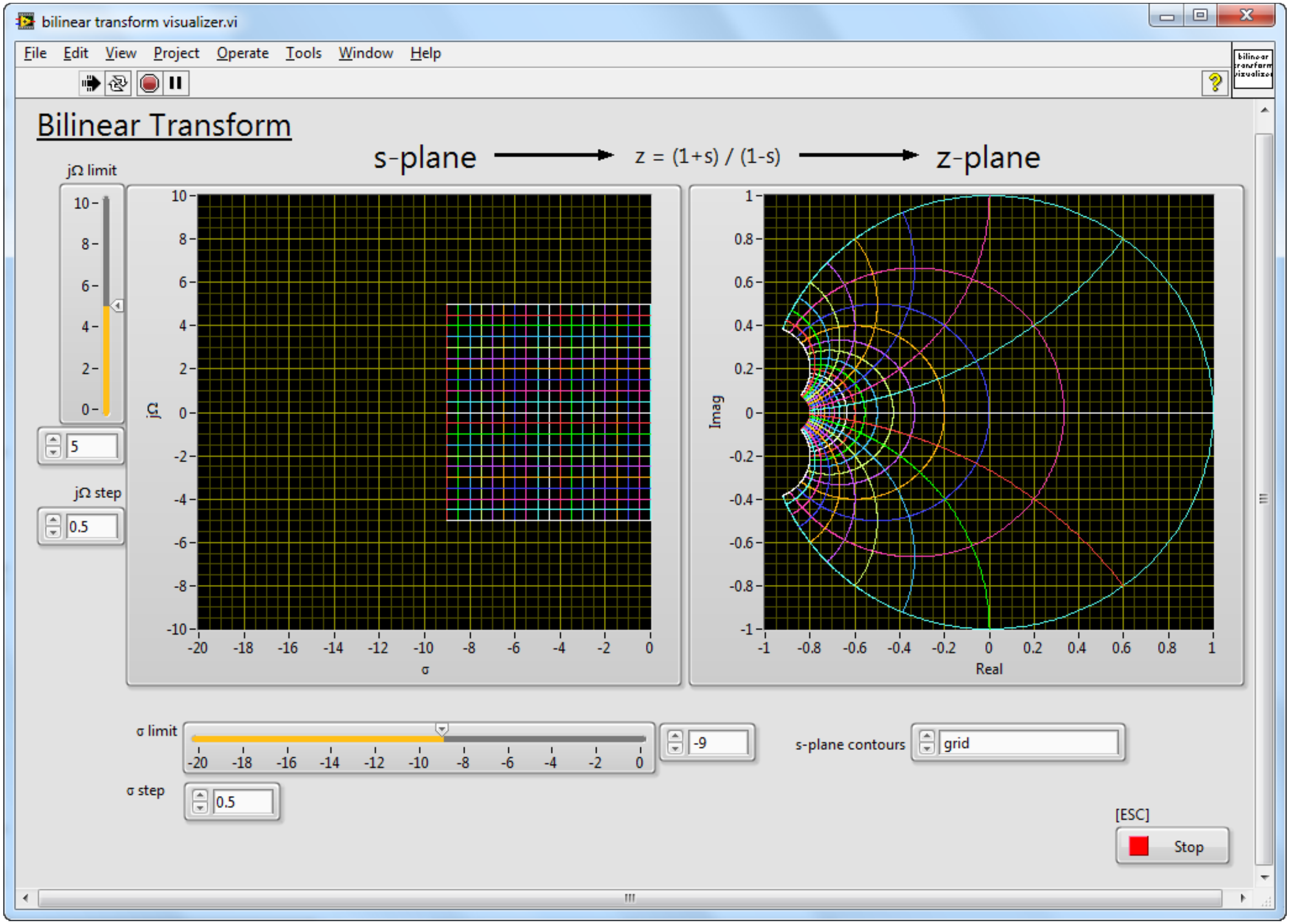
$$\omega = 2\arctan(\Omega T_d/2).$$

Matlab Demo



Based off: https://www.youtube.com/watch?v=NRbGPgcLhU0

Bilinear Mapping of S-plane to Z-plane



https://forums.ni.com/t5/Example-Code/Bilinear-Transform-Visualizer/ta-p/3504082

CT Filters

- Butterworth
  - Monotonic in pass and stop bands
- Chebyshev, Type I
  - Equiripple in pass band and monotonic in stop band
- Chebyshev, Type II
  - Monotonic in pass band and equiripple in stop band
- Elliptic
  - Equiripple in pass and stop bands

- Appendix B in textbook

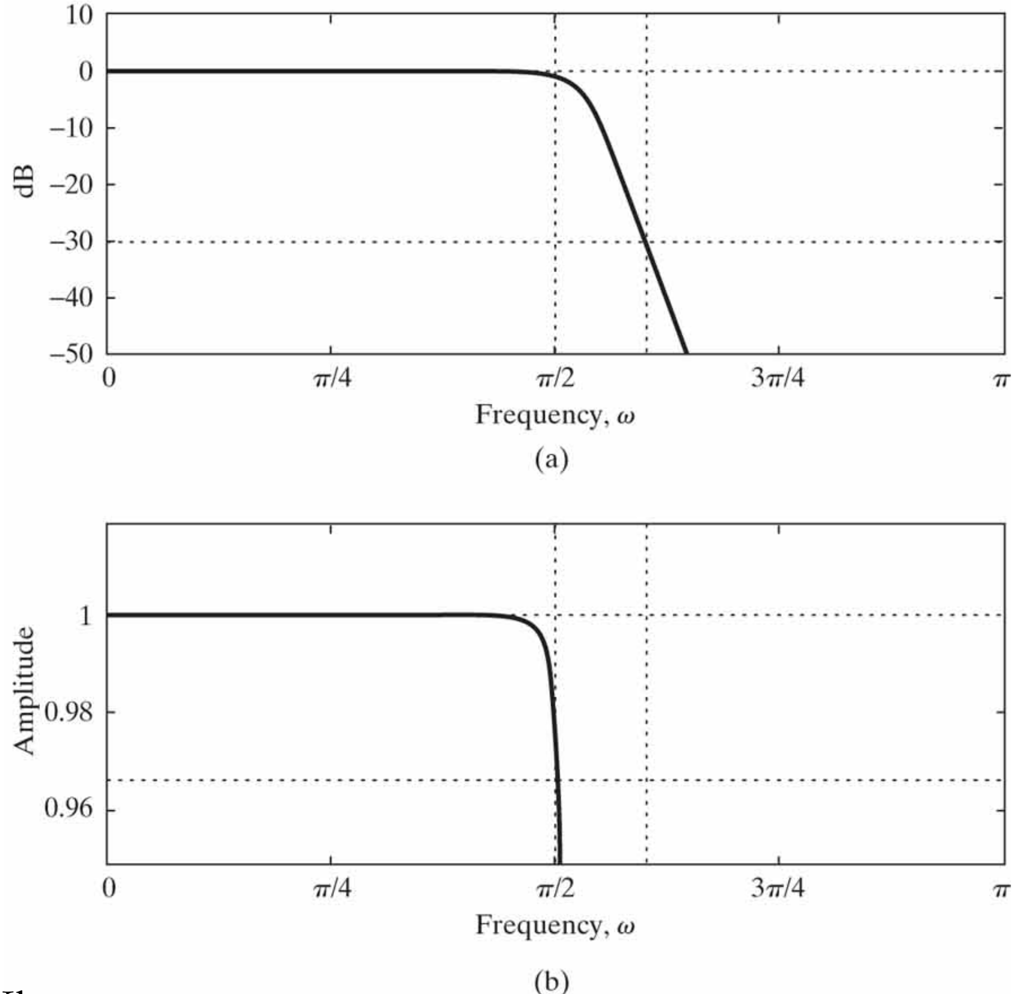
Design Comparison

- Design specifications
  - passband edge frequency $\omega_p = 0.5\pi$
  - stopband edge frequency $\omega_s = 0.6\pi$
  - maximum passband gain = 0 dB
  - minimum passband gain = -0.3dB
  - maximum stopband gain =-30dB

