ESE 5310: Digital Signal Processing

Lecture 16: March 28, 2024

FIR Filters (con't), Filter Transformations

Linear Filter Design

- Used to be an ambiguous process
  - Now, lots of tools to design optimal filters
- For DSP there are two common classes
  - Infinite impulse response IIR
  - Finite impulse response FIR
- Both classes use finite order of parameters for design
  - Filter order (ie. Length) restricts filter design

What is a Linear Filter?

- Attenuates certain frequencies
- Passes certain frequencies
- Affects both phase and magnitude

- IIR
  - Mostly non-linear phase response
  - Could be linear over a range of frequencies
- FIR
  - Much easier to control the phase
  - Both non-linear and linear phase

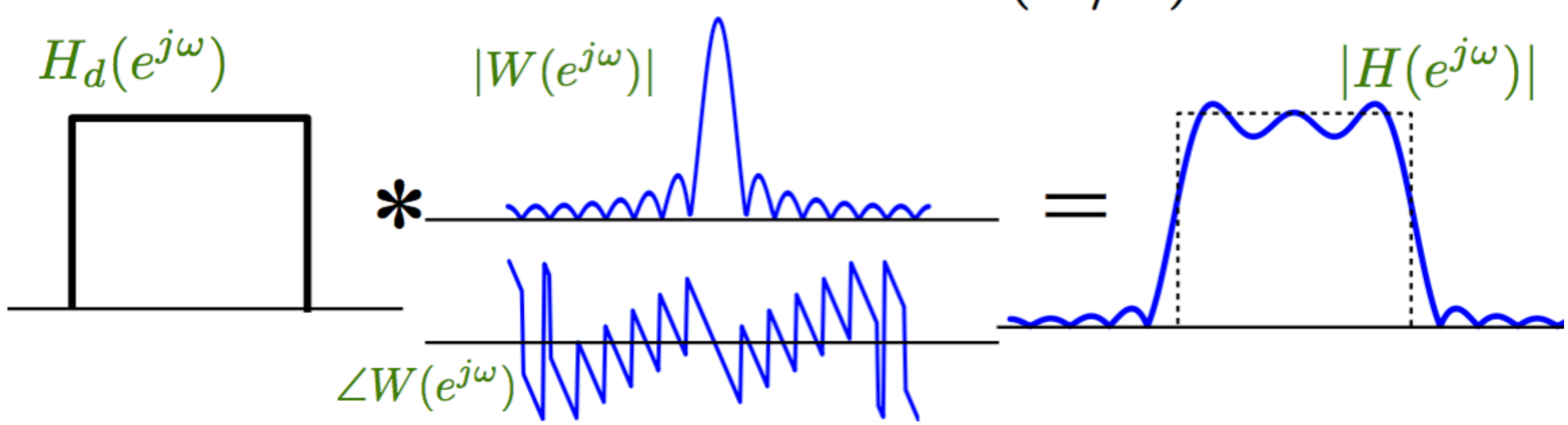
FIR Design by Windowing

- With multiplication in time property,

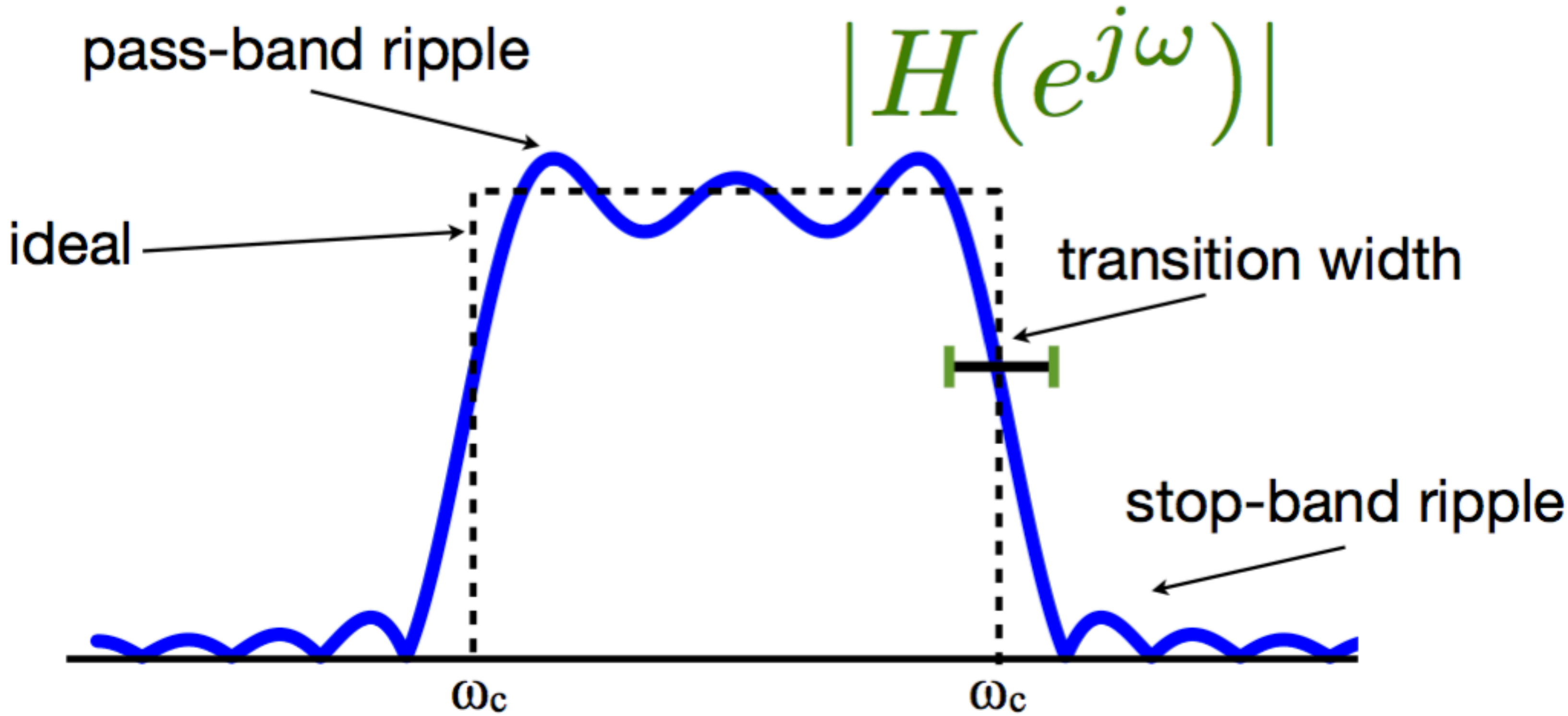
$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