- Use bilinear transformation to design DT low pass filter for each type

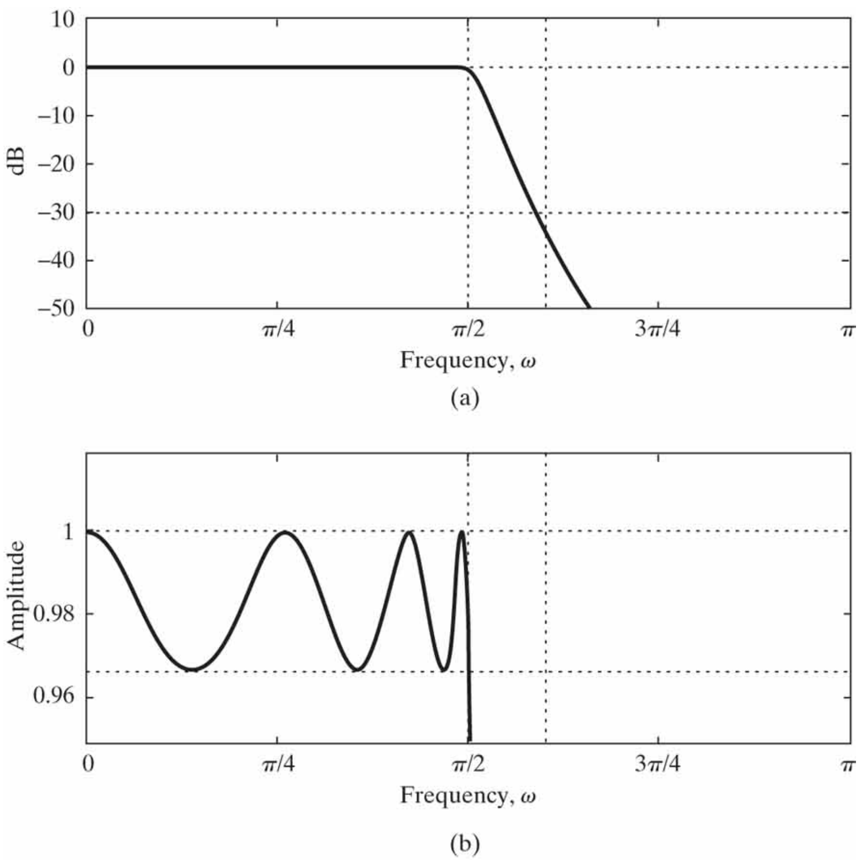
Butterworth

- Butterworth
  - Monotonic in pass and stop bands



Chebyshev

- Type I
  - Equiripple in pass band and monotonic in stop band



Chebyshev

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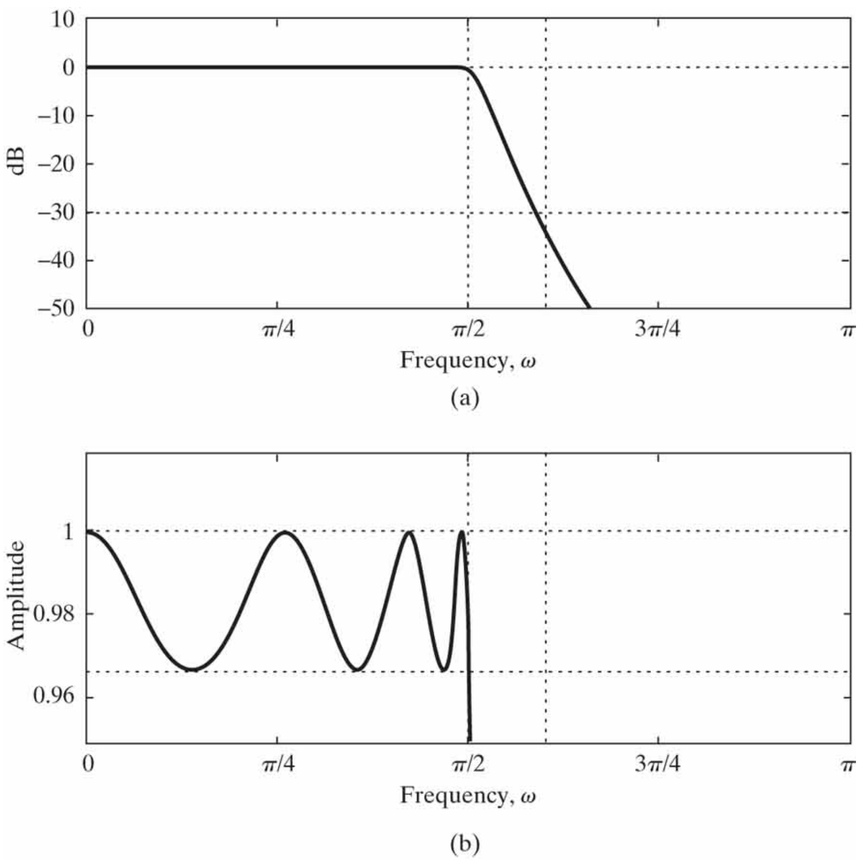
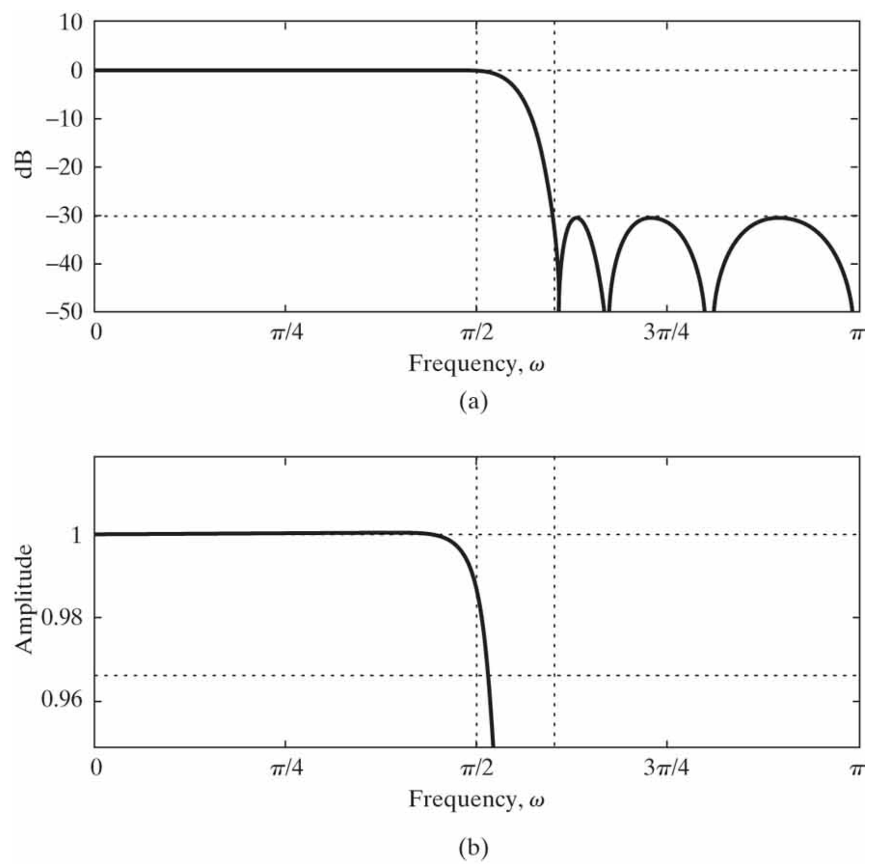
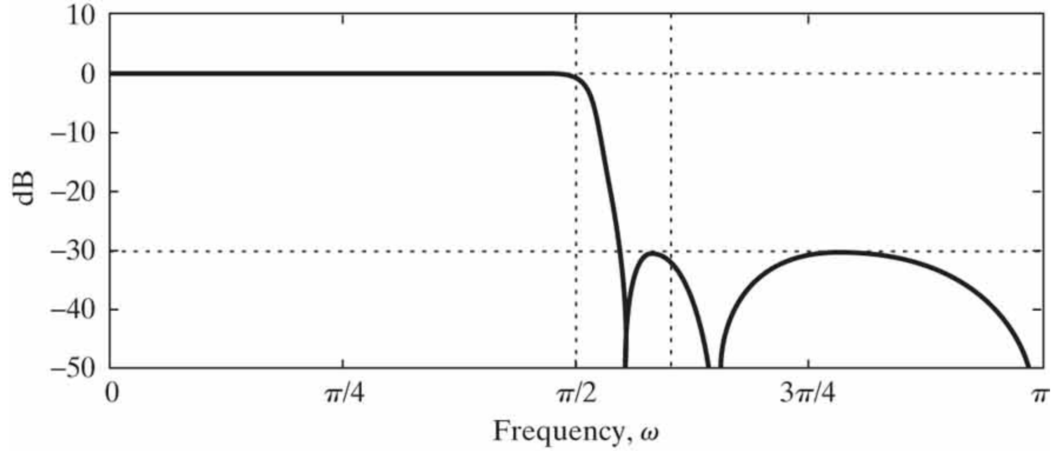