- For Boxcar (rectangular) window

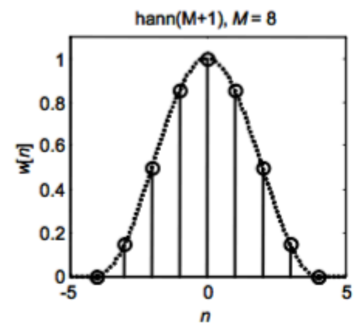
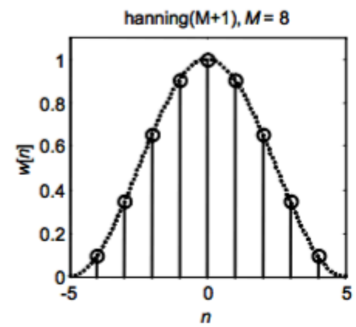
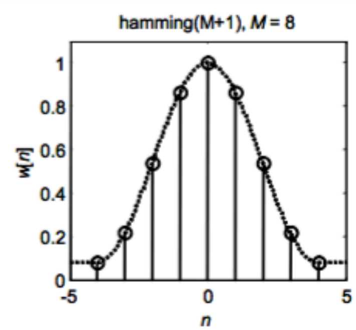
$$W(e^{j\omega}) = e^{-j\omega\frac{M}{2}} \frac{\sin(w(M+1)/2)}{\sin(w/2)}$$



FIR Design by Windowing



Tapered Windows

Name(s)	Definition	MATLAB Command	Graph ($M = 8$)
Hann	$w[n] = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{\pi n}{M/2}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hann(M+1)`	
Hanning	$w[n] = \begin{cases} \frac{1}{2}\left[1 + \cos\left(\frac{\pi n}{M/2+1}\right)\right] & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hanning(M+1)`	
Hamming	$w[n] = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi n}{M/2}\right) & \lvert n\rvert \le M/2 \\ 0 & \lvert n\rvert > M/2 \end{cases}$	`hamming(M+1)`	

Commonly Used Windows

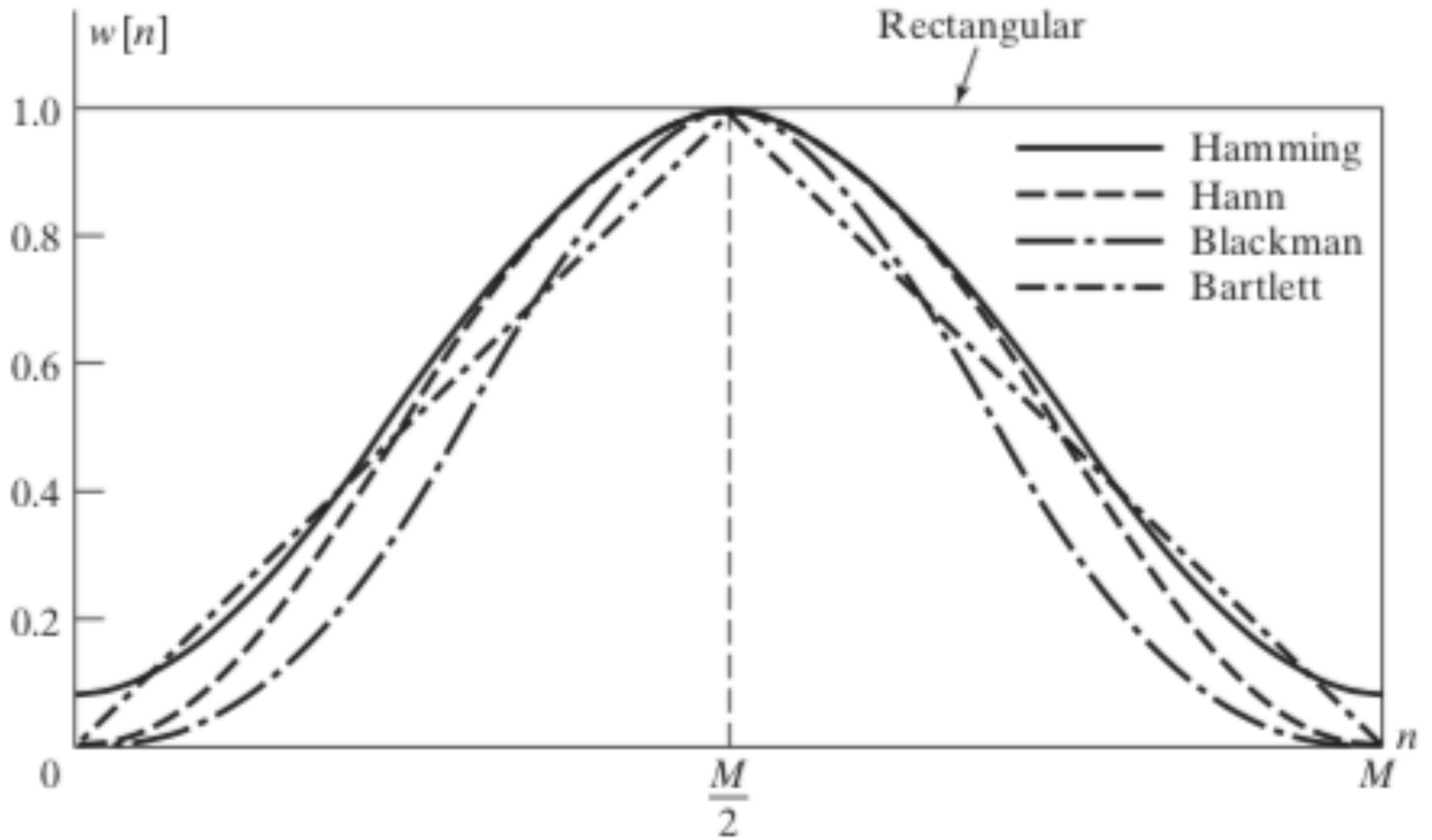
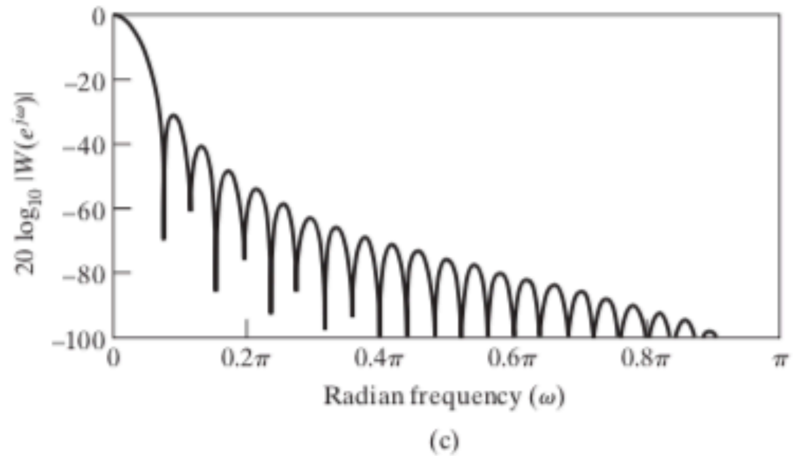
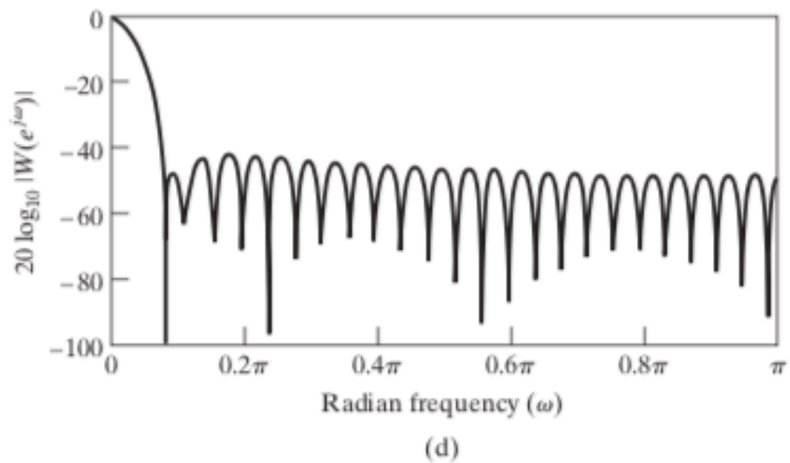