Elliptic

- Elliptic
  - Equiripple in pass and stop bands



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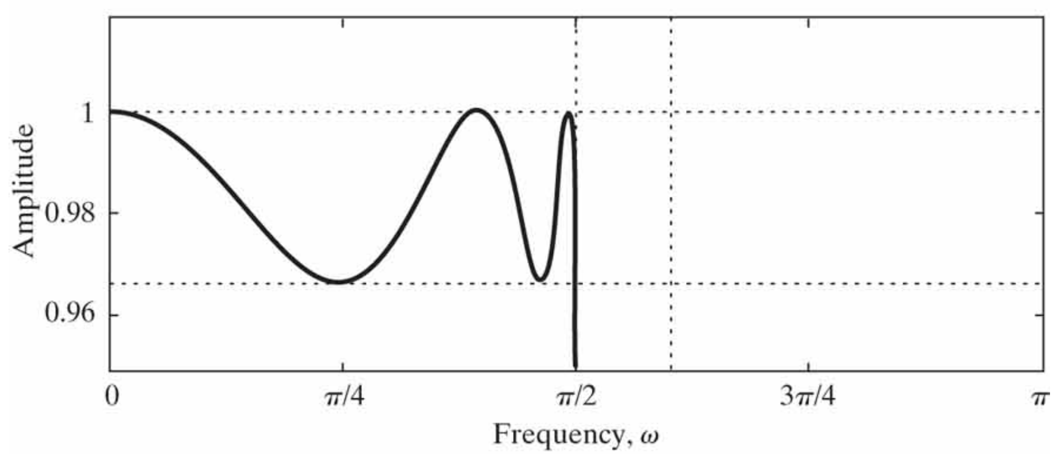
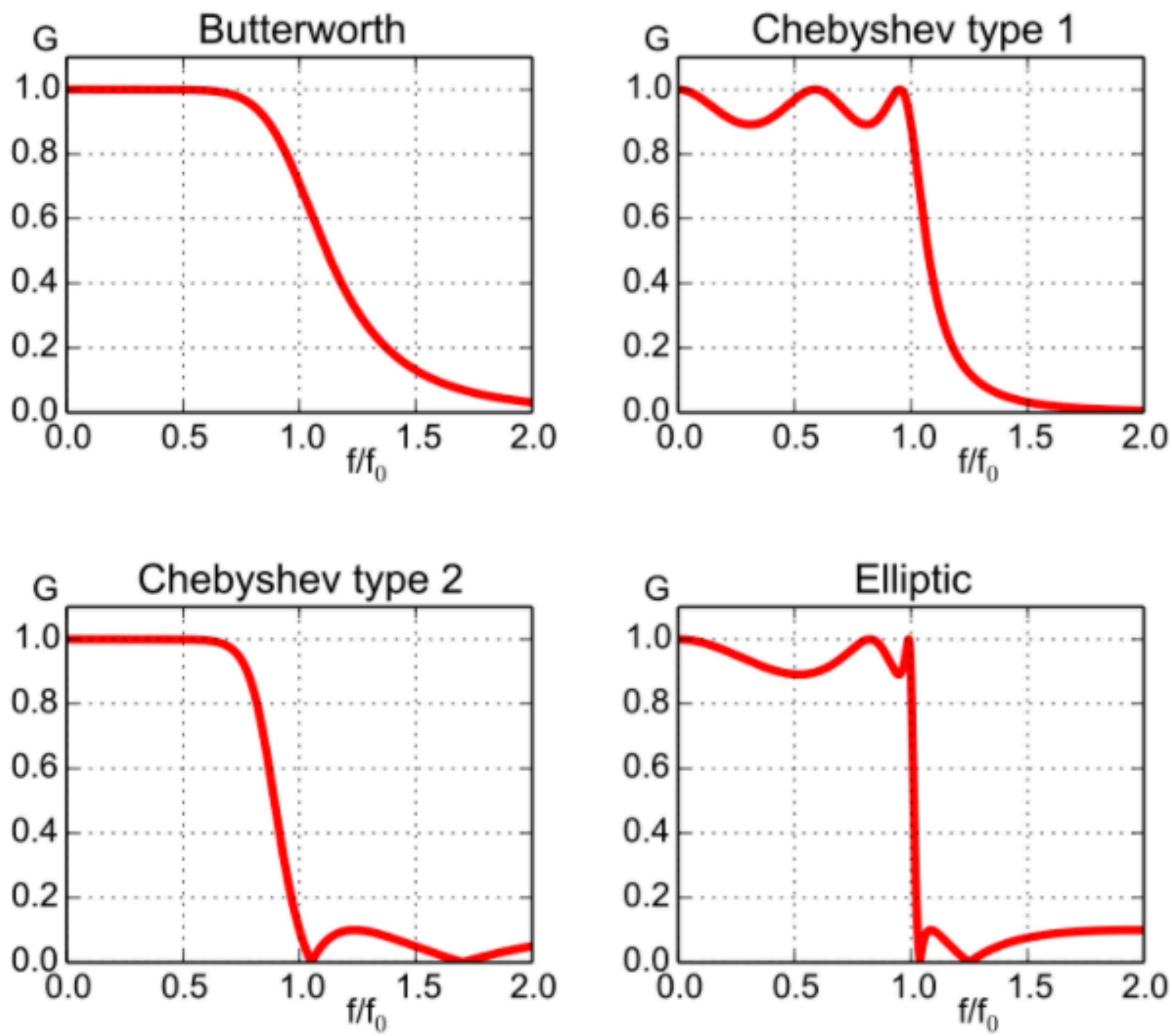


(b)

Comparisons



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- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
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FIR Design by Windowing

- Given desired frequency response, $H_d(e^{j\omega})$, find an impulse response

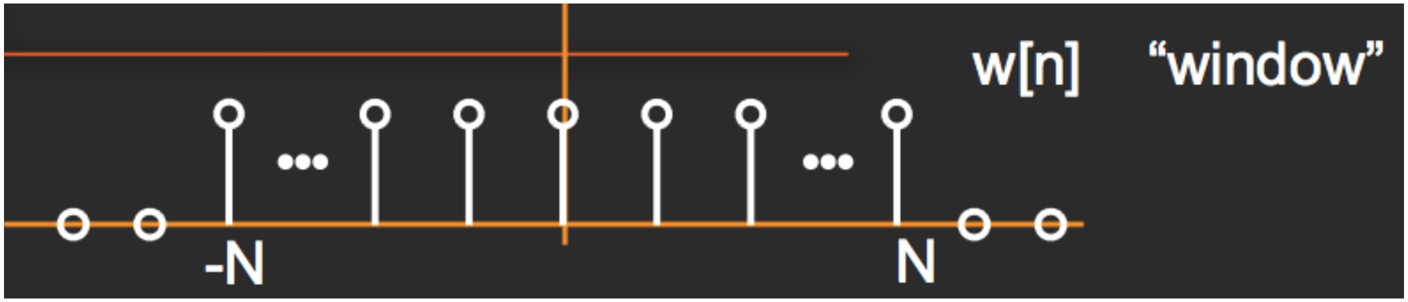
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

ideal

- Obtain the M^{th} order causal FIR filter by truncating/windowing it

$$h[n] = \left\{ \begin{array}{ll} h_d[n]w[n] & 0 \le n \le M \\ 0 & \text{otherwise} \end{array} \right\}$$

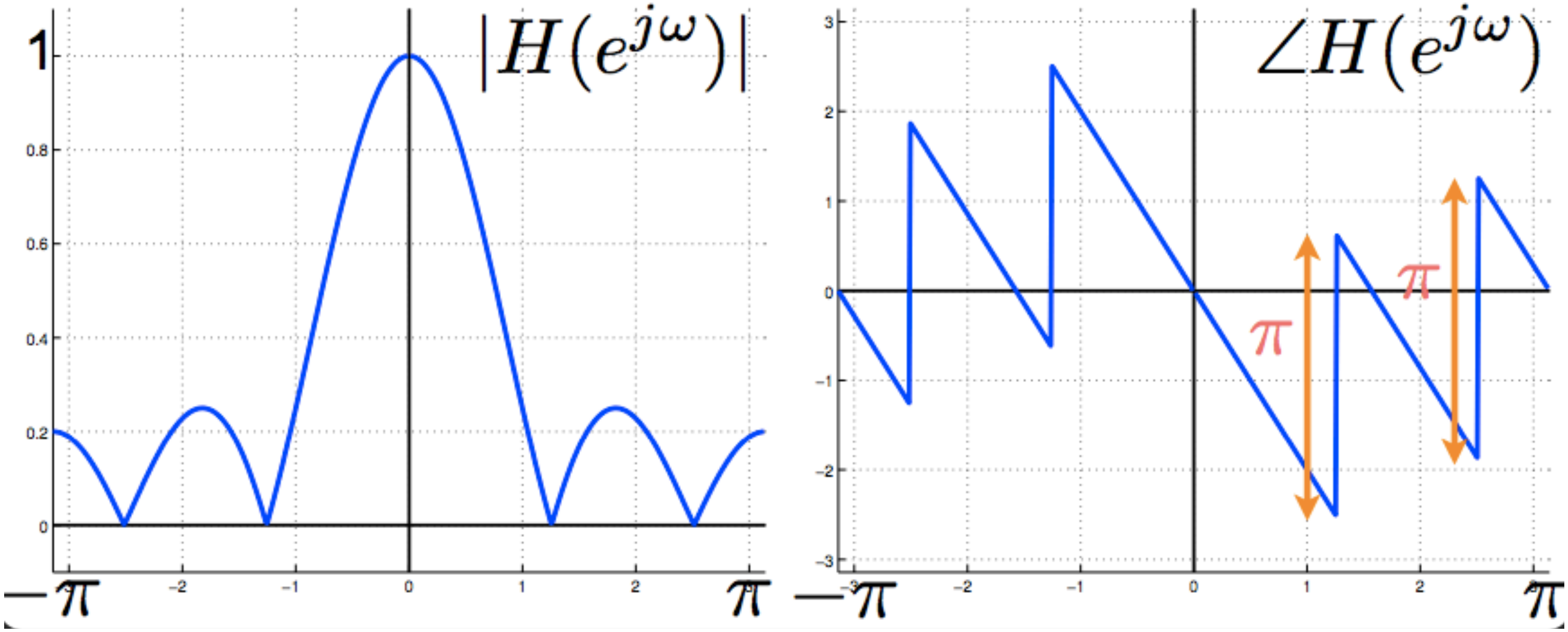
Example: Moving Average



$$w[n] \leftrightarrow W(e^{j\omega}) = \frac{\sin\left((N+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

$$\frac{1}{M+1} w[n-M/2] \leftrightarrow W(e^{j\omega}) = \frac{e^{-j\omega M/2}}{M+1} \frac{\sin\left((M/2+1/2)\omega\right)}{\sin\left(\omega/2\right)}$$

Example: Moving Average



FIR Design by Windowing

- With multiplication in time property,

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

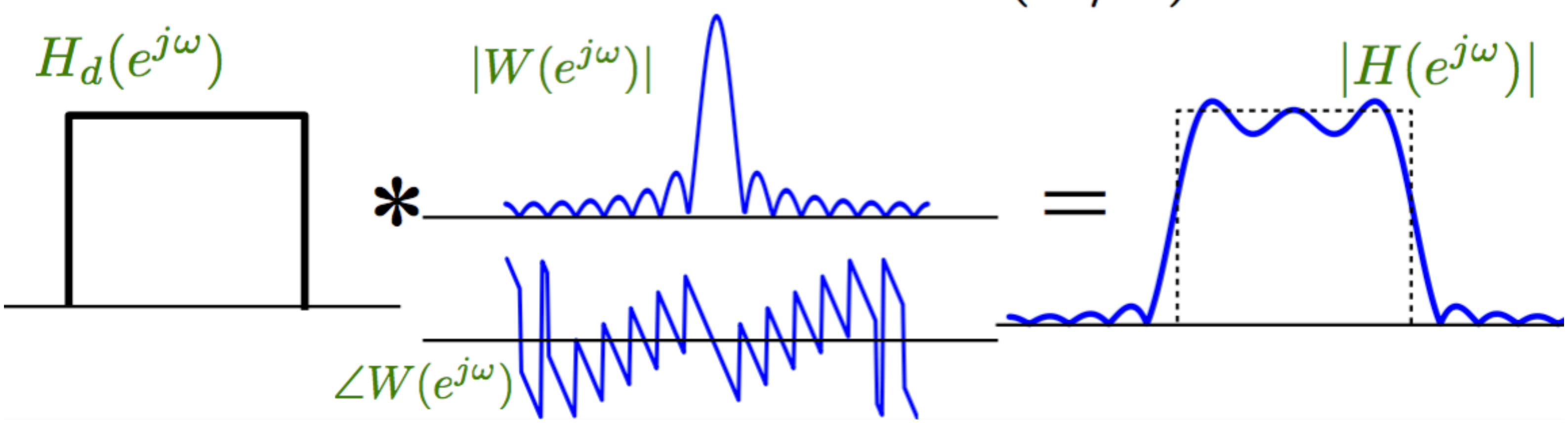
FIR Design by Windowing

- With multiplication in time property,

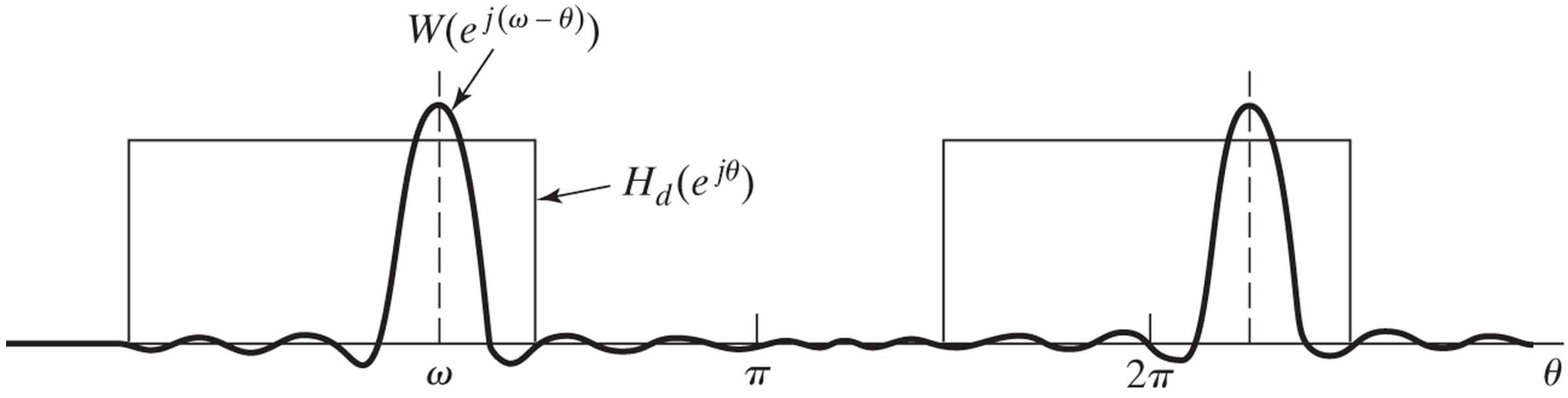
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

$$W(e^{j\omega}) = e^{-j\omega\frac{M}{2}} \frac{\sin(w(M+1)/2)}{\sin(w/2)}$$



FIR Design by Windowing



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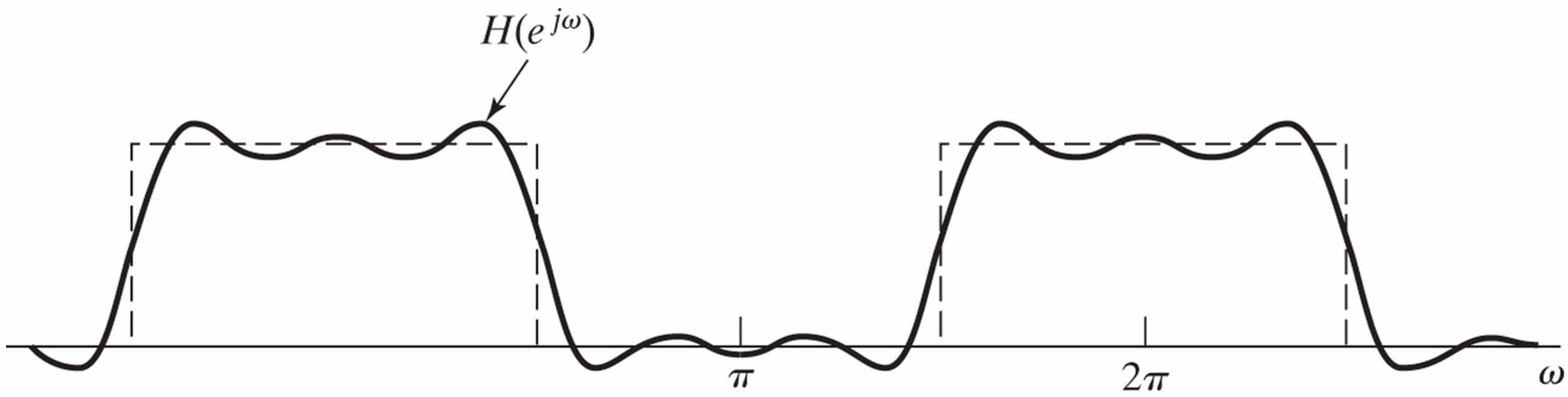
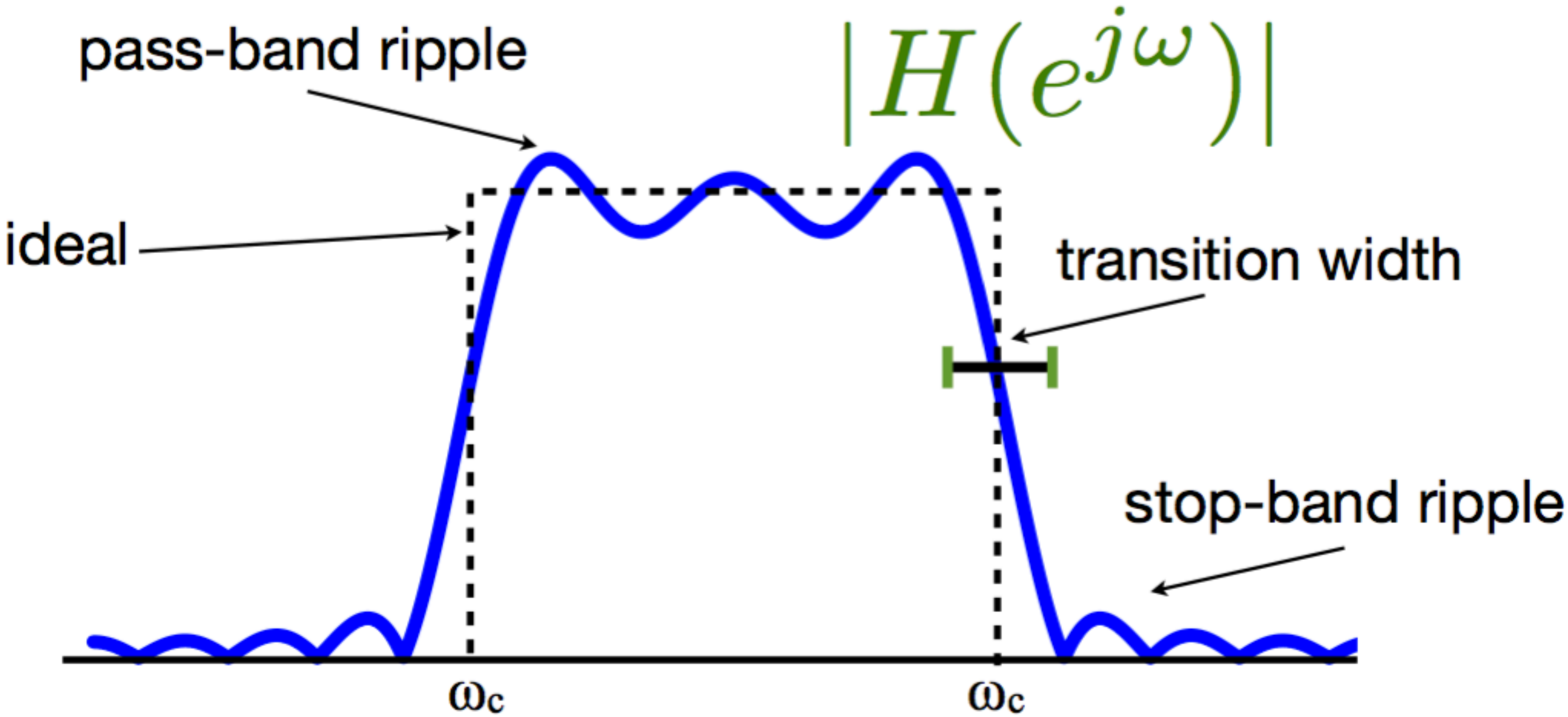
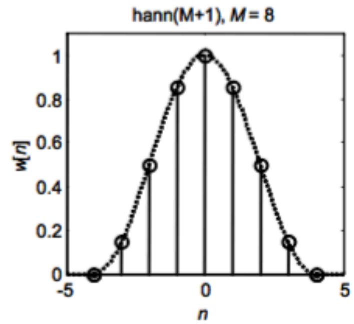
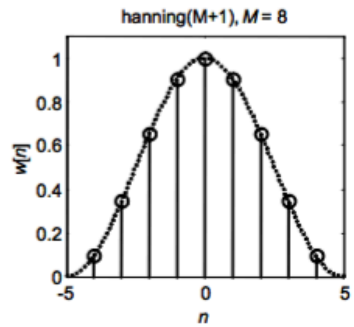
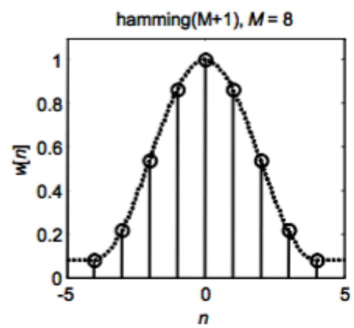