Figure 7.29 Commonly used windows.

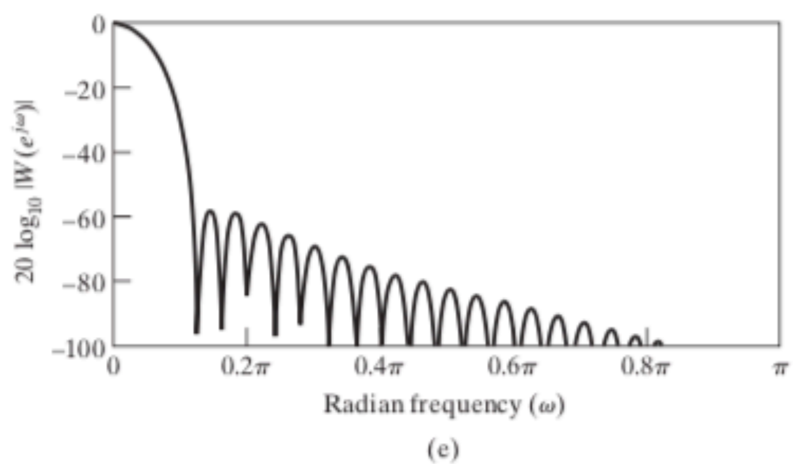
Hann



Hamming



Blackman



Kaiser Window

- Near optimal window quantified as the window maximally concentrated around ω=0

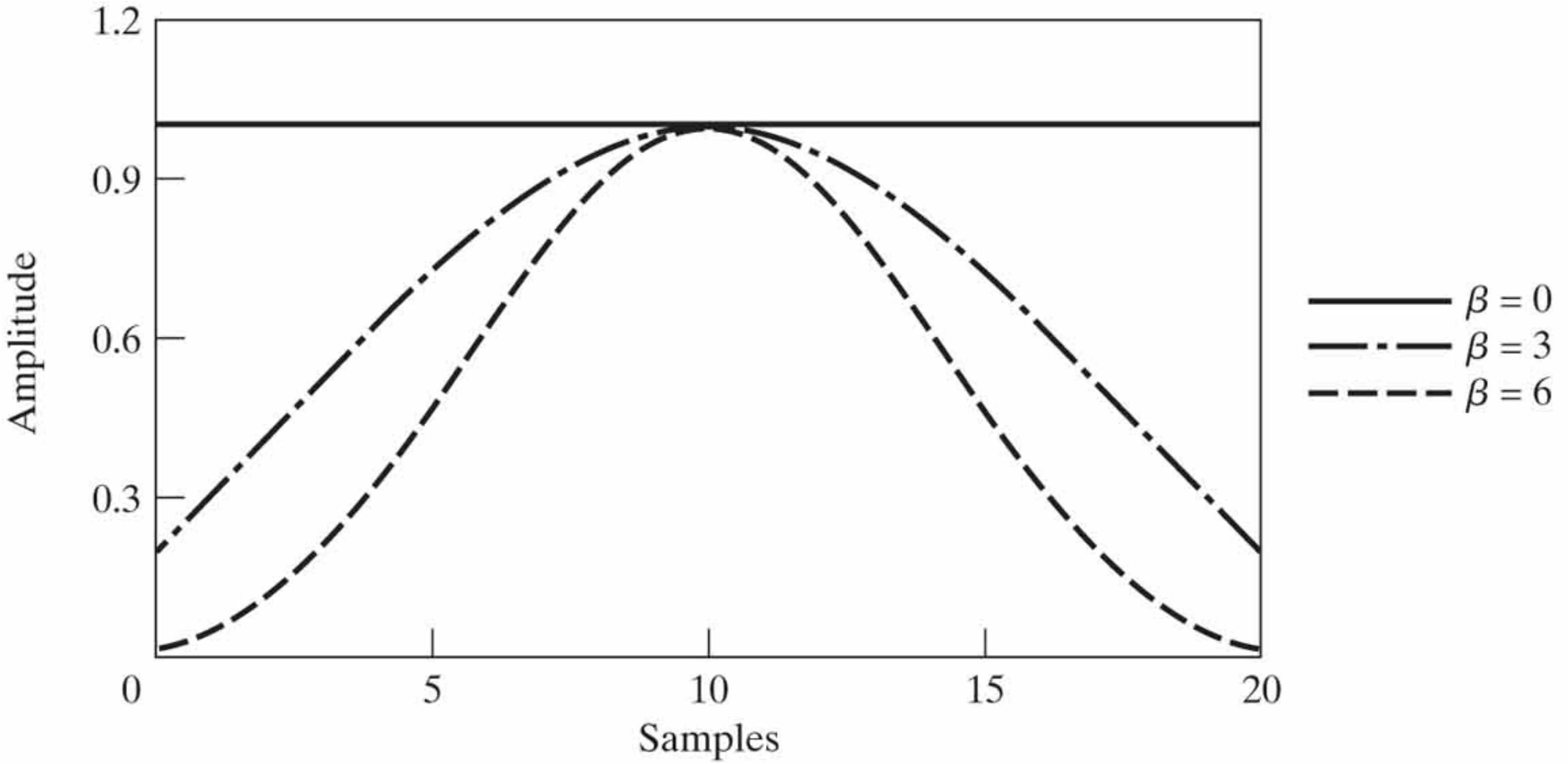
$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M, \\ 0, & \text{otherwise,} \end{cases}$$

- Two parameters – M and β

- $\alpha = M/2$
- $I_0(x)$ – zero[th] order Bessel function of the first kind

Kaiser Window

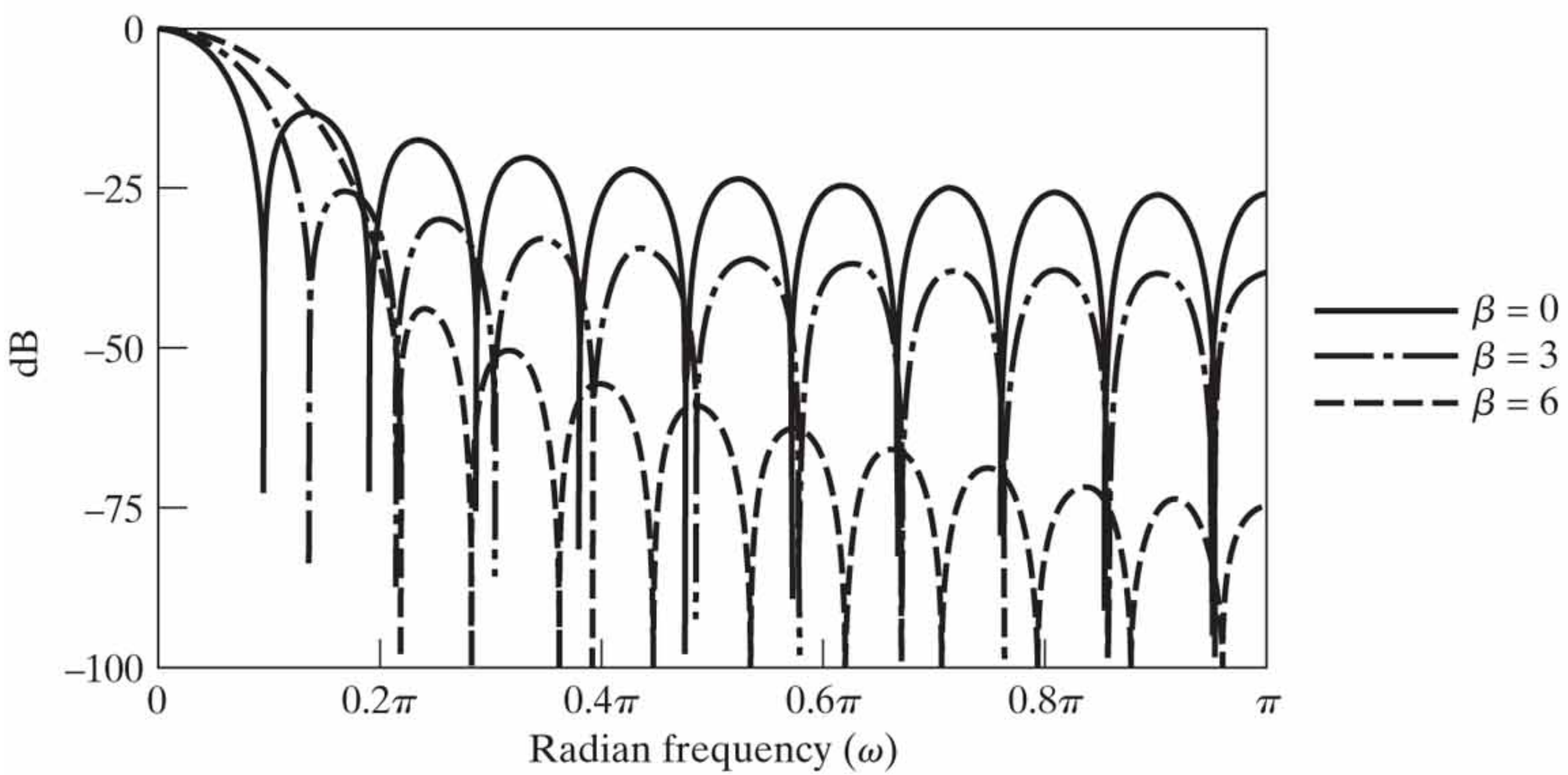
- M=20



(a)

Kaiser Window

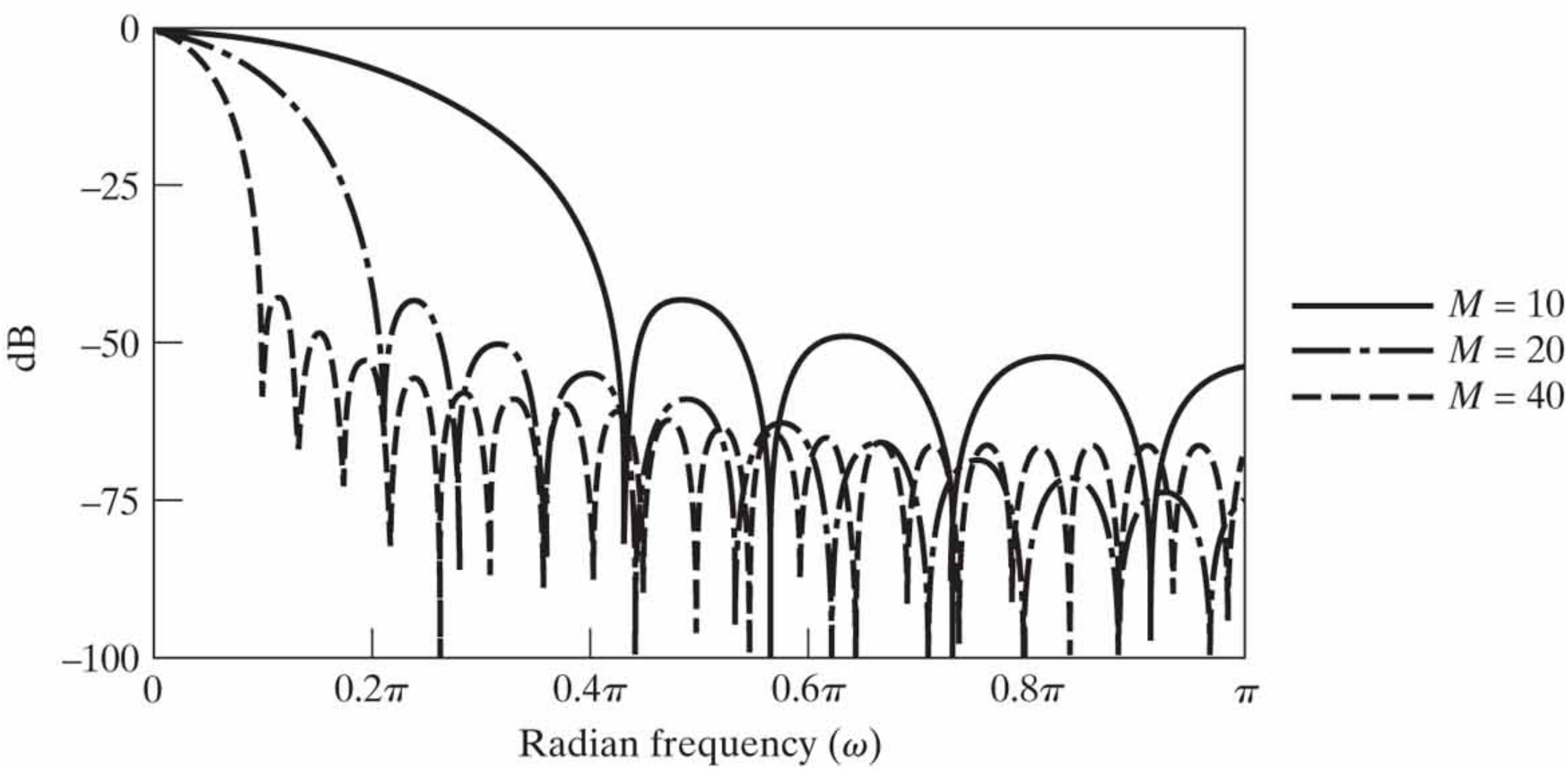
- M=20



(b)