(b)

FIR Design by Windowing



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1+\cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54+0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n \rvert \le M/2 \\ 0 & \lvert n \rvert > M/2 \end{cases}$	`hamming(M+1)`	

Commonly Used Windows

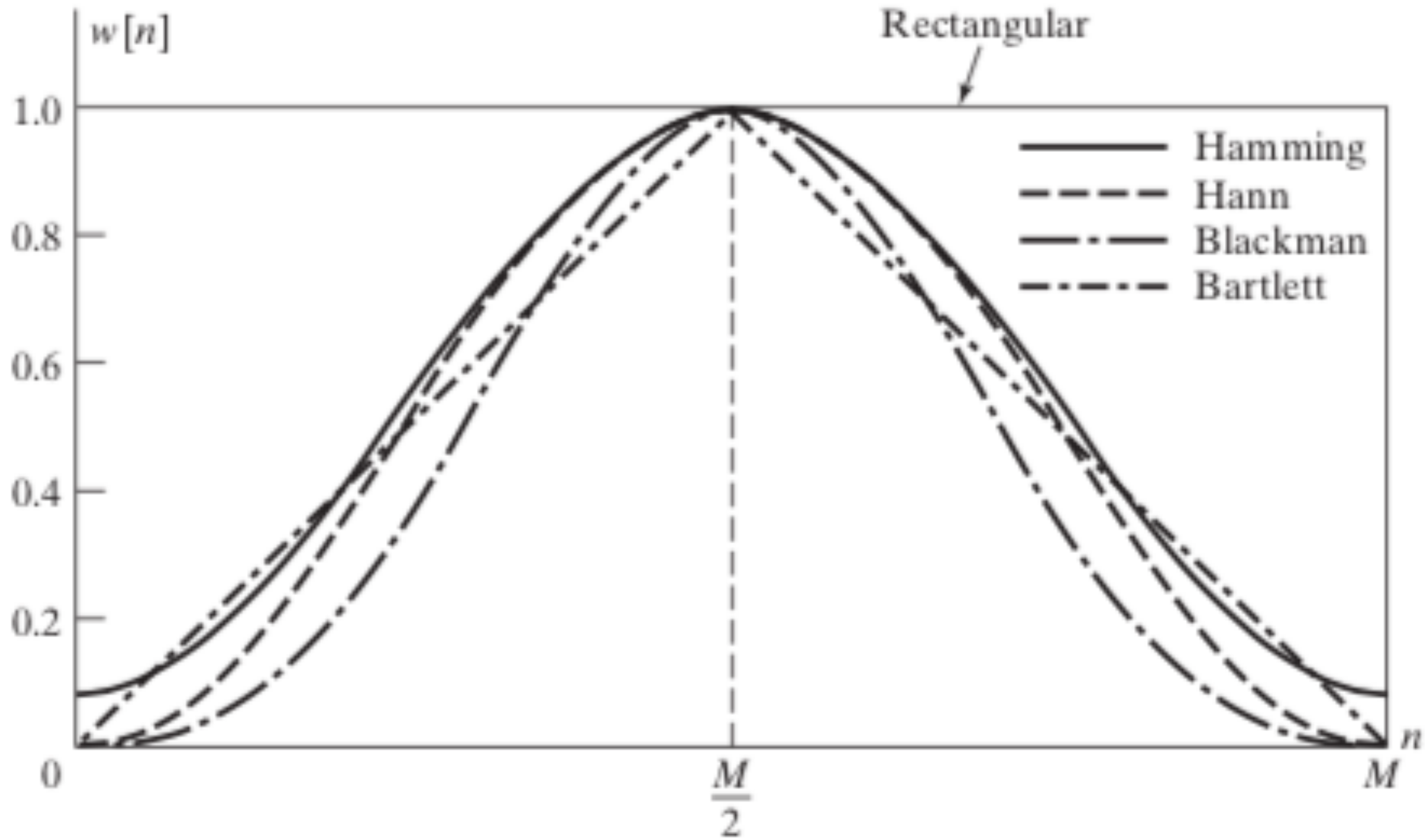