Kaiser Window

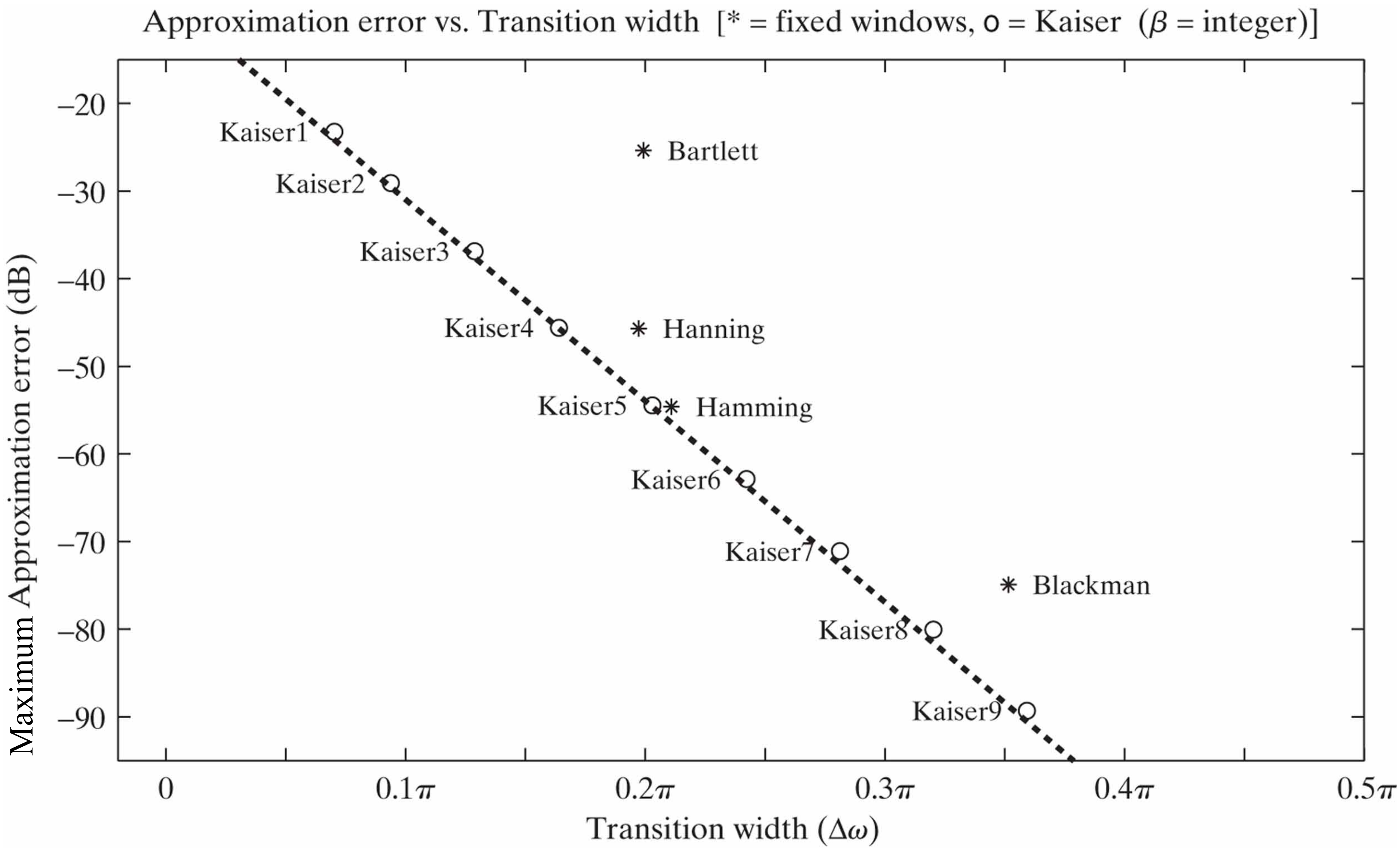
- β=6



(c)

Approximation Error

LPF, M=32, ω_c=0.5π



FIR Filter Design Process

- Choose a desired frequency response $H_d(e^{j\omega})$
  - non causal (zero-delay), and infinite imp. response
  - If derived from C.T, choose T and use:

$$H_d(e^{j\omega}) = H_c(j\frac{\Omega}{T})$$

- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple

FIR Filter Design Process

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- Window:
  - Length M+1 ⇔ affects transition width
  - Type of window ⇔ transition-width/ ripple
  - Modulate to shift impulse response
    - Why?

$$H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

FIR Filter Design Process

- Determine truncated impulse response $h_1[n]$

$$h_1[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}e^{j\omega n} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Apply window

$$h_w[n] = w[n]h_1[n]$$

- Check:
  - Compute $H_w(e^{j\omega})$, if does not meet specs increase M or change window

Example: FIR Low-Pass Filter Design

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Choose M ⇒ Window length and set

$$H_1(e^{j\omega}) = H_d(e^{j\omega})e^{-j\omega\frac{M}{2}}$$

$$h_1[n] = \begin{cases} \frac{\sin(\omega_c(n-M/2))}{\pi(n-M/2)} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c}{\pi}(n - M/2)\right)$$

Example: FIR Low-Pass Filter Design

- The result is a windowed sinc function

$$h_w[n] = w[n]h_1[n]$$

$$\frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c}{\pi}(n - M/2)\right)$$

- High Pass Design:
  - Design low pass
  - Transform to $h_w[n](-1)^n$

- General bandpass
  - Transform to $2h_w[n]\cos(\omega_0 n)$ or $2h_w[n]\sin(\omega_0 n)$

Design through FFT

- To design order M filter:
- Over-Sample/discretize the frequency response at P points where P >> M (P=15M is good)

$$H_1(e^{j\omega_k}) = H_d(e^{j\omega_k})e^{-j\omega_k \frac{M}{2}}$$

- Sampled at: $\omega_k = k\frac{2\pi}{P} \qquad |k = [0, \cdots, P-1]$
- Compute $h_1[n] = \text{IDFT}_P(H_1[k])$
- Apply M+1 length window:

$$h_w[n] = w[n]h_1[n]$$

Example

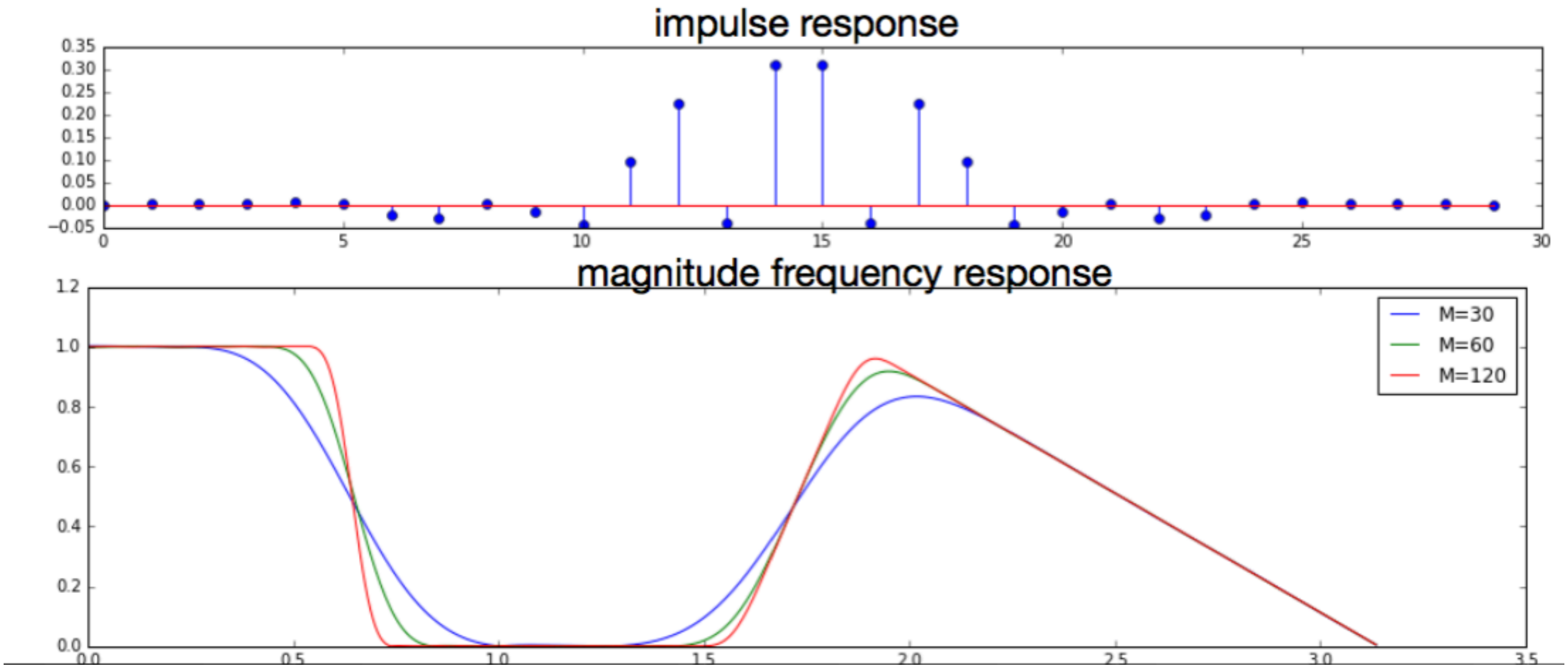
- signal.firwin2(M+1,omega_vec/pi, amp_vec)
- taps1 = signal.firwin2(30, [0.0,0.2,0.21,0.5, 0.6, 1.0], [1.0, 1.0, 0.0,0.0,1.0,0.0])

[0.0 0.2 0.21 0.5 0.6 1.0]

[1.0 1.0 0.0   0.0 1.0 0.0]

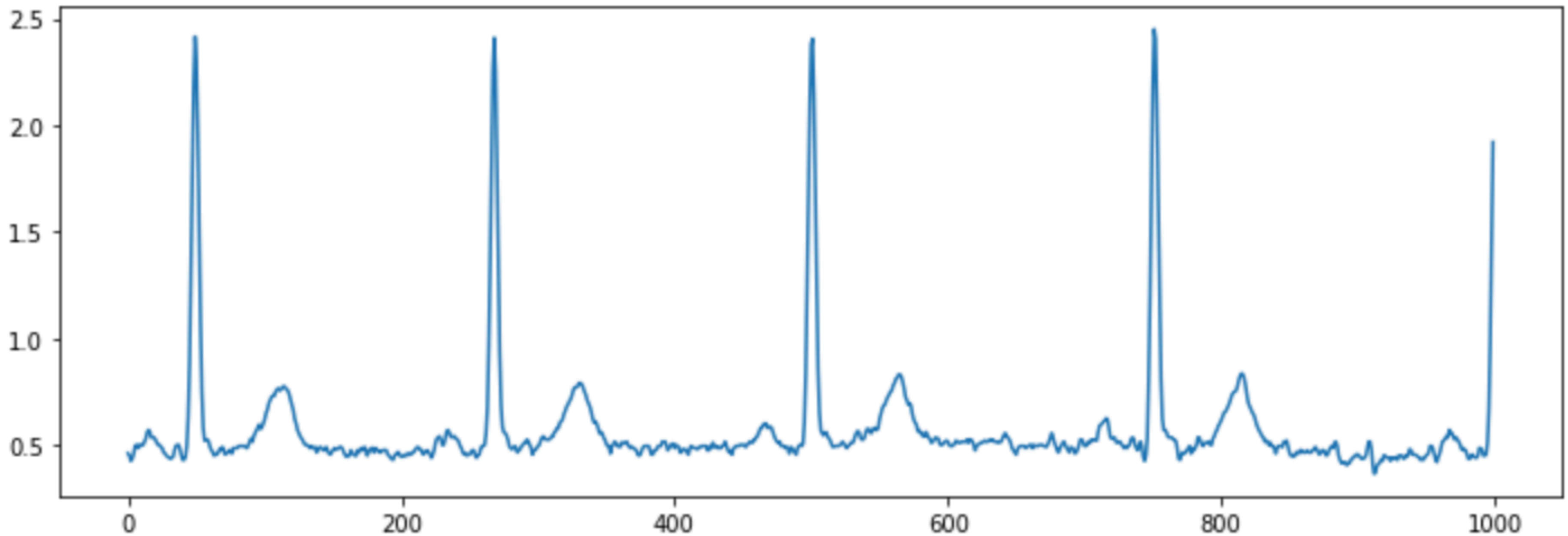
Example

- signal.firwin2(M+1,omega_vec/pi, amp_vec)
- taps1 = signal.firwin2(30, [0.0,0.2,0.21,0.5, 0.6, 1.0], [1.0, 1.0, 0.0,0.0,1.0,0.0])