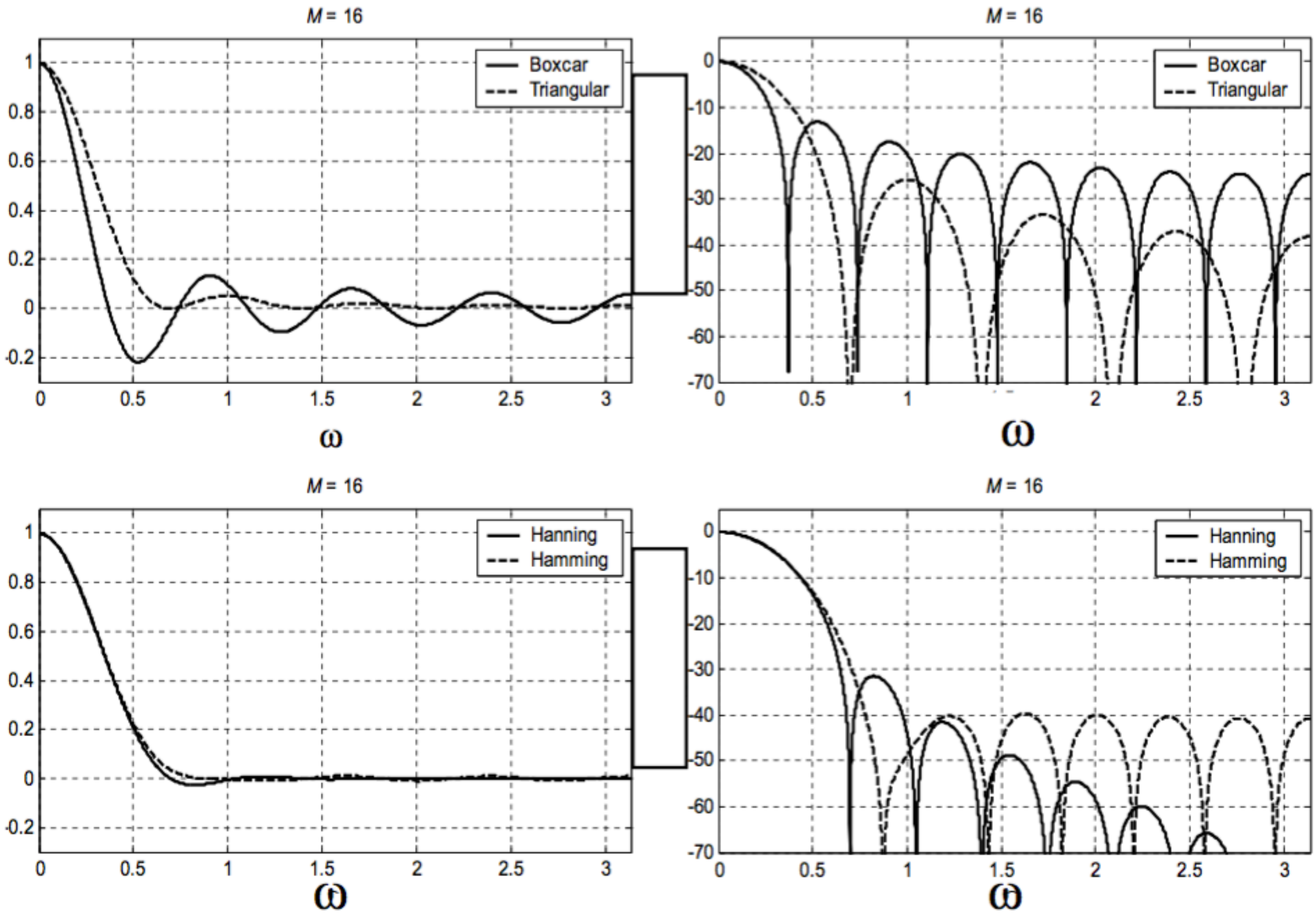
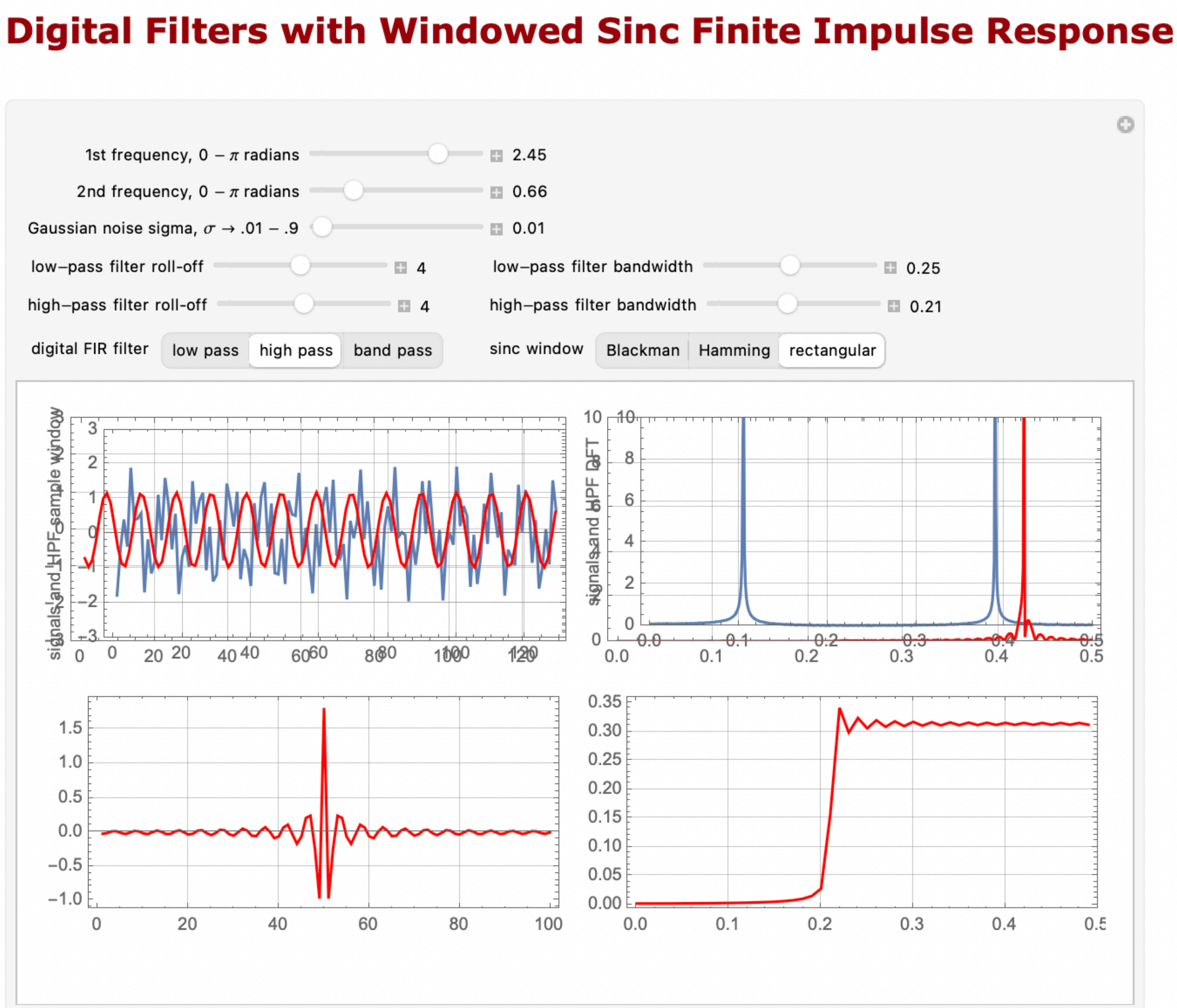


Figure 7.29 Commonly used windows.

Tradeoff – Ripple vs. Transition Width



Filter Design Demo

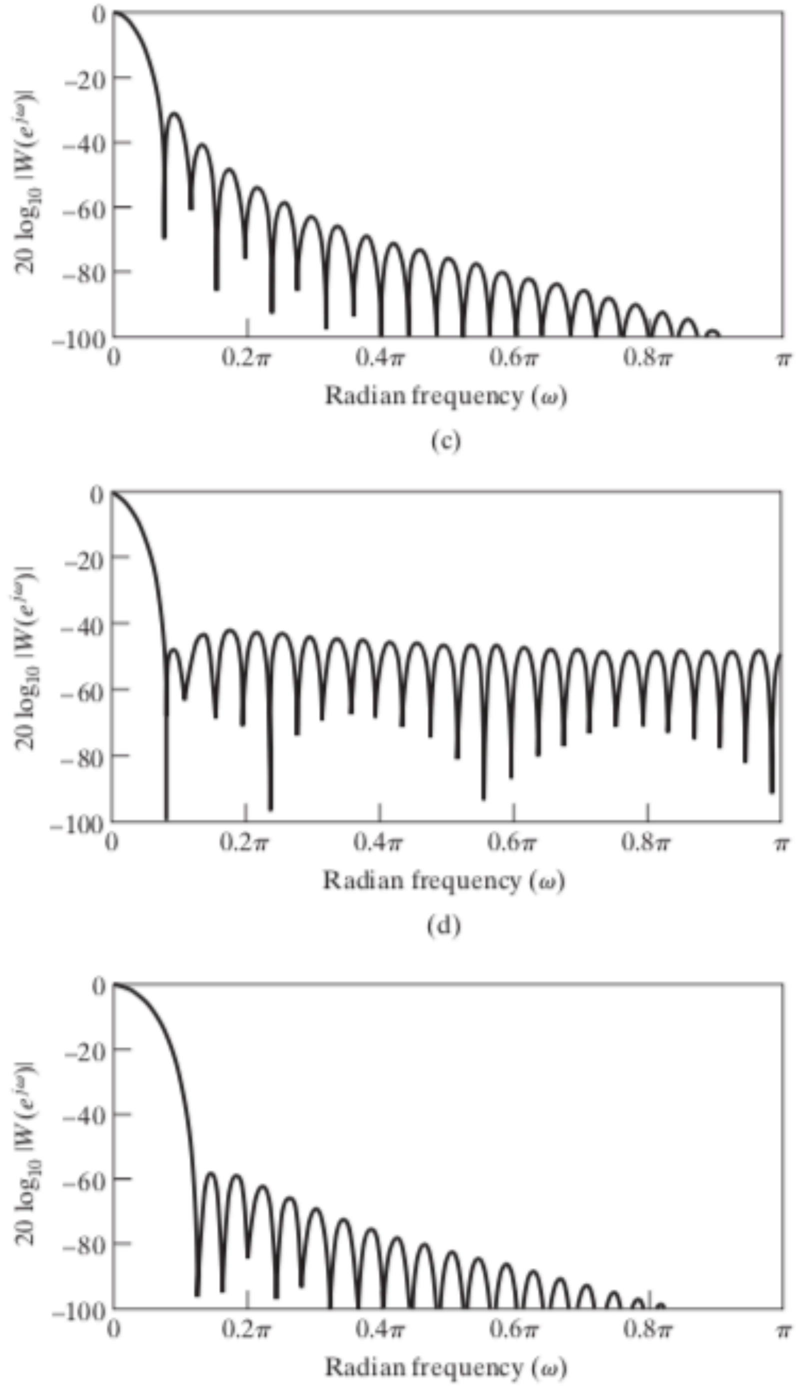


https://demonstrations.wolfram.com/DigitalFiltersWithWindowedSincFiniteImpulseResponse/