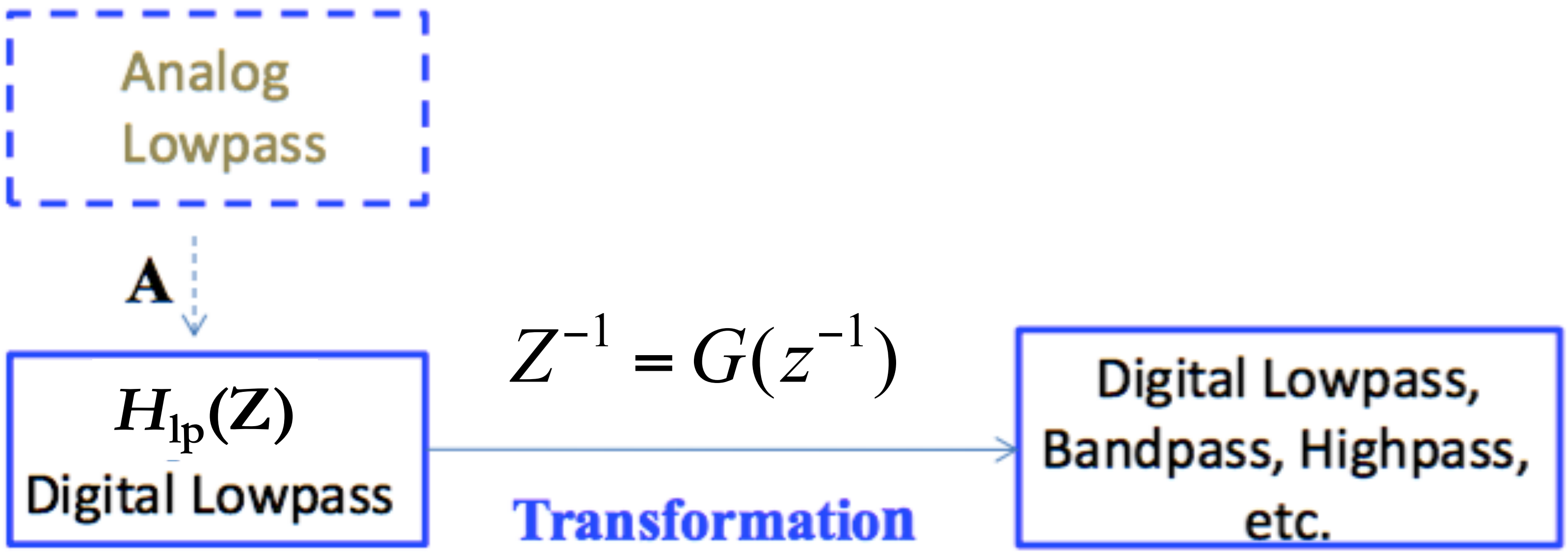


Python Filter/Re-sample Example

- Heartrate detection of ECG signal



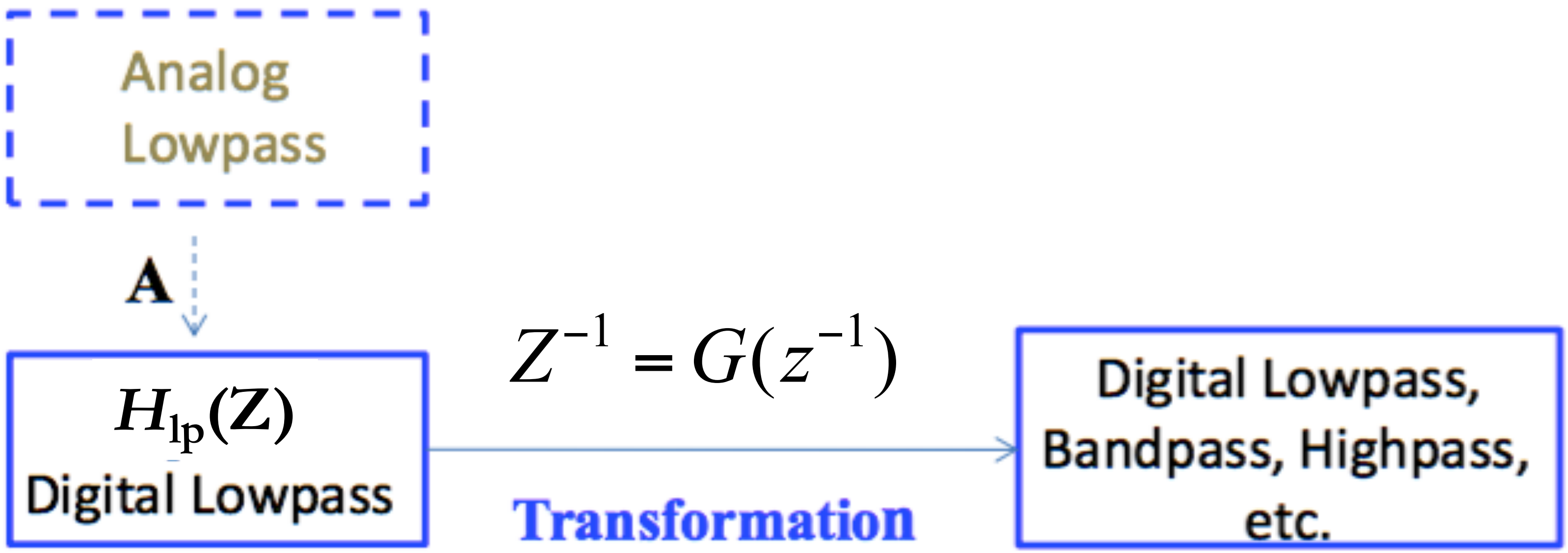
Transformation of DT Filters



- Z – complex variable for the LP filter
- z – complex variable for the transformed filter

- Map Z-plane→z-plane with transformation G

Transformation of DT Filters



- Map Z-plane→z-plane with transformation G

$$H(z) = H_{lp}(Z)\big|_{Z^{-1}=G(z^{-1})}$$

Example 1:

- Lowpass→highpass
  - Shift frequency by π

so $\omega \rightarrow \omega - \pi$ (Lowpass to highpass)

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$G(z^{-1}) = -z^{-1}$ or $e^{-j\omega} \rightarrow e^{-j(\omega - \pi)}$

Example 1:

- Lowpass→highpass
  - Shift frequency by π

$$G(z^{-1}) = -z^{-1}$$

ω	z	$\lvert H_{lp}(z)\rvert = \left\lvert\frac{0.1}{1-0.9z^{-1}}\right\rvert$	$\lvert H_{hp}(z)\rvert = \left\lvert\frac{0.1}{1+0.9z^{-1}}\right\rvert$
0			
$\frac{\pi}{2}$			
π			
$\frac{3\pi}{2}$			
2π			