Hann

Hamming

Blackman



(c)
(d)
(e)

Ended here

Kaiser Window

- Near optimal window quantified as the window maximally concentrated around $\omega=0$

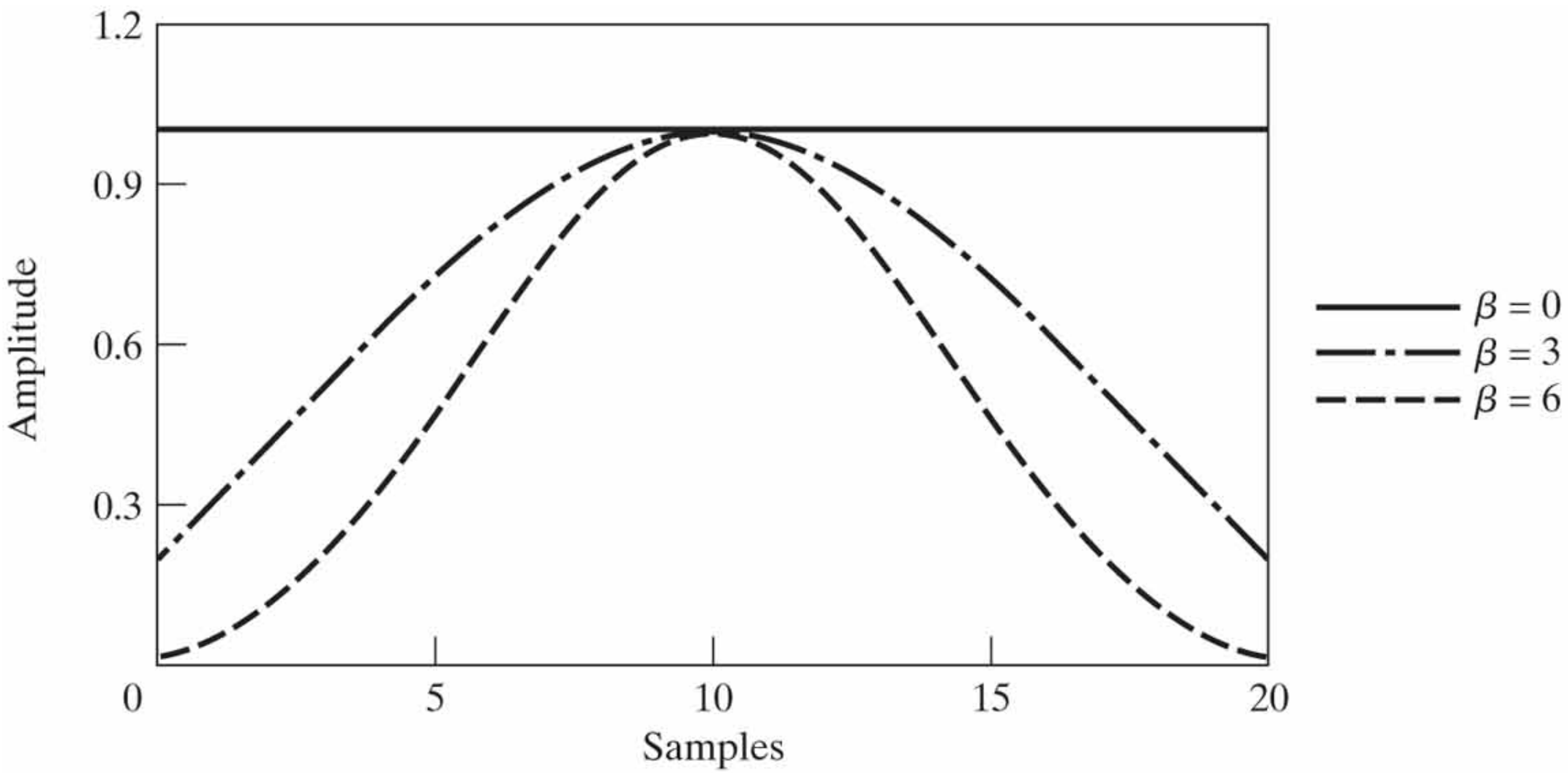
$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

- Two parameters – M and β

- $\alpha=M/2$
- $I_0(x)$ – zeroth order Bessel function of the first kind

Kaiser Window

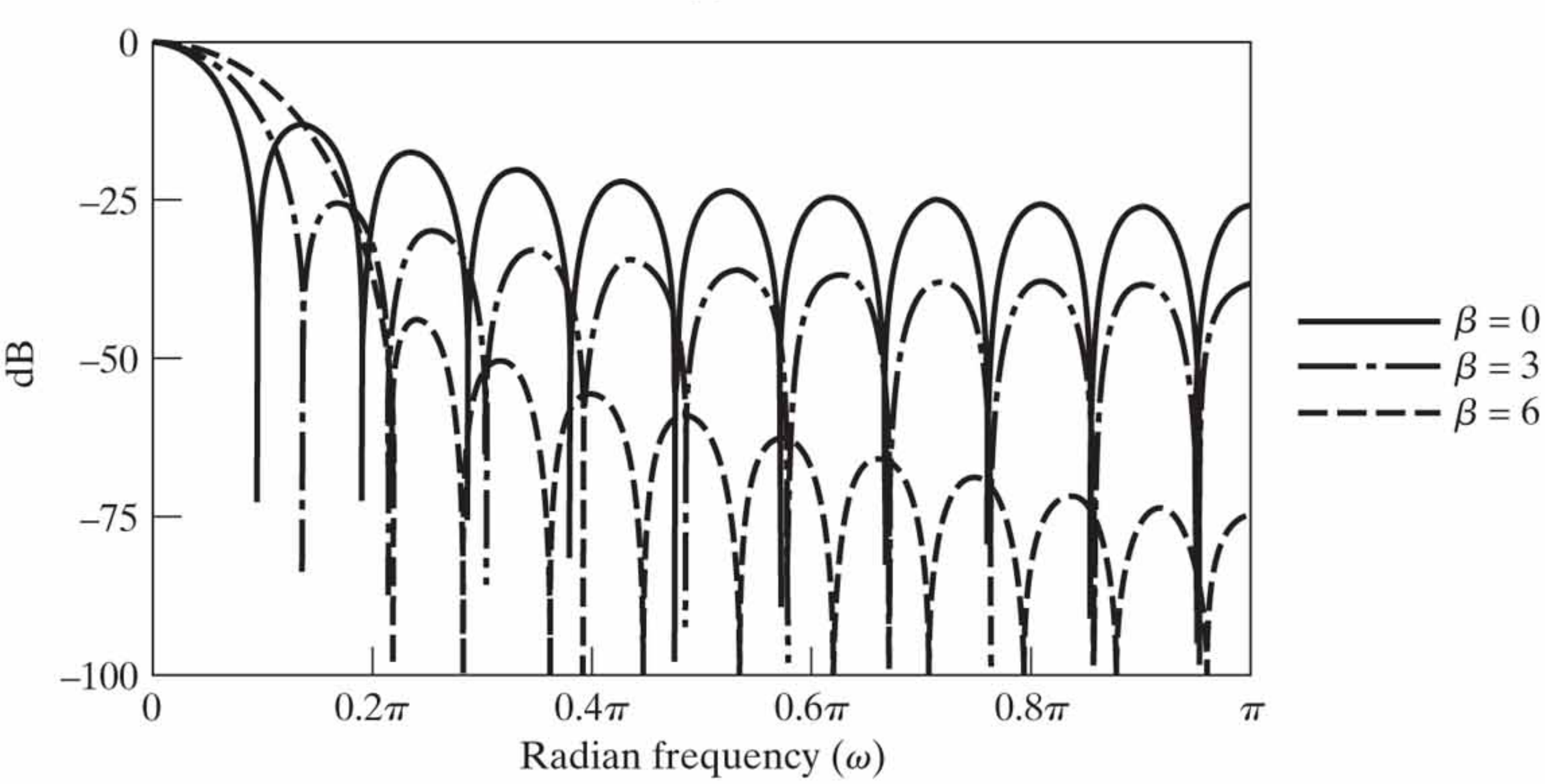
- M=20



(a)

Kaiser Window

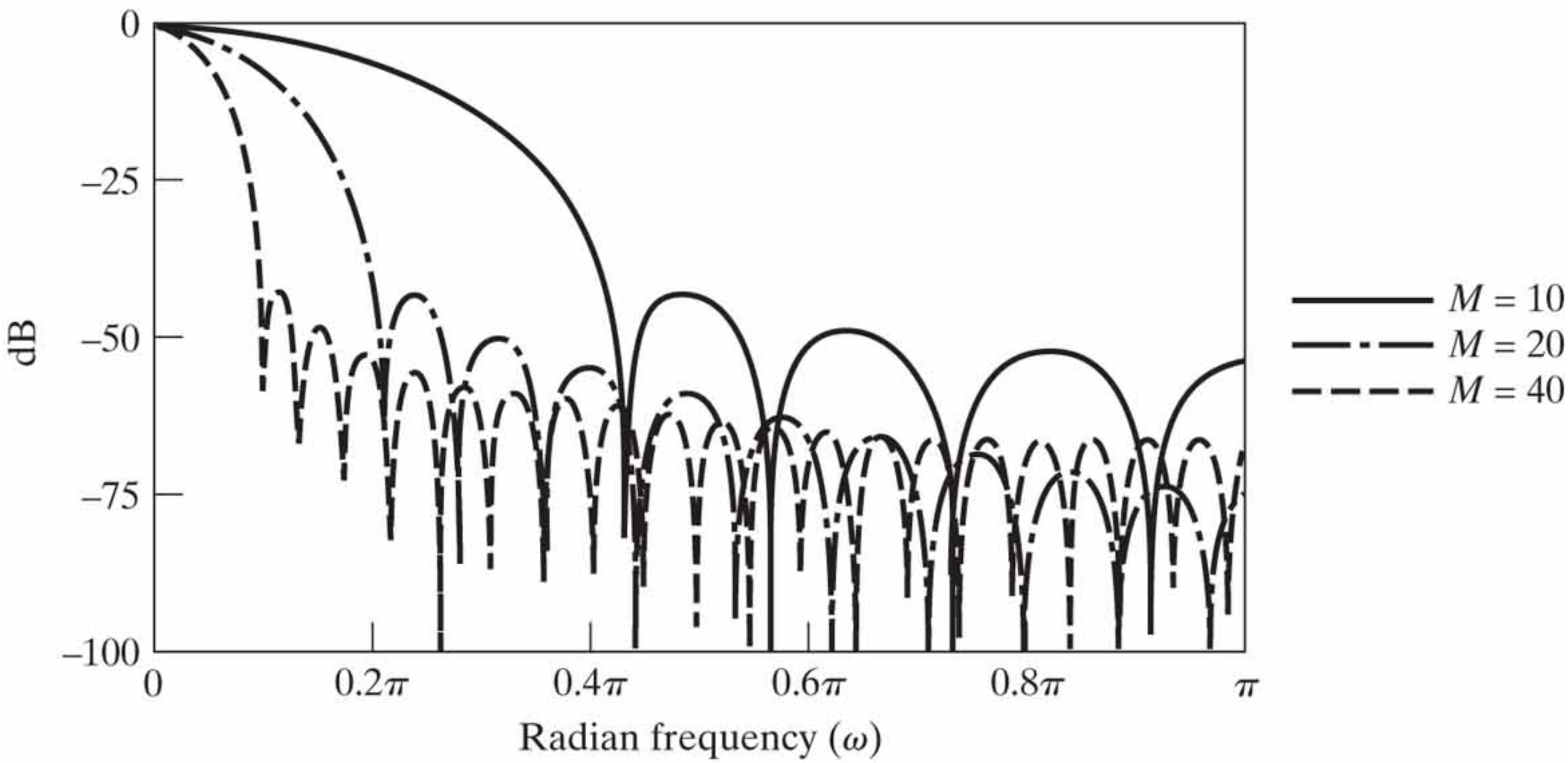
- M=20



(b)

Kaiser Window

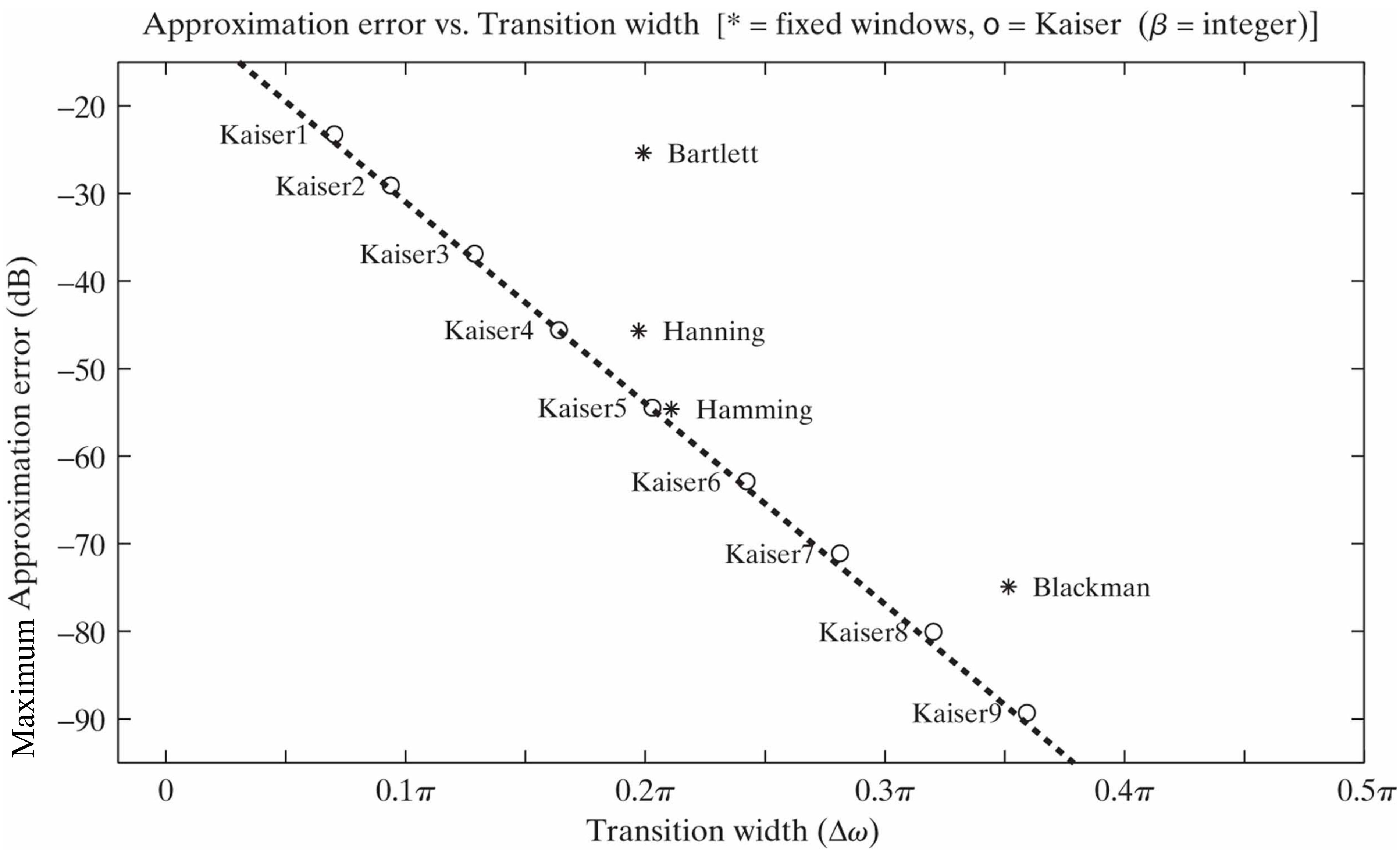
- β=6



(c)

Approximation Error

LPF, M=32, ω_c=0.5π



FIR Filter Design

- Choose a desired frequency response $H_d(e^{j\omega})$
  - non causal (zero-delay), and infinite imp. response
  - If derived from C.T, choose T and use:

$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple

FIR Filter Design

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- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple
  - Modulate to shift impulse response
    - Why?

$$H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

FIR Filter Design

- Determine truncated impulse response $h_1[n]$

$$h_1[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}e^{j\omega n} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Apply window

$$h_w[n] = w[n]h_1[n]$$

- Check:
  - Compute $H_w(e^{j\omega})$, if does not meet specs increase M or change window

Example: FIR Low-Pass Filter Design

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Choose M ⇒ Window length and set

$$H_1(e^{j\omega}) = H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

$$h_1[n] = \begin{cases} \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)} & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}(n - M/2))$$

Example: FIR Low-Pass Filter Design

- The result is a windowed sinc function

$$h_w[n] = w[n]h_1[n]$$

$$\frac{\omega_c}{\pi}\text{sinc}(\frac{\omega_c}{\pi}(n - M/2))$$

- High Pass Design:
  - Design low pass
  - Transform to $h_w[n](-1)^n$

- General bandpass
  - Transform to $2h_w[n]\cos(\omega_0 n)$ or $2h_w[n]\sin(\omega_0 n)$

Big Idea

- IIR
  - Design from continuous time filters with mapping of s-plane onto z-plane
    - Linear mapping – impulse invariance
    - Non-linear mapping – bilinear transformation
- FIR
  - Use desired frequency response to generate desired impulse respons
  - Use filter order and window type to meet specs

Admin

- Proj 1 due Tuesday 4/2