Example 2:

- Lowpass→bandpass

$$G(z^{-1}) = -z^{-2}$$

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- Lowpass→bandpass

$$G(z^{-1}) = -z^{-2}$$

$$H_{lp}(z) = \frac{1}{1 - az^{-1}} \longrightarrow H_{bp}(z) = \frac{1}{1 + az^{-2}}$$

Pole at z=a

Pole at z=±j√a

Example 2:

- Lowpass→bandpass

$$G(z^{-1}) = -z^{-2}$$

$\boldsymbol{\omega}$	**z**	$\lvert H_{lp}(z)\rvert = \left\lvert\frac{0.1}{1-0.9z^{-1}}\right\rvert$	$\lvert H_{\mathrm{bp}}(z)\rvert = \left\lvert\frac{0.1}{1+0.9z^{-2}}\right\rvert$
0	1	1	
$\frac{\pi}{2}$	j	0.074	
π	-1	0.05	
$\frac{3\pi}{2}$	-j	0.074	
2π	1	1	

Example 3:

- Lowpass→bandstop

$$Z^{-1} = G(z^{-1}) = z^{-2}$$

Example 3:

- Lowpass→bandstop

$$Z^{-1} = G(z^{-1}) = z^{-2}$$

$$H_{lp}(z) = \frac{1}{1 - az^{-1}} \longrightarrow H_{bs}(z) = \frac{1}{1 - az^{-2}}$$

Pole at $z = \pm\sqrt{a}$

Example 2:

- Lowpass→bandstop

$$G(z^{-1}) = z^{-2}$$

ω	z	$\lvert H_{lp}(z)\rvert = \left\lvert\frac{0.1}{1-0.9z^{-1}}\right\rvert$	$\lvert H_{bs}(z)\rvert = \left\lvert\frac{0.1}{1-0.9z^{-2}}\right\rvert$
0	1	1	
$\frac{\pi}{2}$	j	0.074	
π	-1	0.05	
$\frac{3\pi}{2}$	-j	0.074	
2π	1	1	

Transformation Constraints on $G(z^{-1})$

- If $H_{lp}(Z)$ is the rational system function of a causal and stable system, we naturally require that the transformed system function *H(z)* be a rational function and that the system also be causal and stable.
  - $G(Z^{-1})$ must be a rational function of z^{-1}
  - The inside of the unit circle of the Z-plane must map to the inside of the unit circle of the z-plane
  - The unit circle of the Z-plane must map onto the unit circle of the z-plane.

Transformation Constraints on $G(z^{-1})$

- Respective unit circles in both planes

$$Z = e^{j\theta} \text{ and } z = e^{j\omega}$$

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- Respective unit circles in both planes

$$Z = e^{j\theta} \text{ and } z = e^{j\omega}$$

$$Z^{-1} = G(z^{-1})$$

$$e^{-j\theta} = G(e^{-j\omega})$$

$$e^{-j\theta} = \left|G(e^{-j\omega})\right| e^{j\angle G(e^{-j\omega})}$$

Transformation Constraints on $G(z^{-1})$

- Respective unit circles in both planes

$$Z = e^{j\theta} \text{ and } z = e^{j\omega}$$

$$Z^{-1} = G(z^{-1})$$

$$e^{-j\theta} = G(e^{-j\omega})$$

$$e^{-j\theta} = \left|G(e^{-j\omega})\right| e^{j\angle G(e^{-j\omega})}$$

$$1 = \left|G(e^{-j\omega})\right| \qquad -\theta = \angle G(e^{-j\omega})$$

Transformation Constraints on $G(z^{-1})$

- General form that meets all constraints:
  - a_k real and $|a_k|<1$

$$G(z^{-1}) = \pm\prod_{k=1}^{N}\frac{z^{-1}-\alpha_k}{1-\alpha_k z^{-1}}$$

General Transformation

- Lowpass→lowpass

$$G(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

- Changes passband/stopband edge frequencies

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$$\text{From } e^{-j\theta} = \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}}, \text{ get}$$

$$\omega(\theta) = \tan^{-1}\left(\frac{(1-\alpha^2)\sin(\theta)}{2\alpha + (1+\alpha^2)\cos(\theta)}\right)$$

General Transformation

- Lowpass→lowpass

$$G(z^{-1}) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

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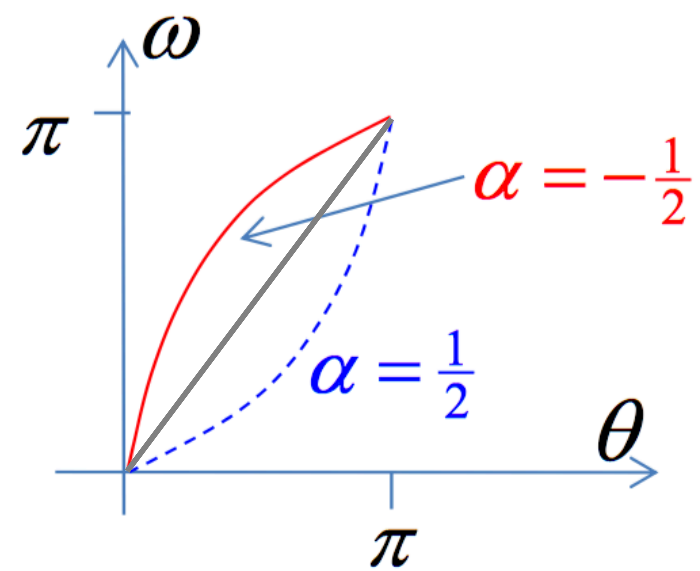

General Transformations

TABLE 7.1 TRANSFORMATIONS FROM A LOWPASS DIGITAL FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p TO HIGHPASS, BANDPASS, AND BANDSTOP FILTERS

Filter Type	Transformations	Associated Design Formulas
Lowpass	$Z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$Z^{-1} = -\dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\dfrac{\cos\left(\frac{\theta_p + \omega_p}{2}\right)}{\cos\left(\frac{\theta_p - \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$Z^{-1} = -\dfrac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \cot\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency
Bandstop	$Z^{-1} = \dfrac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-2} - \frac{2\alpha}{1+k} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\frac{\omega_{p2} + \omega_{p1}}{2}\right)}{\cos\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right)}$ $k = \tan\left(\dfrac{\omega_{p2} - \omega_{p1}}{2}\right)\tan\left(\dfrac{\theta_p}{2}\right)$ ω_{p1} = desired lower cutoff frequency ω_{p2} = desired upper cutoff frequency

Big Idea

- FIR
  - Use desired frequency response to generate desired impulse respons
  - Use filter order and window type to meet specs
- DT filter transformations
  - Transform z-plane with rational function $G(z^{-1})$
    - Constraints on G for causal/stable systems

Admin

- Proj 1 due Tuesday 4/2